Stochastic Volatility and Asset Pricing Puzzles

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Abstract

This paper builds a real-options, term structure model of the firm to shed new light on the value premium, financial distress, momentum, and credit spread puzzles. The model incorporates stochastic volatility in the firm productivity process and a negative market price of volatility risk. Since the equity of growth firms and financially distressed firms have embedded options, such securities hedge against volatility risk in the market and thus command lower volatility risk premia than the equities of value or financially healthy firms. Abnormal risk-adjusted momentum profits are concentrated among low credit-rating firms for similar reasons. Conversely, since increases in volatility generally reduce the value of debt, corporate debt will tend to command large volatility risk premia, allowing the model to generate higher credit spreads than existing structural models. The paper illustrates that allowing for endogenous default by equityholders is necessary for the model to account for the credit spreads of both investment grade and junk debt. The model is extended to include rare disasters to better account for the expected default frequencies and credit spreads of short maturity debt. Finally, the paper develops a novel methodology based on asymptotic expansions to solve the model.

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1 Introduction

An enormous body of work in financial econometrics has documented that volatility in market returns is time-varying and has attempted to model its dynamics. Chacko and Viceira (2005) and Liu (2007) point out that this time-varying volatility implies long-term investors will value assets for their ability to hedge against market volatility risk in addition to their expected returns and other hedging properties. In market equilibrium, then, assets which hedge against persistent volatility risk should require lower average returns, all else equal. Campbell, Giglio, Polk, and Turley (2016) formally model this point, constructing an intertemporal capital asset pricing model (ICAPM) incorporating stochastic volatility and documenting the existence of low-frequency movements in market volatility.

This paper builds on these insights by constructing a real options, term-structure model of the firm that includes persistent, negatively priced shocks to volatility. In contrast to Campbell et al. (2016), the primary focus is to understand how differences in financing, productivity, and investment options across firms and their relationship with volatility generate cross-sectional variability in asset prices. The paper makes a number of contributions. I first demonstrate that the model’s quantitative asset pricing predictions offer a novel perspective on a variety of so-called anomalies documented in the literature, including the value premium, financial distress, and momentum puzzles. I furthermore demonstrate that the model, when calibrated to match historical leverage ratios and recovery rates, can generate empirically observed levels of credit spreads and default probabilities across ratings categories if and only if equityholders are allowed to optimally decide when to default. Thus, the model quantitatively delivers a resolution of the credit spread puzzle documented by Huang and Huang (2012) that existing structural models with only a single sourced price of risk significantly underpredict credit spreads when calibrated to match historical recovery rates, leverage ratios, and empirical default frequencies.

The key insight of the paper is that, structurally, firms are comprised of a variety of embedded options. These include, for example, the option to default on existing debt li-

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1 Much of the literature provides models similar to the ARCH/GARCH models of Engle (1982) and Bollerslev (1986) in which volatility is function of past return shocks and its own lags. More recent literature has used high-frequency data to directly estimate the stochastic volatility process. Papers include Barndorff-Nielsen and Shephard (2002), Bollerslev and Zhou (2002), and Andersen et al. (2003).

2 This will be the case if, in particular, asset prices are priced according to the first-order conditions of a long-term representative investor with Epstein-Zin recursive preferences.

3 The value premium puzzle, due to Basu (1977,1983) and Fama and French (1993), refers to the greater risk-adjusted return of high book-to-market stocks over low book-to-market stocks. The momentum puzzle, attributed to Jagadeesh and Titman (1993), is the finding that a portfolio long winners and short losers generates positive CAPM alpha. The financial distress puzzle, uncovered by Dichev (1998) and Campbell, Hilscher, and Szilagyi (2008), refers to the positive CAPM alpha on a portfolio long financially healthy firms and short firms close to default.
abilities and growth options. Furthermore, these options increase in value when volatility rises. Cross-sectional differences in the presence of these options, therefore, naturally generate cross-sectional differences in firm exposure to fluctuations in volatility. If exposure to volatility risk is priced, these cross-sectional differences can account for a number of empirical asset pricing facts.

