An Equilibrium Analysis of Real Estate Leases

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This article provides a unified equilibrium approach to valuing a wide variety of commercial real estate lease contracts. Using a game-theoretic variant of real options analysis, the underlying real estate asset market is modeled as a continuous-time Nash equilibrium in which developers make construction decisions under demand uncertainty. Then, using the economic notion that leasing simply represents the purchase of the use of the asset over a specified time frame, I use a contingent-claims approach to value many of the most common real estate leasing arrangements. In particular, the model provides closed-form solutions for the equilibrium valuation of leases with options to purchase, pre-leasing, gross and net leases, leases with cancellation options, ground leases, escalation clauses, lease concessions and sale-leasebacks.

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1 Introduction

Commercial real estate, which includes office, retail, industrial, apartment and hotel properties, represents a significant fraction of the investment universe.[1] The ultimate value of commercial real estate emanates from its rental flow, which reflects the price the market is willing to pay for the use of space.[2] While real estate leasing contracts come in an almost endless variety, the terms can almost always be reduced to the fundamental building blocks of financial economics. Examples of analogies to traditional financial contracts abound. Just as there exists a term structure of interest rates, so to is there a term structure of lease rates. Debt contracts with call and put options are paralleled in lease contracts with extension and cancellation options. Fixed and floating notes are analogous to flat and indexed leases. Forward contracts have their leasing counterpart in the pre-leasing of space.

A real estate lease is simply the sale of the use of space for a specified period of time. The tenant receives the benefits of using the space and the landlord receives the value of lease payments. While the contractual specifications of leases can be quite complex, in equilibrium the value of the lease payments must equal the value of the use of space. Valuing the use of space is made simple by using the following option pricing analogy: the value of leasing an asset for T years is economically equivalent to a portfolio consisting of buying the building and simultaneously writing a European call option on the building with expiration date T and a zero exercise price. This characterization of leasing is explicitly derived by Smith (1979) using an option-pricing approach to valuing corporate liabilities. Thus, in equilibrium, the value of the stream of lease payments must equal the value of this portfolio of assets. Leases are simply contingent claims on building values.

This paper provides a unified equilibrium approach to valuing leasing contracts. By the term unified equilibrium I refer to the fact that there must be simultaneous equilibrium in the leasing market and the underlying asset market. From the preceding paragraph, the equilibrium value of a lease must equal the equilibrium value of a portfolio whose value is contingent on the underlying building value. At the same time, the value of a building is driven by the equilibrium obtained in the underlying asset market, where developers choose optimal construction strategies in the face of competition and uncertainty. In essence, this paper values leases as a contingent claim on building values, where building values themselves are determined in an industry equilibrium. Simpler models that take building values (or lease rates) as exogenous are unlikely to be consistent with equilibrium in the underlying asset market, and ignore many of the most fundamental drivers of real estate economics. In this pa-

[1] Miles and Tolleson (1997) provide a conservative estimate of the value of U.S. commercial real estate of $4 trillion, as of 1997. As of 1997, the value of commercial real estate was greater than the combined value of publicly traded Treasury securities and corporate bonds.

[2] While about one third of commercial real estate is owner-occupied, the market rent still represents an opportunity cost of using the space.

per, lease rates will reflect critical real estate market variables such as the degree of concentration of developers, uncertainty over the future demand for space, and the current level of construction activity.

While the literature on leasing is an extensive one, the vast majority of work is concerned with the tax implications of leasing versus borrowing.[3] While taxes play an important role in the lease versus buy decision, this paper instead focuses on the economic benefits accruing to the user of the asset. The underlying approach of this paper is in the tradition of Miller and Upton (1976), with the focus on the economic aspects of leasing. Leasing is simply a mechanism for selling the use of the asset for a specified period of time, without necessitating a transfer of ownership.[4] Miller and Upton (1976), in a classic article on the economics of leasing, provide an equilibrium determination of rental rates in a discrete time model. McConnell and Schallheim (1983) use the short-term equilibrium model of Miller and Upton, and apply it towards the pricing of an impressive assortment of options embedded in leases. The model in this article is most similar to Grenadier (1995), that connects an explicit equilibrium in the underlying asset market with the valuation of leases of a variety of forms. However, the present paper differs in several important respects from Grenadier (1995). First, the underlying equilibrium in the present model captures the game-theoretic strategic aspects of the real estate market, while Grenadier (1995) assumes that agents are price takers in a perfectly competitive market. Second, while Grenadier (1995) focuses on general asset leasing contracts, the present paper focuses on real estate leasing, and provides numerous applications that are not present in Grenadier (1995).

The equilibrium in the underlying asset market is modeled using the method of real options. However, unlike the typical real option approach that ignores strategic interactions, this model uses an explicit game-theoretic equilibrium model in order to realistically model the strategic behavior of rational developers.[5] Using a spe-

[3]See Schall (1974), Myers, Dill, and Bautista (1976), Brealey and Young (1980), and Lewis and Schallheim (1992) for a discussion of the importance of the tax implications of leasing.

[4]As Smith and Wakeman (1985) point out, if the right to use the asset is unbundled from the right to the ownership of the residual asset value, then the incentives for the use and maintenance of the asset can change. For assets whose values are sensitive to these decisions, this assumption becomes less realistic.

[5]There have been relatively few real options applications concerned with strategic equilibrium in exercise policies. This is partially a result of the fact that such issues are typically ignored in the financial options literature, as most financial options are widely held side-bets between agents external to the underlying firm, with the notable exception of warrants and convertible securities [Emanuel (1983), Constantinides (1984), Spatt and Sterbenz (1988)]. Grenadier (1996) uses option exercise games to understand real estate development. Smets (1993) uses an option exercise game for an application in international finance. An application to investment in strategic settings is Kulatilaka and Perotti (1998). Grenadier (1999) analyzes the case of a strategic equilibrium in exercise strategies under asymmetric information over the underlying option parameters. Leahy (1993) analyzes the special case of investment strategies in a perfectly competitive industry. Williams (1993) provides the first rigorous derivation of a Nash equilibrium in a real options framework.

cial case of the option exercise game framework developed by Grenadier (2002), a continuous-time Nash equilibrium is derived for an oligopolistic real estate market. Recent empirical work by Bulan, Mayer and Somerville (2001) shows that this strategic real options framework does a decidedly better job of characterizing real estate markets than traditional real options models. Bulan (2000) also finds empirical support for the strategic real options framework by analyzing the investment behavior of a panel of 2,470 U.S. firms. The model is one of a local real estate market consisting of n developers choosing optimal construction strategies. The developers must formulate their construction/exercise strategies in a world in which the demand for space is stochastic, and where the construction strategies of their competitors impact the payoff from their own strategies. Thus, the inputs of the equilibrium are the degree of competition in the market (n), the impact of competitive development (via the downward-sloping demand curve for space), and the stochastic process driving the demand for space. The outputs of the equilibrium are the processes for construction starts, short-term rents, building values and land values. Most importantly, the equilibrium building values serve as the underlying instrument for using the contingent claims approach for valuing long-term leasing contracts.

Given the underlying equilibrium model, the equilibrium rent on leases of any variety can be explicitly derived. The first lease analyzed is the basic fixed payment lease: a T-year lease with constant payment flow $R(T)$. The equilibrium level of $R(T)$ is presented and analyzed. Most notably, by varying the term of the lease T, the entire term structure of lease rates is derived. Of particular interest is the impact of competition on the slope of the term structure of lease rates. For any given initial short-term lease rate, the terms structure may be upward-sloping, downward-sloping, or single-humped. The upward-sloping term structure is most likely in markets with only a few competitors, while the downward-sloping term structure is most likely in markets with many competitors.

Given the basic lease valuation model, I then apply the model to the analysis of many of the most common real world leasing structures. The first extension of the basic real estate leasing contract I analyze is a lease with an option to purchase the building at the end of the lease. Such embedded options are most common in freestanding single tenant buildings. The option to purchase may be included in a lease to align the incentives of the tenant and landlord. The purchase option provides an

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Importantly, Williams (1993) derives the equilibrium exercise strategies in a strategic setting, and finds that increasing competition leads to earlier exercise of options. The structure of the industry in Williams (1993) differs from that of the present model, and the resulting Nash equilibria differ. Lambrecht and Perraudin (1999) provide an example of an exercise game in which firms compete over the exercise of a real option. Their model uses a different solution approach than the present model, and deals with firms competing over a single investment opportunity. In contrast, the present model describes an industry equilibrium with multiple active firms. The equilibrium framework of Baldursson (1997) is very similar to the present model. However, the solution approach offered by Baldursson (1997) is only applicable to a very specific setting in which demand is linear; his methodology will not work for more general specifications.

incentive to the tenant to maintain the building as well as to refrain from defaulting on the contractual lease payments.

The second application is pre-leasing. This is essentially a forward lease contract, where a rent is established today for a lease that begins at a specific time in the future. Pre-leasing is sometimes required by lenders in order to ensure that a building is readily marketable in the event of default. Pre-leasing is also common in shopping centers, where the developer initially signs an anchor tenant in order to facilitate marketing the space to smaller tenants. Pre-leasing may also be a means by which prospective tenants lock in their future rent obligations.

The third leasing contract I examine are leases where some or all of the property's operating expenses are paid by the landlord. While the basic lease in the model is a net lease, where the tenant pays the operating expenses associated with the space, it is also common to find gross leases, where the landlord is responsible for paying some or all of the operating expenses. It is clear that there is an incentive component to the expense provisions of a lease. When a landlord pays operating expenses, the tenant has little incentive to economize on usage. When the tenant pays operating expenses, the tenant will internalize such costs. It is likely that in leases where the tenant has the most discretion over expense usage, the net lease form will be most likely to prevail. A particularly interesting leasing arrangement involves expense stops, where the tenant pays all expenses above a given threshold. The model is applied to determining equilibrium lease rates on leases under a variety of expense-sharing clauses.

The fourth application is the analysis of leases with cancellation clauses. In many real world leases, the expiration of the lease is stochastic, where leases contain clauses that permit the tenant (and sometimes the landlord) to change the length of the lease during the term of the lease. A cancellation option (also known as a "kick-out" clause) allows the lease to be terminated prior to expiration of the lease term, while a renewal option allows the lease to be extended beyond the initial term. Cancellation options and renewal options can be valued in the same manner, since a long-term lease with an option to cancel is economically identical to a short-term lease with an option to renew.

While the basic model focuses on the leasing of developed space, undeveloped land may also be leased under a ground lease. The fifth application of the model is an equilibrium valuation of ground leases. Under a ground lease, the landowner leases the land to a developer. Upon termination of a ground lease, the land and all improvements revert to the landowner. Typically ground leases have long terms (usually more than thirty years) with multiple renewal options. It is noteworthy that unless the ground lease is properly structured, such leasing arrangements result in inefficient development.

The sixth application is the study of lease escalation clauses. The basic lease in this article calls for a constant rent flow over the entire lease term. Such flat rent structures are particularly common on short-term leases. However, on longer-term leases, it is more common to find leases with rents varying over the term. Escalation clauses in leases specify a rule for determining the rent level at varying points in time during the term. I explicitly model three examples of escalation clauses: rents that move with the market, rents that move deterministically, and rents that move with an exogenous index.

The seventh application is the analysis of lease concessions. It is quite common for leases (particularly during market downturns) to contain one or more concessions for the tenant, such as free rent periods, subsidized moving costs and above-normal tenant improvement allowances. In any rational equilibrium, it is clear that leases offering concessions will result in a higher rent. Empirically, this contributes to the well-observed phenomenon of "sticky" quoted rents. During real estate downturns, quoted rents fall quite slowly, even when building values (and true market rents) fall precipitously.

The eighth and final application is the analysis of sale-leaseback agreements. Under a sale-leaseback agreement, the owner of a building (usually the sole user of the building) sells the building and simultaneously signs a lease on the building. Such transactions are typically justified as a form of financing; the seller/tenant uses the sales proceeds for business expansion and the lease payments represent financing payments. The contracted sales price and leaseback price are connected in equilibrium, with the model capable of explicitly characterizing this link.

This paper is organized as follows. Section 2 develops the strategic equilibrium in the underlying real estate asset market. Section 3 characterizes the term structure of lease rates. Sections 4 through 11 extend the basic model of lease valuation to many of the most common leasing variations: purchase options, pre-leasing, gross leases, cancellation options, ground leases, escalation clauses, lease concessions and sale-leaseback contracts. Section 12 concludes.

2 A Model of Equilibrium in the Underlying Real Estate Asset Market

The equilibrium value of any real estate leasing contract will be dependent on the equilibrium obtained in the underlying real estate asset market. In this section I present a model of equilibrium in which the equilibrium values of buildings, spot rents, construction starts and land values are endogenously determined. The model will represent a special case of the strategic industry equilibrium developed in Grenadier (2002). This asset market equilibrium will form the fundamental basis for the lease valuation framework used in all future sections of this article.

The general framework of the model is the real options approach to investment under uncertainty.[6] Each developer/landlord can be envisioned as holding a sequence of

[6] The application of the real option approach to investment is increasingly broad. Brennan and Schwartz (1985) use an option pricing approach to analyze investment in natural resources. Mc-

development opportunities that are analogous to call options on future construction. In the vast majority of option pricing models (both real and financial), the starting point is an exogenous process for the underlying asset value (e.g., the stock price in the Black-Scholes framework) or cash flows [e.g., Brennan and Schwartz (1985) or McDonald and Siegel (1986)]. However, in this case, the value of the payoff from investment (exercise) is endogenous as it depends on the exercise strategies of all developers operating in the market. In such a strategic environment, optimal exercise strategies cannot be derived in isolation, but must be calculated as part of a game-theoretic equilibrium.

