An Eulerian approach for the simulation of soft solids in fluids

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Stanford University

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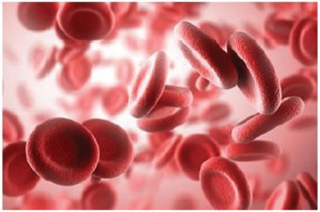
Introduction and Motivation

- System of soft solids in a fluid are ubiquitous
  - Animal tissues, cell membrane.
- Studied for decades - arbitrary Lagrangian-Eulerian (ALE) method - Stiff solids.
- Need for a fully Eulerian approach with "*true solid constitutive laws*".
  - Eulerian Godunov method (Miller & Colella 2001) - unbounded domains
  - Reference Map Technique (RMT) (Kamrin et al. 2012)



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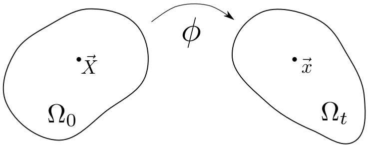
Reference Map Technique



Reference map

$$\vec{\xi}(\vec{x},t) = \vec{X}$$

$$\frac{D\vec{\xi}(\vec{x},t)}{Dt} = 0 \qquad \Rightarrow \qquad \frac{\partial\vec{\xi}(\vec{x},t)}{\partial t} + \vec{u}.\vec{\nabla}\vec{\xi}(\vec{x},t) = 0$$

Deformation gradient

$$\mathbb{F}(\vec{X},t) = \partial\vec{x}/\partial\vec{X} = (\vec{\nabla}\vec{\xi}(\vec{x},t))^{-1}$$

[Valkov et al., *J. Appl. Mech.*, 82, 2015].

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Governing equations for solids and fluids

Momentum balance

$$\frac{\partial(\rho\vec{u})}{\partial t} + \nabla.(\rho\vec{u}\otimes\vec{u}) = \nabla.\underline{\underline{\sigma}}$$

Mass balance

Fluid:

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}.(\vec{u}\rho) = 0 \qquad \Rightarrow \qquad \vec{\nabla}.\vec{u} = 0$$

Solid:

$$\rho = \rho_o(det(\mathbb{F}))^{-1} \qquad \Rightarrow \qquad det(\mathbb{F}) = 1$$

Cauchy stress

Newtonian fluid:

$$\underline{\underline{\sigma}}^f = \mu\left[\left(\vec{\nabla}\vec{u}\right) + \left(\vec{\nabla}\vec{u}\right)^T\right] - \mathbb{1}\left(\vec{\nabla}P\right)$$

Neo-Hookean solid:

$$\underline{\underline{\sigma}}^s = 2(det\mathbb{F})^{-1}\mathbb{F}\frac{\partial\hat{\psi}(\mathbb{C})}{\partial\mathbb{C}}\mathbb{F}^T$$

$$\hat{\psi}(\mathbb{C}) = \mu[tr\mathbb{C} - 3] \qquad \Rightarrow \qquad \underline{\underline{\sigma}}^s = 2\mu^s[(\vec{\nabla}\vec{\xi})^T(\vec{\nabla}\vec{\xi})]^{-1}$$

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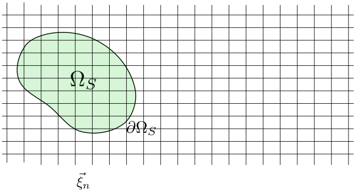
Basic methodology

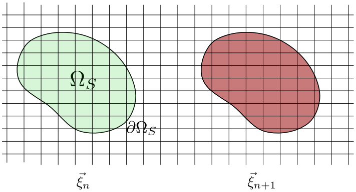
$\vec{\xi}$ defined only within the solid.



Basic methodology

Procedure:

- Solve for $\vec{\xi}_{n+1}$ in Ω_S.



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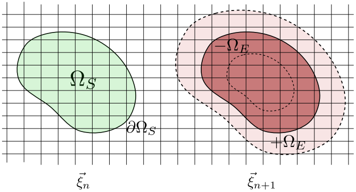
Basic methodology

Procedure:

- Solve for $\vec{\xi}_{n+1}$ in Ω_S.
- $\vec{\xi}_{n+1} \rightarrow$ Extrapolate outside Ω_S.



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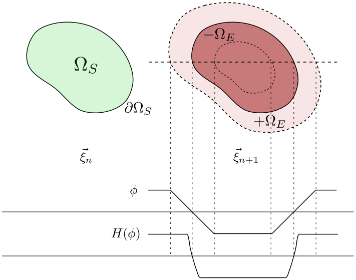
Basic methodology

Procedure:

- Solve for $\vec{\xi}_{n+1}$ in Ω_S.
- $\vec{\xi}_{n+1} \rightarrow$ Extrapolate outside Ω_S.
- $\vec{\xi}_{n+1} \rightarrow$ Construct $\phi \rightarrow$ Construct $H(\phi)$.