Consider, for example, the financial distress puzzle which finds that, empirically, the stocks of firms close to default do not have higher returns than the stocks of financially healthy firms, as would be predicted by standard financial theory. However, a component of equity value is the equityholders’ default option. By the standard logic of option theory, an increase in volatility should raise the value of this embedded option. For example, if uncertainty about demand for the firm’s product increases, the potential downside to equityholders is capped by limited liability, while there is substantial upside potential.4

If the firm is currently in financial distress, i.e. close to default, then most of the equity value is comprised of the default option. Consequently, financially distressed firms serve as an effective hedge against volatility in the market, rising in market value when volatility increases. Conversely, the default option only constitutes a small fraction of the equity value of a financially healthy firm. In fact, increases in volatility slightly lower the equity value of financially healthy firms due to increased debt rollover risk. According to the logic of the ICAPM, these features of the model imply that distressed equities should have smaller variance risk premia than financially healthy firms. Empirically, this mechanism would manifest itself as a positive CAPM alpha in a portfolio which is long healthy firms and short financially distressed firms. Moreover, the variance hedging properties of financially distressed equities counteract the higher market betas of such stocks so that, quantitatively, the model generates a hump-shaped relationship between the equity return and the probability of default, exactly the empirical relationship documented in the Garlappi et al. (2008) study of the financial distress puzzle.

By the same mechanism, the model also predicts a positive CAPM alpha on a portfolio which is long recent winners and short recent losers, consistent with the momentum anomaly. More specifically, the model predicts that abnormal risk-adjusted momentum profits should be concentrated among firms with low credit ratings, which has been empirically confirmed by Avramov, Chordia, Jostova, and Philipov (2007). Intuitively, if a firm in financial distress experiences a string of positive returns, then the variance beta of the equity falls as the firm’s health improves and the importance of the equityholders’ default option decreases. On the other hand, the variance beta of a firm with low credit rating experiencing a string of negative

4Limited liability has been studied in the corporate finance literature in papers such as Hellwig (1981), Innes (1990), Laux (2001), and Biais et al. (2012).
returns should increase as it is likely the firm is becoming more financially distressed. Since shocks to volatility are negatively priced, the variance risk premium and hence risk-adjusted returns of low-credit-rating winners are higher than those of low-credit-rating losers.

Of course, the option to default on debt liabilities is not the only embedded option in the structure of the firm. Firms also vary in their ratio of investment options to book value, i.e. their growth options. Since the growth options become more valuable as volatility rises, all else equal, a firm with a higher ratio of growth options to book value will serve as a better hedge against volatility risk in the market and should earn a lower variance risk premium. But sorting firms of similar productivity by embedded growth options will also sort firms by their book-to-market ratios. Thus, due to the variance risk premium, firms with low book-to-market ratios (growth firms) can earn smaller returns than firms with high book-to-market ratios (value firms). I show that this channel can quantitatively generate a value premium in line with the historical average.

Turning to fixed income, note that while equity is long the option to default, debt is short. An increase in volatility raises the probability of default, which decreases the value of debt. As a result, the required return on debt should be higher in a model with stochastic volatility than a model in which volatility remains constant. This effect lowers prices and increases credit spreads. However, while this intuition is correct, quantitative calibration of the model illustrates that it is in itself insufficient to match empirical credit spreads across ratings categories. If the default barrier is set exogenously, then the model simply creates a credit spread puzzle in the other direction. It is able to better explain the credit spreads on investment grade debt, but ends up substantially overpredicting the credit spreads on junk debt. This result is a reflection of the fact that existing structural models with constant volatility actually do a reasonably good job predicting the credit spreads of junk debt.