2.1 Assumptions and Setup

Consider an oligopolistic local real estate market comprised of n identical firms that lease identical buildings. New supply of space enters the market when these firms choose to develop additional space. At time t, firm i owns $q_i(t)$ units of completed, rentable space. For simplicity, assume that space is infinitely divisible. The process $q_i(t)$ will thus be non-decreasing and continuous in both time and state space.

The essential determinant of value from owning or leasing a building is its underlying service flow: the provision of usable space. The economic benefits from the service flow are realized by the user of the space, while the owner of the building retains the right to sell this service flow to potential tenants. At each point in time, the price of the flow of services from the asset (net of expenses), or the instantaneous lease rate $P(t)$, evolves in order to clear the market.[7] Assume that the market inverse demand function is of a constant-elasticity form:

$$P(t) = X(t) \cdot Q(t)^{-\frac{1}{\gamma}}, \tag{1}$$

where $\gamma > 1/n$ and $Q(t) = \sum_{j=1}^{n} q_j(t)$ is the industry supply process.[8] Such a market is characterized by evolving uncertainty in the state of demand for space. At each point in time, even demand at the next instant is uncertain. $X(t)$ represents a multiplicative demand shock, and evolves as a geometric Brownian motion:

$$dX = \alpha X dt + \sigma X dz, \tag{2}$$

Donald and Siegel (1986) provided the standard continuous-time framework for analysis of a firm's investment in a single project. Majd and Pindyck (1987) enrich the analysis with a time-to-build feature. Dixit (1989) uses the real option approach to examining entry and exit from a productive activity. Triantis and Hodder (1990) analyze manufacturing flexibility as an option. Titman (1985) uses the real options approach to analyze real estate development.

[7] Since this service flow is net of expenses, these spot lease rates represent "net" lease rates. Under a fully net lease, the tenant pays the expenses. In a later section a model of "gross" lease rates is determined in which the landlord pays some or all of the expenses.

[8] The restriction on γ is necessary to ensure a well-defined equilibrium. See Grenadier (2002) for a discussion.

where α is the instantaneous conditional expected percentage change in X per unit time, σ is the instantaneous conditional standard deviation per unit time, and dz is the increment of a standard Wiener process. I assume that cash flows are valued in a risk-neutral framework, where r is the (constant) risk-free rate.

Consider potential examples of representations for the shock term, $X(t)$. For office space the demand might be driven by job growth. For industrial space demand might be driven by changes in industrial production. For hotel space demand might be driven by changes in disposable income.

At any point in time, each firm can develop new rentable units at a cost of K per unit of space. New development represents an increase in space denoted by the infinitesimal increment $dq_i \equiv \frac{dQ}{n}$. Thus, the path of output is continuous, and if all firms increase capacity simultaneously, $Q(t)$ increases by the increment dQ. Denote the supply process of all firms except firm i by $Q_{-i}(t) = \sum_{j=1,j\neq i}^{n} q_j(t)$. Thus, the rental flow for firm i can be expressed as a function of both its own supply and the supply of its competitors by $X(t) \cdot q_i(t) \cdot [q_i(t) + Q_{-i}(t)]^{-\frac{1}{\gamma}}$.

The optimal development decision must be part of an endogenous, Nash equilibrium. Each firm chooses its supply process strategy $q_i(t)$ so as to maximize its value, conditional on the assumed supply processes of its competitors. The n-tuple of strategies $[q_1^*(t), ..., q_n^*(t)]$ constitutes a Nash equilibrium if $q_i^*(t)$ is the optimal strategy for each firm i when it takes the strategies of its competitors, $Q_{-i}^*(t)$, as given. Mathematically, let $\hat{V}^i [X, q_i, Q_{-i}; q_i(t), Q_{-i}(t)]$ denote the value of firm i, for given strategies $q_i(t)$ and $Q_{-i}(t)$, where (X, q_i, Q_{-i}) are the initial values of the state variables. $\hat{V}^i$ can then be written as the discounted expectation of future cash flows:

$$\begin{aligned} \hat{V}^i [X, q_i, Q_{-i}; q_i(t), Q_{-i}(t)] &= E\left\{\int_0^\infty e^{-rt} q_i(t) \cdot X(t) \cdot [q_i(t) + Q_{-i}(t)]^{-\frac{1}{\gamma}} dt\right\} \\ &\quad -E\left[\int_0^\infty e^{-rt} K dq_i(t)\right], \end{aligned} \tag{3}$$

where the expectation operator is conditional on the current state $[X, q_i, Q_{-i}]$. Thus, the strategies $[q_1^*(t), ..., q_n^*(t)]$ constitute a Nash equilibrium if

$$\hat{V}^i \left[X, q_i, Q_{-i}; q_i^*(t), Q_{-i}^*(t)\right] = \sup_{\{q_i(t):t>0\}} \hat{V}^i \left[X, q_i, Q_{-i}; q_i(t), Q_{-i}^*(t)\right], \forall i. \tag{4}$$

I will focus on the case of a symmetric Nash equilibrium. For the case of a symmetric equilibrium, $q_i^*(t) = q_j^*(t)$ for all i,j, and thus $q_i^*(t) = Q^*(t)/n$, $\forall i$.

2.2 Derivation of Equilibrium Development Strategies

I now use the real options approach to solve for the equilibrium exercise (development) strategies. I will provide a mostly heuristic derivation of the Nash equilibrium. For a more rigorous derivation, see Grenadier (2002). I begin by considering firm i's optimal development strategy, where firm i takes all competitors' strategies as given.

Suppose that firm i's competitors are assumed to incrementally increase supply at each moment when $X(t)$ rises to the given trigger function $X^{-i}(q_i, Q_{-i})$, which for ease of notation I write as X^{-i}. Let $F^i[X, q_i, Q_{-i}; X^{-i}]$ denote the value of firm i, contingent on this assumed development strategy of its competitors. Using standard arguments [i.e., Dixit and Pindyck (1994), Grenadier (1995)], over a region in which no new development occurs $F^i[X, q_i, Q_{-i}; X^{-i}]$ must solve the following equilibrium differential equation:

$$0 = \frac{1}{2}\sigma^2 X^2 F^i_{XX} + \alpha X F^i_X - rF^i + X \cdot q_i \cdot (q_i + Q_{-i})^{-\frac{1}{\gamma}}. \tag{5}$$

The solution $F^i[X, q_i, Q_{-i}; X^{-i}]$ to differential equation (5) must then satisfy appropriate boundary conditions.

The first boundary condition that F^i must satisfy is a "value-matching" condition. Suppose firm i exercises its development option at the trigger $X^i(q_i, Q_{-i})$.[9] At the moment of exercise, q_i increases by the infinitesimal increment dq, and the firm pays the construction cost $K \cdot dq$. Thus, at the moment of development, $F^i[X^i(q_i, Q_{-i}), q_i, Q_{-i}; X^{-i}] = F^i[X^i(q_i, Q_{-i}), q_i + dq, Q_{-i}; X^{-i}] - K \cdot dq$, or in derivative form:

$$\frac{\partial F^i}{\partial q_i}\left[X^i(q_i, Q_{-i}), q_i, Q_{-i}; X^{-i}\right] = K. \tag{6}$$

The second boundary condition that F^i must satisfy is a condition that ensures that the trigger $X^i(q_i, Q_{-i})$ is determined optimally. This "smooth-pasting" condition is that $F^i_X[X^i(q_i, Q_{-i}), q_i, Q_{-i}; X^{-i}] = F^i_X[X^i(q_i, Q_{-i}), q_i + dq, Q_{-i}; X^{-i}]$. Writing this in derivative form:

$$\frac{\partial^2 F^i}{\partial q_i \partial X}\left[X^i(q_i, Q_{-i}), q_i, Q_{-i}; X^{-i}\right] = 0. \tag{7}$$

The third boundary condition that F^i must satisfy is a value-matching condition at the competitors' trigger function, $X^{-i}(q_i, Q_{-i})$. At the moment firm i's competitors exercise, Q_{-i} increases by the infinitesimal increment dQ_{-i}. Thus, at the moment of competitive exercise, $F^i[X^{-i}(q_i, Q_{-i}), q_i, Q_{-i}; X^{-i}] = F^i[X^{-i}(q_i, Q_{-i}), q_i, Q_{-i} + dQ_{-i}; X^{-i}]$, or in derivative form:

$$\frac{\partial F^i}{\partial Q_{-i}}\left[X^{-i}(q_i, Q_{-i}), q_i, Q_{-i}; X^{-i}\right] = 0. \tag{8}$$

Finally, F^i must satisfy the regularity condition:

$$F^i\left[0, q_i, Q_{-i}; X^{-i}\right] = 0, \tag{9}$$

[9] See Grenadier (2001) for a discussion of the conditions that ensure that such a trigger strategy is indeed optimal.

since zero is an absorbing barrier for $X(t)$.[10]

I can now fully characterize a symmetric Nash equilibrium in exercise strategies. Since the equilibrium is symmetric, $X^i(q_i, Q_{-i}) = X^{-i}(q_i, Q_{-i})$ for all i, and denote this common equilibrium trigger by $\bar{X}(q_i, Q_{-i})$. Therefore, for $\bar{X}(q_i, Q_{-i})$ to be the symmetric equilibrium trigger, $F^i\left[X, q_i, Q_{-i}; \bar{X}\right]$ must satisfy differential equation (5), subject to boundary conditions (6) - (9), where $X^i(q_i, Q_{-i}) = X^{-i}(q_i, Q_{-i}) = \bar{X}(q_i, Q_{-i})$. Write the equilibrium value of firm i as $V^i(X, q_i, Q_{-i}) \equiv F^i\left[X, q_i, Q_{-i}; \bar{X}\right]$. Such an equilibrium is then represented by the following system:

$$0 = \frac{1}{2}\sigma^2 X^2 V^i_{XX} + \alpha X V^i_X - rV^i + X \cdot q_i \cdot (q_i + Q_{-i})^{-\frac{1}{\gamma}}, \tag{10}$$

subject to:

$$\begin{aligned}
\frac{\partial V^i}{\partial q_i}\left[\bar{X}(q_i, Q_{-i}), q_i, Q_{-i}\right] &= K, \\
\frac{\partial^2 V^i}{\partial q_i \partial X}\left[\bar{X}(q_i, Q_{-i}), q_i, Q_{-i}\right] &= 0, \\
\frac{\partial V^i}{\partial Q_{-i}}\left[\bar{X}(q_i, Q_{-i}), q_i, Q_{-i}\right] &= 0, \\
V^i(0, q_i, Q_{-i}) &= 0.
\end{aligned} \tag{11}$$

The solution to differential equation (10), subject to boundary conditions (11), can be written as:

$$V^i(X, q_i, Q_{-i}) = A(q_i, Q_{-i}) \cdot X^\beta + \frac{X \cdot q_i \cdot (q_i + Q_{-i})^{-\frac{1}{\gamma}}}{r - \alpha}, \tag{12}$$

where $A(q_i, Q_{-i})$ and $\bar{X}(q_i, Q_{-i})$ satisfy:

$$\begin{aligned}
A_{q_i}(q_i, Q_{-i}) \cdot \bar{X}(q_i, Q_{-i})^\beta + \frac{\bar{X}(q_i, Q_{-i})}{r - \alpha}(q_i + Q_{-i})^{-\frac{1}{\gamma}}\left[1 - \frac{q_i}{\gamma(q_i + Q_{-i})}\right] &= K, \\
\beta A_{q_i}(q_i, Q_{-i}) \cdot \bar{X}(q_i, Q_{-i})^{\beta - 1} + \frac{1}{r - \alpha}(q_i + Q_{-i})^{-\frac{1}{\gamma}}\left[1 - \frac{q_i}{\gamma(q_i + Q_{-i})}\right] &= 0, \\
A_{Q_{-i}}(q_i, Q_{-i}) \cdot \bar{X}(q_i, Q_{-i})^\beta - \frac{\bar{X}(q_i, Q_{-i})}{r - \alpha}\frac{q_i}{\gamma}(q_i + Q_{-i})^{-\frac{\gamma+1}{\gamma}} &= 0,
\end{aligned} \tag{13}$$

with $\beta = \frac{-\left(\alpha - \frac{1}{2}\sigma^2\right) + \sqrt{\left(\alpha - \frac{1}{2}\sigma^2\right)^2 + 2r\sigma^2}}{\sigma^2} > 1$, and $r > \alpha$ to ensure convergence.

Because I focus on a symmetric equilibrium, the state space can be reduced. In equilibrium, $q_i = \frac{1}{n}Q$, and $Q_{-i} = \frac{n-1}{n}Q$. Thus, using the change of variables

[10] F^i must also satisfy regularity conditions that ensure that the value of the option to build approaches zero as the industry capacity approaches infinity. This is needed to ensure finite market values.

