Cournot Competition in Networked Markets

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Abstract
The paper considers a model of competition among firms that produce a homogeneous good in a networked environment. A bipartite graph determines which subset of markets a firm can supply to. Firms compete à la Cournot and decide how to allocate their production output to the markets they are directly connected to. We show that the resulting game has a unique equilibrium for any given network and provide a characterization of the production quantities at equilibrium. Our results identify a close connection between the equilibrium outcome and supply paths in the underlying network structure. We then proceed to study the impact of changes in the competition structure on firms’ profits and consumer welfare. In particular, we examine how profits and welfare are affected by a firm expanding into a new market or by two firms merging. The analysis points to the fact that many of the insights from studying Cournot competition in a single market do not generalize in a networked environment and one may need to take the entire competition structure into account. Even assessing a firm’s market power is not straightforward as it depends on the markets it supplies to as well as the supply profiles of the firms it directly competes with. Our analysis illustrates that the price-impact matrix associated with the market that can be directly computed as a function of the competition structure succinctly summarizes the impact of each firm-market pair on equilibrium quantities, firms’ profits, and consumer welfare. Thus, the modeling framework we propose may find use in assessing whether expanding in a new market is profitable for a firm, identifying opportunities for collaboration, e.g., a merger, joint venture, or acquisition, between competing firms, and guiding regulatory action in the context of market design and antitrust analysis.

Keywords: Cournot competition, non-cooperative games, networks, horizontal mergers.

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1 Introduction

Models of oligopolistic competition typically feature a number of firms operating in a single, well-defined, isolated market. In many settings though, firms compete with one another across several different markets. This is particularly prevalent in regional industries that have distinct geographical markets, such as electricity, natural gas, airlines, the cement industry, healthcare, and banking. For example, constraints imposed by the natural gas pipeline network and the electricity grid imply naturally that firms in these industries compete with one another in several distinct consumer markets. In fact, several recent papers aim to explore firms’ strategic behavior in deregulated power markets by emphasizing the role of network structure on market outcomes (e.g., Borenstein, Bushnell, and Knittel (1999), Wu, Bose, Wierman, and Mohesenian-Rad (2013), Bose, Cai, Low, and Wierman (2014)). Similarly, airlines compete with one another across several origin-destination pairs which can be viewed as distinct markets. In fact, the presence of strong multimarket contact effects has been empirically verified for the US airline industry (Kim and Singal (1993), Evans and Kessides (1994)).

As another example, cement, due to its high weight-to-price ratio, can only be economically transported by land or between export/import terminals by sea. Thus, the set of locations a cement plant can supply to is inherently limited by its distance to potential customers and access to waterways. Typically, cement firms own and operate several plants in different geographical locations competing this way in many consumer markets (Jans and Rosenbaum (1997) provide extensive empirical evidence supporting the effect of multimarket contact on prices in the cement industry). Regional competition is also starting to play an important role in healthcare. Networks of providers compete with one another over customers that may have a strong preference towards easy access to care. The past few years have witnessed an increasing number of mergers, acquisitions, and consolidations in the healthcare space that aim to enable providers expand their geographical reach.

Finally, although we present our model and results for a single homogeneous good that is sold by a set of firms across different markets, our analysis is relevant for large conglomerate firms competing with one another in several distinct product markets.\footnote{Relatedly, an infographic created by French blog Convergence Alimentaire clearly illustrates the extent to which the entire consumer goods industry is dominated by ten multinational firms that compete with one another in several different product categories (http://www.convergencealimentaire.info/map.jpg).}

These examples motivate the study of oligopoly models in which firms strategically interact with one another across several markets. Although, as mentioned above, there is strong empirical evidence that multimarket competition is prevalent in many industries and single-market models are inadequate to provide an accurate description of the strategic interactions in such environments, to the best of our knowledge there is very little analytical work that explicitly accounts for the competition structure among firms. Our paper presents one step towards this direction. In particular, we consider a model where the competition structure is given by a general bipartite graph that represents the set of markets each firm can supply to. We emphasize that we do not require any assump-
tions on the number of firms and markets or the structure of the competition among them. Firms compete à la Cournot in each of the markets, i.e., price is determined as a function of the aggregate production quantity supplied to the market, and their cost of production is convex, thus their supply decisions in different markets are coupled.

We begin our analysis by showing that there exists a unique equilibrium in the setting we study. We provide a characterization of the equilibrium flows, i.e., production quantities, in terms of supply paths in the network. The price-impact matrix that can be written explicitly as a function of the underlying competition structure succinctly summarizes the effect of each firm-market pair on production quantities, firms’ profits, and consumer welfare. In particular, this matrix allows us to compute the impact on the price of any market that results from an increase in the quantity associated with a firm-market pair.

Armed with a characterization of equilibrium supply decisions in terms of the price-impact matrix, we explore the effect of changes in the network structure on firms’ profits and consumer welfare. First, we study the question of a firm entering a new market. We show that entry may not be beneficial for either the firm or consumers on aggregate as such a move affects the entire vector of production quantities. The firm may face aggressive competition in its original markets after expanding into a new market. The extent to which a firm’s competitors respond to the event of entry depends on the paths connecting the new market with the rest of the markets the firm supplies to and, thus, even distant markets (in a network sense) may have a first order impact on the firm’s profits. We explicitly quantify the network effect on the competitors’ response to entry and establish that the net benefit associated with expanding into a new market (even in the absence of fixed entry costs) decreases as the chain of competition increases and may in fact turn out to be negative.2

Furthermore, the effect on other firms and consumers also depends on their location in the network. A subset of firms and consumers may benefit while others may not. This is in stark contrast with standard results in Cournot oligopoly where entry directly implies more competition in the market and thus higher consumer welfare and lower profits for all the firms. Thus, our results have important implications for market design since regulatory measures that facilitate expansion to new markets may not necessarily lead to an increase in consumer surplus.

Similarly, the effect of a merger between two firms on profits and overall welfare largely depends on the structure of competition in the original networked economy. In particular, we show that insights from analyzing mergers in a single market do not carry over in a networked environment. Market concentration indices are insufficient to correctly account for the network effect of a merger and one should not restrict attention only to the set of markets that the firms participating in the merger supply to. We highlight that even if two firms do not share a market in the original pre-

2There are several instances of firms that “spread themselves too thin” by entering seemingly profitable markets and as a result ended up facing fierce competition in their home markets. For example, many claim that Frontier airlines invited aggressive competition in its primary hub, Denver, by expanding to a large number of new routes (see also Bulow, Geanakoplos, and Klemperer (1985)). Similarly, Kmart was not able to fend off competition from its major rival, Walmart, presumably because of over-diversifying its product portfolio.
merger economy they can exert market power by coordinating their supply decisions and lead to a decrease in consumer welfare unlike what traditional merger analysis would predict. Interestingly, Kim and Singal (1993) highlight the importance of taking the competition structure into account when considering the welfare effects of a merger by empirically studying the wave of mergers that followed the Deregulation Act in the airline industry.

**Related Literature**  Bulow, Geanakoplos, and Klemperer (1985) serves as one of the main motivations behind our study. They analyze the strategic interactions between two firms that compete in two markets (a monopoly and a duopoly) and show that strategic complementarity and substitutability between the firms’ actions determine the effect of exogenous changes in the markets on profits. We extend their environment by considering an arbitrary network of firms and markets and properly generalize the notion of strategic complementarity and substitutability to account for the network interactions among the firms. Our multimarket environment allows us to consider firms merging or expanding into new markets as well as to study the role of the competition structure in determining how welfare and profits are affected. Relatedly, Bernheim and Whinston (1990) examine how multimarket contact affects the degree of collusive behavior two firms can sustain when they compete repeatedly with one another in two markets.

In addition to the papers discussed above, a different strand of literature studies bilateral trading of indivisible goods among agents in a network. Kranton and Minehart (2001) model competition among buyers of a single, indivisible good as an ascending price auction and study whether the resulting pattern of trades is efficient. Corominas-Bosch (2004) considers a non-cooperative bargaining game and provides conditions for the equilibrium of the bargaining game to coincide with the Walrasian outcome. More recently, Nakkas and Xu (2014) examine how multimarket contact affects the degree of collusive behavior two firms can sustain when they compete repeatedly with one another in two markets.

Finally, Ashlagi, Kanoria, and Leshno (2014b) study the effect of competition on the size of the core in large matching markets. Much of the focus in this literature is on identifying conditions under which equilibrium outcomes are efficient or stable. In contrast, we are interested in explicitly studying how the underlying network structure determines production at

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3Several other papers, e.g., Blume, Easley, Kleinberg, and Tardos (2009), Condorelli and Galeotti (2012), Choi, Galeotti, and Goyal (2014) study trading among agents that are connected by a network structure.
equilibrium as well as how it affects profits and welfare when a firm expands into a new market or two firms decide to merge.

Relatedly, Guzmán (2011) studies Bertrand competition in a networked environment. His analysis is restricted to a duopoly and a key feature in his model is the assumption that a subset of consumers are locked-in, i.e., they can only purchase from one of the firms. Babaioff, Lucier, and Nisan (2013) extend his model and provide several results regarding the (non)-existence of pure and mixed Nash pricing equilibria along with structural properties of the sellers’ pricing equilibrium support. Independently from our work, Abolhassani, Bateni, Hajiaghayi, Mahini, and Sawant (2014) study a model of Cournot competition in networks similar to ours but their goal is to derive algorithms that compute the pure strategy Nash equilibria (as opposed to providing characterizations of equilibrium production quantities, profits, and welfare, which is the main focus of our study). Finally, on a somewhat different direction, Acemoglu and Ozdaglar (2007) and Chawla and Roughgarden (2008) study the extent of inefficiency in Bertrand competition games over networks.

Also related are the recent contributions by Farahat and Perakis (2011), Kluberg and Perakis (2012), Perakis and Sun (2014), and Federgruen and Hu (2014a) which study competition in price and product assortments in a setting where firms offer multiple products under linear costs and various assumptions on the extent of substitution between the products. The goal of the aforementioned line of work is to provide equilibrium existence results as well as worst-case bounds, i.e., over all possible problem instances, on the extent of efficiency loss due to competition (compared to the socially optimal outcomes). Our starting point is different: as has been empirically verified, firms’ behavior exhibits strong multi market contact effects when they compete with one another in multiple product markets (even when the products are not substitutable). Our focus is on providing an analytical framework to complement the extensive empirical work and on relating the underlying structure of the competition among them with equilibrium prices, profits, and consumer welfare.

On a somewhat different direction, Cho (2013) studies horizontal mergers in multitier supply chains in which firms in tier $i$ supply to all firms in tier $i-1$, i.e., the network between two tiers is complete, and interestingly concludes that the effect of a merger on consumer prices depends on whether the tier in which the merger occurs acts as a supply chain leader. Finally, Corbett and Karmarkar (2001), Adida and DeMiguel (2011), and Federgruen and Hu (2014b) study competition in multi-tier supply chains mainly showing the existence of equilibria and studying their efficiency properties. As in Cho (2013), this line of work focuses on structures where the network between consecutive tiers is complete.

Our analysis of mergers extends the work of Farrel and Shapiro (1990) to the setting where firms compete with one another in multiple markets. Specifically, they study mergers in a single market, whereas our focus is on how the structure of competition affects the way profits and consumer welfare change after the merger. Importantly, our analysis establishes that the entire network structure plays a first-order role in determining whether a merger has a positive or negative effect on overall welfare and market concentration indices that have been widely used in antitrust analysis cannot
accurately capture the welfare effect of a merger.

Finally, our paper is also related to a recent stream of papers that study games among agents that are embedded on a network structure. Ballester, Calvó-Armengol, and Zenou (2006) identify a close relation between an agent’s equilibrium action in a game that features local positive externalities and her position in the network structure as captured by her Katz-Bonacich centrality. Candogan, Bimpikis, and Ozdaglar (2012) study the pricing problem of a monopolist that is selling a divisible good to a population of agents and provide a characterization of the optimal pricing policies as a function of the social network structure of agents.

2 Model

Consider an economy with \( n \) firms \( F = \{f_1, \ldots, f_n\} \) producing a perfectly substitutable good and competing in \( m \) markets \( M = \{m_1, \ldots, m_m\} \). A bipartite graph \( G = (F \cup M, E) \), where \( E \) is a set of edges from the set of firms \( (F) \) to the set of markets \( (M) \), represents the subset of markets a firm can supply to. Finally, \( F_i = \{m_k \in M| (f_i, m_k) \in E\} \) denotes the set of markets firm \( f_i \) supplies to and \( M_k = \{f_i \in F| (f_i, m_k) \in E\} \) denotes the set of firms that supply to market \( m_k \). An example of the economy described above is depicted in Figure 1.

