Solving SDPs for synchronization and MaxCut problems via the Grothendieck inequality

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MaxCut SDP

Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. We consider the SDP arising in the MaxCut problem

$$\begin{aligned} \underset{X\in\mathbb{R}^{n\times n}}{\text{maximize}} \quad & \langle A, X\rangle \\ \text{subject to} \quad & X_{ii} = 1, \quad i \in [n], \\ & X \succeq 0. \end{aligned} \tag{SDP}$$

We can solve it in polynomial time, but it scales badly because of the n^2 dimension and the PSD constraint.

Burer-Monteiro approach

The Burer Monteiro approach changes the variable $X = \sigma\sigma^\top$ to get rid of the PSD constraint and lower the dimension to $n \times k$.

$$\begin{aligned} \underset{\sigma\in\mathbb{R}^{n\times k}}{\text{maximize}} \ & \langle \sigma, A\sigma\rangle \\ \text{subject to } & \sigma = [\sigma_1, \dots, \sigma_n]^\top, \\ & \|\sigma_i\|_2 = 1, \quad i \in [n]. \end{aligned} \tag{k-Ncvx-SDP}$$

What did we know?

- As $k \geq \sqrt{2n}$, the global maxima of (k-Ncvx-SDP) coincide with the global maximizer of (SDP) [Pat98, Bar01, BM03].
- As $k \geq \sqrt{2n}$, any local maxima of (k-Ncvx-SDP) is a global maximizer of (k-Ncvx-SDP) [BVB16].
- What if k remains of order 1, as $n \to \infty$? It also works well!

Geometry of the Non-convex SDP

- The function $f(\sigma) = \langle \sigma, A\sigma\rangle$ is smooth on the manifold $\mathcal{M}_k = \{\sigma : \|\sigma_i\|_2 = 1, i \in [n]\}$.
- Definition: we call $\sigma \in \mathcal{M}_k$ an ε-approximate concave point of f on $\mathcal{M}_k$, if for any tangent vector $u \in T_\sigma\mathcal{M}_k$, we have
$$\langle u, \text{Hess} f(\sigma)[u]\rangle \leq \varepsilon\langle u, u\rangle. \tag{1}$$
- A local maximizer is 0-approximate concave. An ε-approximate concave point is nearly locally optimal.

Main Theorem (A Grothendieck Inequality)

For any ε-approximate concave point $\sigma \in \mathcal{M}_k$ of the rank-k non-convex problem (k-Ncvx-SDP), we have

