Online Appendix (Not for Publication)

Properties of the Combinatorial Clock Auction

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A. Mutually Aggressive Behavior

What happens if both bidders are predatory and attempt to push rival payments up toward the final clock prices? As we saw in Section 4 of the paper, a bidder can relax its final bid constraint and increase its rival's cost by exaggerating demand in the clock phase before dropping demand to clear the market. It is difficult to develop an equilibrium theory if both bidders employ non-proxy strategies, so in this section we take a more direct approach. We assume bidders maximize their own payoffs and focus on proxy clock phase strategies, but also assume that each bidder is able to relax the revealed preference constraint on its final bids, forcing its rival to pay closer to the final clock price for all units. This modeling approach is motivated by the features of CCA sales with multiple categories. Under the multi-category CCA rules that have been used in practice, bidders do in fact have a fair amount of flexibility to relax the final round revealed preference constraints without greatly distorting their clock phase bidding.

Specifically, we focus on strategies in which bidder i uses a (linear) demand $v_i(x) = A_i - B_i x$ in the clock phase, with $A_i \geq B_i > 0$, and then in the final round submits demand

$$s_i(x) = (1 - \eta_i) v_i(x) + \eta_i p^*, \tag{1}$$

that is a linear combination of the clock demand $v_i(x)$ and the final clock price p^*.

The parameter $\eta_i \in [0, 1]$ captures the extent to which bidder i is able to relax the local revealed preference constraints. If $\eta_i = 0$, bidder i is consistent. If $\eta_i > 0$, bidder i is able to increase her stated values in the final bid toward the final clock price. In particular, because $v_i(x)$ is decreasing from p^* above x_i^*, an increase in η_i means that bidder i's final bid function is everywhere steeper, starting from the same bid for x_i^*. So an increase in η_i increases the rival's total payment and marginal prices.

The behavior of the predatory bidder 2 in Section 4 is also captured in this model, by assuming that $\eta_2 = 1$. Indeed, the equilibrium described there will correspond to the one we will identify below, with $\eta_1 = 0$ and $\eta_2 = 1$.

A complete strategy for bidder i is described by (A_i, B_i, η_i). As we did in Section 3, we treat η_1, η_2 as parameters, and solve for an ex post equilibrium in choices of (A_i, B_i). As in the above analysis, these equilibria will have the feature that A_i varies with a_i but B_i does not.

A.1 Proxy Best Responses

We identify a strategy for bidder 1 that is an ex post best response, assuming that bidder 2 follows a linear proxy strategy v_2 with varying intercept, and final round strategy given by η_2.

Suppose bidder 1 uses the proxy demand $v_1(x)$. He will win x_1 units, where x_1 satisfies:

$$v_1(x_1) = v_2(1 - x_1). \tag{2}$$

The ending clock price will be $p = v_2(1 - x_1)$. So bidder 2's final bid will specify:

$$s_2(x) = (1 - \eta_2) v_2(x) + \eta_2 v_2(1 - x_1) \text{ for } x \geq 1 - x_1. \tag{3}$$

The final payment by bidder 1 will be

$$\max_x S_2(x) - S_2(1 - x_1) = \max_x (1 - \eta_2) \int_{1-x_1}^{x} v_2(y)\, dy + \eta_2 v_2(1 - x_1) x_1. \tag{4}$$

Bidder 1's marginal payment for his x_1th unit is $v_2(1 - x_1) + \eta_2 B_2 x_1$. Therefore a necessary condition for bidder 1's behavior to be ex post optimal is that his marginal value for his last unit is just equal to his marginal payment for that unit:

$$u_1(x_1) = v_2(1 - x_1) + \eta_2 B_2 x_1. \tag{5}$$

Substituting the market-clearing condition (2) we get that the optimality condition (5) will hold for all v_2 and corresponding purchase quantities x_1 provided that:

$$v_1(x_1) = u_1(x_1) - \eta_2 B_2 x_1. \tag{6}$$

A subtle issue here is that not only can v_1 or v_2 take negative values, but bidder i's best response may involve ending the auction at a negative clock price. For now, we allow this possibility, and return to it later. With this allowance, bidder 1 has an ex post best response that involves a linear proxy strategy for the clock phase with $A_1 = a_1$ and $B_1 = b_1 + \eta_2 B_2$.