![Figure 1: The structure of competition as a bipartite graph.](image)

Firms compete in quantities, i.e., they decide how to allocate their aggregate production among the markets they supply to. Let \( q_{ik} \) denote the quantity firm \( f_i \) supplies to market \( m_k \) and \( q_i \) denote the vector of production quantities of firm \( f_i \). Then, the price at market \( m_k \) is given by \( P_k\left(\sum_{j \in M_k} q_{jk}\right) \). We assume that \( P_k(\cdot) \) is a twice differentiable concave function for every \( k \). Finally, the cost of production for firm \( f_i \) is given by \( C_i\left(\sum_{j \in F_i} q_{ij}\right) \), where \( C_i \) is a twice differentiable convex function for every \( i \). Thus, firm \( f_i \)'s profit is given by the following expression

\[
\pi_i(q) = \sum_{m_k \in F_i} q_{ik} \cdot P_k\left(\sum_{j \in M_k} q_{jk}\right) - C_i\left(\sum_{j \in F_i} q_{ij}\right).
\]

Quite importantly, firm \( f_i \)'s profit function is not separable in the markets it participates in for a general convex function \( C_i(\cdot) \) and the marginal profit from increasing \( q_{ik} \) is decreasing in firm \( f_i \)'s
aggregate production. This non-separability couples the markets a firm operates in, i.e., if costs were linear the environment would be equivalent to a set of markets that could be studied in isolation.\(^4\)

Given graph \(G\) and the production quantities of its competitors, 
\[ q_{-i} = \{q_{jk} \text{ for } (f_j, m_k) \in E \text{ and } j \neq i \}, \]
firm \(f_i\) solves the following optimization problem:
\[
\begin{align*}
\text{maximize} & \quad \pi_i(q_i, q_{-i}) \\
\text{subject to} & \quad q_{ik} \geq 0 \quad \text{for } k \in F_i \\
& \quad q_{ik} = 0 \quad \text{for } k \notin F_i.
\end{align*}
\]

We denote the resulting game by \(CG\left(\{\mathcal{P}_k\}_{1 \leq k \leq m}, \{\mathcal{C}_i\}_{1 \leq i \leq n}, G\right)\). In section 3 we show that game \(CG\) has a unique equilibrium for general concave \(\mathcal{P}_k\)’s and convex \(\mathcal{C}_i\)’s. Then, we proceed to provide a characterization of the production quantities at equilibrium as a function of the underlying network structure. We state our characterization results for inverse linear demands and quadratic production costs as this allows us to bring out the role of graph \(G\) in the clearest and most transparent way, i.e., we consider
\[
\mathcal{P}_k = \alpha_k - \beta_k \cdot \sum_{j \in M_k} q_{jk} \quad \text{and} \quad \mathcal{C}_i = c_i \cdot \left( \sum_{k \in F_i} q_{ik} \right)^2,
\]
where \(\alpha_k, \beta_k, c_i > 0\). Although many of our qualitative insights carry over to the general concave-convex framework, it is worthwhile to note that both inverse linear demand functions and quadratic costs are fairly common assumptions in the literature (e.g., Singh and Vives (1984), Vives (2011) for general quantity competition models, and Yao, Adler, and Oren (2008), Bose, Cai, Low, and Wierman (2014) for studies specific to electricity networks).

Before proceeding to our equilibrium analysis, it is helpful to briefly summarize the notation we use to characterize the equilibrium quantities. For every matrix \(A\), we let \(\lambda_{\min}(A)\) and \(\lambda_{\max}(A)\) denote the minimum and maximum eigenvalues of \(A\) respectively. For a column vector \(u\) and set \(S\), we let \(u_S\) denote the sub-vector of \(u\) with rows corresponding to set \(S\). Similarly, for matrix \(A\) and sets \(S_R\) and \(S_C\), we let \(A_{S_R,S_C}\) denote the sub-matrix of \(A\) with rows corresponding to set \(S_R\) and columns corresponding to set \(S_C\). Also, throughout the paper, we let \(A_{ik}\) denote the row corresponding to firm-market pair \((f_i, m_k)\) and \(A_{ik,j\ell}\) corresponds to firm-market pairs \((f_i, m_k)\) and \((f_j, m_\ell)\).

Finally, given graph \(G\), we define its line graph (denoted by \(L(G)\)) as the graph that has a node corresponding to every edge in \(G\) and an edge between two nodes in \(L(G)\) if their corresponding edges in the original graph \(G\) share an endpoint (see Figure 2 for an illustrative example). The line graph helps keeping track of how the production quantities on different links influence one another.

\(^4\)Note that although, in the interest of analytical tractability, we incorporate this feature through a convex cost function (which is a standard assumption in many oligopoly models), our analysis provides qualitative insights for settings in which this coupling arises due to the firm having limited resources that it shares among its markets and/or fixed capital to finance its operations.
If the two adjacent links in graph $G$ share market $m_k$ as an endpoint, they will influence one another via the demand function parametrized by $\beta_k$ and if they share firm $f_i$ as an endpoint, then they will influence one another via the production function of the firm parametrized by $c_i$. We incorporate this aspect by defining $L(G)$ as a weighted graph: the weight of an edge between nodes $\ell_1$ and $\ell_2$ in $L(G)$ is equal to $\beta_k$ if the edges in $G$ corresponding to $\ell_1$ and $\ell_2$ have market node $m_k$ as a common endpoint. If, on the other hand, the edges have firm node $f_i$ as a common endpoint, then the edge between $\ell_1$ and $\ell_2$ has weight $2c_i$. Paths between any two edges $(u_1, v_1)$ and $(u_2, v_2)$ in $G$ correspond to a path between their associated vertices in $L(G)$. Furthermore, we let the set of paths from link $(i, k)$ to market $m_\ell$ in $G$ refer to the set of paths in $L(G)$ starting from the node corresponding to link $(i, k)$ and ending in a node corresponding to $(j, \ell)$, for some $j \in M_\ell$. Finally, the weight of a path in $L(G)$ is defined to be equal to the product of the weights of the links that belong to the path.

Figure 2: The line graph associated with the economy depicted in Figure 1. Each node corresponds to a firm-market pair in the original graph $G$. Red dashed edges connect nodes (firm-market pairs) that share a market whereas blue solid edges connect nodes that share a firm. Finally, the weight of a blue solid edge is equal to $2c_i$, where $c_i$ is the cost parameter of the corresponding firm and the weight of a red dashed edge is equal to $\beta_k$. For example, the weight of the edge connecting nodes $(5,B)$ and $(5,D)$ is equal to $2c_5$ and the weight of the edge connecting nodes $(2,A)$ and $(6,A)$ is equal to $\beta_A$.

3 Equilibrium Analysis

We begin our analysis by showing that game $CG$ defined above has a unique equilibrium.

**Theorem 1.** Game $CG\left(\{P_k\}_{1 \leq k \leq m}, \{C_i\}_{1 \leq i \leq n}, G\right)$ has a unique Nash equilibrium when $\{P_k(\cdot)\}_{1 \leq k \leq m}$ are twice differentiable, concave, and strictly decreasing, and $\{C_i(\cdot)\}_{1 \leq i \leq n}$ are twice differentiable, convex, and increasing.

The proof of Theorem 1 builds on the existence and uniqueness results for concave games in Rosen (1965). In particular, it is straightforward to see that $CG$ is a concave game, thus existence of an equilibrium follows immediately from Theorem 1 in Rosen (1965). Uniqueness, on the other hand, is obtained by showing that game $CG$ satisfies a strict diagonal concavity property.

The remainder of the section provides a characterization of the production quantities at equilibrium as a function of the structure of competition among the firms. To clearly bring out the
role of network $G$, we make the stronger assumption that inverse demands are linear and costs are quadratic, i.e.,

$$P_k = \alpha_k - \beta_k \cdot \sum_{j \in M_k} q_{jk} \text{ and } C_i = c_i \cdot \left( \sum_{k \in F_i} q_{ik} \right)^2,$$

and we let $CG(\alpha, \beta, c, G)$ denote the corresponding game where $\alpha = [\alpha_1, \ldots, \alpha_m]^T$, $\beta = [\beta_1, \ldots, \beta_m]^T$, and $c = [c_1, \ldots, c_n]^T$.

First, we provide Lemma 1 which states that the unique equilibrium of $CG(\alpha, \beta, c, G)$ is given by the solution to a linear complementarity problem. In particular, let $LCP(w, Q)$ denote the problem of finding vector $z \geq 0$ such that

$$Qz + w \geq 0 \text{ and } z^T(Qz + w) = 0.$$

Then, we have

**Lemma 1.** The unique equilibrium $q$ of $CG(\alpha, \beta, c, G)$ is given by the unique solution of $LCP(-\alpha, D)$, where $D$ is the following $|E| \times |E|$ matrix:

$$D_{ik, j\ell} = \begin{cases} 2(\beta_k + c_i) & \text{if } i = j, k = \ell \\ 2c_i & \text{if } i = j, k \neq \ell \\ \beta_k & \text{if } i \neq j, k = \ell \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Let $q^*_E$ denote the column vector of equilibrium production quantities for each firm-market pair, i.e., edges of graph $G$, where the edges are ordered lexicographically. Note that a subset of edges may carry zero flow, i.e., $q^*_{ik} = 0$ even though $(f_i, m_k) \in E$. The results that follow are stated for the set of active edges, i.e., the subset of edges for which the corresponding production quantities are strictly positive. Lemma 2 below states that we can ignore the inactive edges, which we denote by $Z(q^*_E) = \{(f_i, m_k) | q^*_{ik} = 0\}$, without any loss of generality since they are strategically redundant and play no role in determining the equilibrium.

**Lemma 2.** Consider game $CG(\alpha, \beta, c, G)$ and let $q^*_E$ denote the vector of production quantities at equilibrium for this game. Also, let $G' = (F \cup M, E')$ with $E' = E \setminus Z(q^*_E)$ and $q'$ be the equilibrium of game $CG(\alpha, \beta, c, G')$. Then,

$$q^*_{E \setminus Z(q^*)} = q'.$$

For the remainder of the paper we focus on networks $G$ for which the resulting equilibria only have active edges (Lemma 4 in the Appendix provides sufficient conditions on the primitives of the game that guarantee this). Finally, although not critical for the results, we state Theorem 2 for symmetric games $CG(\alpha, \beta, c, G)$, i.e., games for which firms have the same production technology ($c_i = c_j = c$, for all $f_i, f_j \in F$), and markets have the same demand slope ($\beta_k = \beta_{\ell} = \beta$, for all
This way asymmetry between the firms arises only due to the structure of $G$. Before stating the result we define $|E| \times |E|$ matrix $W$ as

$$w_{i_1k_1, i_2k_2} = \begin{cases} 2c & \text{if } i_1 = i_2, k_1 \neq k_2 \\ \beta & \text{if } i_1 \neq i_2, k_1 = k_2 \\ 0 & \text{otherwise.} \end{cases}$$  

(2)

Note that matrix $W$ is a weighted adjacency matrix associated with the line graph $L(G)$ of graph $G$. In particular, the rows and columns of $W$ correspond to the links in graph $G$. If two links are connected in $G$ via a firm, then their weight in $W$ is $2c$ whereas if they are connected via a market, then their weight is $\beta$. For the links which do not share a node in $G$, the weight is 0. The non-zero entries of $W$ are equal to the change in the marginal profit from production corresponding to a firm-market pair that results from an infinitesimal increase in the quantity corresponding to another firm-market pair. For example, for links originating from the same firm ($i_1 = i_2$) this value is equal to $2c$ (thus capturing the increase in the marginal cost of production for the firm) whereas for edges ending up in the same market it is equal to $\beta$ (thus capturing the marginal decrease in the market's price).

Given $W$ we obtain,

**Theorem 2.** The unique Nash equilibrium of the symmetric game $CG(\alpha, \beta, c, G)$ is given by

$$q^* = [I + \gamma W]^{-1} \gamma \alpha,$$  

(3)

where $\gamma = (2(c + \beta))^{-1}$. Furthermore, if $\lambda_{\text{max}}(\gamma W) < 1$, Expression (3) can be rewritten as

$$q^* = \left[ \sum_{k=0}^{\infty} (\gamma W)^{2k} - \sum_{k=0}^{\infty} (\gamma W)^{2k+1} \right] \gamma \alpha.$$  

(4)

Theorem 2 implies that production quantity $q_{i_k}^*$, i.e., the quantity that firm $f_i$ supplies to market $m_k$ at equilibrium, can be written as a weighted sum of the sizes of all the markets (vector $\alpha$), where the weights are given by matrix $[I + \gamma W]^{-1}$. In particular, the weights depend on the location of link $(i, k)$ within the network through the paths that start from link $(i, k)$ and end up in the nodes representing the different markets.

As we formally show in the Appendix (see Proposition 6), for the production quantity that firm $f_i$ supplies to market $m_k$, i.e., $q_{i_k}$, the weight that corresponds to market $m_\ell$ (and thus multiplies $\alpha_\ell$) is increasing (decreasing) with the weights of even (odd) paths from link $(i, k)$ to market $m_\ell$. In other words, entry $\psi_{i_k, j_\ell}$ of matrix $\Psi = [I + \gamma W]^{-1}$ is increasing (decreasing) with the weights of even (odd) paths from link $(i, k)$ to every link $(j, \ell)$ where $f_j \in M_\ell$. This insight is most apparent in Equation (4), as the even (odd) power terms correspond directly to paths of even (odd) length. In informal terms, Equation (4) is driven by the intuition that “the enemy of my enemy is my friend”. As a side remark, we provide in the Appendix (Lemma 5) conditions under which $\lambda_{\text{max}}(\gamma W) < 1$ and thus Equation (4) is valid. The conditions essentially require that the network is sufficiently sparse.
Bulow, Geanakoplos, and Klemperer (1985) find that the main driver in determining how changes in one market affect a firm’s prospects in a second market is whether competitors view their actions as strategic substitutes or complements. Theorem 2 provides a way to relate the degree of strategic substitutability or complementarity between the actions of firms in two markets with the supply paths that connect them, thus appropriately generalizing these notions to a networked environment. Furthermore, the theorem implies that in order to determine whether a pair of actions are strategic substitutes or complements, one may need to consider the entire competition structure (and not just focus on the pair in isolation).

We can further exploit the simplicity of the equilibrium characterization in Equation (4) by using the weighted adjacency matrix $W$ of the line graph $L(G)$. As we show below, the production quantities at equilibrium are closely related to the following measure of network centrality of the nodes in graph $L(G)$.