$G(X,Q) = V^i\left(X, \frac{1}{n}Q, \frac{n-1}{n}Q\right)$ and $X^*(Q) = \bar{X}(\frac{1}{n}Q, \frac{n-1}{n}Q)$, I obtain the equilibrium solution in closed-form:

$$G(X,Q) = B(Q) \cdot X^\beta + \frac{X \cdot Q^{\frac{\gamma-1}{\gamma}}}{n(r-\alpha)}, \tag{14}$$

where:

$$\begin{aligned} B(Q) &= \left(\frac{v_n^{-\beta}}{n}\right)\left(\frac{\gamma}{\gamma-\beta}\right)\left[K - \left(\frac{v_n}{r-\alpha}\right)\left(\frac{\gamma-1}{\gamma}\right)\right] \cdot Q^{\frac{\gamma-\beta}{\gamma}}, \\ X^*(Q) &= v_n Q^{\frac{1}{\gamma}}, \end{aligned} \tag{15}$$

and where $v_n = \left(\frac{\beta}{\beta-1}\right)\left(\frac{n\gamma}{n\gamma-1}\right)(r-\alpha) \cdot K$.

Therefore, the equilibrium development strategy is for each firm to develop an incremental unit whenever the state variable $X(t)$ rises to the trigger $X^*[Q(t)]$. The trigger function is an increasing function of Q. The explicit dependence of the equilibrium on the degree of competition is through the function v_n. The equilibrium trigger is a decreasing and convex function of n:

$$\begin{aligned} \frac{\partial X^*(Q)}{\partial n} &= -\frac{X^*(Q)}{n\,(n\gamma-1)} < 0, \\ \frac{\partial^2 X^*(Q)}{\partial n^2} &= \frac{2\gamma X^*(Q)}{n\,(n\gamma-1)^2} > 0. \end{aligned} \tag{16}$$

Increasing competition leads firms to develop sooner, as the fear of preemption diminishes the value of their "options to wait."

The Nash equilibrium exercise strategies coincide with the standard, non-strategic solutions that appear in the real options literature for two special cases: the case of monopoly ($n = 1$) and the case of perfect competition ($n \to \infty$). For a monopolist, exercise is triggered at $v_1 Q^{\frac{1}{\gamma}} = \left(\frac{\beta}{\beta-1}\right)\left(\frac{\gamma}{\gamma-1}\right)(r-\alpha) \cdot K$. For a perfectly competitive firm, exercise is triggered at $\lim_{n\to\infty} v_n Q^{\frac{1}{\gamma}} = \left(\frac{\beta}{\beta-1}\right)(r-\alpha) \cdot K$. For $1 < n < \infty$, the Nash equilibrium solution lies somewhere in between these non-strategic cases.

2.3 The Equilibrium Instantaneous Lease Rate

The equilibrium instantaneous lease rate, $P(t) = X(t) \cdot Q^{-\frac{1}{\gamma}}$, takes on a particularly simple form. Since $Q(t)$ increases incrementally only when $X(t) = v_n \cdot Q^{\frac{1}{\gamma}}$, $P(t)$ will follow a geometric Brownian motion with an upper reflecting barrier at v_n. That is, when $P(t) < v_n$, $dP(t) = dX(t)$, but whenever $P(t)$ rises to v_n, $Q(t)$ increases just enough so as to reflect $P(t)$ off the barrier v_n.

The distribution function for the equilibrium instantaneous lease rate at time t, conditional on $P(0) = P$, can be written as:

$$L_t(p;P) \equiv \Pr\left[P(t) \le p\right] \tag{17}$$

$$
\begin{aligned}
&= \Pr\left[\ln\left[\frac{v_n}{P(t)}\right] \geq \ln\left[\frac{v_n}{p}\right]\right] \\
&= \Phi\left[\frac{\ln(p/P) - (\alpha - \sigma^2/2)\cdot t}{\sigma\sqrt{t}}\right] \\
&\quad + (p/v_n)^{\frac{2(\alpha-\sigma^2/2)}{\sigma^2}} \cdot \Phi\left[\frac{\ln(p/v_n) + \ln(P/v_n) + (\alpha - \sigma^2/2)\cdot t}{\sigma\sqrt{t}}\right],
\end{aligned}
$$

for $0 \leq p \leq v_n$, and where "Pr" represents probabilities conditional on $P(0) = P$. Note that $\ln\left[\frac{v_n}{P(t)}\right]$ is a Brownian motion with a lower reflecting barrier at zero, with a drift parameter $(1/2\sigma^2 - \alpha)$ and a variance parameter σ^2. Thus, I can use the distribution function provided in Harrison (1985, Chapter 3, Equation 6.1) to calculate the result in the third equation of the derivation.

Provided $\alpha > \sigma^2/2$, $P(t)$ has a long-run stationary distribution. That is,

$$
\lim_{t\to\infty} L_t(p;P) = (p/v_n)^{\frac{2(\alpha-\sigma^2/2)}{\sigma^2}}, \text{ for } \alpha > \sigma^2/2. \tag{18}
$$

The mean and variance of this stationary distribution are:

$$
\begin{aligned}
\lim_{t\to\infty} E[p(t)] &= \frac{\alpha - \sigma^2/2}{\alpha} v_n, \\
\lim_{t\to\infty} Var[p(t)] &= \frac{\sigma^4 v_n^2}{4\alpha^2}\left(\frac{\alpha - \sigma^2/2}{\alpha + \sigma^2/2}\right).
\end{aligned} \tag{19}
$$

Since $\frac{\partial v_n}{\partial n} < 0$, markets with more competition will have instantaneous rent distributions with lower long-run means and variances.

2.4 Equilibrium Building Values

Given the equilibrium rent process, $P(t)$, I can express the value of an existing building (providing a single unit of space) by the function $H(P)$. Analogous to the equilibrium differential equation derived above, $H(P)$ must solve the following differential equation:

$$
0 = \frac{1}{2}\sigma^2 P^2 H'' + \alpha P H' - rH + P, \tag{20}
$$

subject to:

$$
\begin{aligned}
H'(v_n) &= 0, \\
H(0) &= 0.
\end{aligned} \tag{21}
$$

The first boundary condition conveys the impact of the reflecting barrier at $P = v_n$ and is known as a boundary condition of the Neumann type.[11] The second boundary

[11] See Farnsworth and Bass (1998) for a discussion of the Neumann condition for processes with reflecting barriers.

condition conveys the impact of the absorbing barrier at $P = 0$. A closed-form solution for $H(P)$ can be written as:

$$H(P) = -\frac{v_n^{1-\beta}}{\beta \cdot (r-\alpha)} \cdot P^{\beta} + \frac{P}{r-\alpha}. \tag{22}$$

In order to understand the impact of competition on the equilibrium building values, consider the net present value of development at the moment of construction (when $P = v_n$):

$$H(v_n) - K = \frac{K}{n \cdot \gamma - 1}. \tag{23}$$

Because $n \cdot \gamma > 1$, the net present value is positive (for finite n). This positive net present value is a familiar result in real options analysis resulting from the value of the "option to wait." However, for industries with greater competition (increasing n), this net present value falls. In the limit of perfect competition ($n \to \infty$), the net present value of development is precisely zero, just as one would expect.

2.5 Equilibrium Land Values

Finally, I can express the equilibrium value of a unit of undeveloped land (upon which one building can be constructed) by the function $L(P)$.[12] $L(P)$ must solve the following differential equation:

$$0 = \frac{1}{2}\sigma^2 P^2 L'' + \alpha P L' - rL, \tag{24}$$

subject to:

$$\begin{aligned} L(v_n) &= H(v_n) - K, \\ L(0) &= 0. \end{aligned} \tag{25}$$

The first boundary condition reflects the fact that construction begins at the trigger $P = v_n$, and thus the value of land equals the value of a building minus the cost of construction.[13] Again, the second boundary condition reflects the absorbing barrier

[12] Note that this represents the value of the option to produce an incremental unit of rentable space. One can also value the total land holdings of a firm (assumed to be infinite, although a finite supply can also be accomplished in a slightly different setting.) Suppose the firm currently has q completed buildings, the current demand level is X, and the industry supply equals $Q = n \cdot q$. Let $l(X, \varepsilon)$ denote the value of the option to produce the ε'th unit of rentable space, for $\varepsilon \geq q$. The value of the land holdings of a firm is equal to the integral of all of these options, $\int_q^\infty l(X, \varepsilon) d\varepsilon$. This integral can be shown to equal $q\frac{\gamma}{\beta-\gamma}L(P)$, where $L(P)$ equals the incremental land value in equation (26).

[13] Note that the Neumann condition for a reflecting barrier, $L'(v_n) = 0$, is automatically satisfied by combining the first line in (25) with the first line in (21).

at zero. A closed-form solution for $L(P)$ can be written as:

$$L(P) = \begin{cases} \frac{K}{n\gamma - 1}\left(\frac{P}{v_n}\right)^{\beta}, & \text{for } P < v_n, \\ H(P) - K, & \text{otherwise.} \end{cases} \tag{26}$$

3 Equilibrium Term Structure of Lease Rates

While the study of the term structure of interest rates is a cornerstone of both finance theory and practice, an analogous, but much less studied term structure exists in lease contracts.[14] A term structure of lease rates specifies the equilibrium lease rate on a T-year lease for all choices of T. While the underlying structure of the leasing model in this article differs from the standard term structure of interest rates models, many similarities exist. First, just as standard term structure models express the equilibrium yield on a T-year zero coupon bond as a function of the instantaneous interest rate, maturity, and the parameters of the instantaneous interest rate process, the equilibrium rent on a T-year lease will be a function of the instantaneous lease rate, maturity, and the parameters of the instantaneous rent process. Second, while long term bond yields embody information concerning the expected path of future short term interest rates, long term lease rates reflect expectations of future short term lease rates. In particular, since the present model is based on an underlying industry equilibrium model, one can use the term structure of lease rates to infer expectations of the level of future supply and demand in the local real estate market. Third, just as the term structure of interest rates provides the foundation for pricing a seemingly endless variety of debt contracts (e.g., coupon bonds, interest rate swaps, bond futures contracts, etc.), the term structure of lease rates provides the basis for pricing the assorted lease contract variations that appear in future sections of this article.

Having established the equilibrium in the underlying real estate asset market, the equilibrium rent on leases of any term is derived. For all $T > 0$, define a T-year fixed lease as a contract in which the tenant obtains the use of the asset for T years beginning at time zero and in return, the landlord receives the flow of rental payments of R until time T, with the first payment made immediately upon the signing of the lease. The goal is to derive the equilibrium characterization of R as a function of T.

The simplest means of deriving the term structure is through the following economic characterization of the leasing process. A lease of term T gives the tenant the use of the asset for T years and nothing thereafter. The same service flow can be achieved by forming a portfolio which involves purchasing the underlying building and simultaneously writing a call option on the underlying building with expiration date T and an exercise price of zero. This portfolio provides the same economic benefits as leasing the building over the term of the lease. To avoid dominance, the value

[14] Brennan and Kraus (1982) also develop a term structure of lease rates, but in a framework with an exogenous, log-normal short-term rent process.

of the lease must equal the value of the portfolio. Thus, the value of a T-year lease must be equal to $H(P)$ less the value of a European call option on $H(P)$ with an exercise price of zero and expiration T.

The value of $H(P)$ was obtained in the previous section in equation (22). What remains is the valuation of the call option. Let $C(P,t;T)$ denote the value of a call option on $H(P)$, where t is the current date, $P = P(t)$, and T is the expiration date. The call option, $C(P,t;T)$, must solve the following equilibrium partial differential equation:

$$0 = \frac{1}{2}\sigma^2 P^2 C_{PP} + \alpha P C_P + C_t - rC, \tag{27}$$

subject to:

$$\begin{aligned} C(0,t;T) &= 0, \\ C(P,T;T) &= H(P), \\ C_P(v_n,t;T) &= 0. \end{aligned} \tag{28}$$

The first boundary condition accounts for $P(t)$ having an absorbing barrier at zero. The second boundary condition represents the option's payoff at expiration (recall that this option has a zero exercise price). The third boundary condition is the Neumann condition that accounts for $P(t)$ having a reflecting barrier at v_n.

The solution for $C(P,t;T)$ is obtained in closed-form, albeit quite messy. Since the initial value of the option is all that is relevant for contracting purposes, I will focus on the value of the option at the initiation of the lease, $C(P,0;T)$. This option value can be written as:

$$C(P,0;T) = \frac{v_n e^{-rT}}{r-\alpha}\left[f(P,T,-1) - f(P,T,-\beta)/\beta\right] \tag{29}$$

where:

$$\begin{aligned} f(P,T,w) &= g_1(P,T,w)g_3(P,T,w) + [1-h(w)]g_2(P,T,w)g_4(P,T,w) \\ &\quad -h(w)g_2(P,T,-2\mu/\sigma^2), \\ g_1(P,T,w) &= N\left[\frac{(w\sigma^2+\mu)T + \ln(v_n/P)}{\sigma\sqrt{T}}\right], \\ g_2(P,T,w) &= 1 - g_1(P,T,-w-2\mu/\sigma^2), \\ g_3(P,T,w) &= \left(\frac{P}{v_n}\right)^{-w} \exp\left[\left(w^2\sigma^2/2 + w\mu\right)T\right], \\ g_4(P,T,w) &= g_3(P,T,-w-2\mu/\sigma^2), \\ h(w) &= \frac{-2\mu}{w\sigma^2+2\mu}, \\ \mu &= \sigma^2/2 - \alpha, \end{aligned} \tag{30}$$

and where $N(\cdot)$ denotes the cumulative standard normal distribution function. Since a T-year lease is equivalent to a portfolio which is long one building and short one call on the building, the value for a T-year lease is $H(P) - C(P, 0; T)$.