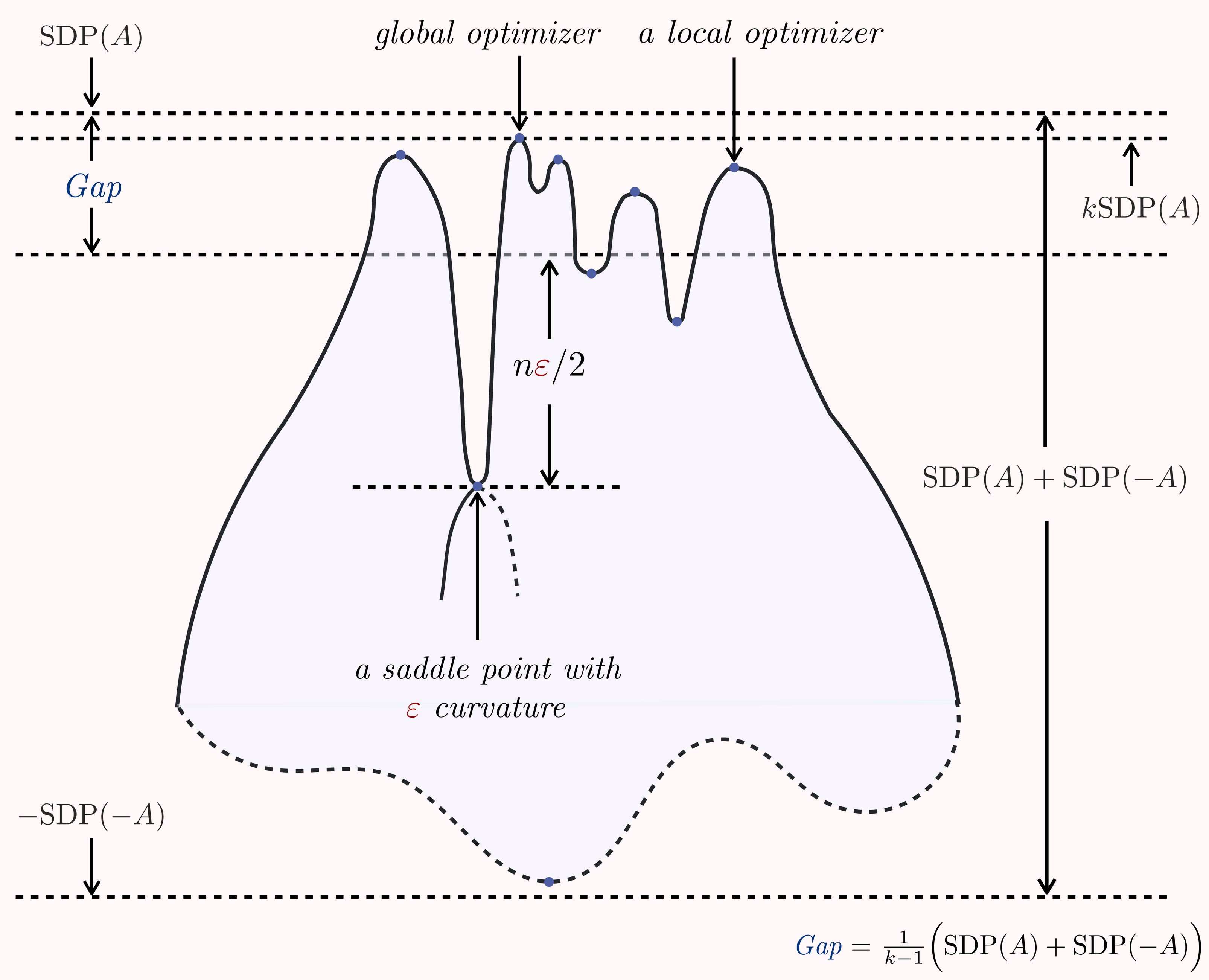
$$f(\sigma) \geq \text{SDP}(A) - \frac{1}{k-1}(\text{SDP}(A) + \text{SDP}(-A)) - \frac{n}{2}\varepsilon. \tag{2}$$

Geometrically: the function value for all local maxima are within a gap of order $O(1/k)$ within the global maximum.
Proof strategy: Approximate concave condition + random projection.

Provably efficient algorithm

- Riemannian trust region method is guaranteed to converge to a point with absolute error $n\varepsilon + (\text{SDP}(A) + \text{SDP}(-A))/(k-1)$ in $c \cdot n\|A\|_1^2/\varepsilon^2$ trust region steps.
- $\text{SDP}(-A)$ typically has the same order as $\text{SDP}(A)$.
- Empirically, gradient descent converges faster than what is guaranteed.

Landscape of Non-convex SDP



MaxCut problem

Let G be a graph and A_G be its adjacency matrix. The MaxCut of a graph G solves the optimization problem

$$\underset{x\in\{\pm1\}^n}{\text{maximize}} \quad \frac{1}{4}\sum_{i,j=1}^{n} A_{G,ij}(1 - x_i x_j). \tag{MaxCut}$$

This optimization problem is known to be NP hard. Goemans and Williamson [GW95] showed that if we solve the problem (SDP) by taking $A = -A_G$, the optimal solution $X^\star$ gives an 0.878-approximate solution of the MaxCut problem (MaxCut).

Theorem (Application to MaxCut)

For any $k \geq 3$, if $\sigma^\star$ is a local maximizer of the rank-k non-convex SDP problem (k-Ncvx-SDP) by taking $A = -A_G$, then using $\sigma^\star$ we can find an $0.878 \times (1 - 1/(k-1))$-approximate solution of the MaxCut problem (MaxCut).

Further results

- Application to $\mathbb{Z}_2$ synchronization problem.
- A similar Grothendieck inequality for the $\text{SO}(d)$ synchronization SDP problem.
- Potentially generalizable to general SDP problems.

References

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