**Definition 1.** Given a weighted adjacency matrix $W$ and a scalar $\rho$, the *Katz-Bonacich centrality* of the nodes in the network is defined as the following vector

$$b(W, \rho) = \sum_{t=0}^{\infty} (\rho W)^t 1.$$

In the case that all markets have the same size $\alpha_1 = \alpha_2 = \cdots = \alpha_m = \alpha$, Equation (4) can be rewritten as

$$q^* = b(W, -\gamma) \gamma \alpha,$$

where $q^*$ denotes the equilibrium production quantities.

It is important to note that equilibrium production quantities are written in terms of centrality vector $b(W, -\gamma)$ that features a negative scalar. Thus, unlike traditional notions of centrality for which nodes that are connected to central nodes are themselves central, this intuition does not hold in our setting. Bonacich (1987) discusses the implications of a negative scalar $\gamma$ in the context of trading networks and notes that having many direct connections (trading opportunities) contributes to centrality (bargaining power), however if one’s connections themselves have many connections, centrality is reduced (as the agent’s potential trading partners have many outside options). Similarly, in our setting a firm’s direct connections (and more generally paths of odd length) contribute negatively to the firm’s profits while paths of length two (and more generally paths of even length) have a positive contribution.

Finally, we conclude this section with an alternative characterization of the equilibrium production quantities that highlights their dependence on the “importance” of a firm-market pair, as captured by the *price-impact* matrix $\Lambda$ defined below. This characterization also illustrates that even distant links (in a network sense) may have a considerable impact on the price that the product is sold in a market. In particular, let $\Lambda$ denote the following $|E| \times m$ matrix

$$\Lambda_{ik,\ell} = -\beta \sum_{j \in M_{\ell}} \frac{\psi_{j,\ell,ik}}{\psi_{ik,ik}}.$$
As becomes apparent from Proposition 1 below, entry \((ik,\ell)\) of matrix \(\Lambda\) is equal to the change in market \(m_\ell\)'s price that results from a marginal increase in the production that firm \(f_i\) supplies to market \(m_k\). One can view the entries of \(\Lambda\) as a measure of the firms’ market power, i.e., the larger their absolute values, the higher market power the corresponding firms have in the underlying networked environment as changes in their actions have a large impact on market prices (and consequently firms’ profits and consumer surplus).

Corollary 1. The equilibrium production quantities can be expressed as

\[
q^* = -V\Lambda\alpha \gamma / \beta,
\]

where \(V = \text{Diag}(\Psi)\).

Interestingly, since \(V \geq 0\) and \(\alpha \geq 0\), Corollary 1 implies that \(q^* \propto -\Lambda\).

4 Changing the Structure of Competition

This section explores the effects on the firms’ profits and consumer welfare of changes in the structure of competition among the firms, i.e., changes in graph \(G\). In particular, we study changes in welfare when a firm enters a new market as well as when two firms merge and choose their production quantities with the goal of maximizing their joint profit. Our main focus is on highlighting the role of the underlying network structure and identifying how insights derived from the analysis of a single market differ due to (second order) network effects. For most of the section, we focus on the case where firms supply to all the markets they operate in at equilibrium. This is a natural assumption in this setting and allows us to express equilibrium quantities in closed form.

The following proposition is central in our subsequent analysis. It describes how firms adjust their production quantities in response to an exogenous change \(\Delta q_{ik}\) in \(q_{ik}\), i.e., the production quantity that firm \(f_i\) supplies to market \(m_k\).

**Proposition 1.** Consider an exogenous change \(\Delta q_{ik}\) in the quantity firm \(f_i\) supplies to market \(m_k\). Then, in the new equilibrium firms adjust their production quantities according to the following expression

\[
\Delta q_{j\ell} = \frac{\psi_{ik,j\ell}}{\psi_{ik,ik}} \Delta q_{ik},
\]

where recall that \(\Psi = [I + \gamma W]^{-1}\).

In particular, Proposition 1 implies that a change in the quantity firm \(f_i\) supplies to market \(m_k\) has ripple effects to the entire network, i.e., it affects the supply decisions of the entire set of competitors (even those that are not directly in competition with firm \(f_i\)). Note that since \(\Psi\) is a symmetric, positive semidefinite matrix with positive diagonal entries, according to Proposition 1 firms \(f_i\) and
$f_j$ view their actions in markets $m_k$ and $m_\ell$ respectively as strategic complements or substitutes depending on the sign of $\psi_{ik,j\ell}$. The latter is equal to the difference of the sums of the weights of even and odd paths from link $(i,k)$ to link $(j,\ell)$. Thus, matrix $\Psi$ allows us to determine the level of strategic complementarity or substitutability between the actions corresponding to two firm-market pairs and relate it to supply paths in the underlying network structure of competition. This result extends the work of Bulow, Geanakoplos, and Klemperer (1985) by identifying the role of the structure of competition on whether firms view their actions as strategic substitutes or complements.

The following example clearly demonstrates the second order network effects associated with a marginal change in a firm’s output.

**Example 1.** Consider the game defined over the graph in Figure 3(a) and assume that the production quantity corresponding to edge $(1, A)$ decreases by $\epsilon_1$ (e.g., due to small changes in firm 1’s cost of production). Then, firm $f_2$ would respond by increasing its output in market $A$ by $\epsilon_2$. Next, consider the game defined over Figure 3(b) and assume again that the production quantity corresponding to edge $(1, A)$ decreases by $\epsilon_1$. Then, clearly firm $f_2$ would increase its supply to market $A$. One would expect that the increase is smaller than $\epsilon_2$ in this case due to the fact that firm $f_2$ is also supplying to market $B$ (and production costs are convex). However, this is not the case: firm $f_2$ ends up responding more aggressively to the same change in firm $f_1$’s output. In Figure 3(a), firm $f_2$ can increase its supply to market $A$ only by increasing its production and hence its marginal cost whereas in Figure 3(b) firm $f_2$ can divert supply from market $B$ to market $A$, without increasing its marginal cost. This aspect of multimarket competition has been empirically demonstrated in Jans and Rosenbaum (1997). As they show for the U.S. cement industry, a firm’s ability to divert production from a market that it has “enough control over the price” to another market allows the firm to respond more aggressively to its competitors.

![Figure 3](image.jpg)

**Figure 3**: A change in firm $f_1$’s production quantity leads to different responses from $f_2$.

Thus, as Example 1 illustrates, the response from a firm’s competitors to changes in the quantity it supplies to any of its markets may be amplified relative to the case of a single-isolated market due to network effects.
4.1 Expanding into a New Market

This section explores the question of how equilibrium production quantities change when firm $f_i$ enters market $m_k$, which in our setting is equivalent to adding edge $(i, k)$ to graph $G$. Entry has a direct effect on firm $f_i$’s profit as the firm has to adjust the allocation of its production to the markets it supplies to. In addition, there is a second order effect that relates to how changes in firm $f_i$’s production quantities across its markets affect the actions of its competitors and their propagation through the network.

The following example illustrates how the competition structure among the firms may affect the profits associated with entry and consequently determine whether entry is beneficial for the firm.

**Example 2.** Let firm $f_1$ enter market $B$ for the two network structures illustrated in Figure 4. Then, in the first case (Figure 4(a)) it is profitable for firm $f_1$ to enter market $B$, whereas in the second (Figure 4(b)) it is not. This is due to the fact that when firm $f_1$ enters market $B$, it shifts part of its production to $B$ and thus decreases its supply to market $A$. Firm $f_2$, on the other hand, responds to a decrease in the level of competition in $A$ by increasing its supply to this market which results in a decrease in the profit for firm $f_1$ in market $A$. The increase in $f_2$’s output in $A$ depends on the level of strategic complementarity between edges $(f_1, B)$ and $(f_2, A)$. Utilizing the results from Section 3, we obtain that firm $f_2$ responds more aggressively, i.e., increases its supply to market $A$ by a higher amount, in the network of Figure 4(b) and as a consequence entry is less profitable for $f_1$ in this network (for the parameters of the example it is actually not profitable for firm $f_1$ to enter market $B$). It is important to note that the difference between the two networks depicted in Figure 4 is not local for firm $f_1$. It is actually straightforward to extend this example in such a way that the difference between the two networks is arbitrarily far (in terms of network distance) from firm $f_1$, i.e., it involves firms and markets that seemingly should not affect firm $f_1$’s profits. Thus, one has to take into account the entire network topology in order to determine the effect on the firm’s profits of expanding into a new market. Interestingly, if the economy takes the form of a chain (e.g., Figure 4) then one can show that the profits associated with expanding on a new market decrease with the length of the chain.

![Figure 4](image-url)

Figure 4: In both games $\alpha = [0.5, 1, 1]^T$, $\beta = 1$, and $c = 1$. In (a) adding edge $(1, B)$ leads to an increase in firm $f_1$’s aggregate profit, however in (b) the profit decreases.

The remainder of this subsection expands on the discussion above and provides a characterization of the change in firm $f_i$’s profits when it enters market $m_k$, i.e., edge $(i, k)$ is added to graph.
G. We restrict attention to edges \((i, k)\) that have positive marginal profit for firm \(f_i\). As one would expect, this is without any loss of generality according to Lemma 3 below.

\textbf{Lemma 3.} If edge \((i, k)\) has a negative marginal profit for \(q^*\), i.e.,
\begin{equation}
P_k = \alpha_k - \beta_k \left( \sum_j q_{jk}^* \right) < 2c,
\end{equation}
then the equilibrium does not change if firm \(f_i\) enters market \(m_k\).

The lemma follows directly from the fact that the optimality conditions for the actions of all agents remain unchanged if the new edge \((i, k)\) is such that inequality (8) holds and thus the equilibrium remains the same.

In the remaining of the section, we restrict our attention to cases in which the new link \((i, k)\) has a positive flow. Let \(W'\), \(q'\), and \(P'\) denote the \(|E| + 1 \times (|E| + 1)\) weight matrix defined over graph \(G \cup (i, k)\), the equilibrium for the new competition structure \(G \cup (i, k)\), and the price in market \(m_k\) in the new equilibrium respectively. Also, let \(\Psi' = [I + \gamma W']^{-1}\) and \(\Lambda'\) denote the price-impact matrix corresponding to \(\Psi'\). Note that by directly applying Proposition 1 we obtain
\begin{equation}
q_{j \ell}' = q_{j \ell} + \psi_{j \ell, ik}' \psi_{i \ell, ik}' q_{ik}.
\end{equation}
Given this, we obtain Proposition 2 below that characterizes the effect of entry into a new market on the profits of the entrant.

\textbf{Proposition 2.} Consider firm \(f_i\) entering market \(m_k\) and let \(q_{ik}'\) denote the production quantity that \(f_i\) supplies to \(m_k\) in the resulting equilibrium. Then, the total supply of \(f_i\) across its markets increases and the profit of firm \(f_i\) changes as follows
\begin{equation}
\Delta \pi_i = q_{ik}' (P_k + \Lambda_{ik,k}' q_{ik}') + \sum_{\ell \in F_i} \left( \frac{\psi_{i \ell, ik}'}{\psi_{i \ell, ik}} q_{ik}' + q_{i \ell}' \right) - \Delta C(S_i),
\end{equation}
where \(S_i = \sum_{k \in F_i} q_{ik}\).

Proposition 2 states that the effect of entry on firm \(f_i\)'s profits can be decomposed in three terms. First, term (*) illustrates the direct effect of entry as \(q_{ik}' (P_k + \Lambda_{ik,k}' q_{ik}')\) is equal to the profit that firm \(f_i\) obtains in market \(m_k\) (since \(P_k' = P_k + \Lambda_{ik,k}' q_{ik}'\)). Note that when firm \(f_i\) enters market \(m_k\) its competitors respond by changing the production quantities they supply to \(m_k\). This further results in a change of the price in market \(m_k\) which is summarized by \(\Lambda_{ik,k}'\). Larger values of \(\Lambda_{ik,k}'\) indicate that firm \(f_i\) obtains a larger direct profit when entering market \(m_k\).\(^5\)

\(^5\)Large values of \(\Lambda_{ik,k}'\) indicate that firm-market pair \(f_i - m_k\) has a high level of strategic substitutability with the rest of the links supplying to market \(m_k\).
On the other hand, term (●) captures the change in firm $f_i$’s profit from its operations in markets other than $m_k$. Note that firm $f_i$’s competitors respond to the event of entry by increasing their production to the markets other than $m_k$ which typically implies that the network effect captured by term (●) is negative for firm $f_i$. However, its magnitude depends on how aggressively the competitors respond, which in turn determines how much the prices in markets other than $m_k$ change.

The extent to which competitors respond to the event of firm $f_i$ entering market $m_k$ is encapsulated in the price-impact entry $\Lambda'_{ik,\ell}$ for $m_\ell \in F_i$. A small value for $\Lambda'_{ik,\ell}$ implies that firms respond aggressively in market $m_\ell$. This may affect firm $f_i$’s profit adversely, and thus make entry less profitable than a setting where network effects were absent. For instance, entry for firm $f_1$ to market $B$ is profitable for the example depicted in Figure 4(a) whereas it is unprofitable for the one in Figure 4(b). The difference between the two examples is edge $(f_2, C)$ which creates an additional path of even length between edges $(f_1, B)$ and $(f_2, A)$, thus increasing the level of strategic complementarity between them (and leading to a more aggressive response from $f_2$ in Figure 4(b)).

Finally, term ($‡$) is equal to the difference in production costs before and after entry and as we show in the first part of Proposition 2 it is always positive, since the aggregate quantity that firm $f_i$ supplies to the markets increases in the post-entry equilibrium.