Finally, it is simple to solve for the equilibrium T-year lease payment, $R(T)$, that provides an annuity value equal to the equilibrium lease value. The risk of default for the stream of lease payments is ignored in this case; for tenants with poor creditworthiness this simplification may be unrealistic, but such a complication would not alter the basic structure of the model. Thus, the equilibrium term structure of lease rates can be expressed as:

$$R(P,T) = \left[\frac{r}{1 - \exp(-r \cdot T)}\right] \cdot \left[H(P) - C(P, 0; T))\right]. \tag{31}$$

Just as with the Cox, Ingersoll and Ross (1985) or Vasicek (1977) models of the term structure of interest rates, the term structure of lease rates converges to a well-defined perpetual lease rate. Note that a perpetual lease is economically equivalent to ownership of the building. Thus, the equilibrium value of the underlying building, $H(P)$, must equal the discounted value of the equilibrium perpetual lease rate, $R(P, \infty)$:

$$R(P, \infty) = r \cdot H(P). \tag{32}$$

Grenadier (1995) provides a detailed analysis of the term structure of lease rates. Here, I shall confine myself to examining the impact of competition on the shape of the term structure of lease rates. Just as with the Cox, Ingersoll, and Ross or the Vasicek models of the term structure of interest rates, the yield curve for rents may take on three possible shapes: downward-sloping, upward-sloping, and single-humped. Figure 1 demonstrates that the degree of competition in a real estate market can determine the shape of the term structure of rents. Figure 1 displays three term structures, each with the same parameters with the exception of the number of firms competing in the market. The top curve, which is upward-sloping, corresponds to a market with only four competitors. The middle curve, which is initially upward-sloping and then downward-sloping, corresponds to a market with six competitors. The bottom curve, which is downward-sloping, corresponds to a market with ten competitors.

Figure 1 demonstrates a result that holds across a wide range of parameter assumptions. For a given instantaneous rent rate $P(0)$, the term structure for markets with many competitors will be more likely to be downward-sloping, while the term structure for markets with few competitors will be more likely to be upward-sloping. For markets with intermediate levels of competition, the term structure will be single-humped. The intuition for this result is fairly simple. First, consider an industry with a large number of competitors. Given the high degree of competition, short-term lease rates cannot rise much in the future, as any significant rent increases will be met by increases in construction. If the term structure is not downward-sloping, tenants would prefer to roll over a series of short-term leases rather than accept a long-term lease. However, in equilibrium tenants and landlords must be indifferent to the form of financing of the use of the building. Thus, the term structure adjusts to allow long-term rents to fall. Second, consider an industry with only a few competitors. With less competition, the supply response to increasing rents will be muted, permitting future short-term rents to grow with demand. In this case, if the term structure were not upward-sloping, landlords would prefer rolling over a series of short-term leases to accepting a single long-term lease. Once again, to ensure indifference in equilibrium, the term structure adjusts to an upward-sloping shape. Finally, for intermediate levels of competition, short-term rents may increase for a period, with moderate competitive pressure leading to increased supply in the future. Thus, short-term rents are expected to increase for a period and then moderate when new construction ensues. As a result, the term structure takes on an upward slope for short and intermediate-term leases, and a downward slope for long-term leases.

4 Leases with an Option to Purchase the Building

A common clause that appears in commercial real estate leases is the option for the tenant to purchase the asset at the end of the lease term. Such clauses are most common in free-standing single tenant buildings. The option to purchase is valued by McConnell and Schallheim (1983), Grenadier (1996), and Buetow and Albert (1998), all in the context of an exogenous market rent. Here, of course, the lease option is valued as part of a unified equilibrium.

The option to purchase may be included in a lease to align the incentives of the tenant and landlord. In a lease without a purchase option, the tenant may have little incentive to maintain the premises beyond the provisions specified in the lease. Since the building reverts to the landlord at the end of the lease, the tenant will typically deviate from policies that maximize the value of the building. However, when the purchase option is included in a lease, the tenant has a potential stake in the value of the underlying building, and may choose policies that are closer to value maximizing. A similar instance of incentive alignment deals with credit risk. A tenant will be less willing to default on a lease with a purchase option, since a purchase option is only viable if the tenant pays the contracted rent throughout the lease term.

First, I consider a purchase option with a fixed exercise price. The terms of a lease with such a purchase option are as follows. Consider a T-year lease, with contractual rental payment flow of R^{op}, where at the end of the term the tenant may purchase the building for an exercise price of E.[15] I now determine the equilibrium rent on such a lease, $R^{op}(P;T,E)$.

I begin by valuing the embedded purchase option. The payoff on this option is

[15] A common manner of setting the exercise price is to set it at the expected time T building value. The expected value of a building at T, $E\left\{H\left[P\left(T\right)\right]\right\}$, can be written as $e^{rT}C[P(0),0,T]$, where the function C appears in (29).

$\max\{H[P(T)]-E,0\}$. In order for the option to have value it must be the case that $E<H(v_n)$, as v_n is an upper reflecting barrier for $P(T)$ and hence $H[P(T)]$ is the maximum attainable value of H. Let $\Phi(P,t;T,E)$ denote the value of the purchase option at time t, where $P(t)=P$. Φ must satisfy the following partial differential equation:

$$0=\frac{1}{2}\sigma^2P^2\Phi_{PP}+\alpha P\Phi_P+\Phi_t-r\Phi, \tag{33}$$

subject to:

$$\begin{aligned}\Phi(0,t;T,E) &= 0, \\ \Phi(P,T;T,E) &= \max[H(P)-E,0], \\ \Phi_P(v_n,t;T,E) &= 0.\end{aligned} \tag{34}$$

The first boundary condition accounts for $P(t)$ having an absorbing barrier at zero. The second boundary condition represents the option's payoff at expiration. The third boundary condition is the Neumann condition that accounts for $P(t)$ having a reflecting barrier at v_n.

The solution for $\Phi(P,t;T,E)$ is obtained in closed-form. For our purposes, I will focus on the value of the option at the initiation of the lease, $\Phi(P,0;T,E)$. This option value can be written as:

$$\Phi(P,0;T,E)=\frac{v_ne^{-rT}}{r-\alpha}\left[f_1(P,T,-1,E)-f_1(P,T,-\beta,E)/\beta\right]-Ee^{-rT}f_2(P,T,E) \tag{35}$$

where:

$$\begin{aligned}f_1(P,T,w,E) &= g_3(P,T,w)\left[g_1(P,T,w)-g_1(P\cdot\frac{v_n}{q(E)},T,w)\right]+ \\ &\quad [1-h(w)]g_4(P,T,w)\left[g_2(P,T,w)-g_2(P\cdot\frac{q(E)}{v_n},T,w)\right]- \\ &\quad h(w)\left[g_2(P,T,-2\mu/\sigma^2)-g_2(P\cdot\frac{q(E)}{v_n},T,-2\mu/\sigma^2)\left(\frac{q(E)}{v_n}\right)^{-(w+2\mu)/\sigma^2}\right], \\ f_2(P,T,E) &= N\left[\frac{\ln(P/q(E))-\mu T}{\sigma\sqrt{T}}\right]-N\left[\frac{\ln(P/v_n)+\ln(q(E/v_n)-\mu T}{\sigma\sqrt{T}}\right]\left(\frac{q(E)}{v_n}\right)^{-2\mu/\sigma^2},\end{aligned} \tag{36}$$

where g_1, g_2, g_3, g_4, h, μ are presented in (30), and $q(E)\in(0,v_n)$ is the unique root of the expression $H[q(E)]-E=0$.[16]

[16] It is simple to demonstrate the properties of $q(E)$. From (22), $H(0)-E=-E<0$, and by assumption (in order for the option to have value) $H(v_n)-E>0$. By continuity, we know at least one root to $H(q)-E=0$ exists in the interval $(0,v_n)$. Uniqueness is demonstrated by the fact that $H'>0$ over this interval.

Given the value of the purchase option, I can present the equilibrium lease payment on a lease with an option to purchase, $R^{op}(P;T,E)$. Under such a lease, the tenant receives the use of the space for T years, plus the value of the option. In return, the landlord receives the rent flow of $R^{op}(P;T,E)$. In equilibrium, these two values must be equal:

$$\frac{1-e^{-rT}}{r}R(P,T)+\Phi(P,0;T,E)=\frac{1-e^{-rT}}{r}R^{op}(P;T,E), \tag{37}$$

where (31) demonstrates that $\frac{1-e^{-rT}}{r}R(P,T)$ equals the value of using the space for T years and $\frac{1-e^{-rT}}{r}R^{op}(P;T,E)$ equals the present value of the rental flow under this lease. Solving for $R^{op}(P;T,E)$ gives the equilibrium rent:

$$R^{op}(P;T,E)=R(P,T)+\frac{r}{1-e^{-rT}}\Phi(P,0;T,E). \tag{38}$$

Figure 2 plots the equilibrium rent on a three-year lease with a purchase option as a function of the exercise price. First, consider the two extremes of the graph. At a zero exercise price, the lease becomes equivalent to full ownership of the building, since the lease will always be exercised. Therefore, $R^{op}(P;T,0)=\frac{r}{1-e^{-rT}}H(P)$, as the present value of the payment flow $R^{op}(P;T,0)$ over T-years must equal the value of the building. At an exercise price of $H(v_n)$ or above, the value of the purchase option equals zero, since v_n is a reflecting barrier and thus $P(T)<v_n$. Thus, the rent on a lease with a purchase option with an exercise price in this range must equal the rent on a lease without a purchase option: $R^{op}(P;T,E)=R(P,T)$ for $E\geq H(v_n)$. Finally, since the option value is decreasing in the exercise price, $\frac{\partial R^{op}(P;T,E)}{\partial E}<0$ for $E<H(v_n)$.

I now briefly consider another form of lease purchase option clause in which the exercise price, rather than being a fixed constant, is instead a fixed fraction of the property's market value at the end of the lease term. While this "bargain purchase" structure is rare for traditional financial option contracts, it is not uncommon for real estate purchase option contracts. For example, a lease could specify that at the end of the term the tenant can purchase the property for 95% of its market value (typically determined through a professional appraisal).[17] By definition, such options are always in-the-money. In return, the contracted rent must be higher to reflect the value of the option granted to the tenant.

In order to value a purchase option with proportional exercise price, consider a T-year lease, with contractual rental payment flow of $\hat{R}^{op}$, where at the end of the term the tenant may purchase the building for an exercise price of $\chi\cdot H[P(T)]$, with $\chi\in[0,1]$. The value of this purchase option is simply $(1-\chi)\cdot C(P,0;T)$, where $C(P,0;T)$ is from (29) and represents the value of a call option on the building with

[17] In a world with transactions costs, the landlord would prefer selling the property to the tenant for 95% of the market value if a sale to a third party would entail transactions costs greater than 5% (e.g., sales commissions, legal fees, etc.).

a zero exercise price. The equilibrium rent on such a lease, $\hat{R}^{op}(P;T,\chi)$, must satisfy:

$$\begin{aligned}\hat{R}^{op}(P;T,\chi) &= R(P,T)+\frac{r}{1-e^{-rT}}(1-\chi)\cdot C(P,0;T), \\ &= \left[\frac{r}{1-\exp(-r\cdot T)}\right]\cdot\left[H(P)-\chi\cdot C(P,0;T)\right],\end{aligned} \tag{39}$$

where the second equation follows from (31). Note that for $\chi=0$, the value to the tenant is equivalent to ownership of the building, and thus the value of the lease payments must equal $H(P)$. In addition, for $\chi=1$ the option has no value and the lease payment is equal to that on a standard $T-$year lease.

5 Pre-Leasing

It is not uncommon for space to be pre-leased. This practice is called pre-leasing, and is essentially a forward lease contract. Under pre-leasing, a rent is established today for a lease term that begins at a specific time in the future.

There are several reasons why such pre-leasing contracts exist. First, either the construction lender or the take-out lender (the lender who pays off the construction lender upon completion of construction) may insist upon pre-leasing some portion of a building prior to lending. This is generally done in order to make sure that the collateral (the building) is or will be in marketable condition. Second, pre-leasing space to a quality tenant can be a useful marketing tool for attracting other tenants. This is most common in shopping malls where pre-signing a reputable anchor tenant can make future leasing to other mall tenants much easier. Third, just as in the case of traditional forward contracts, tenants who foresee the need to lease space in the future may wish to lock in rents and hedge against future market rent increases (just as developers may wish to hedge against future rent decreases).

The equilibrium valuation of pre-leasing contracts is very simple in this setting. Suppose a lease is signed at time zero that permits the tenant to lease space from time T_1 to T_2, where $T_1<T_2$. The lease specifies a rent flow of $R^F(P;T_1,T_2)$ to be paid over this leasing period.

