1 Capital Taxation: Numerical Explorations

This question asks you to numerically explore the welfare implications of capital taxation in the Ramsey setup discussed in class, following Lucas (1990). There is a representative agent with preferences

\[ u(c, l) = \frac{1}{1 - \sigma} \left[ c^\theta (1 - l)^{1-\theta} \right]^{1-\sigma} \]

over consumption \( c \) and labor \( l \). The discount factor is \( \beta \). Technology is Cobb-Douglas with

\[ F(k, l) = Ak^\alpha l^{1-\alpha} \]

and capital \( k \) depreciates at a rate \( \delta \). There is no uncertainty. Perform your calculations for \( \sigma = 2, 3 \) and \( 4 \).

(a) Consider a steady state competitive equilibrium of this economy where the tax on capital is \( \kappa = 0 \) and there is also no tax on labor income. What is the steady state gross interest rate \( R \)? Using the steady state resource constraint and the first order conditions from the consumer’s utility maximization problem and the firms’ profit maximization problem, find 4 equations that implicitly determine the steady state wage \( w \), steady state consumption \( c \) and labor supply \( l \) and the steady state capital stock \( k \) as a function of parameters. Calibrate the economy by choosing parameter values for \( \theta, A, \alpha, \delta \) and \( \beta \) that imply empirically reasonable values for the endogenous variables in the steady state (for instance, see Cooley (1995), chapter 1, posted on coursework). Solve your equations for \( c, l, k, w \).

(b) Using the parameters found in (a), compute the steady state for a capital tax \( \kappa \in [-.5, .5] \), assuming that tax revenue is returned in a lump sum fashion to consumers. Plot steady state output, capital, consumption and labor \( y(\kappa), k(\kappa), c(\kappa) \) and \( l(\kappa) \) as a function of the capital tax \( \kappa \). Compute the value of the steady state as

\[ \tilde{V}(k(\kappa)) = \frac{1}{(1-\beta)(1-\sigma)} \left[ c(\kappa)^\theta (1 - l(\kappa))^{1-\theta} \right]^{1-\sigma} \]
for given \(\kappa\).

We want to evaluate the welfare cost of this capital tax distortion by comparing this to the value that consumers would obtain if the capital tax were set to zero. A naive approach to this would be to compare \(\tilde{V}(k(\kappa))\) with \(\tilde{V}(k(0))\). Compute \(\lambda(\kappa)\) such that

\[
\frac{1}{(1-\beta)(1-\sigma)} \left[ ((1 + \lambda(\kappa))c(\kappa))^{\theta} (1 - l(\kappa))^ {1-\theta} \right]^{1-\sigma} = \tilde{V}(k(0)).
\]

Plot \(\lambda(\kappa)\). How does your result depend on \(\sigma\)? Interpret.

(c) The comparison in (b) ignores the transition from the capital stock \(k(\kappa)\) to \(k(0)\) once the capital tax \(\kappa\) is reduced to zero. The value function accounting for this solves the Bellman equation

\[
\tilde{V}(k) = \max_{c,l,k'} u(c,l) + \beta \tilde{V}(k')
\]

subject to

\[
c + k' = F(k,l) + (1 - \delta)k.
\]

Solve numerically for \(\tilde{V}(k)\) and evaluate it at \(k(\kappa)\). Again, find \(\lambda(\kappa)\) such that

\[
\frac{1}{(1-\beta)(1-\sigma)} \left[ ((1 + \lambda(\kappa))c(\kappa))^{\theta} (1 - l(\kappa))^ {1-\theta} \right]^{1-\sigma} = \tilde{V}(k(\kappa)).
\]

Plot \(\lambda(\kappa)\). How does your result depend on \(\sigma\)? How does it compare to your result in (b)? Interpret.

(d) Suppose that \(\kappa = .3\). Compute the tax revenue generated by the capital tax \(\kappa = .3\) in the steady state in (c) and assume that, contrary to what we assumed so far, this revenue is spent for constant government consumption \(g\). Find the steady state capital stock, consumption and labor supply in this economy and denote it by \(k_0, c_0\) and \(l_0\). Find the optimal policy of labor taxes \(\tau_t\) and capital taxes \(\kappa_t\) starting from capital stock \(k_0\) that finances government consumption \(g\) every period. Assume that bond holdings in the initial steady state are \(B_0 = 0\) and that the capital tax in the initial period is restricted to be \(\kappa_0 \leq 1\). Find the value associated with this optimal policy \(V^R(k_0)\) numerically. Compute \(\lambda\) such that

\[
\frac{1}{(1-\beta)(1-\sigma)} \left[ ((1 + \lambda)c_0)^{\theta} (1 - l_0)^ {1-\theta} \right]^{1-\sigma} = V^R((k_0)).
\]

