

Ambiguous Business Cycles*

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Abstract

This paper considers business cycle models with agents who are averse not only to risk, but also to ambiguity (Knightian uncertainty). Ambiguity aversion is described by recursive multiple priors preferences that capture agents' lack of confidence in probability assessments. While modeling changes in risk typically calls for higher order approximations, changes in ambiguity in our models work like changes in conditional means. Our models thus allow for uncertainty shocks but can still be solved and estimated using simple 1st order approximations. In an otherwise standard business cycle model, an increase in ambiguity (that is, a loss of confidence in probability assessments), acts like an 'unrealized' news shock: it generates a large recession accompanied by ex-post positive excess returns.

1 Introduction

Recent events have generated renewed interest in the effects of changing uncertainty on macroeconomic aggregates. The standard framework of quantitative macroeconomics is based on expected utility preferences and rational expectations. Changes in uncertainty are typically modeled as *expected and realized changes in risk*. Indeed, expected utility agents think about the uncertain future in terms of probabilities. An increase in uncertainty is described by the expected increase in a measure of risk (for example, the conditional volatility of a shock or the probability of a disaster). Moreover, rational expectations implies that agents' beliefs coincide with those of the econometrician (or model builder). An expected increase in risk must on average be followed by a realized increase in the volatility of shocks or the likelihood of disasters.

This paper studies business cycle models with agents who are averse to ambiguity (Knightian uncertainty). Ambiguity averse agents do not think in terms of probabilities

*Preliminary and incomplete.

– they lack the confidence to assign probabilities to all relevant events. An increase in uncertainty may then correspond to a loss of confidence that makes it more difficult to assign probabilities. Formally, we describe preferences using multiple priors utility (Gilboa and Schmeidler (1989)). Agents act *as if* they evaluate plans using a worst case probability drawn from a set of multiple beliefs. A loss of confidence is captured by an increase in the set of beliefs. It could be triggered, for example, by worrisome news about the future. Agents respond to a loss of confidence as their worst case probability changes.

The paper proposes a simple and tractable way to incorporate ambiguity and shocks to confidence into a business cycle model. Agents’ set of beliefs is parametrized by an interval of means for exogenous shocks, such as innovations to productivity. A loss of confidence is captured by an increase in the width of such an interval; in particular it makes the “worst case” mean even worse. Intuitively, a shock to confidence thus works like a news shock: an agent who loses confidence responds *as if* he had received bad news about the future. The difference between a loss of confidence and bad news is that the latter is followed, on average, by the realization of a bad outcome. This is not the case for a confidence shock.

We study ambiguity and confidence shocks in economies that are essentially linear. The key property is that the worst case means supporting agents’ equilibrium choices can be written as a linear function of the state variables. It implies that equilibria can be accurately characterized using first order approximations. In particular, we can study agents’ responses to changes in uncertainty, as well as time variation in uncertainty premia on assets, without resorting to higher order approximations. This is in sharp contrast to the case of changes in risk, where higher order solutions are critical. We illustrate the tractability of our method by estimating a medium scale DSGE model with ambiguity about productivity shocks.

The effects of a lack of confidence are intuitive. On average, less confident agents engage in precautionary savings and, other things equal, accumulate more steady state capital. A sudden loss of confidence about productivity generates a wealth effect – due to more uncertain wage and capital income in the future, and also a substitution effect since the return on capital has become more uncertain. The net effect on macroeconomic aggregates depends on the details of the economy. In our estimated medium scale DSGE model, a loss of confidence generates a recession in which consumption, investment and hours decline together. In addition, a loss of confidence generates increased demand for safe assets, and opens up a spread between the returns on ambiguous assets (such as capital) and safe assets. Business cycles driven by changes in confidence thus give rise to countercyclical spreads or premia on uncertain assets.

Our paper is related to several strands of literature. The decision theoretic literature on ambiguity aversion is motivated by the Ellsberg Paradox. Ellsberg’s experiments suggest

that decision makers' actions depend on their confidence in probability assessments – they treat lotteries with known odds differently from bets with unknown odds. The multiple priors model describe such behavior as a rational response to a lack of information about the odds. To model intertemporal decision making by agents in a business cycle model, we use a recursive version of the multiple priors model that was proposed by Epstein and Wang (1994) and has recently been applied in finance (see Epstein and Schneider (2010) for a discussion and a comparison to other models of ambiguity aversion). Axiomatic foundations for recursive multiple priors were provided by Epstein and Schneider (2003).

Hansen et al. (1999) and Cagetti et al. (2002) study business cycles models with robust control preferences. Under the robust control approach, preferences are smooth – utility contains a smooth penalty function for deviations of beliefs from some reference belief. Smoothness rules out first order effects of uncertainty. Models of changes in uncertainty with robust control thus rely on higher order approximations, as do models with expected utility. In contrast, multiple priors utility is not smooth when belief sets differ in means. As a result, there are first order effects of uncertainty – this is exactly what our approach exploits to generate linear dynamics in response to uncertainty shocks.

The mechanics of our model are related to the literature on news shocks (for example Beaudry and Portier (2006), Christiano et al. (2008), Schmitt-Grohe and Uribe (2008), Jaimovich and Rebelo (2009), Christiano et al. (2010a) and Barsky and Sims (2011)). In particular, Christiano et al. (2008) have considered the response to temporary unrealized good news about productivity to study stock market booms. When we apply our approach to news about productivity, a loss confidence works like a (possibly persistent) unrealized decline in productivity. Recent work on changes in uncertainty in business cycle models has focused on changes in realized risk – looking either at stochastic volatility of aggregate shocks (see for example Fernández-Villaverde and Rubio-Ramírez (2007), Justiniano and Primiceri (2008), Fernández-Villaverde et al. (2010) and the review in Fernández-Villaverde and Rubio-Ramírez (2010)) or at changes in idiosyncratic volatility in models with heterogeneous firms (Bloom et al. (2009), Bachmann et al. (2010)). We view our work as complementary to these approaches. In particular, confidence shocks can generate responses to uncertainty – triggered by news, for example – that is not connected to later realized changes in risk.

The paper proceeds as follows. Section 2 presents a general framework for adapting business cycle models to incorporate ambiguity aversion. Section 3 discusses the applicability of linear methods. Section 4 describes the estimation of a DSGE model for the US.

2 Ambiguous business cycles: a general framework

Uncertainty is represented by a period state space S . One element $s \in S$ is realized every period, and the history of states up to date t is denote $s^t = (s_0, \dots, s_t)$.

Preferences

Preferences order uncertain streams of consumption $C = (C_t)_{t=0}^\infty$, where $C_t : S^t \rightarrow \mathbb{R}^n$ and n is the number of goods. Utility for a consumption process $C = \{C_t\}$ is defined recursively by

$$U_t(C; s^t) = u(C_t) + \beta \min_{p \in \mathcal{P}(s^t)} E^p[U_{t+1}(C; s_t, s_{t+1})], \quad (2.1)$$

where $\mathcal{P}(s^t)$ is a set of probabilities on S .

Utility after history s^t is given by felicity from current consumption plus expected continuation utility evaluated under a “worst case” belief. The worst case belief is drawn from a set $\mathcal{P}_t(s^t)$ that may depend on the history s^t . The primitives of the model are the felicity u , the discount factor β and the entire process of one-step-ahead belief sets $\mathcal{P}_t(s^t)$. Expected utility obtains as a special case if all sets $\mathcal{P}_t(s^t)$ contain only one belief. More generally, a nondegenerate set of beliefs captures the agent’s lack of confidence in probability assessments; a larger set describes a less confident agent.

For our applications, we assume Markovian dynamics. In particular, we restrict belief sets to depend only on the last state and write $\mathcal{P}_t(s_t)$. We also assume that the true law of motion – from the perspective of an observer – is given by a transition probability $p^*(s_t) \in \mathcal{P}_t(s_t)$. The assumption here is that agents know that the dynamics are Markov, but they are not confident in what transition probabilities to assign. However, they are not systematically wrong in the sense that they view the true transition possibility as impossible. The true DGP is not relevant for decision making, but it matters for characterizing the equilibrium dynamics.

Environment & equilibrium

We consider economies with many agents $i \in I$. Agent i ’s preferences are of the form (2.1) with primitives $(\beta_i, u^i, \{\mathcal{P}_t^i(s_t)\})$. Given preferences, it is helpful to write the rest of the economy in fairly general notation that many typical problems can be mapped into. Consider a recursive competitive equilibrium that is described using a vector X of endogenous state variables. Let A^i denote a vector of actions taken by agent i . Among those actions is the choice of consumption – we write $c^i(A^i)$ for agent i ’s consumption bundle implied when the action is A^i . Finally, let Y denote a vector of endogenous variables not chosen by the agent – this vector will typically include prices, but also variables such as government transfers that are endogenous, but are neither part of the state space nor actions or prices.

The technology and market structure are summarized by a set of reduced form functions or correspondences. A recursive competitive equilibrium consists of action and value functions A^i and V^i , respectively, for all agents $i \in I$, as well as a function describing the other endogenous variables Y . We also write A for the collection of all actions $(A^i)_{i \in I}$ and A^{-i} for the collection of all actions except that of agent i . All functions are defined on the state (X, s) and satisfy

$$W^i(A, X, s; p) = u^i(c^i(A^i)) + \beta_i E^p [V^i(x'(X, A, Y(X, s), s, s'), s')] \quad ; i \in I \quad (2.2)$$

$$A^i(X, s) = \arg \max_{A^i \in B^i(Y(X, s), A^{-i}, X, s)} \min_{p \in \mathcal{P}^i(s)} W(A, X, s; p) \quad i \in I \quad (2.3)$$

$$V^i(X, s) = \min_{p \in \mathcal{P}^i(s)} W^i(A(X, s), X, s; p) \quad (2.4)$$

$$0 = G(A(X, s), Y(X, s), X, s) \quad (2.5)$$

The first equation simply defines the agent's objective in state (X, s) , while the second and third equation provide the optimal policy and value. Here B^i is the agent's budget set correspondence and the function x' describes the transition of the endogenous state variables. The function G summarizes all other contemporaneous relationships such as market clearing or the government budget constraint – there are enough equations in (2.5) to determine the endogenous variables Y .