Consider the following two economically equivalent means of purchasing the use of the space over the period $[T_1,T_2]$. The first is through pre-leasing at the rate $R^F(P;T_1,T_2)$. The present value of this stream of payments is $\frac{e^{-rT_1}-e^{-rT_2}}{r}R^F(P;T_1,T_2)$. The second is to form the following portfolio of call options: long one call option on the building with a zero exercise price and expiration T_1 and short one call option on the building with a zero exercise price and expiration T_2. From (29), the value of this portfolio equals $C(P,0;T_1)-C(P,0;T_2)$. In equilibrium the forward rent flow of $R^F(P;T_1,T_2)$ must be such that these alternative means of purchasing the use of the space are equal:

$$R^F(P;T_1,T_2)=\frac{r}{e^{-rT_1}-e^{-rT_2}}\left[C(P,0;T_1)-C(P,0;T_2)\right]. \tag{40}$$

Using the forward rent flow function $R^F(P;T_1,T_2)$, I can derive the instantaneous forward rate, analogous to the forward interest rate. Let $f(P;T) \equiv \lim_{\delta\to 0} R^F(P;T,T+\delta)$, the forward lease rate over the infinitesimal period $[T,T+\delta]$. Taking this limit yields:

$$f(P;T) = -e^{rT} \cdot \frac{\partial C(P,0;T)}{\partial T}. \tag{41}$$

Note that $\frac{\partial C(P,0;T)}{\partial T} = -e^{-rT}E[P(T)]$.[18] Thus, $f(P;T) = E[P(T)]$, and thus (unlike with term structure of interest rate models) the forward rent is an unbiased estimator of the future spot rent.

Figure 3 plots the instantaneous forward rent curve, for three levels of industry competition: $n = 4$, 6, and 10. Several features are worth discussing. First, by definition, $f(P;0)$ equals the current spot lease rate $P(0)$, which is 5.0 in this graph. Second, at the other extreme of the forward rent curve, $f(P;\infty)$ represents the expected spot rent in the infinite future. As derived in (19), provided $\alpha > \sigma^2/2$, $P(t)$ has a long-run stationary distribution with mean $\frac{\alpha-\sigma^2/2}{\alpha}v_n$. Thus, the long-run forward rent converges to a level determined by the degree of industry competitiveness. Third, the forward rent curve may take on three shapes: upward-sloping, downward-sloping and single-humped. For industries with large numbers of competitors, the future rents are expected to decrease (due to new supply increases outstripping demand growth) leading to a downward-sloping forward rent curve. Conversely, for industries with small numbers of competitors, the future rents are expected to increase (due to demand growth outstripping supply growth) leading to an upward-sloping forward rent curve. For industries with intermediate levels of competitors, the forward rent curve will have a single hump. Notably, the forward rent curve takes on the same shape as the underlying term structure of lease rates curve. This result is also shared by the Cox, Ingersoll and Ross model of the term structure of interest rates.

6 Gross Leases and Expense Stops

The leases I have considered thus far are net leases. Under a fully net lease the tenant pays all (or virtually all) of the operating expenses associated with the space. For example, a very common leasing arrangement is known as a triple net lease, where the tenant is responsible for maintenance, insurance, and property taxes. In this section I consider gross leases, where the landlord is responsible for paying some or all of the operating expenses associated with the property. Most leases fall somewhere in between the extreme cases of the landlord or the tenant paying all of the operating

[18] Consider the intuition for this mathematical result. $C(P,0;T)$ equals the present value of the building's rent flow from time T onwards. Thus, using the definition of a partial derivative, $\frac{\partial C}{\partial T}$ equals the value of the rent flow from $T+\varepsilon$ onwards, minus the value of the rent flow from T onwards, divided by ε, where $\varepsilon \to 0$. This is simply equal to the negative of the present value of the rent flow at time T, or $-e^{-rT}E[P(T)]$.

expenses. Such leases, termed hybrid leases, involve both the landlord and tenant paying a portion of the operating expenses. A particularly interesting case of a hybrid lease is a lease with expense stops, where the tenant pays all expenses above a given threshold.

It is clear that there is an incentive element to the expense provisions of a lease. When a landlord pays operating expenses, the tenant has little incentive to economize on usage. When the tenant pays operating expenses, the tenant will internalize such costs. It is likely that in leases where the tenant has the most discretion over expense usage, the net lease form will be the most likely to prevail.

In this section, I begin by considering the general framework for determining the equilibrium gross lease payment. I then move on to two specific examples: a full service lease and a lease with expense stops.

The general framework for determining the equilibrium gross lease payment is simple. Since the landlord must be indifferent between two methods of leasing the space over a period of time, the present value of the rent flow on a net lease must equal the present value of the rent flow on a gross lease minus the present value of the expenses over the period. Consider a T-year gross lease where the present value of expenses over the next T-years is denoted by $Z(T)$. Thus, the equilibrium gross lease payment flow, $R^G[P,T,Z(T)]$, can be written as:

$$\begin{aligned} \frac{1-e^{-rT}}{r} R^G[P,T,Z(T)] &= \frac{1-e^{-rT}}{r} R(P,T) + Z(T), \\ &\quad or, \\ R^G[P,T,Z(T)] &= R(P,T) + \frac{r}{1-e^{-rT}} Z(T). \end{aligned} \tag{42}$$

For purposes of the following examples, assume that the flow of expenses, $c(t)$, follows a geometric Brownian motion:

$$dc = \alpha_c c dt + \sigma_c c dz_c, \tag{43}$$

where α_c is the instantaneous conditional expected percentage change in c per unit time, σ_c is the instantaneous conditional standard deviation per unit time, and dz_c is the increment of a standard Wiener process.

Consider the example of a full service lease, where all operating expenses are paid for by the landlord. The present value of expenses, $Z(T)$, is simply:

$$\begin{aligned} Z(T) &= E\left[\int_0^T e^{-rt} c(t) dt\right] \\ &= \frac{c}{r-\alpha_c}\left[1-e^{-(r-\alpha_c)T}\right], \end{aligned} \tag{44}$$

where $c = c(0)$. Thus, under a full service lease, the equilibrium gross lease payment is obtained by substituting the expression for $Z(T)$ in (44) into (42).

Now, consider the more interesting example of a lease with expense stops. Under a lease with expense stops, the tenant and landlord share expenses by having the tenant pay all expenses above a specified level known as the "stop." Often, but not always, the stop is set at the level of actual expenses at the time the lease is signed. Thus, under a lease with expense stops the landlord is protected against inflation in expenses while the tenant has an incentive to keep expenses under control.

Consider a T-year lease with an expense stop at $\bar{c}$. The expense reimbursement payment of the tenant at time $t \in [0,T]$ is $\max[c(t)-\bar{c},0]$. This is precisely the payoff of a call option on c with expiration t and exercise price $\bar{c}$. Denote the value of this call option by $\Upsilon[c,t,\bar{c}]$. Given the log-normality of $c(t)$, this call value is simply the Black-Scholes value (with proportional dividend parameter $r-\alpha_c$):

$$\Upsilon[c,t,\bar{c}] = ce^{-(r-\alpha_c)t}N(d_1) - \bar{c}e^{-rt}N(d_2), \tag{45}$$

where:

$$d_1 = \frac{[\ln(c/\bar{c}) + (\alpha_c + \sigma_c^2/2)t]}{\sigma_c\sqrt{t}}, \qquad d_2 = d_1 - \sigma_c\sqrt{t}. \tag{46}$$

Therefore, the present value of the flow of expense reimbursements is equal to a time-integral of Black-Scholes values:

$$Z(T) = \int_0^T \Upsilon[c,t,\bar{c}]dt. \tag{47}$$

A solution to this integral, albeit a complicated one, can be written as:

$$\begin{aligned} Z(T) \;=\; & \bar{c}\left(\frac{c}{\bar{c}}\right)^{\delta_1}\left[\left(\frac{c}{\bar{c}}\right)^{\delta_2}\delta_3 N(b_1) + \left(\frac{c}{\bar{c}}\right)^{-\delta_2}\delta_4 N(b_2)\right] \\ & -\frac{c}{r-\alpha_c}e^{-(r-\alpha_c)T}N(b_3) + \frac{\bar{c}}{r}e^{-rT}N(b_4) \\ & +1_{c>\bar{c}}\left\{\frac{c}{r-\alpha_c} - \frac{\bar{c}}{r} - \bar{c}\left(\frac{c}{\bar{c}}\right)^{\delta_1}\left[\left(\frac{c}{\bar{c}}\right)^{\delta_2}\delta_3 + \left(\frac{c}{\bar{c}}\right)^{-\delta_2}\delta_4\right]\right\}, \end{aligned} \tag{48}$$

with:

$$\begin{array}{ll} \delta_1 = \frac{1/2\sigma_c^2-\alpha_c}{\sigma_c^2}, & \delta_2 = \sqrt{\frac{r+1/2\sigma_c^2\delta_1^2}{1/2\sigma_c^2}}, \\ \delta_3 = \frac{r-\alpha_c(\delta_1-\delta_2)}{2r(r-\alpha_c)\delta_2}, & \delta_4 = \frac{\alpha_c(\delta_1+\delta_2)-r}{2r(r-\alpha_c)\delta_2}, \\ b_1 = \frac{\ln(c/\bar{c})}{\sigma_c\sqrt{T}} + \sigma_c\delta_2\sqrt{T}, & b_2 = \frac{\ln(c/\bar{c})}{\sigma_c\sqrt{T}} - \sigma_c\delta_2\sqrt{T}, \\ b_3 = \frac{[\ln(c/\bar{c})+(\alpha_c+\sigma_c^2/2)T]}{\sigma_c\sqrt{T}} & b_4 = b_3 - \sigma_c\sqrt{T}, \end{array} \tag{49}$$

and $1_{c>\bar{c}}$ is an indicator function taking the value 1 if $c > \bar{c}$, and 0 otherwise.[19] Thus, under a lease with expense stops, the equilibrium lease payment is obtained by substituting the expression for $Z(T)$ in (48) into (42).

[19] The solution is continuous and continuously differentiable at $c = \bar{c}$.

7 Leases with Cancellation Options

Many leases contain clauses that permit the tenant (and sometimes the landlord) to alter the length of the lease during the term of the lease. A cancellation option (also known as a "kick-out" clause) allows the tenant to terminate the lease prior to expiration. Typically the exercise of the cancellation option results in a pre-determined penalty fee being paid to the landlord. A renewal option allows the tenant to extend the length of the lease at a specified renewal rent, after paying a pre-determined renewal fee. Cancellation options and renewal options can be valued in the same manner. A long-term lease with an option to cancel is economically equivalent to a short-term lease with an option to renew. Thus, in this section I focus on leases with cancellation options. An excellent analysis of cancellation and renewal options appears in McConnell and Schallheim (1983). This section extends their analysis in two ways: embedding the valuation in an industry equilibrium framework and by using a continuous-time American option methodology.

Cancellation options are clearly valuable to the tenant, as they permit the tenant to respond to changing market conditions. Suppose a tenant signs a ten-year lease with a fixed rent of \$20 per square foot. The lease contains a cancellation clause that permits the tenant to cancel the lease at any time after paying a penalty fee. If after three years the prevailing market rent on a seven-year lease is significantly less than \$20 per square foot, the tenant may find it optimal to pay the penalty fee and sign a new lease. A cancellation option thus provides insurance against declining market rents, so that if rents fall over the course of the lease term, the tenant has the option of signing a new lease at the lower market rent. Some tenants may be particularly concerned about hedging against being stuck at an above-market rent if its competitors may be signing new leases in the future and could therefore achieve a cost advantage. As in all cases of option valuation, the underlying volatility is a key determinant of value. Insuring against rental downturns is likely to be more desirable (and hence more costly) in markets with volatile rents.

Consider the equilibrium valuation of a lease with a cancellation option. The lease term is T-years, with a rental payment flow of R^c. The tenant may choose to cancel at any time by paying a penalty fee of F^c. Let $(P, t; R^c, T, F^c)$ denote the value of the payment flow the landlord receives over the period from t to T. must solve the following partial differential equation:

$$0 = \frac{1}{2}\sigma^2 P^2 \quad {}_{PP} + \alpha P \quad {}_{P} + \quad {}_{t} - r \quad + R^c, \tag{50}$$

subject to:

$$\begin{aligned} (P_L(t), t; R^c, T, F^c) &= \frac{1 - e^{-r(T-t)}}{r} R(P_L(t), T-t) + F^c, & (51) \\ {}_{P}(P_L(t), t; R^c, T, F^c) &= \frac{1 - e^{-r(T-t)}}{r} R_P(P_L(t), T-t), & \end{aligned}$$

$$
\begin{aligned}
(P, T; R^c, T, F^c) &= 0, \\
{}_P(v_n, t; R^c, T, F^c) &= 0.
\end{aligned}
$$

The function $P_L(t)$ is a trigger function such that whenever $P(t)$ falls to $P_L(t)$ the cancellation option is exercised. The first boundary condition ensures that when the cancellation option is exercised the landlord receives the current market value of a lease from t to T (see Section 3), plus the penalty fee. The second boundary condition is a smooth-pasting condition that ensures that the trigger function $P_L(t)$ is chosen by the tenant so as to minimize . The third boundary condition represents the termination of the lease at T, and the fourth boundary condition is the Neumann condition that accounts for the reflecting barrier at v_n.

In equilibrium, two means of selling the use of the asset for T years must have the same value. Thus, the equilibrium rent on a lease with a cancellation clause, $R^c(P; T, F^c)$, must solve:

$$
[P, 0; R^c(P; T, F^c), T, F^c] = \frac{1 - e^{-rT}}{r} R(P, T). \tag{52}
$$

That is, when the lease is signed at time zero, the value of the cash flow to the landlord under the lease with cancellation option (the left side of the equality) must equal the value the rental payment flow under a standard T-year lease (the right side of the equality).