A.2 Proxy Equilibria

We now solve for an ex post equilibrium in linear proxy strategies. Since we already found that i's best-response has $A_i = a_i$, we need only to solve for the equilibrium B_i. Using the best response conditions $B_i = b_i + \eta_j B_j$, we obtain

$$B_i = \left(b_i + \eta_j b_j\right) / \left(1 - \eta_i \eta_j\right). \tag{7}$$

Defining

$$\lambda_i = \eta_j B_j = \eta_j \frac{b_j + \eta_i b_i}{1 - \eta_i \eta_j}, \tag{8}$$

allows us to describe the ex post equilibrium as follows:

Proposition 3. *If bidders relax the local revealed preference constraints in the final bid round, so that they bid $s_i(x) = (1 - \eta_i) v_i(x) + \eta_i p^*$ in the final round, where p^* is the final clock phase price, then for any η_1, η_2 with $\eta_1 \eta_2 < 1$, there is an ex post equilibrium of the CCA in which each bidder i uses a linear proxy strategy $v_i(x) = u_i(x) - \lambda_i x$.*

As noted above, our analysis allows for a negative market clearing price in the clock phase. This can be ruled out by making an assumption about parameter values. For instance in the symmetric case with $b_1 = b_2$ and $\eta_1 = \eta_2$, negative prices are ruled out so long as $\underline{a}(1 - \eta) > b/2$. For large values of η, there will be no linear ex post equilibrium if the auction starts at a clock price of zero.

A.3 Properties of the Equilibria

Bidding Behavior. In the mutually aggressive equilibrium, demand in the clock phase is lower than under truthful bidding:

$$v_1(x) - u_1(x) = -\lambda_1 x \leq 0.$$

This contrasts with the model in Section 3, where the equilibrium response in the clock phase to bidders being quiet in their final bids was demand expansion.

Allocation and Revenue. Again, the equilibrium allocation and revenue generally differ from the truthful Vickrey outcomes. Consider the symmetric example (the extreme asymmetric case $\eta_1 = 0$ and $\eta_2 = 1$ is outcome-equivalent to the predatory player model discussed in Section 4). That is, suppose $\eta_1 = \eta_2 = \eta$ and $b_1 = b_2 = b$, which implies $\lambda_1 = \lambda_2 = \frac{\eta}{1-\eta} b$. Then the allocation is distorted toward 1/2 compared to the efficient allocation x_1^e:

$$x_1^* = \frac{1}{2}\eta + (1 - \eta) x_1^e. \tag{9}$$

As for revenue, in the parameter range where $s_i(x) \geq 0$ for all x, we have:

$$R_{CCA} = \frac{a_1 + a_2}{2} - \frac{1}{4} b \frac{\eta - 3}{\eta - 1} - \frac{1}{4} (a_1 - a_2)^2 \frac{(\eta - 1)^2}{b}. \tag{10}$$

Under our maintained assumption $|a_1 - a_2| < b$, the revenue is decreasing in η (so despite the more aggressive final round bidding when $\eta > 0$, revenues are lower than under truthful bidding since the demand reduction effect dominates).

B. Omitted Calculations from Section 3

In Section 3, we solve for a range of ex post equilibria of the CCA in proxy strategies. We then consider the resulting allocation and revenue for symmetric equilibria. Here we fill in some omitted calculations.

Symmetric Equilibria.

In this case, $\gamma_1 = \gamma_2 = \gamma$ and $b_1 = b_2 = b$. Bidder i's equilibrium clock round strategy is

$$v_i(x) = A_i - Bx.$$

Bidder i's equilibrium final bid strategy is

$$s_i(x) = A_i - B(1+\gamma)x,$$

so that

$$S_i(x) = A_i x - \frac{1}{2} B(1+\gamma)x^2.$$

The equilibrium bid parameters are

$$A_i = a_i + b\frac{\gamma}{1-\gamma} \qquad \text{and} \qquad B = b\frac{1}{1-\gamma}.$$

The equilibrium outcome x^* solves $v_1(x) = v_2(1-x)$, which means

$$\begin{aligned} x^* &= \frac{1}{2} + \frac{a_1 - a_2}{2b}(1-\gamma), \\ 1 - x^* &= \frac{1}{2} - \frac{a_1 - a_2}{2b}(1-\gamma). \end{aligned}$$

Here we omit the 1 subscript to slightly simplify notation.