Characterizing optimal actions & equilibrium dynamics

For every state (X, s) , there is a measure $p^{0i}(X, s)$ that achieves the minimum for agent i in (2.3). Since the minimization problem is linear in probabilities, we can replace \mathcal{P}_t^i by its convex hull without changing the solution. The minimax theorem then implies that we can exchange the order of minimization and maximization in the problem (2.3). It follows that the optimal action A^i is the same as the optimal action if the agent held the probabilistic belief $p^{0i}(X, s)$ to begin with. In other words, for every equilibrium of our economy, there exists an economy with expected utility agents holding beliefs p^{0i} that has the same equilibrium.

The observational equivalence just described suggests the following guess-and-verify procedure to compute an equilibrium with ambiguity aversion:

1. guess the worst case beliefs p^{0i}
2. solve the model assuming that the agents have expected utility and beliefs p^{0i}
3. compute the value functions V^i
4. verify that the guesses p^{0i} indeed achieves the minimum in (2.4) for every i .

Suppose we have found the optimal action functions A as well as the response of the endogenous variables Y and hence the transition for the states X . We are interested in stochastic properties of the equilibrium dynamics that can be compared to the data. We characterize the dynamics in the standard way by calculating (for example or simulating) moments of the economy under the true distribution of the exogenous shocks p^* . The only unusual feature is that this true distribution need not coincide with the distribution p^{0i} that is used to compute optimal actions.

Shocks to confidence

We now specialize a process of belief sets \mathcal{P}_t to capture random changes in confidence. For simplicity, assume that there is a single exogenous process z that directly affects the economy, for example productivity or a monetary policy shock. In addition to z , the exogenous state s has a second component v_t that captures time variation in confidence. Suppose the true dynamics of exogenous state s can be represented by an AR(1) process

$$\begin{aligned} z_{t+1} &= \rho_z z_t + \varepsilon_{t+1}^z, \\ v_{t+1} &= (1 - \rho_v) \bar{v} + \rho_v v_t + \varepsilon_{t+1}^v \end{aligned}$$

where the shocks ε^z and ε^v are iid and normally distributed shock with mean zero and variances σ_z^2 and σ_v^2 , respectively.

The component v_t is a tool to describe the evolution of the belief set \mathcal{P}_t . In particular, we assume that the agent knows the evolution of v_t , but that he is not sure whether the conditional mean of z_{t+1} is really $\rho_z z_t$. Instead, he allows for a range of intercepts. The set \mathcal{P}_t can thus be represented by the family of processes

$$\begin{aligned} z_{t+1} &= \rho_z z_t + a_t + \varepsilon_{t+1}^z, \\ a_t &\in [-v_t, -v_t + 2|v_t|] \\ v_{t+1} &= (1 - \rho_v) \bar{v} + \rho_v v_t + \varepsilon_{t+1}^v \end{aligned} \tag{2.6}$$

If v_t is higher, then the agent is less confident about the mean of z_{t+1} – his belief set is larger. The worst case belief p^0 is now described by a worst case intercept v_t^0 . Changes in ambiguity parametrized by v_t change the worst case conditional mean $\rho_z z_t + v_t^0$ and therefore have 1st order effects on behavior.

As long as $v_t > 0$, the interval of intercepts contemplated by the agent is centered around zero – it can equivalently be described by the condition $a_t \leq |v_t|$. In contrast, if v_t becomes negative, then all intercepts are positive – the agent is thus optimistic relative to the truth. In applications, it is thus useful to parametrize the dynamics such that v does not become

negative very often. The advantage of the present parametrization is that the lower bound, which plays a special role in the computation, is guaranteed to have linear dynamics.

3 Computation in essentially linear economies

The computation and interpretation of equilibria is particularly simple if all dynamics is approximately linear. We start from the assumption that the environment (given by B^i , x' u^i and G) is such that, *under expected utility and rational expectations*, a first order solution provides a satisfactory approximation to the equilibrium dynamics. Under rational expectations, the innovations to v_t are news shocks that provide information about z_t one period ahead. In many applications, standard tools will reveal whether a first order approximation is satisfactory.

It is now natural to look for an equilibrium such that the worst case intercepts v_t^{0i} are all linear in the state variables. If such an equilibrium exists, we call the economy *essentially linear*. In an essentially linear economy, the choices A and other endogenous variables Y will all be well approximated by linear functions of the state variables. The equilibrium dynamics is thus described by a linear state space system with all shocks – including uncertainty shocks – driven by the (true) linear laws of motion. In many interesting economies, it is straightforward to establish essential linearity. Consider, for example, a baseline stochastic growth model with a representative agent who perceives ambiguity about productivity. The worst case is that the mean of productivity innovations is always as low as possible, so $v_t^0 = -v_t$.

More generally, to check whether some general economy is essentially linear, we specialize the guess-and-verify procedure described above. Step 2 of the procedure consists of solving the model using the guesses p^{0i} that sets $v_t^{0i} = -v_t$ or $v_t^{0i} = v_t$. In other words, the worst case for agent i is always either the highest or lowest bound of the interval. Since these guesses just implies a linear shift of the shock, this step can also be done by first order methods. In particular, we propose to linearize the model around a “zero risk” steady state that sets the variance of the shocks to a very small number while retaining the effect of ambiguity on decisions. Properties of zero risk steady states are described in more detail below.

To implement step 3 of the procedure, let V^{0i} denote the value function for the problem with expected utility and a_t at the guessed worst case mean for agent i in (2.6). For example, consider an agent with $v_t^{0i} = -v_t$. For this agent, we verify the guess by checking whether for any $X, A, s = (z, v)$ and v' , the function

$$\tilde{V}(z') := V^0(x'(X, A, Y(X, s), s, z', v'), z', v')$$

is strictly increasing. We do the opposite check for an agent with $v_t^{0i} = v_t$. At this stage, nonlinearity of the value function could be important. We thus compute the value function using higher order approximations and form the function \tilde{V} accordingly.

In what follows, we first describe the dynamics and then the calculation of the zero risk steady state.

3.1 Dynamics

Let x_t denote the endogenous variables of interest. Posit a linear equilibrium law of motion:

$$x_t = Ax_{t-1} + Bs_t,$$

where s_t are the exogenous variables. For notational purposes, split the vector s_t into the technology shock $\hat{z}_t \equiv \log z_t$, mean distortion (or news) v_t and the rest of the exogenous variables, s_t^* .

Posit another linear relation for the exogenous variables:

$$s_t = \begin{bmatrix} s_t^* \\ \hat{z}_t \\ v_t \end{bmatrix} = P \begin{bmatrix} s_{t-1}^* \\ \hat{z}_{t-1} \\ v_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_t \\ \varepsilon_t^z \\ \varepsilon_t^v \end{bmatrix}$$

$$\Xi_t = \begin{bmatrix} \varepsilon_t \\ \varepsilon_t^z \\ \varepsilon_t^v \end{bmatrix} \sim N(0, \Sigma)$$

Ξ_t denotes the innovations to the exogenous variables s_t . They are defined to be zero mean.

We follow the method of undetermined coefficients of Christiano (2002) to solving a system of linear equations with rational expectations. Let the linearized equilibrium conditions be restated in general as:

$$\tilde{E}_t[\alpha_0 x_{t+1} + \alpha_1 x_t + \alpha_2 x_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t] = 0 \quad (3.1)$$

where $\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1$ are constants determined by the equilibrium conditions. Importantly

for us,

$$\begin{aligned}\tilde{E}_t \begin{bmatrix} s_{t+1}^* \\ \hat{z}_{t+1} \\ v_{t+1} \end{bmatrix} &= P \begin{bmatrix} s_t^* \\ \hat{z}_t \\ v_t \end{bmatrix} + \tilde{E}_t \begin{bmatrix} \varepsilon_{t+1} \\ \varepsilon_{t+1}^z \\ \varepsilon_{t+1}^v \end{bmatrix} \\ \tilde{E}_t \begin{bmatrix} \varepsilon_{t+1} \\ \varepsilon_{t+1}^z \\ \varepsilon_{t+1}^v \end{bmatrix} &= 0\end{aligned}$$

To reflect the time t information (news, or mean distortion) about \hat{z}_{t+1} we remember that

$$\tilde{E}_t \hat{z}_{t+1} = \rho_z \hat{z}_t - v_t$$

so the matrix P satisfies the restriction:

$$\begin{aligned}P &= \begin{bmatrix} \rho & 0 & 0 \\ 0 & \rho_z & -1 \\ 0 & 0 & \rho_v \end{bmatrix} \\ \tilde{E}_t \begin{bmatrix} s_{t+1}^* \\ \hat{z}_{t+1} \\ v_{t+1} \end{bmatrix} &= \begin{bmatrix} \rho & 0 & 0 \\ 0 & \rho_z & -1 \\ 0 & 0 & \rho_v \end{bmatrix} \begin{bmatrix} s_t^* \\ \hat{z}_t \\ v_t \end{bmatrix} + \tilde{E}_t \begin{bmatrix} \varepsilon_{t+1} \\ \varepsilon_{t+1}^z \\ \varepsilon_{t+1}^v \end{bmatrix}\end{aligned}$$

where ρ is a diagonal matrix reflecting the autocorrelation structure of the elements in s_t^* . Notice that without the "news" part, the standard form for P is:

$$P = \begin{bmatrix} \rho & 0 & 0 \\ 0 & \rho_z & 0 \\ 0 & 0 & \rho_v \end{bmatrix}$$

Substitute the posited policy rule into linearized equilibrium conditions:

$$0 = \tilde{E}_t [\alpha_0 (Ax_t + Bs_{t+1}) + \alpha_1 (Ax_{t-1} + Bs_t) + \alpha_2 x_{t-1} + \beta_0 (Ps_t + \Xi_{t+1}) + \beta_1 s_t]$$

to get:

$$\begin{aligned}0 &= (\alpha_0 A^2 + \alpha_1 A + \alpha_2) x_{t-1} + (\alpha_0 AB + \alpha_0 BP + \alpha_1 B + \beta_0 P + \beta_1) s_t + (\alpha_0 B + \beta_0) \tilde{E}_t \Xi_{t+1} \\ 0 &= \tilde{E}_t \Xi_{t+1}\end{aligned}$$

Thus, A is the matrix eigenvalue of matrix polynomial:

$$\alpha(A) = \alpha_0 A^2 + \alpha_1 A + \alpha_2 = 0$$

and B satisfies the system of linear equations:

$$F = (\beta_0 + \alpha_0 B)P + [\beta_1 + (\alpha_0 A + \alpha_1) B] = 0$$

The solution to the model obtained so far is one in which the mean distortion v_t to the process for z_{t+1} is realized.