The following example based on Figure 5 further illustrates the intuition behind Proposition 2. In this example, the difference between the two networks is a single edge, i.e., the one connecting firm $f_2$ with market $A$. Edge $(f_2, A)$ increases the level of strategic complementarity between $(f_2, C)$ and $(f_3, D)$, however it decreases the level of strategic complementarity between $(f_1, B)$ and $(f_3, D)$. Market $B$ is much larger than $C$ ($\alpha_2 > \alpha_3$) and thus the latter effect dominates the former making it profitable for firm $f_3$ to enter market $D$ only for the setting of Figure 5(b).

![Figure 5](image-url)

Figure 5: For both examples, $\alpha = [1, 4, 1, 1.8]^T$, $\beta = [1, 2, 1, 1]^T$, and $c = [1, 1, 1]^T$. Entering market $D$ is not profitable for firm $f_3$ for the setting in Figure 5(a), whereas it is profitable for the one in Figure 5(b).

Finally, we turn our attention on the effect of entry to the aggregate consumer surplus. Not sur-

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6Small values for $\Lambda'_{ik,\ell} \neq k$ imply high degree of strategic complementarity between the quantity firm $f_i$ supplies to market $m_k$ and the quantities $f_i$’s competitors supply to market $m_\ell$.

7Edge $(f_2, A)$ creates a path of odd length between edges $(f_3, D)$ and $(f_1, B)$: $(f_3, D) \rightarrow (f_3, C) \rightarrow (f_2, C) \rightarrow (f_2, A) \rightarrow (A, f_1) \rightarrow (f_1, B)$. 
prisingly when firm $f_i$ enters market $m_k$ consumer surplus increases in market $m_k$ as the price in the market decreases. However, entry has an ambiguous effect for the consumer surplus in the rest of the markets. First, let us formally define our quantity of interest, i.e., the aggregate consumer surplus in the environment.

**Definition 2.** The aggregate consumer surplus for game $CG(\alpha, \beta, c, G)$ is defined as the sum of the consumer surplus in all the markets

$$CS = \sum_{k=1}^{m} \frac{(\alpha_k - P_k)^2}{2\beta}.$$ 

Interestingly, we show below that increasing the level of competition in the environment by adding edges, i.e., by giving firms the option to compete in a larger subset of markets, can actually lead to a decrease in the aggregate consumer surplus. The reason behind this can be roughly explained as follows: a new edge $(i, k)$ may “spread” the competition along the network structure, i.e., firms shift (part of) their production away from an area of the network where consumers benefit from intense competition to an area where competition is less intense. Proposition 3 formalizes this intuition. The result follows by noting that if we add edge $(i, k)$ to $G$ then the price in market $\ell$ in the resulting equilibrium is given by

$$P'_\ell = P_\ell + \Lambda'_{ik,\ell} q'_{ik}.$$ 

So the price in any market $m_\ell$ with positive $\Lambda'_{ik,\ell}$ increases, whereas the price in the rest of the markets decrease.

**Proposition 3.** Consider firm $f_i$ entering market $m_k$ and let $q'_{ik}$ denote the production quantity that $f_i$ supplies to $m_k$ in the resulting equilibrium. Then, the aggregate consumer surplus changes as follows

$$\Delta CS = -\frac{q'_{ik}}{2\beta} \sum_{\ell=1}^{m} \Lambda'_{ik,\ell} (2(\alpha_\ell - P_\ell) - \Lambda'_{ik,\ell} q'_{ik}). \quad (11)$$

Proposition 3 implies that if link $(i, k)$ has a positive price-impact in markets where the difference between market size and price, i.e. $\alpha_\ell - P_\ell$, is large then increasing firm $f_i$’s production quantity in market $m_k$ might lead to a decrease in the aggregate consumer surplus. The example given in Figure 6 provides an illustration of the effect described in Proposition 3. Firm $f_4$ upon entering market $C$ shifts a large part of its production from $B$ to $C$. This benefits consumers in market $C$ but has adverse effects for consumers in markets $B$ and $A$ as firm $f_2$ in turn shifts part of its production from $A$ to $B$.

### 4.2 Horizontal Mergers

This subsection studies horizontal mergers between firms in the networked environment described in Section 2. As the effect on the profits of the merging firms, the “insiders”\(^8\), should presumably be

\(^8\)We borrow this terminology from Farrel and Shapiro (1990).
positive (otherwise the firms would have no incentive to initiate the merger), the analysis is mostly
concerned with the profits of outsiders and the welfare of consumers in the equilibrium that is est-
established after the merger. Farrel and Shapiro (1990) study the same question in a single Cournot
market and provide general conditions under which mergers that are profitable for insider firms also
lead to an increase in aggregate welfare.

Much of the antitrust analysis in real-world markets is centered around changes in the level of
market concentration that can be attributed to the merger. A reasonable way to extend the analysis
to a networked environment is to consider each of the markets in which both firms participate and
conclude that a merger should be allowed when the predicted change in concentration in any of the
markets is not too high. However, such an approach would essentially treat each market in isolation
and potentially overlook second order network effects. Example 3 below illustrates this pitfall.

**Example 3.** In the three market environment depicted in Figure 7, considering market $A$ in isolation
(which is the only market that both insider firms supply to) would likely lead to a favorable response
regarding a potential merger between firms $f_3$ and $f_4$, since the market is sufficiently competitive.
However, this reasoning is somewhat misleading. Firms $f_5$ and $f_6$ would react to less (more) aggres-
sive competition in markets $A$ ($C$) respectively and potentially create a captive market in $B$. This
second order network effect illustrates that considering each market in isolation may be incomplete
and motivates our discussion on mergers in a networked environment.

![Figure 6](image)

**Figure 6:** The consumer surplus in markets $A$ and $B$ significantly decreases when firm $f_4$ enters market $C$.

![Figure 7](image)

**Figure 7:** Let $c = 1$, $\beta = 1$, $\alpha_A = 1$, $\alpha_B = 0.3$, and $\alpha_C = 1$. Firms $f_5$ and $f_6$ supply the same quantity to
market $B$ before the merger due to symmetry. If firms $f_3$ and $f_4$ merge then (i) their joint production
decreases in market $A$; (ii) $f_4$ increases its production in $C$; (iii) $f_6$ shifts a fraction of its production
from $C$ to $B$; (iv) $f_5$ finds that market $A$ is more profitable than $B$. Thus, although consumer welfare
in markets $A$ and $C$ does not decrease substantially, competition in market $B$ is significantly lower
in the post-merger equilibrium and thus the overall effect of the merger on welfare may be negative.

As should be evident from the example above, measuring the overall effect of a merger on total
welfare is not a straightforward task when firms compete across several markets. One would potentially need to study how changes in firms’ actions propagate across the network. The goal in the remainder of this section is to provide some insights towards this direction. First, following Farrel and Shapiro (1990), we impose no assumptions on how a merger affects the insider firms, i.e., their production costs, as this is typically hard to observe or predict. Instead, Proposition 4 provides an expression for the change in the production quantities of outsider firms in response to a given change in the output of insiders. Let $I$ denote the set of insider firms and assume that their merger results in a change of their total output in market $m_k$ given by $\Delta q_{I,k}$. Moreover, let $O$ and $G^O = G \setminus I$ denote the rest of the firms (the outsiders), and their subnetwork respectively. Also let $W^G^O$ be equal to $W_{O,O}$, i.e., the sub-matrix of $W$ corresponding to the rows and columns of the outsiders.

**Proposition 4.** Assume that insider firms, i.e., firms in set $I$, change the total output they supply to market $m_\ell$ by $\Delta q_{I,\ell}$. Then, the production quantity that outsider firm $f_i$ supplies to market $m_k$ changes as follows in the post-merger equilibrium

$$\Delta q_{ik} = -\beta \sum_{m_\ell | m_\ell \in F_n \text{ for } n \in I} \sum_{j \in O \text{ and } j \in M_\ell} \Psi_{ik,j,\ell} G^O q_{I,\ell},$$

where $\Psi^O = [I + \gamma W^G^O]^{-1}$.

Proposition 4 provides a relation between the post-merger production quantities of the insider firms, i.e., the firms that participate in the merger, and those of the outsider firms. This relation can be helpful for assessing the overall effect of a merger on welfare as it provides a closed form expression for the changes in both prices and market concentration\(^9\). Concretely, the regulator can use this relation to provide a set of constraints on the post-merger equilibrium supply of insider firms that any merger has to satisfy. For instance, such constraints were imposed in the merger between US-airways and American Airlines in 2013.\(^{10}\) In particular, the Department of Justice, as a condition for allowing the merger, required that the two airlines gave up landing and takeoff slots and gates at “seven key constrained airports.”, thus effectively limiting their post-merger presence in those airports. The slots were to go to low cost airlines, “resulting in more choices and more competitive airfares for consumers.”. We strongly believe that Proposition 4 can be useful in such a setting as it allows one to quantify the effects of a merger taking also the network interactions into account.

Finally, for the remainder of this section we consider the benchmark setting in which mergers do not result in cost synergies between the two insider firms $f_i$ and $f_j$. Rather they decide on their production quantities so as to maximize their joint profit denoted by $\pi_{ij}(\cdot)$, i.e.,

$$\pi_{ij}(\hat{q}) = \pi_i(\hat{q}) + \pi_j(\hat{q}),$$

\(^9\)In particular, following Farrel and Shapiro (1990), this proposition enables us to study the *external* effect of a merger on consumers and outsider firms.

\(^{10}\)For more information, see http://www.justice.gov/opa/pr/2013/November/13-at-1202.html.
where $\hat{q}$ denotes the vector of post-merger equilibrium production quantities. Although we assume that there are no cost synergies, i.e., their production functions do not change after the merger, they can still benefit from jointly deciding how much to supply to each of the markets they participate in.

Note that the post-merger payoff structure is different from our original framework. In particular, the firm that results from the merger between $f_i$ and $f_j$ chooses its production quantities in order to maximize $\pi_{ij}(\cdot)$. Thus, Theorem 1 does not apply directly to guarantee equilibrium uniqueness. In fact, there are examples where the number of equilibria may be infinite. However, we can show that an equilibrium always exists and it is generically unique. Moreover, when there are multiple equilibria, they are all equivalent in the sense that they result in the same prices for all the markets as well as the same profits for all the firms.

**Definition 3.** Post-merger equilibria $q$ and $q'$ are called equivalent if and only if for every market $m_k$ the following holds

$$q_{ik} + q_{jk} = q'_{ik} + q'_{jk},$$

where $i, j$ are the insider firms.

Note that given the aggregate supply of insider firms in a market the response of outsider firms is unique. Therefore, Definition 3 implies that if two post-merger equilibria $q$ and $q'$ are equivalent, then for any outsider firm $f_t$ and every market $m_k$ we have $q_{tk} = q'_{tk}$. The following theorem extends our equilibrium existence and uniqueness results from Section 3 to the setting where firms $i, j$ merge their operations, i.e., they choose their production quantities so as to maximize their joint profit.

**Theorem 3.** Suppose that firms $f_i$ and $f_j$ merge and their joint profit is given by

$$\pi_{ij}(\hat{q}) = \pi_i(\hat{q}) + \pi_j(\hat{q}).$$

Then the following hold,

(i) A post-merger equilibrium always exists.

(ii) If $f_i$ and $f_j$ do not share a market, then the post-merger equilibrium is unique and coincides with the pre-merger equilibrium.

(iii) If $\lambda_{\min}(W) \neq -(2c + \beta)$ the post-merger equilibrium is unique.

(iv) If multiple equilibria exist, they are all equivalent.

Given that the post-merger equilibrium exists and it is essentially unique, our goal in the remainder of this section is to provide a characterization of its properties. In particular, we show that it is equivalent to the equilibrium of a game that has the form described in Section 2 and it is directly related to the original pre-merger game. Specifically, as we formally state in Proposition 5 it turns out that the post-merger equilibrium can be computed as the equilibrium of one of two distinct games (see also Figure 8 for illustrations of both types of post-merger equilibria).
Proposition 5. The post merger equilibrium is equivalent to the equilibrium of one of the following two games:

(i) Firms $f_i$ and $f_j$ can be thought of a single firm $f_{ij}$ (single node in the competition graph) that is connected to the union of the markets that $f_i$ and $f_j$ were originally connected to. The cost function of $f_{ij}$ is given by the following expression

$$C_{ij}(x) = 2C\left(\frac{x}{2}\right).$$

(ii) Firm $f_i$ and $f_j$ operate so as to maximize their own profit (as in the pre-merger environment) and the competition graph is given as

$$G = \{F \cup M, E - \{(f_i, m_k) | (f_i, m_k) \land (f_j, m_k) \in E}\},$$

i.e., the competition graph is the same as the original $G$ without the links from $f_i$ to the markets it shares with $f_j$.

![Figure 8](image)

Figure 8: For this example $\alpha = \beta = c = 1$. The equilibria of the mergers corresponding to Figures 8(a), 8(c) are the same as the equilibria of the games in Figures 8(b), 8(d).

Proposition 5 highlights that firms after a merger may behave in one of two distinct ways. In particular, part (i) of the Proposition describes a post-merger equilibrium outcome where the insider firms can be thought of as a single firm that has access to the union of the markets the original firms have access to. Part (ii) instead states that in some settings the post-merger equilibrium is equivalent to the outcome of the competition where the insider firms act independently but only one of them has access to the markets they share in the original network. This is a nice characterization result that allows us to compute the supply quantities, profits, and welfare associated with the post-merger equilibrium in a straightforward way given our characterization results from Section 3.