To solve for revenue, we consider three cases.

CASE 1: Suppose the parameters are such that $s_1(1), s_2(1) \geq 0$, which is equivalent to $a_i \geq \frac{b}{1-\gamma}$.

Then the CCA revenue, given a final allocation $(x^*, 1-x^*)$ is

$$\begin{aligned} R_{CCA} &= S_1(1) - S_1(x^*) + S_2(1) - S_2(1-x^*) \\ &= A_1(1-x^*) + A_2 x - \frac{1}{2}(1+\gamma)B\left[2 - (x^*)^2 - (1-x^*)^2\right]. \end{aligned}$$

Substituting for A_1, A_2, B, and x^* we get:

$$\begin{aligned} R_{CCA} &= \left(a_1 + b\frac{\gamma}{1-\gamma}\right)\left(\frac{1}{2} - \frac{a_1 - a_2}{2b}(1-\gamma)\right) + \left(a_2 + b\frac{\gamma}{1-\gamma}\right)\left(\frac{1}{2} + \frac{a_1 - a_2}{2b}(1-\gamma)\right) \\ &\quad - \frac{1}{2}\frac{1+\gamma}{1-\gamma}b\left[2 - \left(\frac{1}{2} + \frac{a_1 - a_2}{2b}(1-\gamma)\right)^2 - \left(\frac{1}{2} - \frac{a_1 - a_2}{2b}(1-\gamma)\right)^2\right]. \end{aligned}$$

Simplifying

$$\begin{aligned} R_{CCA} &= \frac{a_1 + a_2}{2} + b\frac{\gamma}{1-\gamma} + \frac{(a_1 - a_2)^2}{2b}(1-\gamma) \\ &\quad - \frac{1+\gamma}{1-\gamma}b + \frac{1+\gamma}{1-\gamma}b\frac{1}{4} - \frac{1}{2}\frac{1+\gamma}{1-\gamma}b\left[\frac{(a_1 - a_2)^2}{2b^2}(1-\gamma)^2\right], \end{aligned}$$

which leads to the expression

$$R_{CCA} = \frac{a_1 + a_2}{2} - \frac{b}{4}\frac{3-\gamma}{1-\gamma} - \frac{(a_1 - a_2)^2}{4b}(1-\gamma)^2.$$

It follows that

$$\begin{aligned} \frac{dR_{CCA}}{d\gamma} &= -\frac{b}{4}\frac{-(1-\gamma) + (3-\gamma)}{(1-\gamma)^2} + 2(1-\gamma)\frac{(a_1 - a_2)^2}{4b} \\ &= -\frac{b}{2(1-\gamma)^2} + (1-\gamma)\frac{(a_1 - a_2)^2}{2b} \\ &< 0 \end{aligned}$$

where the last inequality follows because $b > |a_1 - a_2|$ by assumption, and $\gamma \leq 1$.

Now, if $\gamma = 0$ so that both bidders are consistent, the equilibrium outcome of the CCA is exactly the same as a truthful Vickrey auction. If $\gamma > 0$, the revenue is lower. So the equilibrium CCA revenue is less than the truthful Vickrey revenue.

CASE 2: Suppose the parameters are such that $s_1(1), s_2(1) < 0$, which is equivalent to $a_i < \frac{b}{1-\gamma}$.

In that case the revenue is

$$R_{CCA} = \int_{x^*}^{1} \max\{s_1(x), 0\}\, dx + \int_{1-x^*}^{1} \max\{s_2(x), 0\}\, dx.$$

Since $s_i(x) = 0$ at $x = \frac{1}{b(\gamma+1)}(a_i(1-\gamma) + b\gamma)$, we can write the revenue as

$$\begin{aligned}
R_{CCA} &= \int_{x^*}^{\frac{1}{b(\gamma+1)}(a_1(1-\gamma)+b\gamma)} s_1(x)\,dx + \int_{1-x^*}^{\frac{1}{b(\gamma+1)}(a_2(1-\gamma)+b\gamma)} s_2(x)\,dx \\
&= \frac{1}{8b}\frac{1-\gamma}{\gamma+1}(-b + a_1(1-\gamma) + a_2(1+\gamma))^2 + \frac{1}{8b}\frac{1-\gamma}{\gamma+1}(-b + a_2(1-\gamma) + a_1(1+\gamma))^2 \\
&= \frac{1}{4b}\frac{1-\gamma}{\gamma+1}\left((b-a_1-a_2)^2 + \gamma^2(a_1-a_2)^2\right).
\end{aligned}$$