Since the solution to (50) subject to (51) would be numerically quite intensive, I will focus on the more tractable problem of an infinitely-lived lease where $T \to \infty$. Let the value of the payment flow the landlord receives over the infinite horizon under the cancelable lease be denoted by ${}^{\infty}(P; R^{c,\infty}, F^c) = \lim_{T\to\infty} (P, 0; R^{c,\infty}, T, F^c)$, where $R^{c,\infty}$ is the rent on an infinitely-lived cancelable lease. For a given rent $R^{c,\infty}$, the solution can be shown to be:

$$
{}^{\infty}(P; R^{c,\infty}, F^c) = A_1(P_L)P^{-\beta_1} + A_2(P_L)P^{\beta_2} + \frac{R^{c,\infty}}{r}, \tag{53}
$$

where:

$$
\begin{aligned}
A_1(P_L) &= -\frac{\frac{R^{c,\infty}}{r} - F^c + \frac{v_n^{1-\beta_2}}{\beta_2(r-\alpha)} P_L^{\beta_2} - \frac{P_L}{r-\alpha}}{P_L^{-\beta_1} + \frac{\beta_1}{\beta_2} v_n^{-(\beta_1+\beta_2)} P_L^{\beta_2}}, \\
A_2(P_L) &= \frac{\beta_1}{\beta_2} v_n^{-(\beta_1+\beta_2)} A_1(P_L), \\
P_L &= \arg\min_{0<\omega<v_n} A_1(\omega),
\end{aligned} \tag{54}
$$

and $\beta_1 = \frac{(\alpha - \frac{1}{2}\sigma^2) + \sqrt{(\alpha - \frac{1}{2}\sigma^2)^2 + 2r\sigma^2}}{\sigma^2}$, $\beta_2 = \frac{-(\alpha - \frac{1}{2}\sigma^2) + \sqrt{(\alpha - \frac{1}{2}\sigma^2)^2 + 2r\sigma^2}}{\sigma^2}$, and where P_L is the exercise trigger value for this infinite lease.

In equilibrium the value of this infinite stream of lease payments must equal the value of the underlying building. Thus, the equilibrium lease payment on the infinitely-lived cancelable lease, $R^{c,\infty}(P;F^c)$, must satisfy:

$$^{\infty}(P; R^{c,\infty}(P;F^c), F^c) = H(P), \tag{55}$$

which can easily be solved numerically. At this equilibrium rent, $R^{c,\infty}(P;F^c)$, landlords and tenants are indifferent between a lease with a cancellation option and a lease without a cancellation option. Note that by substituting the equilibrium rent $R^{c,\infty}(P;F^c)$ into (54), the equilibrium cancellation option exercise trigger, P_L, can be determined.

Since the cancellation clause represents a form of tenant insurance against falling market rents, tenants should be willing to pay more for cancellation options in more volatile markets. Figure 4 confirms this intuition. Figure 4 plots the percentage rent premium that must be paid on an infinite-lived cancelable lease for a range of demand volatilities. For example, for a market with a demand volatility of 0.05, the equilibrium rent on a cancelable lease is only 0.5% greater than that on a lease without a cancellation clause. The value of the insurance feature of the cancellation clause is thus quite low. However, for a market with a demand volatility of 0.40, the equilibrium rent on a cancelable lease is 131% greater than that on a lease without a cancellation clause. In volatile markets, the cancellation option is much more likely to be exercised, and is thus much more costly to purchase.

8 Ground Leases

While the basic leasing arrangement in this paper deals with the leasing of a developed building, another leasing arrangement pertains to the leasing of undeveloped land. Under a ground lease, the landowner leases the land to a developer. Upon termination of a ground lease, the land and all improvements (if any) revert to the landowner. Typically ground leases have long terms (usually more than thirty years) with multiple renewal options. Ground leases also typically dictate the details of the construction that can take place on the leased land.

Ground leases introduce a form of inefficiency that is difficult to deal with in the context of the present model.[20] The Nash equilibrium in the underlying property market is based on the premise of agents maximizing the value of the underlying assets (land values). However, under a ground lease with a finite term, the developer will under-build since any building developed during the term will revert to the landlord at the end of the lease. Economically, while a developer that owns the land obtains an infinite stream of rent upon construction (e.g., a building), a ground lessee obtains

[20] Articles that address the incentive compatibility problem regarding the redevelopment options (rather than initial development option) of long-term leases are Capozza and Sick (1991) and Dale-Johnson (2001).

a finite stream of rent upon construction (e.g., the cash flows during the term of the finite lease).

It is reasonable to wonder why ground leases exist, when it is more efficient to simply sell the land to a developer, assuming the current landowner does not have development expertise. One explanation is that developers are liquidity constrained, and ground leases may be the most efficient means of financing the land. A second explanation could be tax-based, where the sale might trigger an immediate recognition of a large gain. A third explanation is simply

However, there are various features of ground leases that mitigate much of the inefficiency introduced by the finite term of the ground lease. First, ground leases are commonly long term and contain multiple extension options, thus approximating infinite leases. In addition, ground leases are often signed at or near the point at which development begins. Thus, in terms of the value received upon development, the value of a very long term lease will closely approximate the value of the underlying building. Second, ground leases are frequently renegotiated. Since the overall surplus is maximized by developing at the equilibrium trigger point, the parties may find it mutually beneficial to make side payments to provide an incentive for optimal development. Third, ground leases often contain strict provisions concerning development, including the timing of development.[21] All of these factors serve to diminish (although not eliminate) the under-development problem of ground leases.

The simplest ground lease specification, and most consistent with the underlying equilibrium model, is one of an infinite term. Under an infinite ground lease, the ground lessor is essentially selling the land in exchange for a flow of lease payments. Recall that the value of the land, $L(P)$, was derived in (26), where it is assumed that $P < v_n$, as no development has yet occurred. The ground rent, $R^G(P)$, must be such that its perpetuity value equals $L(P)$, and therefore:

$$R^G(P) = \frac{rK}{n\gamma - 1}\left(\frac{P}{v_n}\right)^{\beta}. \tag{56}$$

The difference between a perpetual ground lease and a perpetual lease of a building is twofold. First, the ground lessee does not receive the service flow from a building until the first passage time of $P(t)$ to the trigger v_n. Second, the ground lessee must make the cash outflow of K upon construction, at the aforementioned first passage time. The difference in the rent between a perpetual lease of a building and a perpetual ground lease is thus:

$$R(P,\infty) - R^G(P) = r\left[\frac{P - v_n\left(\frac{P}{v_n}\right)^{\beta}}{r-a} + K\left(\frac{P}{v_n}\right)^{\beta}\right]. \tag{57}$$

[21] While it would seem that the ground lessor would be indifferent to the timing of development during the lease term, in a richer model that included credit risk, the lessor would indeed care. Should the lessee default on the ground rent, the lessor would like to ensure that the land value is maximized.

The first term in brackets equals the value of the service flow of the building from the initiation of the lease until the first passage time of $P(t)$ to the trigger v_n. The second term in brackets equals the cost of construction at the first passage time of $P(t)$ to the trigger v_n. These two terms are multiplied by r in order to obtain the rent flow difference between the two perpetual leases.

9 Escalation Clauses

The basic lease structure in this article calls for a constant rent flow over the entire lease term. Such flat rent structures are particularly common on short-term leases. However, on longer-term leases, it is common to find rents that vary over the term. Escalation clauses may provide the landlord with a hedge against such factors as unexpected inflation or cost fluctuations. Escalation clauses in leases specify a rule for determining the rent level at varying points in time. Examples considered in this section include rents that move with overall real estate market rents, rents that move deterministically, and rents that move with an exogenous index. Clearly, the relationship between leases with fixed rent and leases with variable rents is analogous to the relationship between fixed rate mortgages and adjustable rate mortgages.

Escalation clauses specify an initial rent, various points during the lease where the rent is re-set, and the rules by which the rent is re-set. To simplify the modeling, I will assume that the lease is re-set at only one point during the lease. Let the lease be signed at time zero with a base rent of R_0, re-set at time T_1, and end at time T_2.

9.1 Revaluated Rent

One payment structure calls for future lease levels to be marked-to-market. Under such a lease (termed a revaluated lease), the rent at specified points in time is re-set to market rent levels. Some leases specify that the rent is re-set only if the market rent has increases, while others allow the rent to both rise and fall with the market.[22] I will consider both specifications.

First, suppose that the rent is re-set with the market at time T_1, whether the market rent has increased or decreased. This case is obviously very simple. The rent at time T_1 will be re-set to $R[P(T_1), T_2 - T_1]$, the equilibrium market rent on a $(T_2 - T_1)$-year lease prevailing at time T_1. In equilibrium, the initial rent R_0 must equal the market rent on a T_1-year lease, $R[P, T_1]$. In the more general case of rents being reset at various points of time throughout the lease, the initial rent will remain equal to the market rent on a lease with term equal to the first re-setting point. In the limit, as the resetting points occur every instant, the initial lease rate will simply equal the instantaneous rate $P(0)$.

[22] For a detailed treatment of the former specification, sometimes known as upward-only adjusting leases, see Ambrose, Hendershott and Klosek (2001).

Second, suppose that the rent is re-set to the market rent at time T_1 if rents have increased, but kept constant otherwise. Thus, the rent will be re-set at time T_1 to equal $\max\{R[P(T_1), T_2 - T_1], R_0\}$. The value of the rental payments under such a lease can be expressed in terms of various leasing structures previously analyzed in this article, notably cancellation options and pre-leasing contracts. Since

$$\max\{R[P(T_1), T_2 - T_1], R_0\} = R_0 + R[P(T_1), T_2 - T_1] - \min\{R[P(T_1), T_2 - T_1], R_0\}, \tag{58}$$

the value of the payment stream on this lease over the period $[T_1, T_2]$ equals the value of the three right-hand side cash flows. The value of the constant payment flow of R_0 is simply $\frac{R_0}{r}(e^{-rT_1} - e^{-rT_2})$. The value of the market rental stream of $R[P(T_1), T_2 - T_1]$ must equal the value of the pre-leasing rent of $R^F(P; T_1, T_2)$, or $\frac{e^{-rT_1} - e^{-rT_2}}{r} R^F(P; T_1, T_2)$, since these are two equivalent ways of leasing space over this period. The value of the rental stream of $\min\{R[P(T_1), T_2 - T_1], R_0\}$ is equal to the value of a lease cancellation option (of the European variety), with initial rent R_0 and with a zero penalty fee. Denote the value of this particular lease cancellation option by $\Lambda(P, R_0, T_1, T_2)$.

Let $M(P, T_1, T_2, R_0)$ equal the value of the cash flows under this revaluation lease. This must equal the value of the constant payment flow of R_0 over the first T_1 years, plus the sum of the three values in the previous paragraph. Simplifying, this yields:

$$M(P, T_1, T_2, R_0) = \frac{R_0}{r}(1 - e^{-rT_2}) + \frac{e^{-rT_1} - e^{-rT_2}}{r} R^F(P; T_1, T_2) - \Lambda(P, R_0, T_1, T_2). \tag{59}$$

In equilibrium, the equilibrium initial rent value, R_0^*, must be set so that the value of the rent flows equals that on a standard T_2-year lease:

$$M(P, T_1, T_2, R_0^*) = \frac{R(P, T_2)}{r}(1 - e^{-rT_2}). \tag{60}$$

9.2 Graduated Rent

A second payment structure calls for simple, pre-determined rent increases. Under such a lease (termed a graduated lease), the lease specifies non-stochastic rent changes (step-ups) at various points in time. Valuing such leases is a trivial matter.

Consider the example of a lease with initial rent R_0, stepped up at T_1 by the growth rate of g, and ending at T_2. Thus, the stepped-up rent at T_1 is $e^{gT_1} R_0$. The present value of these lease payments is:

$$\frac{R_0}{r}\left[1 - e^{-rT_1} + e^{gT_1}(e^{-rT_1} - e^{-rT_2})\right]. \tag{61}$$

In equilibrium, the equilibrium initial rent value, R_0^*, must be set so that the value of the rent flows equals that on a standard T_2-year lease:

$$\frac{R_0^*}{r}\left[1 - e^{-rT_1} + e^{gT_1}(e^{-rT_1} - e^{-rT_2})\right] = \frac{R(P, T_2)}{r}(1 - e^{-rT_2}) \tag{62}$$

$$R_0^* = \frac{R(P,T_2)(1-e^{-rT_2})}{\left[1-e^{-rT_1}+e^{gT_1}(e^{-rT_1}-e^{-rT_2})\right]}.$$

or,

9.3 Indexed Rent

A third payment structure calls for the rent to move with an exogenous, publicly observable index. Common indices include the CPI and PPI. The rent may increase one-for-one with the index, or move with a fraction of the index.

Consider the example of a lease that is adjusted to $\rho\%$ of the growth of the index $I(t)$, where the index follows the geometric Brownian motion:

$$dI = \alpha_I I dt + \sigma_I I dz_I. \tag{63}$$

Specifically, let the initial rent equal R_0, and let the rent be changed to $\rho \cdot \frac{I(T_1)}{I(0)} R_0$ at T_1. The present value of these lease payments is:

$$R_0 \left[\frac{1-e^{-rT_1}}{r} + \rho \cdot \frac{e^{-(r-\alpha_I)T_1} - e^{-(r-\alpha_I)T_2}}{r-\alpha_I} \right]. \tag{64}$$

In equilibrium, the equilibrium initial rent value, R_0^*, must be set so that the value of the rent flows equals that on a standard T_2-year lease:

$$R_0^* \left[\frac{1-e^{-rT_1}}{r} + \rho \cdot \frac{e^{-(r-\alpha_I)T_1} - e^{-(r-\alpha_I)T_2}}{r-\alpha_I} \right] = \frac{R(P,T_2)}{r}(1-e^{-rT_2}) \tag{65}$$

or,

$$R_0^* = \frac{R(P,T_2)(1-e^{-rT_2})}{\left[1-e^{-rT_1}+\rho r \cdot \frac{e^{-(r-\alpha_I)T_1}-e^{-(r-\alpha_I)T_2}}{r-\alpha_I}\right]}.$$

10 Leases with Concessions

It is not unusual for leases to contain one or more inducements to attract tenants. Such inducements are known as concessions and are most common during real estate market downturns. Lease concessions may include allowing free rent periods, paying moving costs and providing above-normal tenant improvement allowances. A free rent provision allows the tenant to use the space for an initial period without paying rent. The free rent period can be anywhere from one month to over a year on a long-term lease. Tenant improvement allowances are cash allowances provided by the landlord to pay for the costs of interior improvements in the leased space. While some base level of tenant improvement allowances is typically granted in a lease, anything beyond this base level is considered a concession.