With this solution in hand, look at the variables when the negative mean distortion is not realized at each period t . In the equilibrium defined above:

$$\begin{aligned} x_t &= Ax_{t-1} + Bs_t \\ s_t &= \begin{bmatrix} s_t^* \\ \widehat{z}_t \\ v_t \end{bmatrix} = \begin{bmatrix} \rho & 0 & 0 \\ 0 & \rho_z & -1 \\ 0 & 0 & \rho_v \end{bmatrix} \begin{bmatrix} s_{t-1}^* \\ \widehat{z}_{t-1} \\ v_{t-1} \end{bmatrix} + \Xi_t \end{aligned} \quad (3.2)$$

but now we have:

$$\begin{aligned} x_t &= Ax_{t-1} + Bs_t \\ s_t &= \begin{bmatrix} s_t^* \\ \widehat{z}_t \\ v_t \end{bmatrix} = \begin{bmatrix} \rho & 0 & 0 \\ 0 & \rho_z & 0 \\ 0 & 0 & \rho_v \end{bmatrix} \begin{bmatrix} s_{t-1}^* \\ \widehat{z}_{t-1} \\ v_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_t \\ \varepsilon_t^z \\ \varepsilon_t^v \end{bmatrix} \\ s_t &= \begin{bmatrix} \rho & 0 & 0 \\ 0 & \rho_z & -1 \\ 0 & 0 & \rho_v \end{bmatrix} \begin{bmatrix} s_{t-1}^* \\ z_{t-1} \\ v_{t-1} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s_{t-1}^* \\ z_{t-1} \\ v_{t-1} \end{bmatrix} + \Xi_t \\ s_t &= P \begin{bmatrix} s_{t-1}^* \\ \widehat{z}_{t-1} \\ v_{t-1} \end{bmatrix} + C \begin{bmatrix} s_{t-1}^* \\ \widehat{z}_{t-1} \\ v_{t-1} \end{bmatrix} + \Xi_t, \quad C \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

so:

$$\begin{aligned} x_t &= Ax_{t-1} + Bs_t \\ x_t &= Ax_{t-1} + BP \begin{bmatrix} s_{t-1}^* \\ \widehat{z}_{t-1} \\ v_{t-1} \end{bmatrix} + B \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s_{t-1}^* \\ \widehat{z}_{t-1} \\ v_{t-1} \end{bmatrix} + B\Xi_t \end{aligned}$$

or in other words, for every j element in the vector x_t , where the superscript j refers to the j th row of the corresponding matrix, we have:

$$x_t^j = A^j x_{t-1} + B^j P s_{t-1} + B^j \Xi_t + B_z^j v_{t-1} \quad (3.3)$$

where the element B_z^j refers to the coefficient of the matrix B that reflects the response of the element x_t^j to the realized state \widehat{z}_t .

Notice that the evolution in (3.3) defines the equilibrium law of motion for our economy. From the perspective of the economy in (3.2) it is interpreted as the response to an "unusual" innovation to ε_t^z whose value is not zero (on average) but rather v_{t-1} . However, conditional on this "innovation" (state of the economy x_t) the expectations are still governed by the equation (3.1) where for example the expectation about z_{t+1} is:

$$\widetilde{E}_t \widehat{z}_{t+1} = \rho_z \widehat{z}_t - v_t$$

3.1.1 Zero risk steady state

We now describe an approach to find the stochastic steady state of our model using the linearized law of motion of the endogenous variables. Take the perceived law of motion:

$$\widehat{z}_{t+1} = \rho_z \widehat{z}_t + \sigma \varepsilon_{t+1}^z - \bar{v}$$

where \bar{v} is the steady state level of v_t . We can summarize our procedure in the following steps:

1. Find the deterministic 'distorted' steady state in which the intercept is actually $-\bar{v}$. The steady state technology level is then

$$z^o = \exp \left(\frac{-\bar{v}}{1 - \rho_z} \right)$$

Using z^o , one can compute the deterministic 'distorted' steady state, by analyzing the FOC of these economy. Denote these steady state values of the m variables as a vector x_o

$$x_{m \times 1}^o$$

2. Linearize the model around this deterministic 'distorted' steady state.

For example, in the above notations, looks for matrices A, B that describe the evolution

of the variables as:

$$x_t - x_o = A(x_{t-1} - x_o) + BP(s_{t-1} - s_o) + B(\Xi_t - \Xi_o) \quad (3.4)$$

$$s_t = \begin{bmatrix} s_t^* \\ \hat{z}_t \\ v_t \end{bmatrix} = \begin{bmatrix} \rho & 0 & 0 \\ 0 & \rho_z & -1 \\ 0 & 0 & \rho_v \end{bmatrix} \begin{bmatrix} s_{t-1}^* \\ \hat{z}_{t-1} \\ v_{t-1} \end{bmatrix} + \Xi_t$$

where Ξ_t are the innovations to the stochastic shock processes. Let the size of this vector be $n \times 1$, where the first element refers to the innovation of the technology shock.

Notice for example that when innovations are equal to their expected values, set to zero, i.e. that $\Xi_o = 0_{n \times 1}$ and $s_{t-1} = s_o$, the law of motion recovers that $x_t = x_o$.

So, in step 2 we need to find the matrices A and B . This is done by standard solution techniques of forward looking rational expectations model.

3. Correct for the fact that, from the perspective of the agent's ex-ante beliefs, the average innovation of the technology shock at time t is not equal to 0. The average innovation is equal to \bar{v}/σ . Indeed:

$$\tilde{E}_{t-1}\hat{z}_t = \rho_z \hat{z}_{t-1} - \bar{v}$$

but the realized average log z_t is

$$\hat{z}_t = \rho_z \hat{z}_{t-1}$$

Thus, from the perspective of the time $t - 1$ expectation:

$$\begin{aligned} \hat{z}_t &= \tilde{E}_{t-1}\hat{z}_t + \sigma \bar{\varepsilon}_t \\ \bar{\varepsilon}_t &= \bar{v}/\sigma \end{aligned}$$

So, take the law of motion in (3.4) and impose that the first element of Ξ_t is equal to \bar{v}/σ , while keeping the rest equal to 0 :

$$\hat{\Xi} = \begin{bmatrix} \bar{v}/\sigma \\ 0_{(n-1) \times 1} \end{bmatrix}$$

4. Find the steady state of the variables x_t , given that the law of motion is:

$$x_t - x_o = A(x_{t-1} - x_o) + BP(s_{t-1} - s_o) + B(\hat{\Xi}_t - \Xi_o)$$

The steady state version for the exogenous variables is:

$$s^{SS} - s_o = P(s^{SS} - s_o) + (\hat{\Xi} - \Xi_o)$$

where s^{SS} are the steady state values of the exogenous variables under their true DGP. So:

$$s^{SS} = s_o + \begin{bmatrix} \bar{v}/\sigma \\ 0_{(n-1) \times 1} \end{bmatrix} (I - P)^{-1}$$

Then solve for the rest of the endogenous variables by using:

$$x^{SS} - x_o = A(x^{SS} - x_o) + B(s^{SS} - s_o)$$

where x^{SS} are the steady state values of the variables under ambiguity. So, x^{SS} can be found as:

$$x^{SS} = x_o + B(s^{SS} - s_o) (I - A)^{-1}$$

4 An estimated model with ambiguity

This section describes the model that we use to describe the US business cycles. The model is based on a standard medium scale DSGE model along the lines of Christiano et al. (2005) and Smets and Wouters (2007). The key difference in our model is that decision makers are ambiguity-averse. We now describe the model structure and the shocks.

4.1 The model

4.1.1 The goods sector

The final output in this economy is produced by a representative final good firm that combines a continuum of intermediate goods $Y_{j,t}$ in the unit interval by using the following linear homogeneous technology:

$$Y_t = \left[\int_0^1 Y_{j,t}^{\frac{1}{\lambda_{f,t}}} dj \right]^{\lambda_{f,t}},$$

where $\lambda_{f,t}$ is the markup of price over marginal cost for intermediate goods firms. The markup shock evolves as:

$$\log(\lambda_{f,t}/\lambda_f) = \rho_{\lambda_f} \log(\lambda_{f,t-1}/\lambda_f) + \lambda_{f,t}^x,$$

where $\lambda_{f,t}^x$ is *i.i.d.* $N(0, \sigma_{\lambda_f}^2)$. Profit maximization and the zero profit condition leads to the following demand function for good j :

$$Y_{j,t} = Y_t \left(\frac{P_t}{P_{j,t}} \right)^{\frac{\lambda_{f,t}}{\lambda_{f,t}-1}} \quad (4.1)$$

The price of final goods is:

$$P_t = \left[\int_0^1 P_{j,t}^{\frac{1}{1-\lambda_{f,t}}} dj \right]^{(1-\lambda_{f,t})}.$$

The intermediate good j is produced by a price-setting monopolist using the following production function:

$$Y_{j,t} = \max\{\epsilon_t K_{j,t}^\alpha (z_t L_{j,t})^{1-\alpha} - \Phi z_t^*, 0\},$$

where Φ is a fixed cost and $K_{j,t}$ and $L_{j,t}$ denote the services of capital and homogeneous labor employed by firm j . Φ is chosen so that steady state profits are equal to zero. The intermediate goods firms are competitive in factor markets, where they confront a rental rate, $P_t r_t^k$, on capital services and a wage rate, W_t , on labor services.

