Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix. We consider the SDP arising in the MaxCut problem

$$\begin{align*}
\max_{X \in \mathbb{R}^{n \times n}} & \langle A, X \rangle \\
\text{subject to} & \quad X_{ii} = 1, \ i \in [n], \\
& \quad X \succeq 0.
\end{align*}$$

(SDP)

We can solve it in polynomial time, but it scales badly because of the $n^2$ dimension and the PSD constraint.

The Burer Monteiro approach changes the variable $X = \sigma \sigma^T$ to get rid of the PSD constraint and lower the dimension to $n \times k$:

$$\begin{align*}
\max_{\sigma \in \mathbb{R}^{n \times k}} & \langle \sigma, A \sigma \rangle \\
\text{subject to} & \quad \sigma = [\sigma_1, \ldots, \sigma_n]^T, \\
& \quad \|\sigma_i\|_2 = 1, \ i \in [n],
\end{align*}$$

($k$-Ncvx-SDP)

$\|u, \text{Hess} f(\sigma)[u]\| \leq \epsilon(u, u)$.

A local maximizer is $O$-approximate concave. An $\epsilon$-approximate concave point is nearly locally optimal.

For any $\epsilon$-approximate concave point $\sigma \in \mathcal{M}_k$ of the rank-$k$ non-convex problem ($k$-Ncvx-SDP), we have

$$f(\sigma) \geq \text{SDP}(A) - \frac{1}{k-1}(\text{SDP}(A) + \text{SDP}(-A)) - \frac{n}{2} \epsilon.$$  

Geometrically: the function value for all local maxima are within a gap of order $O(1/k)$ within the global maximum. Proof strategy: Approximate concave condition + random projection.

Application to $Z_2$ synchronization problem.

A similar Grothendieck inequality for the $SO(d)$ synchronization SDP problem.

Potentially generalizable to general SDP problems.

References