Finally, we conclude this section by providing an example (see Figure 9) that illustrates that even if firms do not share any markets in the original graph $G$ (and in the absence of any cost synergies), in the post-merger equilibrium aggregate consumer welfare may actually decrease. This is an extreme illustration of the network effect in multimarket competition and thus further highlights the inadequacy of market concentration type indices in assessing the effect of a merger in overall welfare.
Figure 9: For this example $\beta = c = 1$ and $\alpha = [10, 1, 2]$. Firms 4 and 5 do not share any markets in the original economy. After they merge, firm 4 shifts a fraction of its production to market $C$. This results in less aggressive competition in market $B$ which incentivizes firms 1 and 2 to shift part of their production to market $B$ resulting in a sharp decrease in welfare for consumers in market $A$.

5 Conclusion

This paper studies a model of competition in a networked environment. A bipartite graph determines the set of potential supply relationships. We provide a characterization of the unique equilibrium that highlights the relation between production quantities and supply paths in the underlying network structure. Using this characterization we derive several comparative statics results regarding the effect on quantities, prices, and welfare of changes in the network structure that may be the result of a firm expanding into a new market or a merger between two firms. Our results illustrate that qualitative insights from the analysis of a single market do not necessarily generalize when firms compete with one another in multiple markets. We believe that the modeling framework we propose may find use in assessing whether expanding in a new market is profitable for a firm, identifying opportunities for collaboration, e.g., a merger, joint venture, or acquisition, between competing firms, and guiding regulatory action in the context of market design and antitrust analysis.

There are a number of directions that are worth pursuing. First, we purposefully abstracted from several details of the industries that serve as our main motivation since our primary goal was to present in the most transparent and general way how the competition structure affects equilibrium outcomes in a setting of multimarket competition. For example, although we expressed our results for general structures most major airlines use hub-and-spoke networks to route their passenger traffic. Similarly, we assumed that firms can in principle supply any amount of the good to a market (albeit at at an increasingly higher marginal production cost). Competition in electricity networks though is subject to constraints on the flows that can use a given transmission line, thus the aggregate amount of electricity that can actually reach a market is inherently dependent on the physical capacity of the transmission network. Incorporating these important features into our modeling framework would allow us to explore how our insights specialize to each of those industries and it is obviously a fruitful area for future research.

Second, following Bulow, Geanakoplos, and Klemperer (1985) we focused on competition among a set of firms in a static setting. Bulow, Geanakoplos, and Klemperer (1985) study an economy with
only two markets and show that the demand structure of the markets and the production technology of the firms determine whether firms view their production quantities as strategic substitutes or complements. Our analysis shows that in a setting with multiple firms and markets the network structure is another factor that must be taken into account when determining how a firm’s competitors may respond to changes in its production decisions for a market. A natural direction for extending our results is to consider an environment where firms interact repeatedly with one another. Bernheim and Whinston (1990) examine the effect of multimarket contact on the degree of cooperation that two firms can sustain when they repeatedly compete in two markets over an infinite time horizon. Although Bernheim and Whinston (1990) analyze a repeated game, both in ours and their model the presence of multiple markets increases the incentives to collude, e.g., the firms can engage in a reciprocal market sharing agreement (as in Belleflamme and Bloch (2004)). Understanding how the structure of competition may facilitate or deter collusive behavior is another question which warrants further investigation.
Appendix

Proof of Theorem 1

First, note that each agent’s action space is convex, bounded, and closed. Second, we show that when the strategies of other firms are fixed to $q_{-i}$, then $\pi_i$ is concave in $q_i$. This follows by showing that the Hessian matrix below is negative semi-definite. Define the $|F_i| \times |F_i|$ Hessian matrix $H_i$ as follows:

$$H_i = \begin{bmatrix}
\frac{\partial^2 \pi_i}{\partial q_{i,1} \partial q_{i,1}} & \frac{\partial^2 \pi_i}{\partial q_{i,1} \partial q_{i,2}} & \cdots & \frac{\partial^2 \pi_i}{\partial q_{i,1} \partial q_{i,m}} \\
\frac{\partial^2 \pi_i}{\partial q_{i,2} \partial q_{i,1}} & \frac{\partial^2 \pi_i}{\partial q_{i,2} \partial q_{i,2}} & \cdots & \frac{\partial^2 \pi_i}{\partial q_{i,2} \partial q_{i,m}} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 \pi_i}{\partial q_{i,m} \partial q_{i,1}} & \frac{\partial^2 \pi_i}{\partial q_{i,m} \partial q_{i,2}} & \cdots & \frac{\partial^2 \pi_i}{\partial q_{i,m} \partial q_{i,m}}
\end{bmatrix}.$$  

Also define the $|F_i| \times (m + 1)$ matrix $V$ as follows:

$$V_{ik,\ell} = \begin{cases} 
\sqrt{-\left(2P_i' \left( \sum_{j \in M_k} q_{jk} \right) + q_{ik}P''_i \left( \sum_{j \in M_k} q_{jk} \right) \right)} & \text{if } \ell = k \\
\sqrt{C''_i \left( \sum_{j \in F_i} q_{ij} \right)} & \text{if } \ell = m + 1 \\
0 & \text{otherwise.}
\end{cases}$$

Then, we obtain that $H_i = -VV^T$, and one can conclude that $H_i$ is negative semi-definite. Thus, this is a concave game and has an equilibrium according to Rosen (1965). To prove uniqueness, it is sufficient to show that the $|E| \times |E|$ matrix $[G(x, r) + G^T(x, r)]$ defined below is negative definite for every $x$ in the action space and some fixed and positive $r$. Let $r = 1$ for the remainder of the proof. Then, for every vector of actions $q$ we have

$$G(q, 1) = \begin{bmatrix}
\frac{\partial^2 \pi_1}{\partial q_{1,1} \partial q_{1,1}} & \frac{\partial^2 \pi_1}{\partial q_{1,1} \partial q_{1,2}} & \cdots & \frac{\partial^2 \pi_1}{\partial q_{1,1} \partial q_{1,m}} \\
\frac{\partial^2 \pi_2}{\partial q_{1,2} \partial q_{1,1}} & \frac{\partial^2 \pi_2}{\partial q_{1,2} \partial q_{1,2}} & \cdots & \frac{\partial^2 \pi_2}{\partial q_{1,2} \partial q_{1,m}} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 \pi_n}{\partial q_{1,m} \partial q_{1,1}} & \frac{\partial^2 \pi_n}{\partial q_{1,m} \partial q_{1,2}} & \cdots & \frac{\partial^2 \pi_n}{\partial q_{1,m} \partial q_{1,m}}
\end{bmatrix}.$$  

Each row of $G(q, 1)$ corresponds to an edge in the original competition graph. The entries corresponding to edge $(i, k)$ are equal to the cross derivatives of $\pi_i$ with respect to $q_{ik}$ and $q_{jl}$ for all $(j, l) \in E$, i.e.,

$$\frac{\partial^2 \pi_i}{\partial q_{ik} \partial q_{jl}} = \begin{cases} 
2P'_k \left( \sum_{j \in M_k} q_{jk} \right) - C''_i \left( \sum_{j \in F_i} q_{ij} \right) + q_{ik}P''_k \left( \sum_{j \in M_k} q_{jk} \right) & \text{if } i = j \text{ and } k = \ell \\
- C''_i \left( \sum_{j \in F_i} q_{ij} \right) + q_{ik}P''_k \left( \sum_{j \in M_k} q_{jk} \right) & \text{if } i = j \text{ and } k \neq \ell \\
P'_k \left( \sum_{j \in M_k} q_{jk} \right) + q_{ik}P''_k \left( \sum_{j \in M_k} q_{jk} \right) & \text{if } i \neq j \text{ and } k = \ell \\
0 & \text{otherwise.}
\end{cases}$$
Let $\eta_k \equiv -P'_k \left( \sum_{j \in M_k} q_{jk} \right)$ and $2\theta_i \equiv C''_i \left( \sum_{j \in F_i} q_{ij} \right)$ and define $|E| \times |E|$ matrices $\Omega$ and $\Phi$ as follows:

$$
\Omega_{ik,j\ell} = \begin{cases} 
2(\eta_k + \theta_i) & \text{if } i = j, k = \ell \\
2\theta_i & \text{if } i = j, k \neq \ell \\
\eta_k & \text{if } i \neq j, k = \ell \\
0 & \text{otherwise}
\end{cases}, \quad \text{and } \Phi_{ik,j\ell} = \begin{cases} 
(q_{ik} + q_{jk})P''_k \left( \sum_{j \in M_k} q_{jk} \right) & \text{if } k = \ell \\
0 & \text{otherwise}.
\end{cases}
$$

Then, $G(q, 1) + G^T(q, 1) = -2\Omega + \Phi$. So to complete the proof, it is sufficient to show that $\Omega$ is positive definite and $\Phi$ is negative semi-definite. We show the positive definiteness of $\Omega$ by providing a full rank matrix $R$ such that $\Omega = R^T R$. In particular, let $R$ be a $(|E| + m + n) \times |E|$ matrix that can be written in the form of a block matrix as follows:

$$
R = \begin{bmatrix} A \\ B \end{bmatrix},
$$

where $A$ is an $|E| \times |E|$ diagonal matrix with

$$
A_{ik,j\ell} = \begin{cases} 
\sqrt{\eta_k} & \text{if } i = j, k = \ell \\
0 & \text{otherwise},
\end{cases}
$$

and $B$ is an $(m + n) \times |E|$ matrix with

$$
B_{t,(i,k)} = \begin{cases} 
\sqrt{2\theta_i} & \text{if } t \leq n, t = i \\
\sqrt{\eta_k} & \text{if } t > n, t = n + k \\
0 & \text{otherwise}.
\end{cases}
$$

Matrix $R$ is full rank since $A$ is a diagonal matrix with non-zero entries on its diagonal. It is also straightforward to check that $\Omega = R^T R$. This implies that $\Omega$ is positive-definite. Furthermore, matrix $\Phi$ can be decomposed as follows:

$$
\Phi = \Phi^1 + \Phi^2 + \cdots + \Phi^m,
$$

where $\Phi^k$ is a matrix that corresponds to market $m_k$. For every two edges $(i, k)$ and $(j, k)$ we have

$$
\Phi_{ik,jk}^k = q_{ik}P''_k \left( \sum_{j \in M_k} q_{jk} \right) + q_{jk}P''_k \left( \sum_{j \in M_k} q_{jk} \right),
$$

and so

$$
\Phi^k = P''_k \left( \sum_{j \in M_k} q_{jk} \right) \left( q_k 1^T_k + 1_k q^T_k \right),
$$

where $q_k$ is a $|E| \times 1$ vector, with $q_{k\ell} = q_{\ell k}$ if $\ell = k$ and 0 otherwise. Moreover, $1_k$ is a $|E| \times 1$ vector, with $1_{k\ell} = 1$ if $\ell = k$ and 0 otherwise. Thus, given that $P(\cdot)$ is concave, matrix $\Phi^k$ is negative semi-definite for every $k$ and this completes the proof. \qed
Proof of Lemma 1

First, note that the marginal profit associated with a firm-market pair has to be non-positive at equilibrium, i.e.,
\[
\frac{\partial \pi_i}{\partial q_{ik}} \bigg|_{q_{ik}^*} = \alpha_k - \beta_k q_{ik}^* - \beta_k \sum_{j \in M_k} q_{jk}^* - 2c_i \sum_{l \in F_i} q_{i\ell}^* \leq 0.
\] (12)

This set of equations can be rewritten in matrix form as
\[
-\alpha + Dq^* \geq 0,
\] (13)
where recall that $D$ is a $|E| \times |E|$ matrix defined in Equation (1).

Second, every firm-market pair for which the corresponding production quantity is strictly positive at equilibrium has to satisfy Equation (12) with equality, i.e., if $q_{ik}^* > 0$ then $\frac{\partial \pi_i}{\partial q_{ik}} \bigg|_{q_{ik}^*} = 0$. So given that we assume that all edges in $E$ are active at equilibrium, we obtain the following
\[
q^*(-\alpha + Dq^*) = 0.
\] (14)

Finally, the supply at a firm-market pair has to be non-negative at equilibrium
\[
q^* \geq 0.
\] (15)

Conditions (13), (14), and (15) constitute a linear complementarity problem $LCP(-\alpha, D)$. According to the results in Samelson, Thrall, and Wesler (1958), problem $LCP(-\alpha, D)$ has a unique solution if and only if all the principal minors of $D$ are positive. Positive definite matrices satisfy this condition and thus what remains to be shown is that $D$ is positive definite for every graph $G$, which follows by arguments similar to the ones we used for the proof of the positive definiteness of matrix $\Omega$ in Theorem 1. Thus, the equilibrium of game $CG(\alpha, \beta, c, G)$ can be characterized as the unique solution for $LCP(-\alpha, D)$.

Proof of Lemma 2

Note that as we showed in Lemma 1 the equilibrium of game $CG(\alpha, \beta, c, G)$ is the unique solution of a linear complementarity problem. Furthermore, note that production quantities that take value zero do not have any effect in the linear complementarity problem, i.e., the solution of the $LCP$ remains the same even when they are omitted. Thus, the production quantities that take strictly positive values in the solution of the linear complementarity problem corresponding to $CG(\alpha, \beta, c, G)$ take the same values in the solution of the linear complementarity problem corresponding to $CG(\alpha, \beta, c, G')$.

To complete the discussion on active edges, we state and prove the following lemma that provides a necessary and sufficient condition on graph $G$ so that all edges in $G$ are active at equilibrium.
Lemma 4. The unique equilibrium of game \( CG = (\alpha, \beta, c, G) \) has no inactive edges if and only if there does not exist a vector \( x \in \mathbb{R}^{|E|} \) such that

\[
D^T x > 0 \\
\alpha^T x = -1
\]

where \( D \) is the matrix defined in Equation (1).