The variable, z_t , is a shock to technology, which has a covariance stationary growth rate. The variable, ϵ_t , is a stationary shock to technology. The fixed costs are modeled as growing with the exogenous variable, z_t^* :

$$z_t^* = z_t \Upsilon^{\left(\frac{\alpha}{1-\alpha}t\right)}$$

with $\Upsilon > 1$. If fixed costs were not growing, then they would eventually become irrelevant. We specify that they grow at the same rate as z_t^* , which is the rate at which equilibrium output grows. Note that the growth of z_t^* , i.e. $\mu_{z,t}^* \equiv \Delta \log(z_t^*)$, exceeds that of z_t , i.e. $\mu_{z,t} \equiv \Delta \log(z_t)$:

$$\mu_{z,t}^* = \mu_{z,t} \Upsilon^{\frac{\alpha}{1-\alpha}}.$$

This is because we have another source of growth in this economy, in addition to the upward drift in z_t . In particular, we posit a trend increase in the efficiency of investment. We discuss this process as well as the time series representation for the transitory technology shock further below. The process for the stochastic growth rate is:

$$\log(\mu_{z,t}^*) = (1 - \rho_{\mu_z^*}) \log \mu_z^* + \rho_{\mu_z^*} \log \mu_{z,t-1}^* + \mu_{z,t}^{x*},$$

where $\mu_{z,t}^{x*}$ is *i.i.d.* $N(0, \sigma_{\mu_z^*}^2)$.

We now describe the intermediate good firms pricing opportunities. Following Calvo (1983), a fraction $1 - \xi_p$, randomly chosen, of these firms are permitted to reoptimize their price every period. The other fraction ξ_p cannot reoptimize. Of these, a (randomly selected)

fraction $(1 - \iota_P)$ must set $P_{it} = \bar{\pi}P_{i,t-1}$ and a fraction ι_P set $P_{it} = \pi_{t-1}P_{i,t-1}$, where $\bar{\pi}$ is steady state inflation. The j^{th} firm that has the opportunity to reoptimize its price does so to maximize the expected present discounted value of the future profits:

$$E_t^{p^0} \sum_{s=0}^{\infty} (\beta \xi_p)^s \frac{\lambda_{t+s}}{\lambda_t} [P_{j,t+s} Y_{j,t+s} - W_{t+s} L_{j,t+s} - P_{t+s} r_{t+s}^k K_{j,t+s}] , \quad (4.2)$$

subject to the demand function (4.1), where λ_t is the marginal utility of nominal income for the representative household that owns the firm.

It should be noted that the expectation operator in these equations is, in the notation of the general representation in section 2, the expectation under the worst-case belief p^0 . This is because state prices in the economy reflect ambiguity. We will describe the household's problem further below.

We now describe the problem of the perfectly competitive "employment agencies". The households specialized labor inputs are aggregated by these agencies into a homogeneous labor service according to the following function:

$$L_t = \left[\int_0^1 (l_{i,t})^{\frac{1}{\lambda_w}} di \right]^{\lambda_w} .$$

These employment agencies rent the homogeneous labor service L_t to the intermediate goods firms at the wage rate W_t . In turn, these agencies pay the wage $W_{i,t}$ to the household supplying labor of type i . Similarly as for the final goods producers, the profit maximization and the zero profit condition leads to the following demand function for labor input of type i :

$$l_{i,t} = L_t \left(\frac{W_t}{W_{i,t}} \right)^{\frac{\lambda_w}{\lambda_w - 1}} , \quad (4.3)$$

We follow Erceg et al. (2000) and assume that the household is a monopolist in the supply of labor by providing $l_{i,t}$ and it sets its nominal wage rate, $W_{i,t}$. It does so optimally with probability $1 - \xi_w$ and with probability ξ_w it does not reoptimize its wage. In case it does not reoptimize, it sets the wage as:

$$W_{i,t} = \pi_{t-1}^{\iota_w} \bar{\pi}^{1-\iota_w} \mu_{z^*} W_{i,t-1} ,$$

where μ_{z^*} is the steady state growth rate of the economy. When household i has the chance to reoptimize, it does so by maximizing the expected present discounted value of future net

utility gains of working:

$$E_t^{p^0} \sum_{s=0}^{\infty} \sum_{s=0}^{\infty} (\beta \xi_w)^s \left[\lambda_{t+s} W_{i,t+s} l_{i,t+s} - \zeta_{c,t+s} \zeta_{L,t+s} \frac{\psi_L}{1 + \sigma_L} l_{i,t+s}^{1+\sigma_L} \right]. \quad (4.4)$$

subject to the demand function (4.3).

4.1.2 Households

The model described here is a special case of the general formulation of recursive multiple priors of section 2. In particular, recall the recursive representation in (2.1), where preferences were defined over uncertain streams of consumption $C = (C_t)_{t=0}^{\infty}$, where $C_t : S^t \rightarrow \Re^n$ and n is the number of goods. In the model described here, there are 2 goods, the consumption of the final good Y_t and leisure. We then only have to define the per-period felicity function, which for agent i is:

$$u^i(C_t) = \zeta_{c,t} \left\{ \log(C_{t+s} - \theta C_{t+s-1}) - \zeta_{L,t+s} \frac{\psi_L}{1 + \sigma_L} l_{i,t+s}^{1+\sigma_L} \right\}. \quad (4.5)$$

Here, $\zeta_{c,t}$ denotes an intertemporal preference shock, $\zeta_{L,t}$ is an intratemporal labor supply shock, θ is an internal habit parameter, C_t denotes individual consumption of the final good and $l_{i,t}$ denotes a specialized labor service supplied by the household. Also, $\psi_L > 0$ is a parameter.¹ The two preference shocks follow the time series representation:

$$\begin{aligned} \log(\zeta_{c,t}) &= \rho_{\zeta_c} \log(\zeta_{c,t-1}) + \zeta_{c,t}^x \\ \log(\zeta_{L,t}) &= \rho_{\zeta_L} \log(\zeta_{L,t-1}) + \zeta_{L,t}^x, \end{aligned}$$

where $\zeta_{c,t}^x$ is $i.i.d.N(0, \sigma_{\zeta_c}^2)$ and $\zeta_{L,t}^x$ is $i.i.d.N(0, \sigma_{\zeta_L}^2)$.

Utility follows a recursion similar to (2.1):

$$U_t(c; s^t) = u^i(c_t, c_{t-1}) + \beta \min_{p \in \mathcal{P}(s^t)} E^p[U_{t+1}(c; s_t, s_{t+1})], \quad (4.6)$$

Let the solution to the minimization problem in (4.6) be denoted by p^0 . This minimizing p^0 is the same object that appears in the expectation operator of equations (4.4) and (4.2).

The type of Knightian uncertainty we consider in this model is over the transitory technology level. In section 4.1.1, we described the production side of the economy and showed where technology enters this economy. Ambiguity here is reflected by the set of one-

¹Note that consumption is not indexed by i because we assume the existence of state contingent securities which implies that in equilibrium consumption and asset holdings are identical across households.

step ahead conditional beliefs $\mathcal{P}(s^t)$ about the future transitory technology. We follow the description in section 2 to describe the stochastic process. Specifically, we will assume that there is time-varying ambiguity about the future technology. This time-variation is captured by an exogenous component v_t . The true dynamics of the transitory productivity shock ϵ_t can be represented by an AR(1) process:

$$\log \epsilon_t = \rho_\epsilon \log \epsilon_{t-1} + \sigma_\epsilon \epsilon_t^x. \quad (4.7)$$

The set \mathcal{P}_t of one step conditional beliefs about future technology can then be represented by the family of processes:

$$\log \epsilon_{t+1} = \rho_\epsilon \log \epsilon_t + \sigma_\epsilon \epsilon_{t+1}^x + a_t \quad (4.8)$$

$$a_t \in [-v_t, -v_t + 2|v_t|] \quad (4.9)$$

$$v_{t+1} = (1 - \rho_v) \bar{v} + \rho_v v_t + \sigma_v v_{t+1}^x \quad (4.10)$$

where the shocks ε^z and ε^v are iid and normally distributed shock with mean zero and variances σ_ϵ^2 and σ_v^2 , respectively. As in Section 2, we assume that the agent knows the evolution of v_t , but that he is not sure whether the conditional mean of $\log \epsilon_{t+1}$ is really $\rho_\epsilon \log \epsilon_t$. Instead, the agent allows for a range of intercepts. If v_t is higher, then the agent is less confident about the mean of $\log \epsilon_{t+1}$ – his belief set is larger.

In this model, it is easy to what is the worst-case scenario, i.e. what is the belief about a_t that solves the minimization problem in (4.6). In section 3 we described a general procedure to find the worst-case belief. In the present model, the environment (given by B , x' u and G in the general formulation of section 2) is such that, *under expected utility and rational expectations*, a first order solution provides a satisfactory approximation to the equilibrium dynamics. In this environment, it is easy to check that the value function, under expected utility, is increasing in the innovation ϵ_t^x . This monotonicity implies that the worst-case scenario belief that solves the minimization problem in (4.6) is given by the lower bound of the set $[-v_t, -v_t + 2|v_t|]$. Intuitively, it is natural that the agents take into account that the worst case is always that the mean of productivity innovations is as low as possible.

The laws of motion in (4.8) and (4.10) are linear. To maintain the interpretation that a_t is the worst-case scenario solution to the minimization problem, v_t should be positive. Thus, it is useful not to have v_t become negative very often. We are then guided to parameterize the ambiguity process in the following way. We compute the unconditional variance of the process in (4.10) and insist that the mean level of ambiguity is high enough so that even a large negative shock to v_t of m unconditional standard deviations away from the mean will

remain in the positive domain. We can write this constraint as:

$$\bar{v} \geq m \frac{\sigma_v}{\sqrt{1 - \rho_v^2}} \quad (4.11)$$

The formula in (4.11) provides a constraint on our parameterization. If the constraint is binding, then, for a given m , there is a fixed relationship on our parameterization between \bar{v} , σ_v and ρ_v . In that case, we are left with essentially choosing two out of three parameters.

