Omitted Calculations from Section 5

In Section 5, we solve for a range of ex post equilibria of the CCA in proxy strategies. We then consider the resulting allocation and revenue for some particular cases of interest. Here we fill in some omitted calculations.

*Symmetric Equilibria.*

In this case, \( \gamma_1 = \gamma_2 = \gamma \) and \( b_1 = b_2 = b \). Bidder \( i \)'s equilibrium clock round strategy is

\[
v_i(x) = A_i - Bx.
\]

Bidder \( i \)'s equilibrium final bid strategy is

\[
s_i(x) = A_i - B(1 + \gamma)x,
\]

so that

\[
S_i(x) = A_i x - \frac{1}{2} B (1 + \gamma) x^2.
\]

The equilibrium bid parameters are

\[
A_i = a_i + b \frac{\gamma}{1 - \gamma} \quad \text{and} \quad B = b \frac{1}{1 - \gamma}.
\]

The equilibrium outcome \( x^* \) solves \( v_1(x) = v_2(1 - x) \), which means

\[
x^* = \frac{1}{2} + \frac{a_1 - a_2}{2b} (1 - \gamma),
\]

\[
1 - x^* = \frac{1}{2} - \frac{a_1 - a_2}{2b} (1 - \gamma).
\]

Here we omit the 1 subscript to slightly simplify notation.

To solve for revenue, suppose the parameters are such that \( s_1(1), s_2(1) \geq 0 \). Then the CCA
revenue, given a final allocation \((x^*, 1 - x^*)\) is

\[
R_{CCA} = S_1 (1) - S_1 (x^*) + S_2 (1) - S_2 (1 - x^*)
\]

\[
= A_1 (1 - x^*) + A_2 x - \frac{1}{2} (1 + \gamma) B \left[ 2 - (x^*)^2 - (1 - x^*)^2 \right].
\]

Substituting for \(A_1, A_2, B,\) and \(x^*\) we get:

\[
R_{CCA} = \left( a_1 + b \frac{\gamma}{1 - \gamma} \right) \left( \frac{1}{2} - \frac{a_1 - a_2}{2b} (1 - \gamma) \right) + \left( a_2 + b \frac{\gamma}{1 - \gamma} \right) \left( \frac{1}{2} + \frac{a_1 - a_2}{2b} (1 - \gamma) \right)
\]

\[
- \frac{1 + \gamma}{2} b \left[ 2 - \left( \frac{1}{2} + \frac{a_1 - a_2}{2b} (1 - \gamma) \right)^2 - \left( \frac{1}{2} - \frac{a_1 - a_2}{2b} (1 - \gamma) \right)^2 \right]
\]

Simplifying

\[
R_{CCA} = \frac{a_1 + a_2}{2} b + \frac{\gamma}{1 - \gamma} + \frac{(a_1 - a_2)^2}{2b} (1 - \gamma)
\]

\[
- \frac{1 + \gamma}{2} b \left[ 1 + \frac{\gamma}{1 - \gamma} b \right] - \frac{1}{2} \frac{1 + \gamma}{1 - \gamma} b \left[ \frac{(a_1 - a_2)^2}{2b^2} (1 - \gamma)^2 \right]
\]

which leads to the expression

\[
R_{CCA} = \frac{a_1 + a_2}{2} - \frac{b}{4} \frac{3 - \gamma}{1 - \gamma} - \frac{(a_1 - a_2)^2}{4b} (1 - \gamma)^2.
\]

It follows that

\[
\frac{dR_{CCA}}{d\gamma} = - \frac{b}{4} \frac{(1 - \gamma) + (3 - \gamma)}{(1 - \gamma)^2} + 2 (1 - \gamma) \frac{(a_1 - a_2)^2}{4b}
\]

\[
= - \frac{b}{2} \frac{1}{(1 - \gamma)^2} + (1 - \gamma) \frac{(a_1 - a_2)^2}{2b}
\]

\[
< 0
\]

where the last inequality follows because \(b > |a_1 - a_2|\) by assumption, and \(\gamma \leq 1\).