Basic methodology

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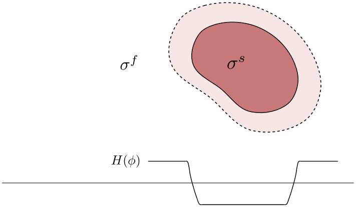
- Compute $\underline{\underline{\sigma}}^s, \underline{\underline{\sigma}}^f$.



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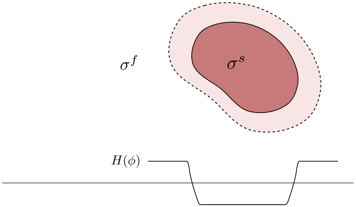
Basic methodology

Procedure:

- Solve for $\vec{\xi}_{n+1}$ in Ω_S.
- $\vec{\xi}_{n+1} \rightarrow$ Extrapolate outside Ω_S.
- $\vec{\xi}_{n+1} \rightarrow$ Construct $\phi \rightarrow$ Construct $H(\phi)$.
- Compute $\underline{\underline{\sigma}}^s, \underline{\underline{\sigma}}^f$.
- $\underline{\underline{\sigma}} \leftarrow$ Blend $f(H(\phi), \underline{\underline{\sigma}}^s, \underline{\underline{\sigma}}^f)$
- $\underline{\underline{\sigma}} \rightarrow$ Compute $\vec{u} \rightarrow$ Project $\vec{\nabla}.\vec{u} = 0$.



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Level-set reconstruction

Analytical expression for $\phi(\vec{x}, t=0)$ is known.

Level-set reconstruction

$$\phi(\vec{x}, t) = \phi(\vec{\xi}, t=0)$$

Reinitialize ϕ using Fast-marching method.

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Closure model

Fluid-Solid coupling is based on the "*one-fluid formulation*".

Smoothed heaviside function

$$H(x) = \begin{cases} 0 & x \leq -w_T \\ \frac{1}{2}\left(1 + \frac{x}{w_T} + \frac{1}{\pi} sin\left(\frac{\pi x}{w_T}\right)\right) & |x| < w_T \\ 1 & x \geq w_T \end{cases}$$

constructed based on the reinitialized level-set field.

Mixture model

$$\underline{\underline{\sigma}} = H(\phi(\vec{x},t))\underline{\underline{\sigma}}^f + (1 - H(\phi(\vec{x},t)))\underline{\underline{\sigma}}^s$$

$$\rho = H(\phi(\vec{x},t))\rho^f + (1 - H(\phi(\vec{x},t)))\rho^s$$

Capability to handle Solid-solid and solid-wall contact.

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Updates and major issues fixed

- Momentum conservation - conservative form of equations.

$$\frac{\partial \rho \vec{u}}{\partial t} + \vec{\nabla}.(\rho \vec{u}\vec{u}) = \vec{\nabla}.(\sum_i \underline{\underline{\sigma}}_i)$$

- Non-dissipative schemes - stress evaluation, convective flux evaluation.
  - Central difference schemes
  - No viscous damping in the solid
- Modified reference map equation - momentum consistent solid advection.
  - Improved robustness

$$\frac{\partial \vec{\xi}(\vec{x},t)}{\partial t} + H(\psi)\vec{u}.\vec{\nabla}\vec{\xi}(\vec{x},t) = 0$$

- Least-squares based extrapolation method.
- Collocated grid.

[Jain & Mani, *CTR Annual Research Briefs*, 2017]

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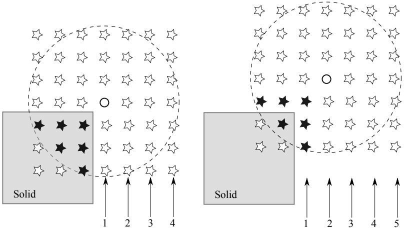
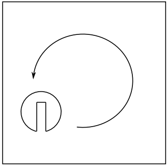
Least-square method for extrapolation



Locally linear approximation

$$\xi = ax + by + c$$

	PDE	least-squares
Error (L_2)	8.32×10^{-4}	6.72×10^{-9}
Cost	≈ 1550ms	≈ 100ms



Zalesak disk

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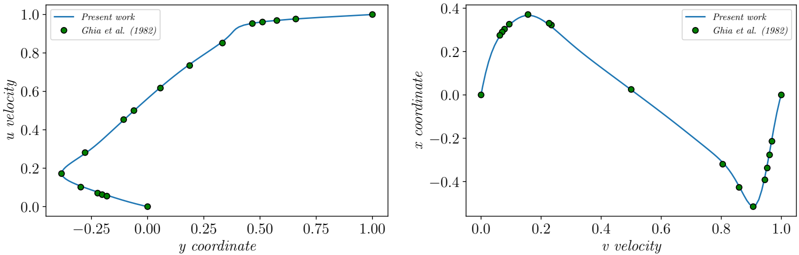
Lid-driven cavity

- $Re = 1000$
- 100x100 grid.