The household accumulates capital subject to the following technology:

$$\bar{K}_{t+1} = (1 - \delta)\bar{K}_t + \left[1 - S\left(\zeta_{I,t} \frac{I_t}{I_{t-1}}\right)\right] I_t,$$

where $\zeta_{I,t}$ is a disturbance to the marginal efficiency of investment with mean unity, \bar{K}_t is the beginning of period t physical stock of capital, and I_t is period t investment. The function S reflects adjustment costs in investment. The function S is convex, with steady state values of $S = S' = 0, S'' > 0$. The specific functional form for $S(\cdot)$ that we use is:

$$S\left(\zeta_{I,t} \frac{I_t}{I_{t-1}}\right) = \exp\left[\sqrt{\frac{S''}{2}}\left(\zeta_{I,t} \frac{I_t}{I_{t-1}} - 1\right)\right] + \exp\left[-\sqrt{\frac{S''}{2}}\left(\zeta_{I,t} \frac{I_t}{I_{t-1}} - 1\right)\right] - 2$$

The marginal efficiency of investment follows the process:

$$\log(\zeta_{i,t}) = \rho_{\zeta_I} \log(\zeta_{i,t-1}) + \zeta_{i,t}^x,$$

where $\zeta_{i,t}^x$ is *i.i.d.* $N(0, \sigma_{\zeta_I}^2)$.

Households own the physical stock of capital and rent out capital services, K_t , to a competitive capital market at the rate $P_t \tilde{r}_t^k$, by selecting the capital utilization rate u_t :

$$K_t = u_t \bar{K}_t,$$

Increased utilization requires increased maintenance costs in terms of investment goods per unit of physical capital measured by the function $a(u_t)$. The function $a(\cdot)$ is increasing and convex, $a(1) = 0$ and u_t is unity in the nonstochastic steady state. We assume that $a''(u) = \sigma_a r^k$, where r^k is the steady state value of the rental rate of capital. Then, $a''(u)/a'(u) = \sigma_a$ is a parameter that controls the degree of convexity of utilization costs.

The i^{th} household's budget constraint is:

$$P_t C_t + P_t \frac{I_t}{\mu_{\Upsilon,t} \Upsilon^t} + B_t = B_{t-1} R_{t-1} + P_t \bar{K}_t [\tilde{r}_t^k u_t - a(u_t) \Upsilon^{-t}] + W_{t,i} l_{t,i} - T_t P_t$$

where B_t are holdings of government bonds, R_t is the gross nominal interest rate and T_t is net lump-sum taxes.

When we specify the budget constraint, we will assume that the cost, in consumption units, of one unit of investment goods, is $(\Upsilon^t \mu_{\Upsilon,t})^{-1}$. Since the currency price of consumption goods is P_t , the currency price of a unit of investment goods is therefore, $P_t (\Upsilon^t \mu_{\Upsilon,t})^{-1}$. The stationary component of the relative price of investment follows the process:

$$\log(\mu_{\Upsilon,t}/\mu_{\Upsilon}) = \rho_{\mu_{\Upsilon}} \log(\mu_{\Upsilon,t-1}/\mu_{\Upsilon}) + \mu_{\Upsilon,t}^x,$$

where $\mu_{\Upsilon,t}^x$ is *i.i.d.* $N(0, \sigma_{\mu_{\Upsilon}}^2)$.

4.1.3 The government

The market clearing condition for this economy is:

$$C_t + \frac{I_t}{\mu_{\Upsilon,t} \Upsilon^t} + G_t = Y_t^G$$

where G_t denotes government expenditures and Y_t^G is our definition of measured *GDP*, i.e. $Y_t^G \equiv Y_t - a(u_t) \Upsilon^{-t} \bar{K}_t$. We model government expenditures as $G_t = g_t z_t^*$, where g_t is a stationary stochastic process. This way of modeling G_t helps to ensure that the model has a balanced growth path. The fiscal policy is Ricardian. The government finances G_t by issuing short term bonds B_t and adjusting lump sum taxes T_t . The law of motion for g_t is:

$$\log(g_t/g) = \rho_g \log(g_{t-1}/g) + g_t^x$$

where g_t^x is *i.i.d.* $N(0, \sigma_g^2)$.

The nominal interest rate R_t is set by a monetary policy authority according to the following feedback rule:

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R} \right)^{\rho_R} \left[\left(\frac{\pi_t}{\pi} \right)^{a_{\pi}} \left(\frac{Y_t^G}{Y_t^*} \right)^{a_y} \left(\frac{Y_t^G}{\mu_z^* Y_{t-1}^G} \right)^{a_{gy}} \right]^{1-\rho_R} \exp(\epsilon_{R,t}),$$

where $\epsilon_{R,t}$ is a monetary policy shock *i.i.d.* $N(0, \sigma_{\epsilon_R}^2)$.

4.1.4 Model Solution

The model economy fluctuates along a stochastic growth path. Some variables are stationary: the nominal interest rates, the long-term interest rate, inflation and hours worked. Consumption, real wages and output grow at the rate determined by z_t^* . The capital stock

and investment grow faster, due to increasing efficiency in the investment producing sector, at a rate determined by $z_t^* \Upsilon^t$, with $\Upsilon > 1$. The solution of our model with ambiguity follows the general steps described in section 3. The solution of our model builds on the standard approach to solve for a rational expectations equilibrium with the difference that we need to take into account that the worst-case scenario expectations do not materialize on average ex-post. Therefore, the solution involves the following procedure. First, we solve the model as a rational expectations model in which expectations are correct on average. Here we follow the standard approach of solving these type models: we rewrite the model in terms of stationary variables by detrending each variable using its specific trend growth rate. Then we find the non-stochastic steady state for this detrended system and construct a log-linear approximation around it. We then solve the resulting linear system of rational expectations equations. With that law of motion in hand, we then correct for the fact that the true dynamics of the productivity process follow the process in (4.7) in which $a_t = 0$.

The model has 9 fundamental shocks:

$$[\epsilon_t^x, \mu_{z,t}^{x,*}, \epsilon_{R,t}^x, g_t^x, \mu_{\Upsilon,t}^x, \lambda_{f,t}^x, \zeta_{I,t}^x, \zeta_{l,t}^x, \zeta_{c,t}^x]$$

and the ambiguity shock v_t^x .

4.1.5 Estimation and Data

The linearity of the state space representation of the model and the assumed normality of the shocks allow us to estimate the model using standard Bayesian methods as discussed for example in An and Schorfheide (2007) and Smets and Wouters (2007). We estimate the posterior distribution of the structural parameters by combining the likelihood function with prior information. The likelihood is based on the following vector of observable variables:

$$[\Delta \log Y_t^G, \Delta \log I_t, \Delta \log C_t, \Delta \log W_t, \log L_t, \log \pi_t, \log R_t, \Delta \log P_{I,t}]$$

where Δ denotes the first difference operator. The vector of observables refers to data for US on GDP growth rate, investment growth rate, consumption growth rate, real wage growth rate, log of hours per capita, log of gross inflation rate, log of gross short term nominal interest rate and price of investment growth rate. The sample period used in the estimation is 1984Q1-2008Q1. In the state space representation we do not allow for a measurement error on any of the observables.

We now discuss the priors on the structural parameters. The only parameter we calibrate is the share of government expenditures in output which is set to match the observed

empirical ratio of 0.22. The rest of the structural parameters are estimated. The priors on the parameters not related to ambiguity and thus already present in the standard medium scale DSGE are broadly in line with those adopted in previous studies (e.g. Justiniano et al. (2010), Christiano et al. (2010b)). The prior for each of the autocorrelation parameter of the shock processes is a Beta distribution with a mean of 0.5 and a standard deviation of 0.15.² The prior distribution for the standard deviation of the 9 fundamental shocks is an Inverse Gamma with a mean of 0.01 and a standard deviation of 0.01.

Regarding the choice over the ambiguity parameters, recall the discussion in 4.1.2. In preliminary estimations of the model, we find that the constraint in (4.11) between the amount of unconditional volatility and mean ambiguity is binding.³ With equality, the constraint implies that we can estimate two parameters out of the three: \bar{v} , σ_v and ρ_v . We choose to work with the following scaling between the mean amount of ambiguity and the standard deviation of the technology shock

$$\bar{v} = \frac{\sigma_\epsilon}{n} \quad (4.12)$$

Given this scaling, the binding constraint in (4.11) implies that:

$$\sigma_v = \frac{\sigma_\epsilon}{n} \frac{\sqrt{1 - \rho_v^2}}{m}$$

We fix $m = 2$ so that a negative shock of two unconditional standard deviations is still in the positive domain. So, the structural parameters to be estimated are n and ρ_v .

Alternatively, instead of the scaling in (4.12), we could estimate directly the parameters ρ_v and σ_v . The mean ambiguity would then be given by the binding constraint in (4.11). We find that the results are very insensitive to this choice. An advantage of the scaling in (4.12) is that it shows a direct link between how much ambiguity about the mean innovation $\sigma_\epsilon \epsilon_t^x$ and the standard deviation of that innovation.

In the benchmark model, the prior on the scaling parameter n is a Gamma distribution with mean 4 and standard deviation equal to 2. The prior is loose and it allows a wide range of plausible values. The mean is based on a statistical consideration related to the concept of detection error probability (see Hansen and Sargent (2008)). A value of 4 corresponds to

²The only exception is the prior on the autocorrelation of the price markup shock, for which the standard deviation is set to 0.05.

³More precisely, if the three parameters characterizing ambiguity are separately estimated we find that the implied unconditional volatility of the v_t process is so large that it implies very frequent negative realizations to v_t . Additionally there are further issues of identification: if we do not fix other structural parameters, such as β , we can easily run into identification problems of \bar{v} . This is a further reason to impose a relationship between the ambiguity parameters.

a detection probability of 10% when the sample size is equal to 200. The prior on ρ_v follows the pattern of the other autocorrelation coefficients and is a Beta distribution with a mean of 0.5 and a standard deviation of 0.15.

The prior and posterior distributions are described in Table 2. The posterior estimates of our structural parameters that are unrelated to ambiguity are in line with previous estimations of such medium scale DSGE models (Del Negro et al. (2007), Smets and Wouters (2007), Justiniano et al. (2010), Christiano et al. (2010b)). These parameters imply that there are significant ‘frictions’ in our model: price and wage stickiness, investment adjustment costs and internal habit formation are all substantial. The estimated policy rule is inertial and responds strongly to inflation but also to output gap and output growth. Given that these parameters have been extensively analyzed in the literature, we now turn attention to the role of ambiguity in our estimated model.