In any rational equilibrium, it is clear that leases offering concessions will result in higher rental rates. Empirically, this contributes to the well-observed phenomenon of "sticky" quoted rents. Quoted rents are much less volatile than effective rents that take into account the impact of concessions during the lease term. There are several potential reasons why leases provide concessions in return for higher rents. The first explanation is that tenants are more likely to be liquidity constrained than landlords. Thus, concessions are a means of providing financing to tenants, especially during a period of high moving and start-up costs. A second explanation is that using concessions may assist landlords in negotiating with multiple tenants. While quoted rents are reported to the public, concessions are typically more private and the result of negotiations between landlords and tenants. Landlords may find that keeping their reservation rent private may improve their multilateral bargaining position with current and future tenants. A third explanation could be that there exists some friction or regulatory feature that makes it in the landlord's interest to report high recurring rentals (even at the expense of concessions granted). For example, a landlord may be able to achieve better refinancing terms if appraisals do not fully take into account concessions, but instead use some multiple of rents for valuation purposes.

Finding the equilibrium rent on a lease with concessions is quite simple in the current context. Consider two types of concessions: lump sum concessions such as moving allowances and free rental periods. Let $\bar{C}$ denote the dollar value of the lump sum concessions, and τ the period of the T-year lease under which no rent is paid (where $\tau < T$). The rent paid on this lease is denoted by $R^{con}(P, T; \bar{C}, \tau)$.

In equilibrium, the landlord must be indifferent between a lease with concessions and a lease without concessions. In a lease with concessions, the landlord receives the rent flow of $R^{con}(P, T; \bar{C}, \tau)$ from τ to T, minus the lump sum cost $\bar{C}$. Under a lease without concessions, the landlord receives a lease flow of $R(P, T)$ from 0 to T. Equating these two lease values and solving for $R^{con}(P, T; \bar{C}, \tau)$ results in:

$$R^{con}(P, T; \bar{C}, \tau) = \frac{1 - e^{-rT}}{e^{-r\tau} - e^{-rT}} R(P, T) + \frac{r}{e^{-r\tau} - e^{-rT}} \bar{C}. \tag{66}$$

As an illustration, suppose that for various reasons the landlord would like to report a quoted rent that is the highest possible rent on a lease without concessions. The maximum rent would be achieved if the instantaneous rent equals its upper reflecting barrier, $R(v_n, T)$. Suppose, however, the current instantaneous rent is less than its upper bound, $P < v_n$. Thus, in order to set a quoted rent of $R^{con}(P, T; \bar{C}, \tau) = R(v_n, T)$, the package of concessions $(\bar{C}, \tau)$ would have to solve:

$$R(v_n, T) = \frac{1 - e^{-rT}}{e^{-r\tau} - e^{-rT}} R(P, T) + \frac{r}{e^{-r\tau} - e^{-rT}} \bar{C}. \tag{67}$$

Obviously there are an infinite number of combinations of $\bar{C}$ and τ that can solve this equality. However, first suppose this is accomplished solely through a free rent period. Then, $\bar{C} = 0$ and τ must equal:

$$\tau = \ln \left[\left(e^{-rT} + \left(1 - e^{-rT}\right) \frac{R(P, T)}{R(v_n, T)} \right)^{-r} \right]. \tag{68}$$

Note that $\frac{\partial\tau}{\partial P} < 0$, and thus the greater the distance of the current rent P from its maximum v_n, the greater the free rent period. Consider the two extremes. If the current rent is at its maximum ($P = v_n$), then no free rent is necessary and $\tau = 0$. If the current rent is at is minimum ($P = 0$), then the free rent period must be the entire lease term ($\tau = T$).

Conversely, suppose equality (67) is accomplished solely through lump-sum concessions. Then, $\tau = 0$ and $\bar{C}$ must equal:

$$\bar{C} = \frac{1 - e^{-rT}}{r} \left[R(v_n, T) - R(P, T) \right]. \tag{69}$$

Note that $\frac{\partial\bar{C}}{\partial P} < 0$, and thus the greater the distance of the current rent P from its maximum v_n, the greater the required lump-sum concession. Consider the two extremes. If the current rent is at its maximum ($P = v_n$), then no concessions are necessary and $\bar{C} = 0$. If the current rent is at is minimum ($P = 0$), then the required lump-sum concession is $\frac{1-e^{-rT}}{r} R(v_n, T)$.

11 Sale-Leaseback Transactions

Under a sale-leaseback agreement, the owner of a building (usually the sole occupant) sells the building and simultaneously signs a lease on the building. Such transactions are typically justified as a form of financing; the seller/tenant uses the sales proceeds for business expansion and the lease payments represent financing payments. This justification makes most sense for small firms that have difficulty obtaining financing, however it may be less compelling for the many large corporations that engage in sale-leasebacks of their corporate headquarters.

Clearly, modeling the sale-leaseback transaction is quite simple. There are two components of the transaction: setting the sales price and setting the lease terms. If the sales price equals the true market value of the building, then the lease rate must equal the equilibrium lease rate on a standard lease. However, if the sales price differs from the market value of the asset, then the lease terms will also differ from the equilibrium lease rate on a standard lease.

Consider the following sale-leaseback arrangement. The building is sold for S, and simultaneously leased back for T-years at a rental rate of Y. The market value of the building is $H(P)$, and the equilibrium rent on a standard T-year lease is $R(P, T)$. In equilibrium, the difference between the market value of the building and the sales price must equal the difference between the market value of the equilibrium lease payments and the agreed upon lease payments:

$$H(P) - S = \frac{1 - e^{-rT}}{r} \left[R(P, T) - Y \right]. \tag{70}$$

Therefore, for a given sales price, S, the equilibrium sale-leaseback rental rate, $Y(P, S, T)$, equals:

$$Y(P,S,T) = R(P,T) + \frac{r}{1-e^{-rT}} \left[S - H(P)\right]. \tag{71}$$

There are two degrees of freedom in setting the contract terms: S and Y. Consider a few examples of possible contract terms. First, suppose that the seller would like to record a sales price that represents the maximum obtainable from a direct sale, $H(v_n)$. Then, using (71), the equilibrium lease rate must be $R(P,T)+\frac{r}{1-e^{-rT}}\left[H(v_n) - H(P)\right]$. Second, suppose the seller wishes to sell the property at the same price at which the property was purchased, where the purchase price was $H(\hat{P})$ and where $\hat{P}$ was the market rent prevailing at the time of purchase. Then, using (71), the equilibrium lease rate must be $R(P,T) + \frac{r}{1-e^{-rT}}\left[H(\hat{P}) - H(P)\right]$. Finally, consider the case in which seller is willing to pay the maximum standard rent of $R(v_n,T)$ in order to achieve a high selling price. Then, the equilibrium sales price must be $H(P) + \frac{1-e^{-rT}}{r}\left[R(v_n,T) - R(P,T)\right]$.

12 Conclusion

I derive a model that provides a unified equilibrium approach to valuing real estate lease contracts. The underlying real estate asset market equilibrium is modeled as a continuous-time Nash equilibrium in which firms choose optimal development strategies in the face of evolving demand uncertainty. Using an option pricing approach, the real estate leasing market is then modeled as a contingent claim on the equilibrium building value. Given the underlying structure of the real estate market (number of developers, the demand curve for space and the properties of the stochastic process underlying demand), endogenous processes for rent, construction starts, building values and land values are derived. This equilibrium framework is flexible enough to value a wide variety of realistic leasing contracts. Examples of leasing contracts valued in this article are purchase options, pre-leasing, net and gross leases, cancellation options, ground leases, escalation clauses, lease concessions and sale-leaseback contracts.

Several extensions of the model would prove interesting. First, the model could be empirically tested using actual lease contracts. The availability of large and reliable samples of individual lease contracts is currently difficult to obtain. The model suggests that equilibrium lease rates will be sensitive to the degree of competition in the local real estate market, the shape of the demand curve for space, the parameters underlying the shock term for demand, and the specific terms of the contract (e.g., maturity, embedded options, operating expense provisions, etc.). Second, the underlying assumption of identical firms could be weakened. It is likely to be the case that in any given real estate markets developers differ in their skill levels, experience, and financial strength. While providing greater realism to the model, the concomitant loss of the simplifying feature of a symmetric equilibrium would greatly diminish the tractability of the model.

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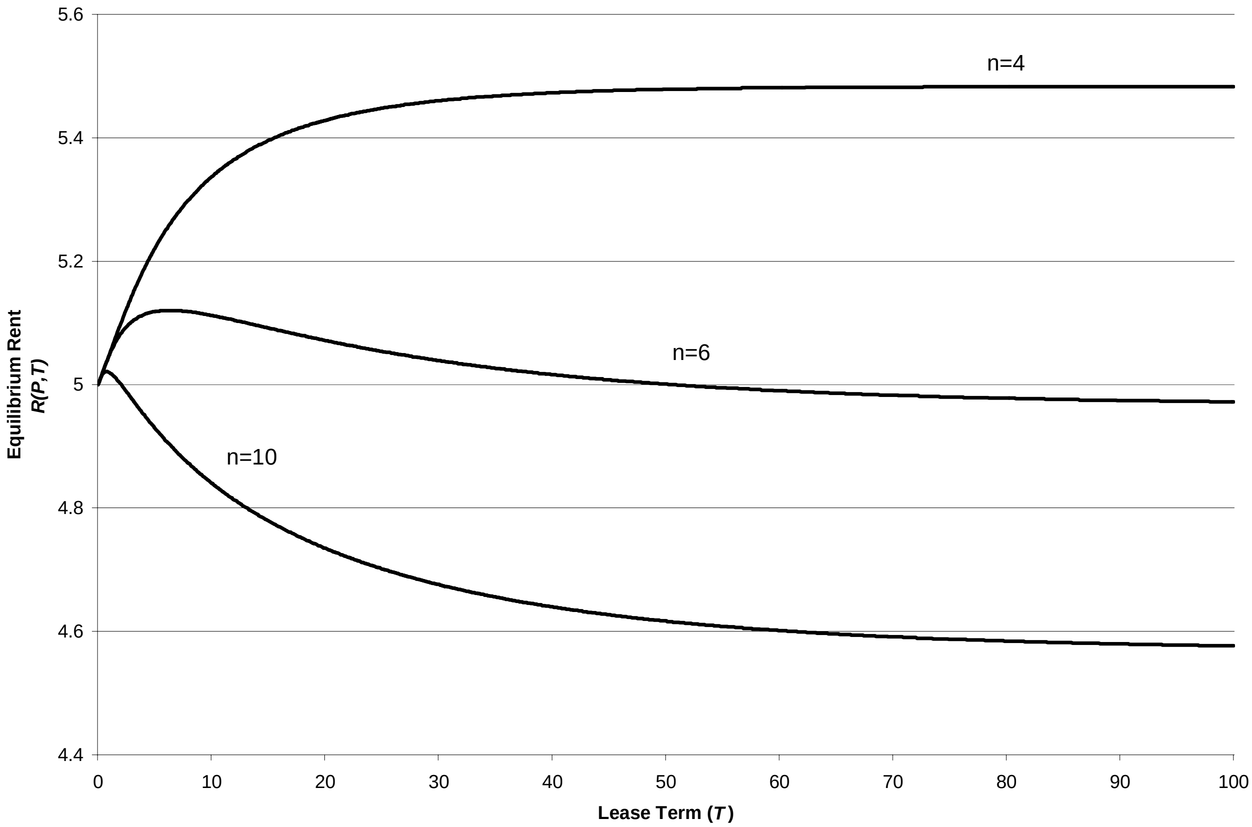

Figure 1. Term Structure of Lease Rates. Each graph in the figure shows the equilibrium rental rate as a function of the lease term, T, for different levels of industry competition, n. The term structure can take on three possible shapes: upward-sloping, downward-sloping and single-humped. Intuition for the slope of the term structure of lease rates in a given market can be developed using a form of the expectations hypothesis: long-term lease rates must leave landlords and tenants indifferent between signing long term leases and the expected outcome of rolling over a series of short-term leases. The top graph is for a real estate market with only four firms, and is upward-sloping. With few competitors, the supply response to increasing rents will be muted, permitting expected future short-term rents to grow with demand. The middle graph is for a real estate market with six firms, and is single-humped. For intermediate levels of competition, expected short-term rents may increase for a period, with moderate competitive pressure leading to increased supply in the future. The bottom graph is for a real estate market with ten firms, and is downward-sloping. Given the high degree of competition, expected short-term lease rates cannot rise much in the future, as any significant rent increases will be met by increases in construction. The default parameter values are $\alpha=0.02$, $\sigma=0.10$, $r=0.04$, $K=100$, $\gamma=0.75$ and $P(0)=5$.