Proof. From the proof of Lemma 2, one can see that the equilibrium does not feature inactive edges if and only if there exists a vector \( q \) such that

\[
-\alpha + Dq \geq 0 \\
q(-\alpha + Dq) = 0 \\
q > 0.
\]

Using Stiemek's lemma, we obtain that the above is equivalent to the following: there exists no vector \( x \in \mathbb{R}^{|E|} \) such that

\[
D^T x > 0 \\
\alpha x = -1,
\]

which concludes the proof of the lemma.

Proof of Theorem 2

For every active edge \((i, k)\) it should be the case that \( \frac{\partial \pi_i(\alpha, \beta, c, G)}{\partial q_{ik}} = 0 \), which implies that

\[
q_{ik}^* = \frac{\alpha_k - 2c \sum_{\ell \in F_i, \ell \neq k} q_{i\ell}^* - \beta \sum_{j \in M_k} q_{jk}^*}{2(\beta + c)} = \frac{\alpha_k}{2(\beta + c)} - \sum_{(j, \ell) \in E(G^*)} (\gamma W)_{ik, j\ell} q_{j\ell}^*.
\]

This further implies that

\[
q^* = \gamma \alpha - \gamma W q^* \Rightarrow q^* = [I + \gamma W]^{-1} \gamma \alpha.
\]

To show the second part of the theorem, first note that Expression (17) can be rewritten as

\[
q^* = [I - (-\gamma W)]^{-1} \gamma \alpha.
\]

Matrix \([I - (-\gamma W)]^{-1}\) can be rewritten as the power series of matrix \((-\gamma W)\) if and only if the spectral radius of \((-\gamma W)\) is less than 1, i.e., if and only if

\[-1 < \lambda_{\text{min}}(\gamma W) \leq \lambda_{\text{max}}(\gamma W) < 1.\]
The final step in the proof involves showing that for every game $\lambda_{\min} > -1$. To this end, define the $(n + m) \times |E|$ edge incident matrix $B$ for graph $G$ as follows

$$B_{v,(i,t,k_t)} = \begin{cases} \sqrt{\frac{2c}{2(c+\beta)}} & \text{if } v \leq n, \ i_t = v, \\ \sqrt{\frac{\beta}{2(c+\beta)}} & \text{if } v > n, \ k_t = v - n, \\ 0 & \text{otherwise}. \end{cases}$$

Then, it is straightforward to see that

$$\gamma W = B^T B - \frac{2c + \beta}{2(c + \beta)} I.$$ 

Note that $B^T B$ is a positive semidefinite matrix and thus all of its eigenvalues are non-negative. Furthermore, $\frac{2c + \beta}{2(c + \beta)} < 1$ thus we conclude that

$$\lambda_{\min} \geq -\frac{2c + \beta}{2(c + \beta)} > -1.$$

This concludes the proof since to be able to rewrite $[I - (-\gamma W)]^{-1}$ as in Expression (2) it suffices that $\lambda_{\max} < 1$. 

Finally, to complete the picture we state and prove Lemma 5 below that provides a sufficient condition for $\lambda_{\max} < 1$. The condition essentially implies that graph $G$ is sufficiently sparse.

**Lemma 5.** Consider a symmetric CG game. Then, $\lambda_{\max}(\gamma W) < 1$ if one of the following two conditions holds

(i) The marginal cost of production is sufficiently low, i.e., $2c < \beta$. Furthermore, each market can have at most 2 suppliers and each firm supply to at most 3 markets.

(ii) The marginal cost of production is sufficiently high, i.e., $2c \geq \beta$. Furthermore, each firm can supply to at most 2 markets and each market can have at most 2 suppliers.

We chose to state the corollary for when the Cournot game is symmetric, but it readily extends to asymmetric games.

**Proof.** We provide a proof for the case where $2c < \beta$ (the proof for the other case is identical). Consider the dual graph $L(G)$. Without loose of generality we can assume that $G$ is connected and as a result $L(G)$ is also connected. As a result, $W$ is an irreducible matrix with non-negative entries. Thus by Perron-Frobenius theorem, we have

$$\lambda_{\max}(\gamma W) \leq \max_{(i,k) \in E} \gamma \sum_{(j,t) \in E} w_{ik,jt}.$$ 

Now consider the case where $2c < \beta$. In this case if each market have at most 2 supplies and each firm has at most 3 suppliers, one can see that for any link $(i, k)$ we have $\gamma \sum_{(j,t) \in E} w_{ik,jt} \leq 2 \frac{2c}{2(c+\beta)} +$
Lemma 5 shows that sparsity is a sufficient condition to have $\lambda_{\text{max}}(\gamma W) < 1$. In the following lemma we show that sparsity is also a necessary condition.

**Lemma 6.** Consider a symmetric CG game. Then, $\lambda_{\text{max}}(\gamma W) < 1$ only if one of the following two conditions holds

(i) If the marginal cost of production is sufficiently high, i.e., $2c \geq \beta$ then each firm can supply to at most 10 markets and the total number of links connecting firms to markets is smaller than $4n$.

(ii) If the marginal cost of production is sufficiently low, i.e., $2c < \beta$ then, each market can have at most 5 suppliers and the total number of links connecting firms to markets is smaller than $3m$.

**Proof.** We provide a proof for the case when $2c \geq \beta$ (the proof for the case when $\beta > 2c$ is identical). Consider the dual graph $L(G)$ and remove the edges that have weight $\beta$ (recall that edges in $L(G)$ have weight equal to $2c$ or $\beta$). Let $W'$ denote the adjacency matrix of the resulting matrix. Every edge in the remaining line graph has weight $2c$ so we have

$$W' = 2c \times H,$$

where $H$ is a binary adjacency matrix. Note that $\lambda_{\text{max}}(\gamma W') \leq \lambda_{\text{max}}(\gamma W) \leq 1$, since removing edges always decreases the maximum eigenvalue of a matrix. Furthermore,

$$\lambda_{\text{max}}(\gamma W') = \frac{2c}{2(c + \beta)} \lambda_{\text{max}}(H) \geq \frac{1}{3} \lambda_{\text{max}}(H).$$

Therefore, $\lambda_{\text{max}}(\gamma W) < 1$ implies that $\lambda_{\text{max}}(H) \leq \frac{1}{3}$. Finally, note that for every unweighted graph corresponding to adjacency matrix $H$ we have

$$\max\{\sqrt{\deg_{\text{max}}(H)}, \overline{\deg}(H)\} \leq \lambda_{\text{max}}(H),$$

where $\deg_{\text{max}}(H)$, $\overline{\deg}(H)$ are the maximum, average degrees of the graph corresponding to $H$ respectively. Thus, we obtain that $\deg_{\text{max}}(H) \leq 9$. Finally, this implies that each firm can supply to at most 10 markets (since it can have at most 9 direct neighbors in the line graph $L(G)$). Similarly, $\overline{\deg}(H) \leq 3$ implies that on average each firm can supply to at most 4 markets. This concludes the proof of the lemma. 

Finally, Proposition 6 below confirms a basic feature of the equilibrium described in Theorem 2: equilibrium production corresponding to a firm-market pair increases (decreases) with the weights of even (odd) paths from the edge corresponding to the pair to any of the markets.

**Proposition 6.** Consider the unique Nash equilibrium $q^*$ of game $CG(a, \beta, c, G)$. Then, the quantity firm $f_i$ supplies to market $m_k$ at equilibrium, i.e., $q^*_{ik}$ is increasing (decreasing) with the weights of even (odd) paths from edge $(i, k)$ to a market $m_\ell$.

We provide a proof for this proposition after the proof of Proposition 1 as we need a lemma that is stated as part of the proof of Proposition 1.
Proof of Corollary 1

According to Theorem 2, we have $q^* = [I + \gamma W]^{-1} \gamma \alpha$. So, for any link $(i,k)$ we have:

$$q^*_{ik} = \gamma \sum_{(j,\ell) \in E} \psi_{ik,j\ell} \alpha_{\ell} = \gamma \sum_{\ell \in M} \alpha_{\ell} \sum_{j \in M_{\ell}} \psi_{ik,j\ell}$$

$$= -\frac{\gamma}{\beta} \psi_{ik,ik} \sum_{\ell \in M} \alpha_{\ell} \sum_{j \in M_{\ell}} -\beta \psi_{ik,j\ell} \psi_{ik,ik}$$

$$= -\frac{\gamma}{\beta} \psi_{ik,ik} \sum_{\ell \in M} \alpha_{\ell} \Lambda_{ik,\ell}. \quad (18)$$

The claim follows by writing Equation (18) in a matrix form. \hfill \square

Proof of Proposition 1

Consider an exogenous change $\Delta q_{ik}$ in the production quantity corresponding to link $(i,k)$ and denote by $\Delta q_{j\ell}$ the change in the production quantity corresponding to link $(j,\ell)$ in the new equilibrium. The first order optimality conditions imply the following equation for links $(j,\ell) \neq (i,k)$

$$q_{j\ell} + \Delta q_{j\ell} = \frac{\alpha_{\ell}}{2(\beta + c)} - \sum_{(j_1,\ell_1) \in E} (\gamma W)_{j_1j_1\ell_1} (q_{j_1\ell_1} + \Delta q_{j_1\ell_1}). \quad (19)$$

By subtracting Equation (16) from Equation (19), we get the following equation:

$$\Delta q_{j\ell} = -\sum_{(j_1,\ell_1) \in E \setminus (i,k)} (\gamma W)_{j_1j_1\ell_1} \Delta q_{j_1\ell_1} - (\gamma W)_{j\ell,ik} \Delta q_{ik}. \quad (20)$$

Let $\tilde{W} = W_{E \setminus (i,k), E \setminus (i,k)}$ and let $\zeta$ denote a vector such that for every link $(j,\ell)$, we have $\zeta_{j\ell} = \gamma w_{j\ell,ik}$. Then, we can rewrite Equation (20) in a matrix form as follows:

$$\Delta q = -[I + \gamma \tilde{W}]^{-1} \zeta \Delta q_{ik}. \quad (21)$$

Finally, let $\tilde{\Psi}' = [I + \gamma \tilde{W}]^{-1}$. In order to make the calculations easier we define matrix $\tilde{\Psi}$ which constructed by attaching one additional row and column corresponding to link $(i,k)$, to the matrix $\tilde{\Psi}'$. The entries of the new row and column are all zero except the entry on diagonal which is equal to 1. The following lemma relates matrices $\Psi$ and $\tilde{\Psi}$.

Lemma 7. Let $\tilde{W} = W_{E \setminus (i,k), E \setminus (i,k)}$. Then, we have

$$\Psi = \tilde{\Psi} + \frac{\Gamma}{1 - C}, \quad (22)$$

where
\[
\Gamma_{j_1,\ell_1, j_2, \ell_2} = \begin{cases}
\left( \sum_{(j,\ell) \in E} \zeta_{j\ell} \bar{\psi}_{j_1\ell_2} \right) \left( \sum_{(j,\ell) \in E} \zeta_{j\ell} \bar{\psi}_{j_2\ell_2} \right) & \text{if } (j_1, \ell_1) \neq (i, k) \text{ and } (j_2, \ell_2) \neq (i, k) \\
- \sum_{(j,\ell) \in E} \zeta_{j\ell} \bar{\psi}_{j_1\ell_2} & \text{if } (j_1, \ell_1) = (i, k) \text{ and } (j_2, \ell_2) \neq (i, k) \\
- \sum_{(j,\ell) \in E} \zeta_{j\ell} \bar{\psi}_{j_1\ell_1} & \text{if } (j_1, \ell_1) \neq (i, k) \text{ and } (j_2, \ell_2) = (i, k) \\
C & \text{if } (j_1, \ell_1) = (i, k) \text{ and } (j_2, \ell_2) = (i, k)
\end{cases}
\]

and 
\[
C = \sum_{(j_1, \ell_1) \in E} \sum_{(j_2, \ell_2) \in E} \zeta_{j_2\ell_2} \bar{\psi}_{j_2\ell_2} \zeta_{j_1, \ell_1} \zeta_{j_1, \ell_1}.
\]

**Proof.** Note that we have:

\[
\Psi = [I + \gamma W]^{-1} = [I + \gamma \tilde{W} + e_{ik} \zeta^T + \zeta e_{ik}^T]^{-1} = \left( \tilde{\Psi} - \frac{\tilde{\Psi} \zeta e_{ik}^T \tilde{\Psi}}{1 + e_{ik}^T \zeta \tilde{\Psi}} \right) - \frac{\left( \tilde{\Psi} - \frac{\tilde{\Psi} \zeta e_{ik}^T \tilde{\Psi}}{1 + e_{ik}^T \zeta \tilde{\Psi}} \right) e_{ik} \zeta^T \left( \tilde{\Psi} - \frac{\tilde{\Psi} \zeta e_{ik}^T \tilde{\Psi}}{1 + e_{ik}^T \zeta \tilde{\Psi}} \right)}{1 + C} = \tilde{\Psi} + \Gamma \frac{1}{1 - C}
\]

where the second equality follows from applying the Sherman-Morrison formula twice. \(\square\)

Equation (22) implies that \(\psi_{ik,ik} = 1 + \frac{C}{1-C} = \frac{1}{1-C}\). Also, again according to Equation (22), we obtain
\[
\psi_{ik,j\ell} = \psi_{j\ell,ik} = \frac{\Gamma_{ik,j\ell}}{1 - C} = \Gamma_{ik,j\ell} \psi_{ik,ik}.
\]