It follows that

$$\frac{dR_{CCA}}{d\gamma} = -\frac{1}{2b(\gamma+1)^2}\left((b-a_1-a_2)^2 + (a_1-a_2)^2\gamma\left(\gamma^2+\gamma-1\right)\right)$$

If $\gamma\left(\gamma^2+\gamma-1\right) \geq 0$, this is unambiguously negative. Otherwise, recall that we assumed $b > |a_1 - a_2|$ and $a_i \geq b$. Hence

$$(b-a_1-a_2)^2 + (a_1-a_2)^2\gamma\left(\gamma^2+\gamma-1\right) \geq b^2\left(1+\gamma\left(\gamma^2+\gamma-1\right)\right)$$

which is positive for all $\gamma \in [0,1]$. Hence, also in this case revenues decline in γ.

CASE 3: For one of the players $s_i(1) \geq 0$ and for the other $s_j(1) < 0$.

Without loss of generality let $a_1 < \frac{b}{1-\gamma}$ and $a_2 \geq \frac{b}{1-\gamma}$. In that case the revenue is

$$\begin{aligned}
R_{CCA} &= \int_{x^*}^{\frac{1}{b(\gamma+1)}(a_1(1-\gamma)+b\gamma)} s_1(x)\,dx + \int_{1-x^*}^{1} s_2(x)\,dx \\
&= \frac{\left(\gamma^2-2\gamma-1\right)b + 2(1-\gamma)(a_2-a_1+\gamma(a_1+a_2))}{4(1-\gamma^2)} \\
&\quad + \frac{a_1^2 - a_2^2 + 2a_1a_2 + \gamma^2(a_1-a_2)^2}{4b}.
\end{aligned}$$

It follows that

$$\frac{dR_{CCA}}{d\gamma} = -\frac{1}{2b(1-\gamma^2)^2}\left(\left(\gamma^2+1\right)b^2 + (1-\gamma)^2\left(2a_1(a_2-b) + a_1^2 - a_2^2\right) + (a_1-a_2)^2\gamma(1-\gamma)^2\left(\gamma+\gamma^2-1\right)\right).$$

We claim that it is negative. To prove it, we need to show that

$$\left(\gamma^2+1\right)b^2 + (1-\gamma)^2\left(2a_1(a_2-b) + a_1^2 - a_2^2\right) + (a_1-a_2)^2\gamma(1-\gamma)^2\left(\gamma+\gamma^2-1\right) \geq 0.$$

Note that since in our case $a_2 > a_1$, the term $\left(2a_1(a_2-b) + a_1^2 - a_2^2\right)$ decreases in a_2 and increases in a_1 so that it is minimized at $a_2 = b + a_1$ and $a_1 = b$ (recall the assumptions $|a_2 - a_1| < b$ and $a_i \geq b$).

Hence,

$$\begin{aligned} & \left(\gamma^2+1\right) b^2+(1-\gamma)^2\left(2 a_1\left(a_2-b\right)+a_1^2-a_2^2\right)+\left(a_1-a_2\right)^2 \gamma(1-\gamma)^2\left(\gamma+\gamma^2-1\right) \\ \geq\ & 2 b^2 \gamma+\left(a_1-a_2\right)^2 \gamma(1-\gamma)^2\left(\gamma+\gamma^2-1\right) . \end{aligned}$$

If $\left(\gamma+\gamma^2-1\right) \geq 0$, the expression is positive. Otherwise, it is minimized when $\left(a_1-a_2\right)^2$ is maximized. Using again $|a_2 - a_1| < b$ we can bound the whole expression from below by

$$b^2 \gamma\left(\gamma^4-\gamma^3-2 \gamma^2+3 \gamma+1\right)$$

which is positive for all $\gamma \in [0,1]$. So revenue is indeed decreasing in this case as well.