4.2 Results

We evaluate the importance of ambiguity in our model along two dimensions: the steady state effect of ambiguity and its role in business cycle fluctuations. We will argue that ambiguity plays an important part along both of these dimensions.

4.2.1 Steady state

The posterior mode of the structural parameters of ambiguity implies that the mean level of ambiguity is

$$\bar{v} = \frac{\sigma_\epsilon}{n} = \frac{0.0029}{0.19} = 0.015$$

which means that the ambiguity averse agent is on average concerned about a mean one-step ahead future technology level that is 1.5% lower than the true technology, normalized to 1. In the long run, the agent expects the technology level to be ϵ^* , which solves:

$$\log \epsilon^* = \rho_\epsilon \log \epsilon^* - \bar{v}.$$

For the estimated $\rho_\epsilon = 0.736$, we get that $\epsilon^* = 0.944$. Thus, the ambiguity-averse agent expects under his worst-case scenario evaluations the long run mean technology to be approximately 6% lower than the true mean.

Based on these estimates we can directly find that the standard deviation of the innovations to ambiguity is

$$\sigma_v = 0.0026$$

which is close to the estimated standard deviation of the technology innovations.

Our interpretation of the reason why we find a relatively large \bar{v} is the following: the estimation prefers to have a large σ_v because the ambiguity shock provides a channel in the model that delivers dynamics that seem to be favored by the data. Indeed, as detailed in the next section, the ambiguity shocks generate comovement between variables that enter in the observation equation. This is a feature that is strongly in the data and is not easily captured by other shocks. Given the large role that the fit of the data places on the ambiguity shock, the implied estimated σ_v is relatively large. Because of the constraint on the size of the mean ambiguity in (4.11), this results also in a large required steady state ambiguity. Thus, given also the estimated σ_ϵ , the posterior mode for n is small. The picture that comes out of these estimates is that ambiguity is large in the steady state, it is volatile and persistent.

The estimated amount of ambiguity has substantial effects on the steady state of endogenous variables. To describe these effects we perform the following calculations. We fix all the estimated parameters of the model but change the estimated standard deviation of the transitory technology shock, σ_ϵ , from its estimated value of $\bar{\sigma}_\epsilon = 0.0029$ to being equal to 0. When $\sigma_\epsilon = 0$, then the level of ambiguity \bar{a} is also equal to 0. By reporting the difference between the steady states with $\sigma_\epsilon = \bar{\sigma}_\epsilon > 0$ and with $\bar{\sigma}_\epsilon = 0$ we calculate the effect on steady states of fluctuations in transitory technology that goes through the estimated amount of ambiguity. In Table 1 we present a net percent difference of some variables of interest between the two cases, i.e. for a variable X we report $100[X_{SS,(\sigma_\epsilon=\bar{\sigma}_\epsilon)}/X_{SS,(\sigma_\epsilon=0)} - 1]$, where $X_{SS,(\sigma_\epsilon=\bar{\sigma}_\epsilon)}$ and $X_{SS,(\sigma_\epsilon=0)}$ are the steady states of variable X under $\sigma_\epsilon = \bar{\sigma}_\epsilon$ and respectively $\bar{\sigma}_\epsilon = 0$.

Table 1: Steady state percent difference from zero fluctuations

Variable	<i>Welfare</i>	<i>Output</i>	<i>Capital</i>	<i>Consumption</i>	<i>Hours</i>	<i>Nom.Rate</i>
	-7.6	-3.4	-0.68	-2.9	-5.7	-38

As evident from Table 1, the effect of fluctuations in the transitory technology shock that goes through ambiguity is very substantial. Output, capital, consumption, hours are all significantly smaller when $\sigma_\epsilon = \bar{\sigma}_\epsilon$. The nominal interest rate is smaller by 38%, which corresponds to the quarterly steady state interest rate being lower by 69 basis points. Importantly, the welfare cost of fluctuations in this economy is also very large, of about 8% of steady state consumption. These effects are much larger than what it is implied by the standard analysis featuring only risk. By standard analysis we mean the strategy of shutting down all the other shocks except the transitory technology and computing a second order approximation of the model in which there is no ambiguity but $\sigma_\epsilon = \bar{\sigma}_\epsilon$. For such

a calculation, we find that the welfare cost of business cycle fluctuations is around 0.01% of steady state consumption. The effects on the steady state values of the other variables reported in Table 1 is negligible. This latter result is consistent with the conclusions of the literature on the effect of business cycle fluctuations and risk.

4.2.2 Business cycle fluctuations

In this section we analyze the role of time-varying ambiguity in generating business cycles. We highlight the role of ambiguity by discussing three main points: a theoretical variance decomposition of variables; a historical variance decomposition based on the smoothed shocks and impulse responses experiments.

Table 3 reports the unconditional variance decomposition of several variables of interest. For each structural shock we compute the share of the total variation in the corresponding variable that is accounted by that shock at an infinite horizon by solving the dynamic Lyapunov equation characterizing the law of motion of the model. We report two values: the first is the share in the benchmark model with ambiguity. The second, the value in squared brackets, is the share in an estimated model without ambiguity.

The main message of this exercise is that ambiguity shocks can be a very influential factor in explaining the variance of key economic variables. This shock accounts for about 46% of GDP variability and explains a large share of real variables such as investment (23%), consumption (41%), hours (44%), wages (44%) but also of inflation (12%), while it does so less for the nominal interest rate (5%). It is interesting that the shock is very influential in explaining simultaneously several variables. This simultaneous effect is relatively different from the one implied by the standard shocks that have been recently used to explain macroeconomic variability in medium scale DSGE models. For a comparison, we can analyze the estimated model without ambiguity. There the largest share of GDP variability is explained by the marginal efficiency of investment shock, confirming the results of many recent studies, such as Justiniano et al. (2010) and Christiano et al. (2010b). But there are several other shocks that are important in explaining a particular variable. For example the intertemporal preference shock helps explain consumption, the price mark-up shock explains inflation, the labor supply largely explains wages. When time-varying ambiguity is introduced, this shock reduces the importance of all of these shocks in explaining the above mentioned variables. The simultaneous large shares of variation explained by ambiguity suggest that time-variation in the confidence of the agents about transitory technology shocks can be a unified source of macroeconomic variability.

We now turn to discussing the historical variance decomposition and the smoothed shocks that result from the estimated model. In Figure 1 we plot the smoothed ambiguity shock,

as a deviation from its steady state value. The figure first shows that ambiguity is very persistent. After an initial increase around 1991, which also corresponds to an economic downturn, the level of ambiguity was low and declining during the 90's, reaching its lowest values around 2000. It then increase back to levels close to steady state until 2005. Following a few years of relatively small upward deviations from its mean, ambiguity spikes starting in 2008. Ambiguity rapidly increases to reach its peak in 2009Q1, when it is 4 times larger than its 2008Q1 value. The figure shows a dotted vertical line at 2008Q3, which corresponds to the Lehman Brothers bankruptcy. Our model interprets the period following 2008Q1 as one in which ambiguity about future productivity has increased dramatically. It thus offers a formal way to think about time variation in agents confidence in their probability assessments.

Based on these smoothed path of ambiguity shocks we can now calculate what the model implies for the historical evolution of endogenous variables. In Figure 2 we compare the observed data with the hypothetical historical evolution for the growth rate of output, consumption, investment and inflation when the ambiguity shock is the only shock active in the model economy. The ambiguity shock implies a path for variables that come close to matching the data, especially for output and consumption. The model implied path of investment is less volatile but the correlation with the observed data is still significantly large. The ambiguity shock also helps explain the historical path for inflation, except for the last recession where it predicts that inflation should have gone up.

The ambiguity shock generates the three large recessions observed in this sample. Indeed, if we analyze the smoothed path of the shock in figure 2, the time-varying ambiguity helps explain the recession of the 1991, the large growth of the 1990's (as a period of low ambiguity), and then the recession of 2001. Given that the estimated ambiguity still continues to rise through 2005, the model misses by predicting a more prolonged recession than in the data, where output picks up quickly. The rise in ambiguity in 2008 predict in the model that output, investment and consumption fall. The model matches the fall in consumption, but fails to generate a large fall in investment.

It is important to highlight that the ambiguity shock implies that in the model consumption and investment comove. Indeed, in the historical decomposition, recessions are times when both of these variables fall. This is an important effect because standard shocks that have been recently found to be quantitatively important, such as the marginal efficiency of investment or intertemporal preference shocks imply a significant negative comovement between these two components. The negative comovement can be clearly seen in the impulses responses to shocks to the intertemporal preference and marginal efficiency, reported in Figures 4 and 5.

We conclude this section by analyzing the impulse responses for the ambiguity shock in the estimated model. As suggested already in the discussion, an increase in ambiguity generates a recession, in which hours worked, consumption and investment fall. The fact that this shock predicts comovement between these variables is an important feature that helps explain why the estimation prefers through the maximization of its likelihood such a shock. Figure 3 plots the responses to a one standard deviation increase in ambiguity for the estimated model. On top of the mentioned comovement, the model also predicts a fall in the price of capital, a fall in the real interest rate and a countercyclical excess return.

We briefly explain these results. The main intuition in understanding the effect of this shock is to relate it to its interpretation of a news shock. An increase in ambiguity makes the agent act under a more cautious forecast of the future technology. From an outside observer that analyzes the agent's behavior, it seems that this agent acts under some negative news about future productivity. This negative news interpretation of the increase in ambiguity helps explain the mechanics and economics of the impulse response. As described in detail in Christiano et al. (2008), Christiano et al. (2010a), in a rational expectations model, a negative news about future productivity can produce a significant bust in real economy while simultaneously generating a fall in the price of capital. This result is reflected in our impulse response. In our model, the negative news is on average not materialized, because nothing changed in the true process for technology, as shown in the first panel of Figure 3. However, because of the persistent effect of ambiguity, the economy continues to go through a prolonged recession. The ex-post excess return, defined as the difference between the realized return on capital and the risk-free rate, is positive following the period of the initial increase in ambiguity. The ex-ante excess return is always equal to zero, as we solve a linearized model. The explanation for the countercyclical excess returns is that the negative expectation about future productivity does not materialize, so ex-post, capital pays more. The ex-post excess return is a rational uncertainty premium that ambiguity-averse agents require to invest in the uncertain asset.