Finally using Equations (21) and (24) we obtain
\[
\Delta q_{j\ell} = - \sum_{(j_1, \ell_1) \in E} \tilde{\psi}_{j\ell,j_1\ell_1} \zeta_{j_1, \ell_1} \Delta q_{ik} = \frac{\psi_{j\ell,ik}}{\psi_{ik,ik}} \Delta q_{ik},
\]

which concludes the proof of the Proposition. \(\square\)

**Proof of Proposition 6**

Note that according to Theorem 2, at equilibrium \(q^*_{ik} = \gamma \sum_{\ell \in M} \alpha_\ell \sum_{j \in M_\ell} \psi_{ik,j\ell} \). So in order to prove the Proposition, it is enough to show the following claim:

**Claim.** In any network and for every two arbitrary links \((i, k)\) and \((j, \ell)\), \(\psi_{ik,j\ell}\) is increasing (decreasing) with the weights of even (odd) paths from edge \((i, k)\) to edge \((j, \ell)\).
Proof. We prove this claim by induction on the number of edges in the network. Obviously the claim holds for the trivial network with only one link. Now assume that the claim holds for any network with $|E| - 1$ links, and we will prove it for a network with $|E|$ links. Remove an arbitrary link $(i, k)$, and let $\tilde{W} = W_{E \setminus (i,k), E \setminus (i,k)}$, then according to Lemma 7 we have

$$
\Psi = \tilde{\Psi} + \frac{\Gamma}{1 - C},
$$

where $\Gamma$ is a matrix defined in equation (23). According to the induction hypothesis entries of $\tilde{\Psi}$ satisfy the claim. Also note that according to the proof of Lemma 7, $\psi_{ik,ik} = \frac{1}{1-C}$, and since $\Psi$ is a positive definite matrix, and thus all the diagonal entries are positive, we should have $\frac{1}{1-C} > 0$. So it is enough to have the entries of $\Gamma$ satisfy the claim as well i.e. for every two arbitrary links $(i, k)$ and $(j, \ell)$, $\Gamma_{ik,j\ell}$ is increasing (decreasing) with the weights of even (odd) paths from edge $(i, k)$ to edge $(j, \ell)$, which is obviously hold by the definition of $\Gamma$. \hfill \Box

Proof of Lemma 3

Let $q$ and $q'$ are equilibrium before and after firm $f_i$ expands to new market $m_k$ respectively. Also recall that according to Lemma 1, the equilibrium of any Cournot game is the unique solution to $LCP(-\alpha, D)$. Then if $\alpha_k - \beta_k\left(\sum_{j \in M_k} q_{jk}^*\right) < 2c$, having $q'_{j\ell} = q_{j\ell}, \forall (j, \ell) \neq (i, k)$ and $q'_{ik} = 0$ is the solution for $LCP$ and thus is an equilibrium.

Proof of Proposition 2

First, we state and prove the following lemma, which is equivalent to the first part of the proposition, i.e., the total supply of firm $f_i$ increases in the equilibrium after firm $f_i$ enters market $m_k$..

Lemma 8. Consider an increase in the production quantity firm $f_i$ supplies to market $m_k$. Then, both the total supply in market $m_k$ as well as the aggregate production by firm $f_i$ increase in the resulting equilibrium.\textsuperscript{11}

Proof. Let $\Delta q_{ik}$ denote the increase in the production quantity firm $f_i$ supplies to market $m_k$. Then, from Proposition 1, we obtain that for any other firm-market pair $(j, \ell)$ we have $\Delta q_{j\ell} = \psi_{j\ell,ik} \frac{\psi_{ik,ik}}{\psi_{ik,ik}} \Delta q_{ik}$. So the lemma follows if the two inequalities below hold

$$
\sum_{j \in M_k} \psi_{jk,ik} > 0,
$$

$$
\sum_{\ell \in F_i} \psi_{i\ell,ik} > 0.
$$

\textsuperscript{11}We would like to thank Will Nelson for helping us prove this lemma. We would also like to thank math.stackexchange.com for providing a forum where we could ask for help regarding this.
For each firm $f_i$, let $u_{f_i}$ denote the following $|E| \times 1$ binary column vector

$$[u_{f_i}]_{j\ell} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise}. \end{cases}$$

Similarly, for each market $m_k$, let $u_{m_k}$ denote the following $|E| \times 1$, binary column vector

$$[u_{m_k}]_{j\ell} = \begin{cases} 1 & \text{if } \ell = k, \\ 0 & \text{otherwise}. \end{cases}$$

Then, we obtain

$$\Psi = [I + \gamma W]^{-1} = \left[ \frac{\beta}{2(\beta + c)} I + \frac{2c}{2(\beta + c)} \sum_{i=1}^{n} u_{f_i} u_{f_i}^T + \frac{\beta}{2(\beta + c)} \sum_{k=1}^{m} u_{m_k} u_{m_k}^T \right]^{-1}. \quad (25)$$

Finally, let $Y$ denote the following $|E| \times (m + n)$ matrix

$$Y = [u_{f_1}, \ldots, u_{f_n}, u_{m_1}, \ldots, u_{m_m}],$$

and let $\mathcal{D}$ denote the following diagonal matrix

$$\mathcal{D} = \text{Diag} \left[ \sqrt{\frac{2c}{2(\beta + c)}}, \ldots, \sqrt{\frac{2c}{2(\beta + c)}}, \sqrt{\frac{\beta}{2(\beta + c)}}, \ldots, \sqrt{\frac{\beta}{2(\beta + c)}} \right].$$

Then, we can rewrite Equation (25) as follows

$$\Psi = \left[ \frac{\beta}{2(c + \beta)} I + Y \mathcal{D}^2 Y^T \right]^{-1}. \quad (26)$$

The lemma is equivalent to the following claim: for any $i$ and $j$ such that $Y_{ij} = 1$ we have $(\Psi Y)_{ij} > 0.$ Using Woodbury’s matrix identity, we obtain

$$\Psi Y = \left[ \frac{\beta}{2(c + \beta)} I + Y \mathcal{D}^2 Y^T \right]^{-1} Y$$

$$= \left( \frac{2(\beta + c)}{\beta} I - \frac{2(\beta + c)}{\beta} Y (\mathcal{D}^2 + Y^T 2(\beta + c) \mathcal{D} - 2Y^T 2(\beta + c) \mathcal{D} - I) Y \right) Y$$

$$= \frac{2(\beta + c)}{\beta} Y - Y \left( \frac{\beta}{2(\beta + c)} \mathcal{D}^2 + Y^T Y \right) \left( \frac{\beta}{2(\beta + c)} \mathcal{D}^2 + Y^T Y \right)^{-1} Y$$

$$= \frac{2(\beta + c)}{\beta} \left( Y - Y \left( \frac{\beta}{2(\beta + c)} \mathcal{D}^2 + Y^T Y \right)^{-1} \left( \frac{\beta}{2(\beta + c)} \mathcal{D}^2 + Y^T Y \right)^{-1} \right)$$

$$= \frac{2(\beta + c)}{\beta} \left( Y - Y (I - \frac{\beta}{2(\beta + c)} \mathcal{D}^2 + Y^T Y \mathcal{D} - 2(\beta + c) \mathcal{D} - I) \right)$$

$$= Y \mathcal{D}^2 \left( \frac{\beta}{2(\beta + c)} \mathcal{D}^2 + Y^T Y \right)^{-1}$$

$$= Y \left( \frac{\beta}{2(\beta + c)} I + Y^T Y \mathcal{D}^2 \right)^{-1}.$$
So according to Equation (26) it is enough to show that for every entry $i,j$ such that $Y_{ij} = 1$, then
\[
\left( Y \left( \frac{\beta}{2(\beta+c)} I + Y^T \mathcal{D}^2 \right)^{-1} \right)_{ij} > 0.
\]
To show this, note that in every row of $Y$ there are exactly two non-zero entries. Consider row $r$ of $Y$ and assume that $Y_{rj_1}$ and $Y_{rj_2}$ are the two non-zero entries. Then, the claim is equivalent to the following two statements
\[
(YP)_{rj_1} = P_{j_1j_1} + P_{j_2j_1} > 0, \quad (YP)_{rj_2} = P_{j_2j_2} + P_{j_1j_2} > 0,
\]
where $P \equiv \left( \frac{\beta}{2(\beta+c)} I + Y^T \mathcal{D}^2 \right)^{-1}$. Note that this follows if we show that for every $i,j$, we have $P_{ii} > |P_{ji}|$. To this end, we use the following lemma from Ostrowski (1952)

**Lemma 9 (Ostrowski (1952)).** If $A$ is a strictly diagonally dominant matrix with non-negative entries and positive diagonal then $B = A^{-1}$ always exists and for every $i,j$ we have:
\[
|B_{ij}| < B_{ii}.
\]
Thus, it is enough to show that matrix $Y^T Y = \begin{bmatrix} A_f & B \\ B^T & A_m \end{bmatrix}$, where $A_f$ and $A_m$ are diagonal matrices with $A_{fii} = \|u_{f_i}\|_1$ and $A_{mii} = \|u_{m_i}\|_1$. Also, for every $i,j$, $B_{ij} = u_{f_i} \cdot u_{m_j}$. Thus, for every $i$ we have:
\[
(Y^T Y)_{ii} = \sum_{j \neq i} (Y^T Y)_{ij},
\]
and since $\mathcal{D}$ is a positive diagonal matrix and $\frac{\beta}{2(\beta+c)} > 0$, we conclude that $\frac{\beta}{2(\beta+c)} I + \mathcal{D}^2 Y^T Y$ is strictly diagonally dominant and the lemma follows. 

Finally, from Equation (9), we have that for every link $(j, \ell)$ the production quantity supplied by firm $f_j$ in market $m_\ell$ is given by
\[
q'_{j\ell} = q_{j\ell} + \frac{\psi'_{j,ik}}{\psi_{ik,ik}} q_{ik}.
\]
Also the price in each market $\ell$ is given by
\[
P'_\ell = P_\ell + \Lambda'_{ik,\ell} q'_l.
\]
So the profit for firm $f_i$ in equilibrium after entry $q'_{i}$ is given by
\[
\pi_i(q') = q'_{ik} P'_k + \sum_{\ell \in F_i} q'_{i\ell} P'_\ell - c S_i'^2
\]
\[
= q'_{ik} P'_k + \sum_{\ell \in F_i} (q_{i\ell} + \frac{\psi'_{ik,ik}}{\psi_{ik,ik}} q_{ik})(P_\ell + \Lambda'_{ik,\ell} q_{ik}) - c(S_i + \Delta S_i)^2
\]
\[
= \pi_i + q'_{ik} \left( P_k + \Lambda'_{ik,k} q_{ik} \right) + q'_{ik} \sum_{\ell \in F_i} \left( \frac{\psi'_{ik,ik}}{\psi_{ik,ik}} P_\ell + \Lambda'_{ik,\ell} \left( \frac{\psi'_{ik,ik}}{\psi_{ik,ik}} q_{ik} + q_{i\ell} \right) \right) - c \left( \Delta S_i'^2 + 2 S_i \Delta S_i \right),
\]

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which completes the proof.

Proof of Proposition 3

The new price in market \( m_\ell \) is equal to

\[
P'_\ell = P_\ell + \Lambda'_{ik,\ell} q'_ik.
\]

So the aggregate consumer surplus \( CS' \) in the equilibrium after entry will be given by

\[
CS' = \sum_{\ell=1}^m \frac{(\alpha_\ell - P'_\ell)^2}{2\beta} = \sum_{\ell=1}^m \frac{(\alpha_\ell - P_\ell - \Lambda'_{ik,\ell} q'_ik)^2}{2\beta} = CS - \frac{q'_ik}{2\beta} \sum_{\ell=1}^m \Lambda'_{ik,\ell} (2(\alpha_\ell - P_\ell) - \Lambda'_{ik,\ell} q'_ik).
\]

Thus, we conclude that

\[
\Delta CS = -\frac{q'_ik}{2\beta} \sum_{\ell=1}^m \Lambda'_{ik,\ell} (2(\alpha_\ell - P_\ell) - \Lambda'_{ik,\ell} q'_ik).
\]

Proof of Proposition 4

The first order optimality conditions for firm \( f_j \) in the post-merger equilibrium imply that

\[
q_{j\ell} + \Delta q_{j\ell} = \frac{\alpha_\ell}{2(\beta + c)} - \sum_{(j_1,\ell_1) \in E} (\gamma W)_{j_\ell,j_1\ell_1} (q_{j_1\ell_1} + \Delta q_{j_1\ell_1}). \quad (27)
\]

Subtracting Equation (16), i.e., the equation that corresponds to the first order optimality conditions for firm \( f_j \) in the original equilibrium, from Equation (27) we obtain

\[
\Delta q_{j\ell} = -\sum_{(j_1,\ell_1) \in E} \Delta q_{j_1\ell_1}. \quad (28)
\]

For each link \((j, \ell)\) we let \( \eta_{j\ell} = -\beta \Delta q_{I2,1} \). We can rewrite Equation (28) as follows

\[
\Delta q^O = -\gamma W^O \Delta q^O + \eta.
\]

Thus we conclude that the changes in the production output for the outsider firms are given by

\[
\Delta q^O = [I + \gamma W^O]^{-1} \eta.
\]
Proof of Theorem 3

First, we prove the following lemma, which states that any equilibrium in the post-merger game is a solution to an appropriately defined linear complementarity problem.