Now, if \(\gamma = 0\) so that both bidders are consistent, the equilibrium outcome of the CCA is exactly the same as a truthful Vickrey auction. If \(\gamma > 0\), the revenue is lower. So the equilibrium CCA revenue is less than the truthful Vickrey revenue.
Asymmetric Equilibria.
Suppose now $\gamma_1 > 0$ and $\gamma_2 = 0$. In equilibrium, bidder 1 is truthful in the clock phase:

$$v_1(x) = u_1(x) = a_1 - b_1 x$$

but does not fully raise his final bids, so that

$$s_1(x) = a_1 - (1 + \gamma_1) b_1 x$$

Bidder 2 expands demand to

$$v_2(x) = a_2 - b_2 x + \lambda_2 (1 - x)$$

in the clock phase, where $\lambda_2 = \gamma_1 b_1$, and then is consistent in the final round, so that:

$$s_2(x) = a_2 - b_2 x + \gamma_1 b_1 (1 - x).$$

We can write the final bid functions as:

$$S_1(x) = a_1 x - \frac{1}{2} (1 + \gamma_1) b_1 x^2$$

$$S_2(x) = (a_2 + \gamma_1 b_1) x - \frac{1}{2} (b_2 + \gamma_1 b_1) x^2$$

Bidder 2’s equilibrium quantity is insufficiently large. Setting $v_1(x) = v_2(1 - x)$ and solving for the equilibrium quantity $x^*$ that goes to bidder 1, we obtain:

$$x^* = \frac{a_1 - a_2 + b_2}{b_2 + b_1 (1 + \gamma_1)}$$

$$1 - x^* = \frac{b_1 (1 + \gamma_1) - (a_1 - a_2)}{b_2 + b_1 (1 + \gamma_1)}$$

The revenue comparisons are more subtle. We have:

$$R_{CCA} = S_1(1) - S_1(x^*) + S_2(1) - S_2(1 - x^*)$$

$$= a_1 (1 - x^*) - \frac{1}{2} (b_1 + \gamma_1 b_1) \left[1 - (x^*)^2\right]$$

$$+ (a_2 + \gamma_1 b_1) x^* - \frac{1}{2} (b_2 + \gamma_1 b_1) \left[1 - (1 - x^*)^2\right].$$
We know that when $\gamma_1 = 0$, the CCA outcome corresponds to the truthful Vickrey outcome. We therefore consider how revenue changes when there is a small increase in $\gamma_1$ that takes the CCA outcome away from the Vickrey outcome.

Differentiating with respect to $\gamma_1$, we obtain

$$\frac{dR_{CCA}}{d\gamma_1} = -\frac{1}{2} b_1 \left[ 1 - (x^*)^2 \right] + b_1 x^* - \frac{1}{2} b_1 \left[ 1 - (1 - x^*)^2 \right]$$

$$+ \frac{dx^*}{d\gamma_1} \left[ a_1 + (b_1 + \gamma_1 b_1) x^* + (a_2 + \gamma_1 b_1) - (b_2 + \gamma_1 b_1) (1 - x^*) \right],$$

and evaluating at $\gamma_1 = 0$:

$$\frac{dR_{CCA}}{d\gamma_1} \bigg|_{\gamma_1=0} = -\frac{1}{2} b_1 \left[ 1 - (x^*)^2 \right] + b_1 x^* - \frac{1}{2} b_1 \left[ 1 - (1 - x^*)^2 \right]$$

$$+ \frac{dx^*}{d\gamma_1} \left[ -a_1 + b_1 x^* + a_2 - b_2 (1 - x^*) \right],$$

The second term disappears because when $\gamma_1 = 0$, then $x^*$ is defined as the solution to:

$$a_1 - b_1 x = a_2 - b_2 (1 - x)$$

So simplifying, we have

$$\frac{dR_{CCA}}{d\gamma_1} \bigg|_{\gamma_1=0} = b_1 \left[ (x^*)^2 - \frac{1}{2} \right].$$

This expression can be either positive or negative, depending on the parameters that determine $x^*$.

The intuition is the following. As $\gamma_1$ increases, bidder 1 gives bidder 2 a discount, while bidder 2 (by expanding demand) makes bidder 1 pay more. The allocation also changes but that has a second-order effect on revenue. The net effect is that revenue increases if $x^*_1$ is sufficiently high, and otherwise decreases.