We can conclude that an increase in Knightian uncertainty (ambiguity) generates in our estimated model a recession, in which consumption, investment and price of capital fall, while producing countercyclical ex-post excess returns. Given these dynamics, we believe that time-varying ambiguity can be an important source of observed business cycle dynamics.

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Figure 1: Smoothed ambiguity shock

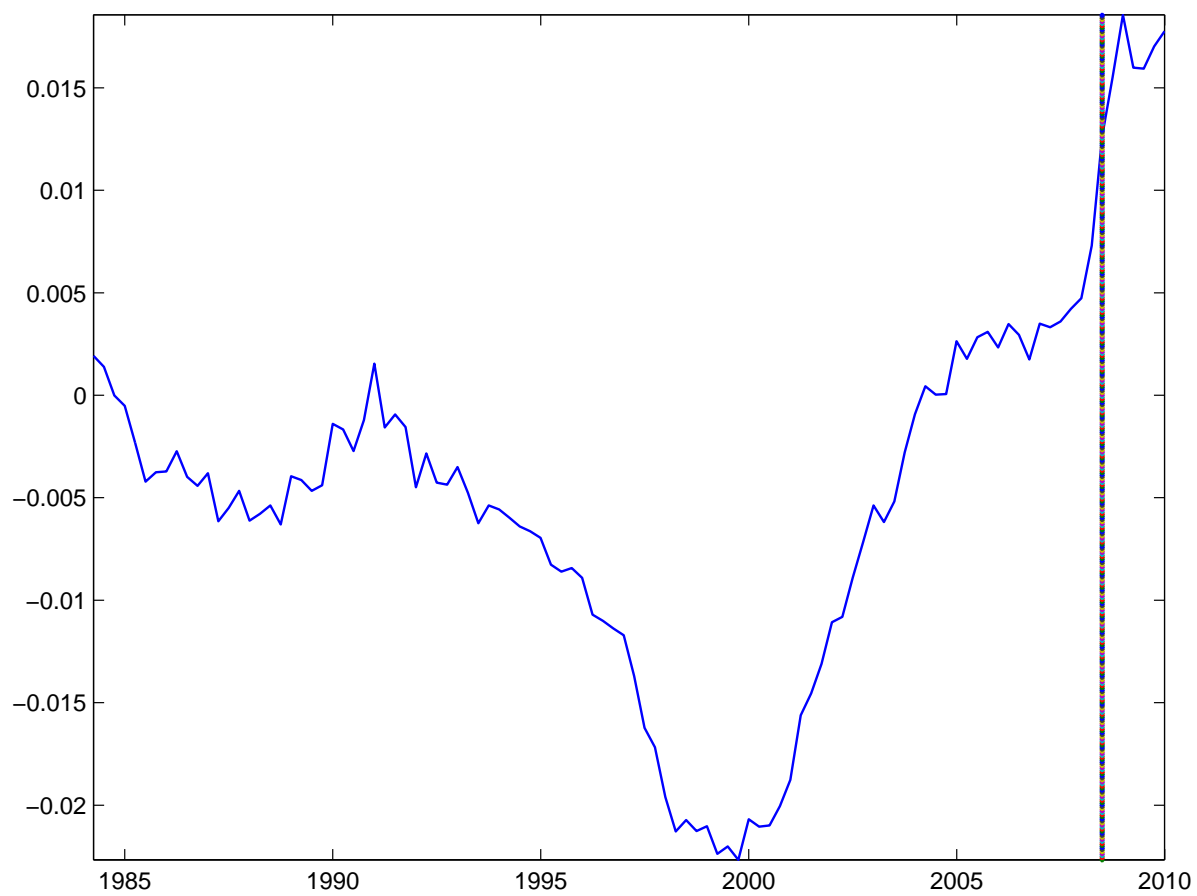


Figure 2: Historical shock decomposition

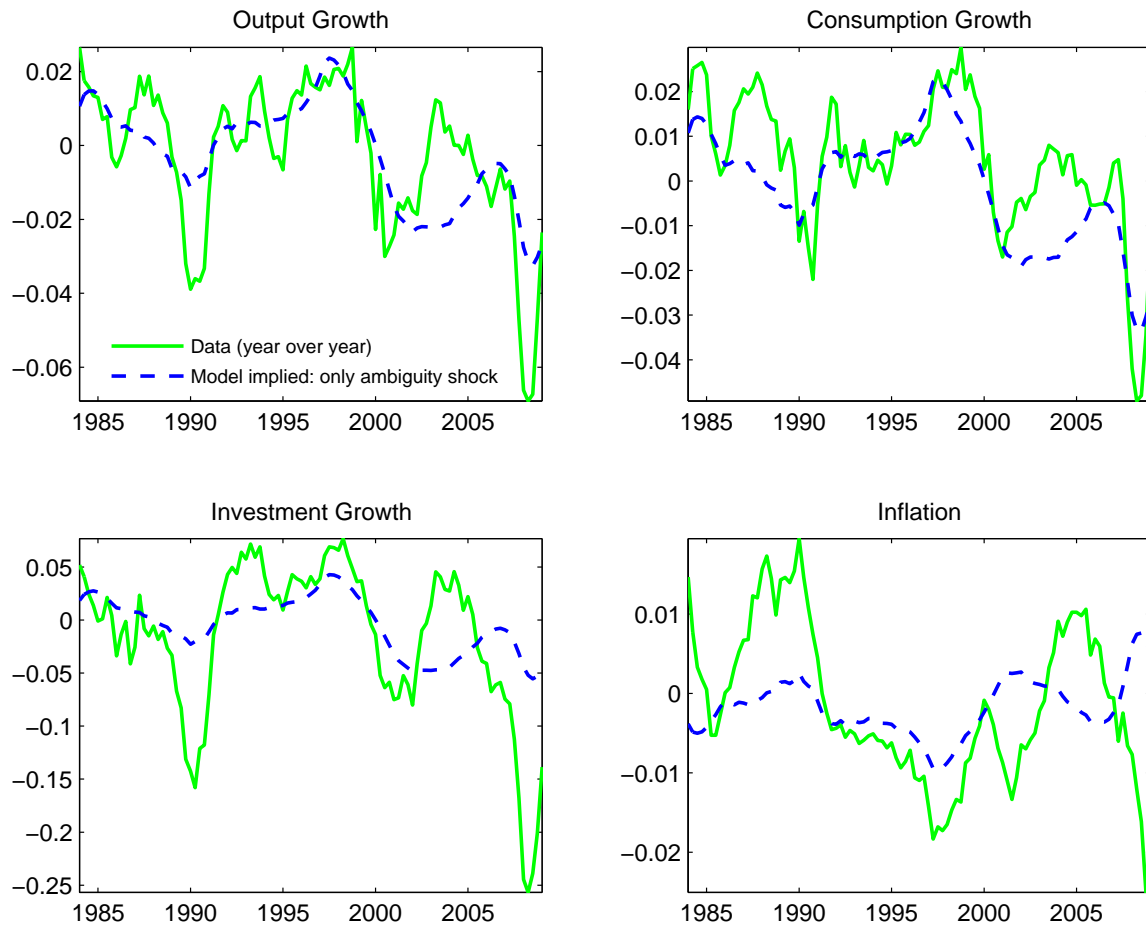


Table 2: Priors and Posteriors for structural parameters

Parameter	Description	Prior			Posterior	
		Type	Mean	St.dev	Mode	St.dev
α	Capital share	B	0.4	0.02	0.318	0.014
δ	Depreciation	B	0.025	0.001	0.0245	0.001
$100(\beta^{-1} - 1)$	Discount factor	G	0.3	0.05	0.356	0.04
$100(\mu_z^* - 1)$	SS net growth rate	N	0.4	0.08	0.45	0.06
$100(\mu_{\Upsilon} - 1)$	SS price of investment net growth rate	N	0.4	0.08	0.45	0.01
$\bar{\pi} - 1$	SS net inflation	N	0.006	0.002	0.01	0.001
ξ_p	Calvo prices	B	0.375	0.1	0.843	0.02
ξ_w	Calvo wages	B	0.375	0.1	0.635	0.06
S''	Investment adjustment cost	G	10	5	10.26	3.47
σ_a	Capacity utilization	G	2	1	1.21	0.64
a_π	Weight on inflation in Taylor rule	N	1.7	0.3	2.2	0.19
a_y	Weight on output in Taylor rule	N	0.15	0.05	0.1	0.04
a_{gy}	Weight on output growth in Taylor rule	N	0.15	0.05	0.23	0.04
ρ_R	Coefficient on lagged interest rate	B	0.5	0.15	0.84	0.01
$\lambda_f - 1$	SS price markup	N	0.2	0.05	0.22	0.04
$\lambda_w - 1$	SS wage markup	N	0.2	0.05	0.156	0.05
θ	Internal habit	B	0.5	0.1	0.78	0.04
σ_L	Curvature on disutility of labor	G	2	1	2.65	0.9
n	Level ambiguity scale parameter	G	4	2	0.19	0.06
ρ_ϵ	Transitory technology	B	0.5	0.15	0.736	0.08
$\rho_{\mu_z^*}$	Persistent technology	B	0.5	0.15	0.439	0.11
ρ_{ζ_i}	Marginal efficiency of investment	B	0.5	0.15	0.657	0.06
ρ_{ζ_L}	Labor supply	B	0.5	0.15	0.42	0.1
ρ_{λ_f}	Price mark-up	B	0.5	0.05	0.465	0.05
ρ_g	Government spending	B	0.5	0.15	0.942	0.01
$\rho_{\mu_{\Upsilon}}$	Price of investment	B	0.5	0.15	0.955	0.01
ρ_{ζ_c}	Intertemporal Preference	B	0.5	0.15	0.727	0.08
ρ_v	Level Ambiguity	B	0.5	0.15	0.937	0.01
σ_ϵ	Transitory technology	IG	0.01	0.01	0.0029	0.0004
$\sigma_{\mu_z^*}$	Persistent technology	IG	0.01	0.01	0.0062	0.001
σ_{ζ_i}	Marginal efficiency of investment	IG	0.01	0.01	0.023	0.002
σ_{ζ_L}	Labor supply	IG	0.01	0.01	0.005	0.002
σ_{λ_f}	Price mark-up	IG	0.005	0.01	0.004	0.001
σ_g	Government spending	IG	0.01	0.01	0.019	0.0015
$\sigma_{\mu_{\Upsilon}}$	Price of investment	IG	0.01	0.01	0.003	0.0002
σ_{ϵ_R}	Monetary policy shock	IG	0.005	0.01	0.0014	0.0001
σ_{ζ_c}	Intertemporal Preference	IG	0.01	0.01	0.02	0.0035

Notes: B refers to the Beta distribution, N to the Normal distribution, G to the Gamma distribution, and IG to the Inverse-gamma distribution.