Lemma 10. Strategy profile \( q \) is an equilibrium for the game that results when firms \( f_i \) and \( f_j \) merge if and only if \( q \) is a solution of the linear complementarity problem \( LCP(-\alpha, D') \), where \( D' \) is an \(|E| \times |E| \) matrix defined as follows

\[
D'_{i_1 k, j_1 l} = \begin{cases} 
2(\beta + c) & \text{if } i_1 = j_1, k = \ell \\
2c & \text{if } i_1 = j_1, k \neq \ell \\
\beta & \text{if } i_1 \neq j_1, k = \ell \text{ and } \{i_1, j_1\} \neq \{i, j\} \text{ or } k \notin F_i \cap F_j \\
2\beta & \text{if } i_1 \neq j_1, k = \ell \text{ and } \{i_1, j_1\} = \{i, j\} \text{ and } k \in F_i \cap F_j \\
0 & \text{otherwise.}
\end{cases}
\]

Proof. We follow the same approach as in the proof of Lemma 1. The first order optimality conditions imply that any equilibrium strategy profile has to be a solution of the linear complementarity problem \( LCP(-\alpha, D') \). Next, we show that the other direction holds as well, i.e., we show that any solution that satisfies the first order optimality conditions is an equilibrium. To this end, first consider firm \( f_{i_1} \neq f_i, f_j \). Define the \(|F_{i_1}| \times |F_{i_1}| \) Hessian matrix \( H_{i_1}^{i_1} \) for \( f_{i_1} \)’s optimization problem as follows

\[
H_{i_1 k, i_1 l}^{i_1} = \begin{cases} 
-2(\beta + c) & \text{if } k = \ell \\
-2c & \text{if } k \neq \ell
\end{cases}
\]

It is straightforward to see that \( H \) is a negative definite matrix and thus outsider firm \( i_1 \) has no profitable deviation. Similarly, for the firm that results from the merger of \( f_i \) and \( f_j \), let \( H_{ij}^{ij} \) denote the Hessian matrix associated with its optimization problem, i.e.,

\[
H_{i_1 k, j_1 l}^{ij} = \begin{cases} 
-2(\beta + c) & \text{if } i_1 = j_1, k = \ell \\
-2\beta & \text{if } i_1 \neq j_1, k = \ell \\
-2c & \text{if } i_1 = j_1, k \neq \ell \\
0 & \text{otherwise.}
\end{cases}
\]

Hessian \( H_{ij}^{ij} \) is negative semi-definite (and furthermore higher order conditions involve 0 matrices), thus we conclude that the insider firms have no incentive to deviate from their strategy and the joint strategy profile that corresponds to the solution of the linear complementarity problem is an equilibrium.

Given Lemma 10 we can proceed with the proofs of parts (i)-(iv) of the Theorem. First, for part (ii) note that when the two insider firms \( f_i \) and \( f_j \) do not share any markets, then \( D' = D \) (matrix \( D \) was defined in Equation (1)) and therefore the linear complementarity problem corresponding to the post-merger game is exactly the same as the one corresponding to the game before the merger. This implies that the equilibrium is unique and it is the same as the equilibrium for the pre-merger game.
Then, we turn our attention to parts (i) and (iii). Lemma 10 implies that if \( D' \) is positive semi-definite, then \( \text{LCP}(−\alpha, D') \) has a solution (albeit not necessarily unique). On the other hand, if \( D' \) is positive definite then \( \text{LCP}(−\alpha, D') \) has a unique solution. Note that \( D' \) and \( D \) are related as follows

\[
D' = D + X,
\]

where \( X \) is the following \(|E| \times |E| \) matrix

\[
X_{i,k,j,\ell} = \begin{cases} 
\beta & \text{if } \{i_1, j_1\} = \{i, j\} \text{ and } k = \ell \in F_i \cap F_j \\
0 & \text{otherwise}.
\end{cases}
\]

According to Weyl’s theorem and noting that \( \lambda_{\text{min}}(X) = −\beta \), we obtain

\[
\lambda_{\text{min}}(D') \geq \lambda_{\text{min}}(D) + \lambda_{\text{min}}(X) = \lambda_{\text{min}}(D) - \beta
\]

So if we show that \( \lambda_{\text{min}}(D) \) is at least equal to \( \beta \) then matrix \( D' \) is positive semi-definite and thus an equilibrium always exists. Furthermore, in the case that \( \lambda_{\text{min}}(D) > \beta \), \( D' \) is positive definite and thus the equilibrium is unique. Note that

\[
D = \frac{1}{\gamma}BB^T + \beta I,
\]

where matrix \( B \) is defined in Equation (5). So since matrix \( \frac{1}{\gamma}BB^T \) is symmetric positive semi-definite, one can conclude that \( \lambda_{\text{min}}(D) \geq \beta \), and equilibrium always exists. Finally, note that

\[
\frac{1}{\gamma}BB^T = W + (2c + \beta)I,
\]

and thus again by Weyl’s theorem we obtain

\[
\lambda_{\text{min}}(\frac{1}{\gamma}BB^T) \geq \lambda_{\text{min}}(W) + (2c + \beta).
\]

So if \( \lambda_{\text{min}}(W) \neq -(2c + \beta) \) then \( \lambda_{\text{min}}(\frac{1}{\gamma}BB^T) > 0 \), matrix \( D' \) is positive definite, and thus the equilibrium is unique. This concludes the proof of parts (i) and (iii) of the Theorem, since the condition \( \lambda_{\text{min}}(W) \neq -(2c + \beta) \) holds generically.

Next we show the last part of the Theorem. We say that the post-merger equilibrium \( q \) is balanced if the aggregate production of firm \( f_i \) is equal to the aggregate production of firm \( f_j \). Also, we call a post-merger equilibrium connected if both firms \( f_i \) and \( f_j \) supply a strictly positive production quantity to at least one of the markets they share. The proof follows from the following three lemmas.

**Lemma 11.** All balanced equilibria are equivalent.

**Proof.** Consider a balanced equilibrium \( q \) and let

\[
G' = \{F \cup M, E \cup \{(f_i, m_k) | m_k \in F_j\} \cup \{(f_j, m_k) | m_k \in F_i\}\},
\]

...
denote the network that results from $G$ when we add the links from both firms $f_i$ and $f_j$ to all the markets that at least one of them participates in the original networked economy represented by graph $G$. Then, we claim that vector $q'$ with

$$q'_{j\ell} = \begin{cases} 0 & \text{if } (j, \ell) \text{ is a link in } G' \text{ but not in } G, \\ q_{j\ell} & \text{otherwise,} \end{cases}$$

is an equilibrium for the game defined over $G'$. Consider first a market $m_v \in F_i \cap F_j$, i.e., a market that the two insider firms share in the original network $G$. Note that

$$\frac{\partial \pi_{ij}}{\partial q_{iv}} = \alpha_v - 2\beta q_{iv} - 2\beta q_{jv} - \beta \sum_{\ell \in M_v, \ell \neq i, j} q_{\ell v} - 2c \sum_{\ell \in F_i} q_{i\ell} \leq 0,$$

$$\frac{\partial \pi_{ij}}{\partial q_{jv}} = \alpha_v - 2\beta q_{jv} - 2\beta q_{iv} - \beta \sum_{\ell \in M_v, \ell \neq i, j} q_{\ell v} - 2c \sum_{\ell \in F_j} q_{j\ell} \leq 0,$$

and since the equilibrium is balanced, we have $\frac{\partial \pi_{ij}}{\partial q_{iv}} = \frac{\partial \pi_{ij}}{\partial q_{jv}}$. Next consider a market $m_v$ such that $m_v \in F_j$ but $m_v \notin F_i$. Since $q$ is an equilibrium for the game defined over $G$ we have

$$\frac{\partial \pi_{ij}}{\partial q_{jv}} = \alpha_v - 2\beta q_{jv} - 2\beta q_{iv} - \beta \sum_{\ell \in M_v, \ell \neq i, j} q_{\ell v} - 2c \sum_{\ell \in F_j} q_{j\ell} \leq 0.$$

Finally, consider the first order condition corresponding to link $(i, v)$ in network $G'$

$$\frac{\partial \pi_{ij}}{\partial q_{iv}} = \alpha_v - 2\beta q_{iv} - 2\beta q_{jv} - \beta \sum_{\ell \in M_v, \ell \neq i, j} q_{\ell v} - 2c \sum_{\ell \in F_i} q_{i\ell},$$

Since $q$ is a balanced equilibrium, we have

$$\left. \frac{\partial \pi_{ij}}{\partial q_{iv}} \right|_{q'} = \left. \frac{\partial \pi_{ij}}{\partial q_{jv}} \right|_{q'} \leq 0,$$

so there is no incentive for firm $f_i$ to produce in market $m_v$. Similarly, we obtain that firm $f_j$ has no incentive to produce in market $m_u$ which is such that $m_u \in F_i$ but $m_u \notin F_j$. Putting all this together, we conclude that vector $q'$ is an equilibrium of the game defined over $G'$.

The second case we need to consider is when the two insider firms participate in all of one another's markets in the original pre-merger economy. Then, we can rewrite the first order conditions that correspond to their optimization problem after the merger as the first order conditions of a single firm that has cost parameter equal to $\frac{c}{2}$. The cost of production is convex, and thus it is straightforward to see that at any equilibrium the aggregate output of firm $f_i$ must be equal to the aggregate output of firm $f_j$. For any market $v \in F_i \cup F_j$, define $q_{xv} = q_{iv} + q_{jv}$, and replace firms $f_i$ and $f_j$ with a single firm $f_x$ connected to $F_x \equiv F_i \cup F_j$ with cost parameter $\frac{c}{2}$. Now consider the first order conditions for firm $f_x$

$$\frac{\partial \pi_{x}}{\partial q_{xv}} = \alpha_v - \beta q_{xv} - \beta \sum_{\ell \in M_v} q_{\ell v} - c \sum_{\ell \in F_x} q_{x\ell},$$

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and note that at any equilibrium $q'$, we have
\[
\frac{\partial \pi}{\partial q_{xv}}|_{q'} = \frac{\partial \pi_{ij}}{\partial q_{iv}}|_{q'} = \frac{\partial \pi_{ij}}{\partial q_{jv}}|_{q'} \leq 0.
\]

This implies that any equilibrium $q'$ for the post-merger game in a network that insider firms share all their markets is equivalent to the unique equilibrium for the case when we replace them by a single firm $f_x$ and can be derived by decomposing $q_{xv}$ into $q_{iv}$ and $q_{jv}$. Consequently, all post-merger equilibria for the case when insider firms share all their markets are equivalent. Finally, since we can convert any equilibrium $q$ of the original network to an equilibrium $q'$ in a network where insider firms share all their markets, we conclude that all equilibria $q$ in the original network are equivalent.

\[\square\]

**Lemma 12.** Every connected equilibrium is balanced.

**Proof.** Assume that firms $f_i$ and $f_j$ both supply strictly positive production quantity to markets $k_i \in F_i \cap F_j$ and $k_j \in F_i \cap F_j$ respectively. Then due to the strict convexity of their productions costs, the aggregate supply of firms $f_i$ and $f_j$ should be equal in the post-merger equilibrium. Otherwise, if for example $\sum_{m_k \in F_i} q_{ik} < \sum_{m_k \in F_j} q_{jk}$, the two firms can reduce their production costs without decreasing their aggregate supply in any of the markets by decreasing $q_{jk}$ by (sufficiently small) $\epsilon$ while increasing $q_{ik}$ by the same amount. So, in any connected equilibrium, the aggregate supply of both insider firms should be the same.

\[\square\]

**Lemma 13.** If there exists a connected equilibrium, then all equilibria are balanced.

**Proof.** Let $q_1$ be a connected equilibrium and $q_2$ be any other equilibrium. Since the solution space of a linear complementarity problem is convex, $q_3 = \gamma q_2 + (1 - \gamma) q_1$ is also a solution to the linear complementarity problem for every $\gamma \in (0, 1)$. Now, note that since $q_1$ is connected, equilibrium $q_3$ should also be connected. Thus, according to Lemmas 11 and 12, we conclude that $q_1$ and $q_3$ are equivalent and balanced and as a result equilibrium strategy profiles $q_1$ and $q_2$ are also equivalent and balanced.

\[\square\]

To conclude the proof it is sufficient to show that even when there is no connected equilibrium in the post-merger game, then still all equilibria are equivalent. To this end, assume that there is no connected post-merger equilibrium. This implies that there cannot exist two equilibria $q_1$ and $q_2$ such that in $q_1$ firm $f_i$ supplies a positive production quantity to a shared market whereas in $q_2$ firm $f_j$ supplies a positive production quantity to a shared market. If this was the case, a convex combination of $q_1$ and $q_2$ (which would also be an equilibrium strategy profile) would be connected. Therefore, it has to be the case that one of the firms, say $f_i$, does not supply to any of markets it shares with firm $f_j$. Equivalently, we can consider the network that results from removing all links from firm $f_i$ to the markets it shares with firm $f_j$, since the equilibria in the post-merger game remain the same. However, according to part (ii) of the Theorem, in this case the post-merger equilibrium is unique and it is same as the equilibrium in the pre-merger game.

\[\square\]
Proof of Proposition 5

The proof follows directly from the proof of Theorem 3. In particular, according to Lemma 13, if there exists a connected equilibrium then all equilibria are balanced and equivalent. Therefore, since in all balanced equilibria, the two firms produce the same quantity on aggregate, it is equivalent to view them as a single firm which is connected to the union of the markets that $f_i$ and $f_j$ originally participate in, and its cost function is equal to $C_{ij}(x) = 2C(x/2)$.

On the other hand, if there exists no connected equilibrium, then by definition one of the firms, say $f_i$, does not supply at equilibrium to any of the markets that the insider firms share. Thus, we can remove the links from firm $f_i$ to the markets that the insider firms share without affecting the equilibrium strategy profile. Part (ii) of Theorem 3 implies that the post-merger equilibrium in this case coincides with the pre-merger equilibrium for a graph $G$ in which the two insider firms do not share any markets.

References


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