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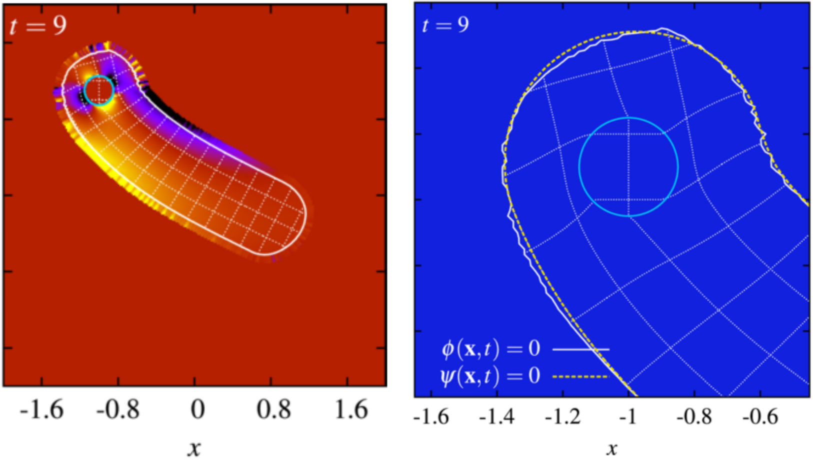
Problems with the previous approach



[Valkov et al., 2015]

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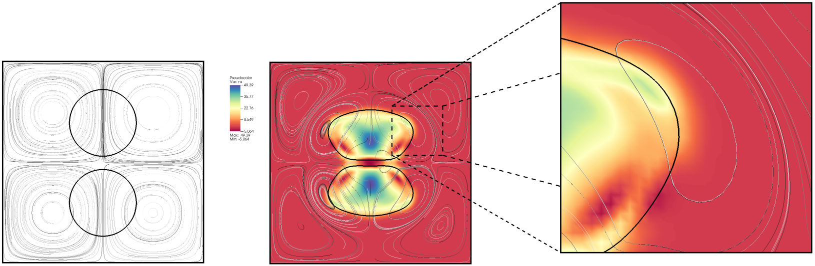
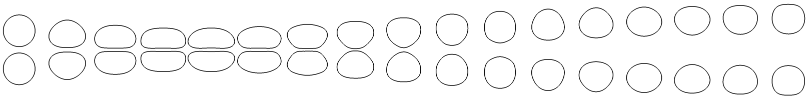
Two solids in a Taylor-Green Vortex



Initial field *After collision* *Smooth interface*



Modified reference map advection equation

$$\frac{\partial \vec{\xi}(\vec{x},t)}{\partial t} + H(\psi)\vec{u}.\vec{\nabla}\vec{\xi}(\vec{x},t) = 0$$

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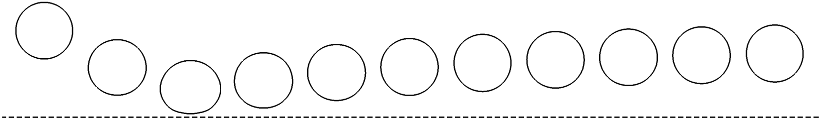
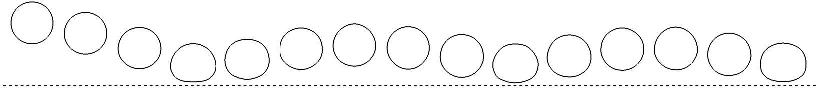
Simulations of a solid in a fluid with wall contact

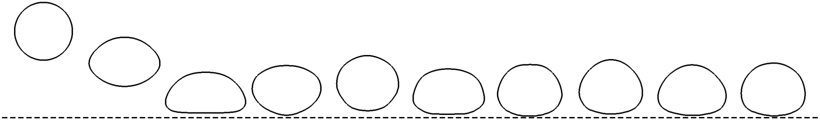
Collision in microgravity:



Collision in gravity - Stiff solid:



Collision in gravity - Soft solid:



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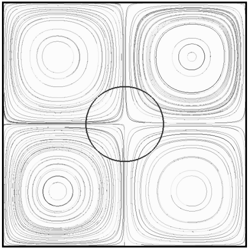
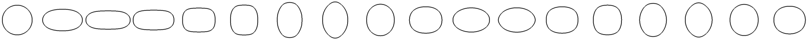
Conservative vs Non-conservative formulation



Initial state

Non-conservative formulation:

Conservative formulation:



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Conclusion

We here presented:

- An incompressible fully Eulerian formulation for soft solids in fluids using an approximate Projection method.
- Advantages:
  - **Momentum conserving formulation.**
  - **Non-dissipative schemes for flux computation - better KE conservation.**
  - **Solid momentum advection is consistent with reference map advection.**
  - **An accurate and cost effective extrapolation procedure.**

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Reference:

Jain, S, S, & Mani, A, 'An incompressible Eulerian formulation for soft solids in fluids', *CTR Annual Research briefs, 2017.*