Table 3: Theoretical variance decomposition

Shock\Variable	Output	Cons.	Investm.	Hours	Inflation	Int. rate	Wage
TFP Ambiguity (v_t)	46.12	41.58	23.58	44	12	4.68	44
	[-]	[-]	[-]	[-]	[-]	[-]	[-]
Transitory technology (ϵ_t)	0.18	0.11	0.13	2.17	1.44	0.75	0.09
	[0.43]	[0.34]	[0.21]	[2.86]	[1.07]	[0.65]	[0.14]
Persistent technology ($\mu_{z,t}^*$)	7.39	23.14	1	5	8.5	5.7	23.29
	[6.5]	[24.33]	[1.3]	[6.08]	[2.88]	[3.12]	[25.3]
Government spending (g_t)	2.84	6	2.2	5.37	1.5	3.75	0.48
	[3.8]	[7.39]	[1.24]	[7.15]	[0.88]	[2.84]	[0.38]
Price mark-up ($\lambda_{f,t}$)	0.4	0.2	0.33	0.5	35.86	9.2	3
	[3]	[1.68]	[1.8]	[3.28]	[62.9]	[19.1]	[11.3]
Monetary policy ($\epsilon_{R,t}$)	1.28	0.75	1	1.6	3.47	22.8	0.55
	[2.7]	[1.8]	[1.47]	[3]	[1.82]	[25.4]	[1.4]
Price of investment ($\mu_{\Upsilon,t}$)	2	1.68	5	0.28	0.32	0.9	1
	[2.7]	[2.24]	[4.66]	[0.57]	[0.37]	[0.7]	[1.33]
Preference ($\zeta_{c,t}$)	2.14	12.8	1.83	3.26	3.64	6.1	0.66
	[4.3]	[34.5]	[2.53]	[6.1]	[4.5]	[8.37]	[2.12]
Efficiency of investment ($\zeta_{I,t}$)	31.56	9.8	60.66	30.34	13.72	32.9	7.93
	[66.8]	[19.8]	[82.15]	[60.13]	[8.3]	[28]	[16.6]
Labor supply ($\zeta_{L,t}$)	6	3.87	4.25	7.57	19.54	13.2	18.8
	[9.7]	[7.9]	[4.6]	[10.8]	[17.2]	[11.8]	[41.29]

Note: For each variable, the first row of numbers refers to the variance decomposition in the estimated model with ambiguity. The numbers in the second row, in squared brackets, refer to the estimated model without ambiguity.

Figure 3: Impulse response: ambiguity

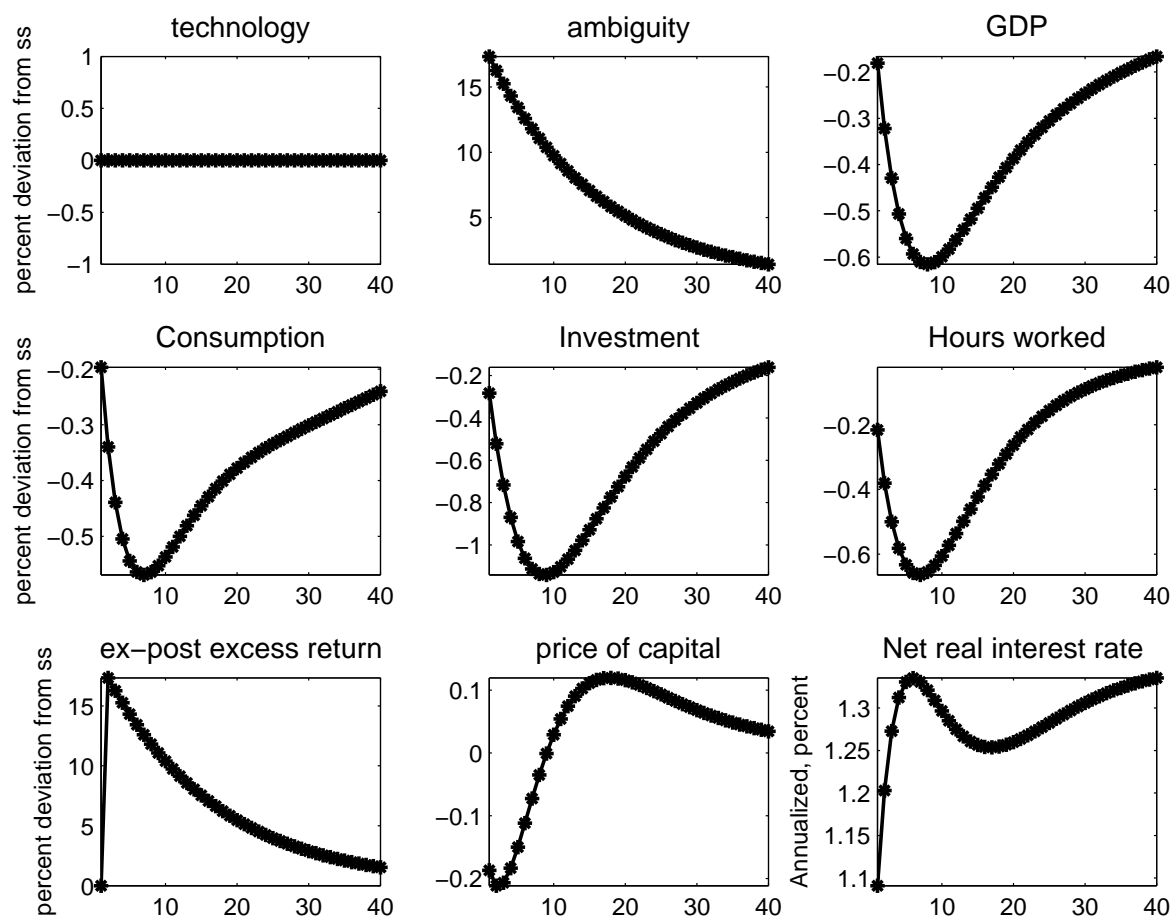


Figure 4: Impulse response: efficiency of investment

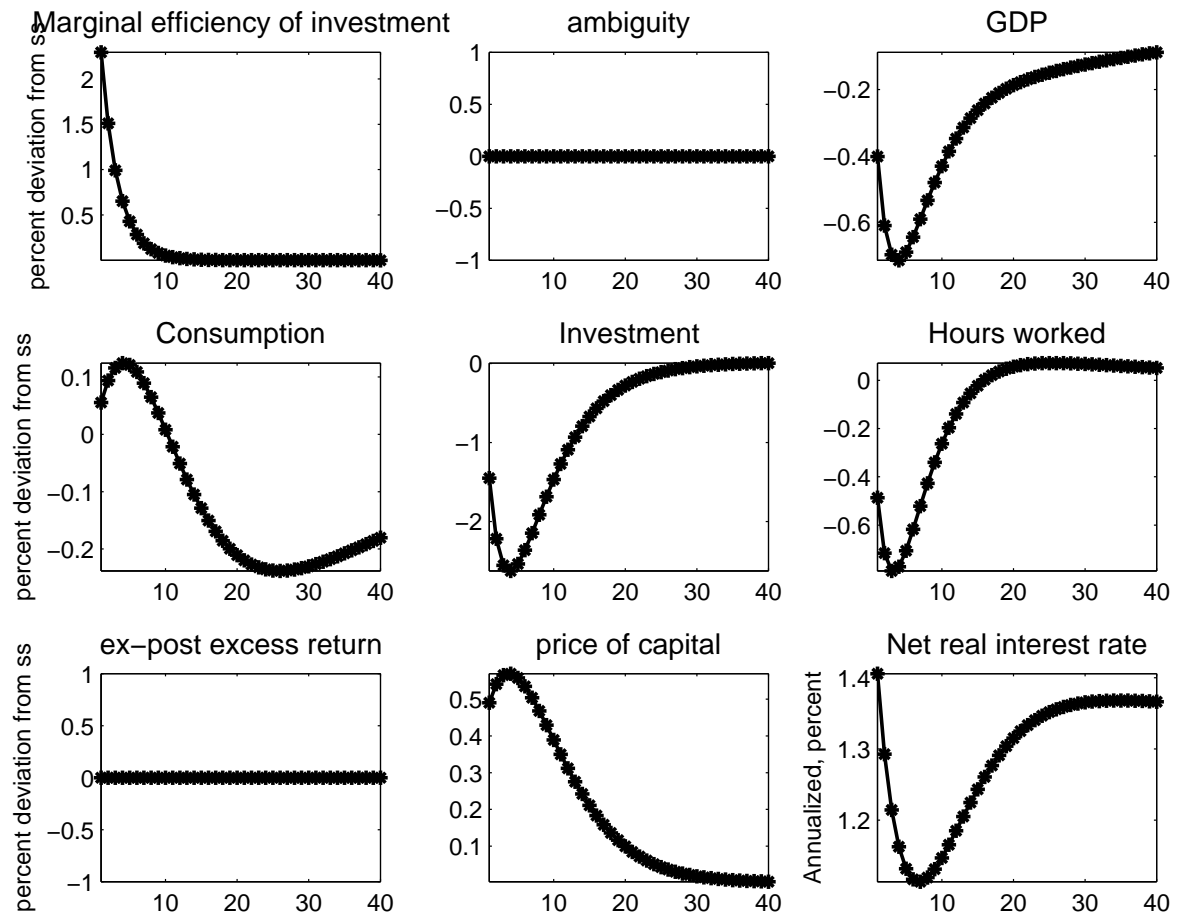


Figure 5: Impulse response: intertemporal preference