In summary, revenue in all three cases decreases in γ. Since revenue is continuous on the boundaries of the three cases, the overall claim is established.

Asymmetric Equilibria.

Suppose now $\gamma_1 > 0$ and $\gamma_2 = 0$. In equilibrium, bidder 1 is truthful in the clock phase:

$$v_1(x) = u_1(x) = a_1 - b_1 x$$

but does not fully raise his final bids, so that

$$s_1(x) = a_1 - (1+\gamma_1) b_1 x$$

Bidder 2 expands demand to

$$v_2(x) = a_2 - b_2 x + \lambda_2 (1-x)$$

in the clock phase, where $\lambda_2 = \gamma_1 b_1$, and then is consistent in the final round, so that:

$$s_2(x) = a_2 - b_2 x + \gamma_1 b_1 (1-x).$$

We can write the final bid functions as:

$$\begin{aligned} S_1(x) &= a_1 x - \frac{1}{2}(1+\gamma_1) b_1 x^2 \\ S_2(x) &= (a_2 + \gamma_1 b_1) x - \frac{1}{2}(b_2 + \gamma_1 b_1) x^2 \end{aligned}$$

Bidder 2's equilibrium quantity is inefficiently large. Setting $v_1(x) = v_2(1-x)$ and solving for the equilibrium quantity x^* that goes to bidder 1, we obtain:

$$\begin{aligned} x^* &= \frac{a_1 - a_2 + b_2}{b_2 + b_1(1+\gamma_1)}. \\ 1 - x^* &= \frac{b_1(1+\gamma_1) - (a_1 - a_2)}{b_2 + b_1(1+\gamma_1)} \end{aligned}$$

The revenue comparisons are more subtle. We have for small γ_1 (so that $s_1(1) \geq 0$):

$$\begin{aligned} R_{CCA} &= S_1(1) - S_1(x^*) + S_2(1) - S_2(1-x^*) \\ &= a_1(1-x^*) - \frac{1}{2}(b_1 + \gamma_1 b_1)\left[1 - (x^*)^2\right] \\ &\quad + (a_2 + \gamma_1 b_1) x^* - \frac{1}{2}(b_2 + \gamma_1 b_1)\left[1 - (1-x^*)^2\right]. \end{aligned}$$

We know that when $\gamma_1 = 0$, the CCA outcome corresponds to the truthful Vickrey outcome. We therefore consider how revenue changes when there is a small increase in γ_1 that takes the CCA outcome away from the Vickrey outcome.

Differentiating with respect to γ_1, we obtain

$$\begin{aligned}\frac{dR_{CCA}}{d\gamma_1} &= -\frac{1}{2}b_1\left[1-(x^*)^2\right]+b_1x^*-\frac{1}{2}b_1\left[1-(1-x^*)^2\right]\\ &\quad +\frac{dx^*}{d\gamma_1}\left[a_1+(b_1+\gamma_1b_1)\,x^*+(a_2+\gamma_1b_1)-(b_2+\gamma_1b_1)\,(1-x^*)\right],\end{aligned}$$

and evaluating at $\gamma_1 = 0$:

$$\begin{aligned}\left.\frac{dR_{CCA}}{d\gamma_1}\right|_{\gamma_1=0} &= -\frac{1}{2}b_1\left[1-(x^*)^2\right]+b_1x^*-\frac{1}{2}b_1\left[1-(1-x^*)^2\right]\\ &\quad +\frac{dx^*}{d\gamma_1}\left[-a_1+b_1x^*+a_2-b_2\,(1-x^*)\right],\end{aligned}$$

The second term disappears because when $\gamma_1 = 0$, then x^* is defined as the solution to:

$$a_1 - b_1x = a_2 - b_2\,(1-x)$$

So simplifying, we have

$$\left.\frac{dR_{CCA}}{d\gamma_1}\right|_{\gamma_1=0} = b_1\left[(x^*)^2-\frac{1}{2}\right].$$

This expression can be either positive or negative, depending on the parameters that determine x^*.

The intuition is the following. As γ_1 increases, bidder 1 gives bidder 2 a discount, while bidder 2 (by expanding demand) makes bidder 1 pay more. The allocation also changes but that has a second-order effect on revenue. The net effect is that revenue increases if x_1^* is sufficiently high, and otherwise decreases.