Compare to your results in (b) and (c).
2 Optimal Saving Distortions and Welfare

Consider the two period Mirrlees model analyzed in class where preferences are given by
\[ E \left[ \frac{c_0^{1-\gamma}}{1-\gamma} + \beta \left( \frac{c_1^{1-\gamma}}{1-\gamma} - v(y/\theta_1) \right) \right]. \]

\( \theta_1 \) is a stochastic skill shock realized in period 1. As in Farhi/Werning (2012), suppose that \{c_0, c_1(\theta_1), y(\theta_1)\} is an incentive compatible allocation where consumption is such that \( c_0 \) is deterministic and \( c_1(\theta_1)/c_0 \) is lognormally distributed with \( \log(c_1(\theta_1)/c_0) \sim N(\mu, \sigma^2) \).

(a) Suppose \{c_0, c_1(\theta_1), y(\theta_1)\} is an optimal allocation that satisfies the agent’s inverse Euler equation. Find the implied technological rate of return \( q_a \) such that
\[ \frac{1}{u'(c_0)} = \frac{q_a}{\beta} \frac{1}{E[u'(c_1(\theta_1))]} \]
holds. (You should be able to solve for \( q_a \) as a function of \( \beta, \gamma, \mu \) and \( \sigma \).) Now suppose the agent can save freely at some fixed rate of return. Find the value of \( q_b \) that would make the agent’s Euler equation
\[ u'(c_0) = \frac{\beta}{q_b} E[u'(c_1(\theta_1))] \]
be satisfied given the allocation \{c_0, c_1(\theta_1), y(\theta_1)\}. Compute the relative wedge \( q_b/q_a \). What is the sign of the wedge in the return to savings and how does its magnitude depend on the parameters? Interpret.

(b) Now suppose that \{c_0, c_1(\theta_1), y(\theta_1)\} is incentive compatible but not necessarily optimal. Consider the special case of log-preferences, i.e. \( \gamma = 1 \), and assume that the technological rate of return is now given by \( q_b \) found in part (a). Compute the optimal distorted consumption path \{c_0^*, c_1^*(\theta_1)\} that leaves incentives for output and expected discounted utility from consumption unchanged and minimizes the expected cost
\[ c_0^* + q_b E[c_1^*(\theta_1)]. \]

Find the welfare gains of the optimal savings distortion by computing the ratio
\[ \frac{c_0 + q_b E[c_1(\theta_1)]}{c_0^* + q_b E[c_1^*(\theta_1)]} \]
where \( c_0 + q_b \mathbb{E}[c_1(\theta_1)] \) is the expected cost of the baseline consumption allocation from (a). In addition, compute the optimal distorted consumption path \( \{\tilde{c}_0, \tilde{c}_1(\theta_1)\} \) that leaves incentives for output and expected cost unchanged and maximizes expected discounted utility from consumption

\[
\log(\tilde{c}_0) + \beta \mathbb{E}[\log(\tilde{c}_1(\theta_1))].
\]

Evaluate the welfare gains by finding the percentage increase of baseline consumption in both periods \( \lambda \) that would make the agent as well off as under the optimally distorted allocation, i.e. compute \( \lambda \) such that

\[
\log((1 + \lambda)c_0) + \beta \mathbb{E}[\log((1 + \lambda)c_1(\theta_1))] = \log(c_0) + \beta \mathbb{E}[\log(\tilde{c}_1(\theta_1))].
\]

Compare. Find the values \( q^* \) and \( \tilde{q} \) that would make the agent’s Euler equation be satisfied at the optimal consumption allocations \( \{c_0^*, c_1^*(\theta_1)\} \) and \( \{\tilde{c}_0, \tilde{c}_1(\theta_1)\} \), respectively, and compare to the wedge found in (a).

(c) Add a third period to the model in which there is no shock (i.e. there is no \( \theta_2 \)) and the agent does not produce output, but only consumes. Show that consumption in the third period, \( c_2(\theta_1) \), is proportional to \( c_1(\theta_1) \). (Argue that we can in fact collapse periods 1 and 2 into one single period.) Think of the three period model as a stylized life-cycle model where the agent is a young worker of age up to 40 in period 0, an old worker (age 40-60) in period 1 and retired (age >60) in period 2. Calibrate the model by finding parameter values for \( \beta, \sigma \) and \( \mu \) that correspond with this interpretation. What does it imply for the welfare gains of the optimal savings distortion in this model? Compute \( \lambda \) and the relative wedge \( \tilde{q}/q_b \) as in (b). Translate the relative wedge into an annual implicit tax on the (net) return to savings \( 1/q_b - 1 \).