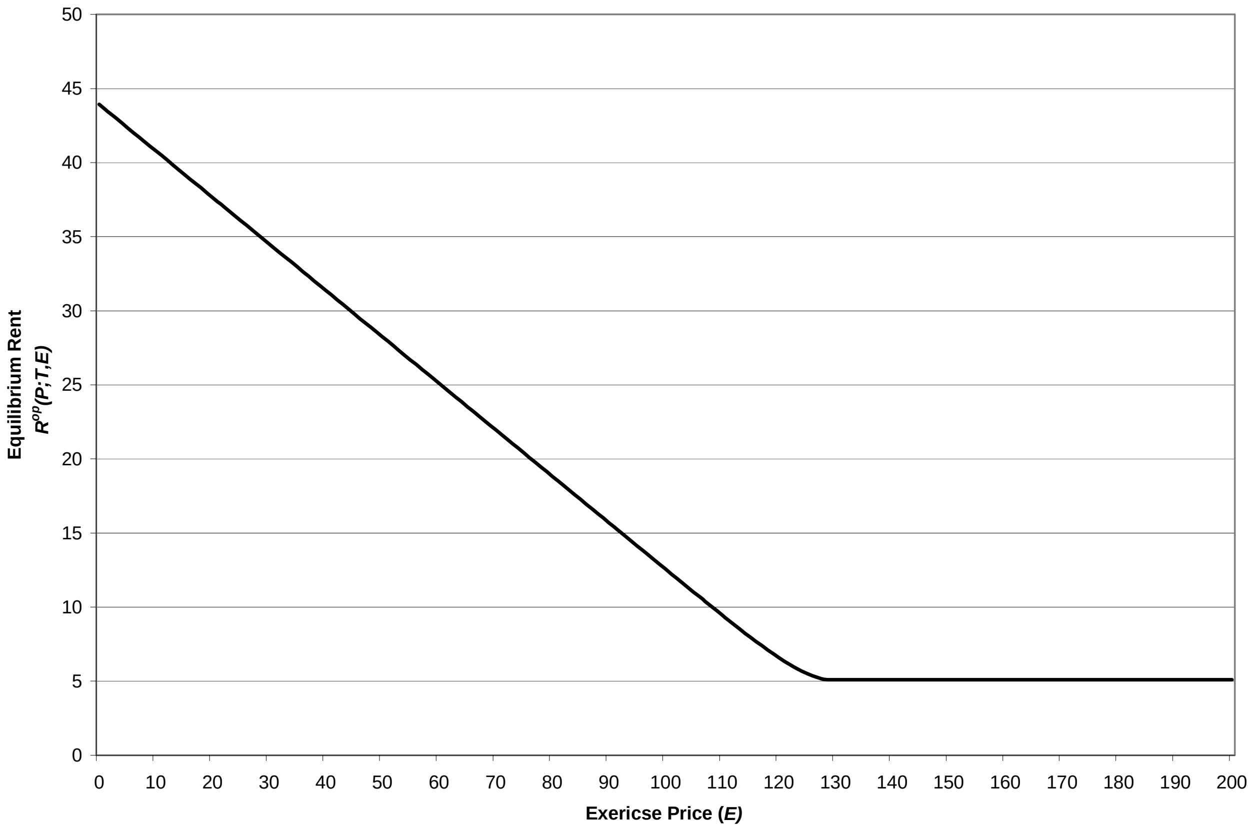

Figure 2. Equilibrium Rent on a Lease With an Option to Purchase. This graph shows the equilibrium rent on a lease with the option to purchase the building as a function of the option exercise price, E. A lease with a purchase option provides the tenant with a European call option on the building with an exercise price of E and an expiration date equal to the term of the lease. Consider the equilibrium rent at two extremes of the range of E. At an exercise price of zero, the lease becomes economically equivalent to outright ownership of the building. At any exercise price greater than or equal to $H(v_n)$, the option will never be exercised, as $H(v_n)$ represents an upper reflecting barrier on the building's value. In this figure, $H(v_n)=128.57$. Thus, for leases with $E> H(v_n)$, the rent must be the same as that on leases without purchase options. For all $0<E< H(v_n)$, the equilibrium rent is decreasing in E. The default parameter values are $\alpha=0.02$, $\sigma=0.10$, $r=0.04$, $K=100$, $\gamma=0.75$, $T=3$ and $P(0)=5$.

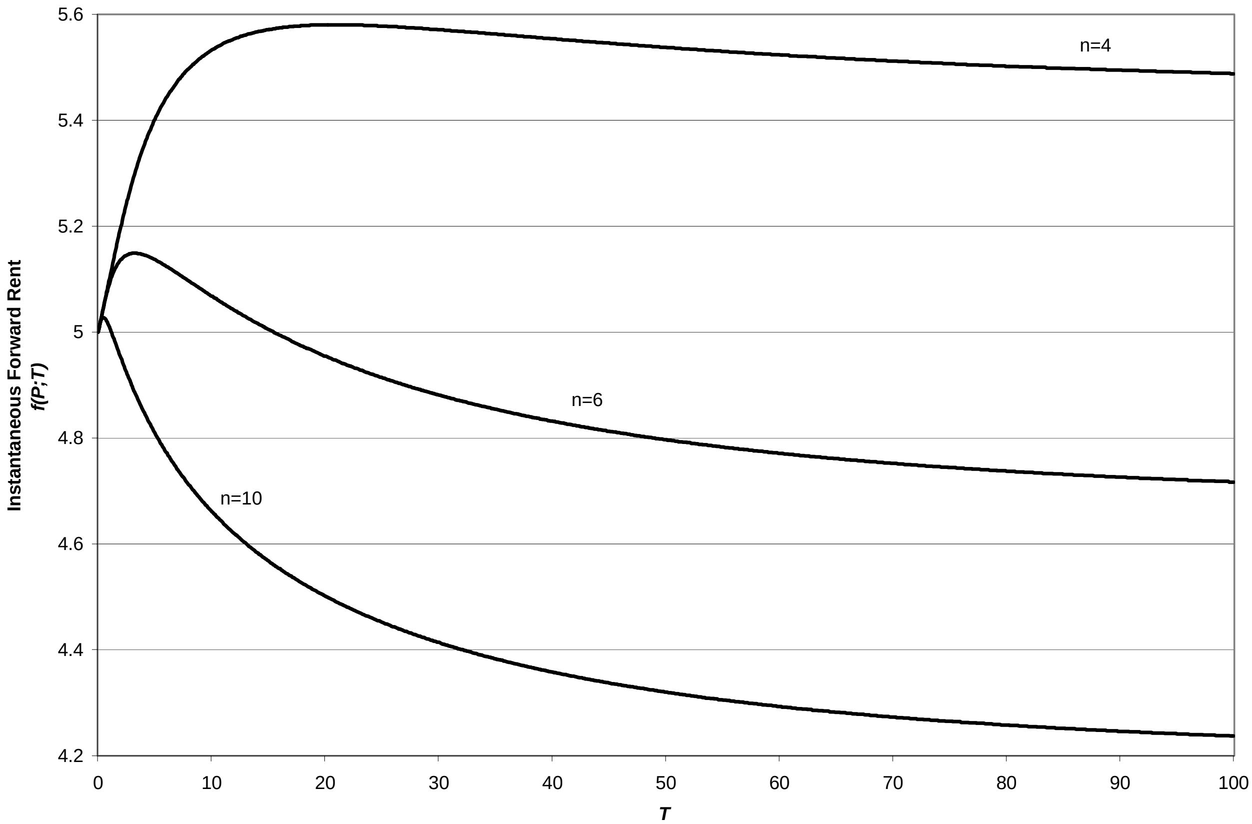

Figure 3. Equilibrium Instantaneous Forward Rents. A forward lease is an agreement to lease the building for a given term τ, but where the lease does not begin until the date *T*. An instantaneous forward lease rate, *f(P;T)*, is a forward lease rate where $\tau \rightarrow 0$. The instantaneous forward rent is an unbiased estimator of the future spot rent: *f(P;T)=E[P(T)]*. Each graph in the figure shows the equilibrium instantaneous forward lease rate as a function of the lease term, *T*, for different levels of industry competition, *n*. Each instantaneous forward rent curve takes on the same shape as the corresponding term structure of lease rates depicted in Figure 1. The top graph is for a real estate market with only four firms, and is upward-sloping. With few competitors, the supply response to increasing rents will be muted, permitting expected future short-term rents to grow with demand. The middle graph is for a real estate market with six firms, and is single-humped. For intermediate levels of competition, expected short-term rents may increase for a period, with moderate competitive pressure leading to increased supply in the future. The bottom graph is for a real estate market with ten firms, and is downward-sloping. Given the high degree of competition, expected short-term lease rates cannot rise much in the future, as any significant rent increases will be met by increases in construction. The default parameter values are $\alpha=0.02$, $\sigma=0.10$, $r=0.04$, $K=100$, $\gamma=0.75$ and $P(0)=5$.

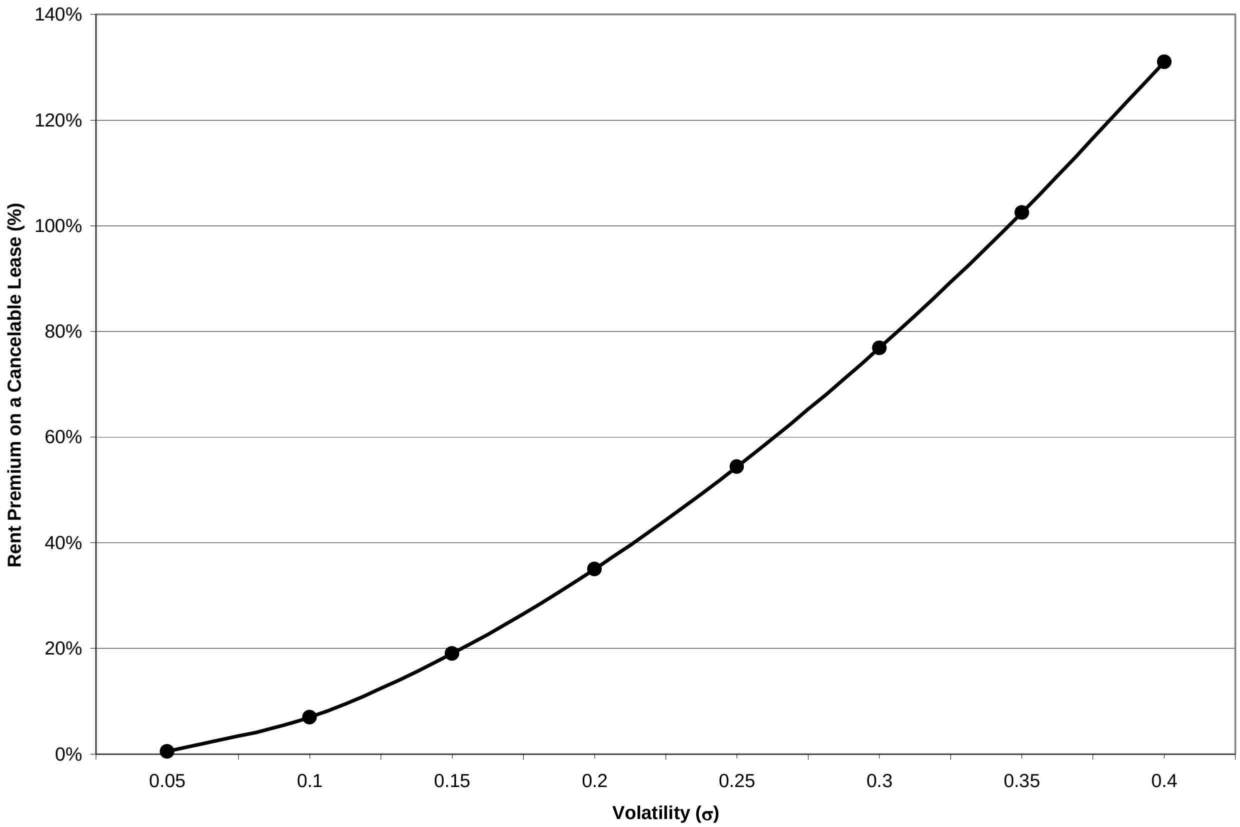

Figure 4. Equilibrium Rent Premium on a Cancelable Lease. A lease with a cancellation option allows the tenant to cancel making the remaining payments on a lease after paying an exercise price of F^c. The rent premium on a cancelable lease equals the percentage difference between lease payments on a cancelable lease and a non-cancelable lease with all other terms being identical. This graph displays the rent premium on a cancelable lease (with infinite term) as a function of demand volatility, σ. Since the cancellation clause represents a form of tenant insurance against falling market rents, tenants should be willing to pay more for cancellation options in more volatile markets. For example, for a market with a demand volatility of 0.05, the equilibrium rent on a cancelable lease is only 0.5% greater than that on a lease without a cancellation clause. However, for a market with a demand volatility of 0.40, the equilibrium rent on a cancelable lease is 131% greater than that on a lease without a cancellation clause. The default parameter values are $\alpha=0.02$, $\sigma=0.10$, $r=0.04$, $K=100$, $\gamma=0.75$, $F^c=2$, $T=\infty$ and $P(0)=5$.