The paper demonstrates that it is exactly the interaction of stochastic volatility and endogenous default which allows the model to quantitatively match credit spreads across credit ratings and therefore resolve the credit spread puzzle. The reasons are twofold. First, as has been discussed, when volatility increases, it raises the value of the equityholders’ default option. If equityholders optimally decide when to default, they respond to an increase in volatility by postponing default. This channel ameliorates the adverse effects of higher volatility on debt value and, in fact, can reverse the sign if the firm is sufficiently financially distressed. That is, the debt of a firm extremely close to default actually benefits from an increase in volatility and thus hedges volatility risk. In a precise sense to be made clear, the price of junk debt reflects these hedging properties such that the model generates lower

\[ ^5 \text{This result implies that the value of the firm increases with volatility close to the default boundary, since the equity of financially distressed firms also increases with volatility.} \]
credit spreads than a model with exogenous default.

Second, the simple fact that volatility is stochastic raises the value of the default option such that at all volatility levels, the endogenous default barrier is lower than in a model with constant volatility. This once again tends to raise debt values, but the effect of the shift is strongest for junk debt since it is closest to default and the future is discounted. With these two effects present, the calibrated model is able to quantitatively match well the target credit spreads and historical default probabilities across all ratings categories for intermediate and long maturity debt, holding fixed the market price of variance risk.

The model is less successful at shorter maturities. While still providing a substantial improvement over a model with constant volatility, it accounts for less of the empirically observed credit spreads than at longer maturities. The calibration further demonstrates, though, that some of this underprediction is due to the model significantly understating credit risk at these maturities. In an extension, I consider a model featuring rare disasters in the form of low-frequency jumps in the firm productivity process. This addition allows the model to generate higher credit risk at short maturities, while only marginally affecting credit risk at longer maturities.

That stochastic volatility has not received much attention in structural credit modeling is likely due to the considerable technical difficulties involved, a fact pointed out by Huang and Huang (2012). To overcome the technical hurdles, I develop novel perturbation techniques to construct accurate, approximate asymptotic series expansions of contingent claim valuations. The fundamental assumption which makes this methodology operative in the primary model is that volatility is slowly-moving and persistent. While the econometric literature has uncovered multiple time scales in volatility dynamics, it is exactly these persistent fluctuations which long-run investors should care about and should therefore be significantly priced in equilibrium. The key advantages of this approach are twofold. First, it provides analytic tractability by transforming the solution of difficult partial differential equations problems with an unknown boundary to recursively solving a standard sS problem, and then a straightforward ordinary differential equations problem involving key comparative statics of the model in which volatility remains constant. Second, an application of the Feynman-Kac formula provides a useful probabilistic interpretation of the first-order correction terms which allows one to cleanly see the various mechanisms at work in the model.

The paper proceeds as follows. Section 2 briefly reviews the relevant literature. Section 3 introduces the baseline model and provides characterizations of contingent claims valuations as solutions to appropriately defined partial differential equations problems. Section 4 discusses the perturbation methodology used to solve the problems. Section 5 considers the asset pricing implications of the model for equities. Section 6 demonstrates how the model
quantitatively resolves the credit spread puzzle. Section 7 considers short maturity debt and the rare disasters extension to the baseline model. Finally, Section 8 concludes.

2 Literature Review

This paper is connected to a substantial literature recognizing the asset pricing implications of stochastic volatility. The long-run risks model of Bansal and Yaron (2004) incorporates time-varying consumption volatility into a consumption-based asset pricing framework. Later calibrations of the model by Beeler and Campbell (2012) and Bansal, Kiku, and Yaron (2012) emphasize the importance of this feature in delivering empirically reasonable results. Coval and Shumway (2001), Ang, Hodrick, Xing, and Zhang (2006), Adrian and Rosenberg (2008), and Carr and Wu (2009) provide empirical evidence that innovations in market volatility are priced risk factors in the cross-section of stock returns. This paper differs from those above by explicitly constructing a structural model of the firm to understand how differences across firms can explain cross-sectional asset pricing patterns observed in the data for both equity and debt.

Previous papers have offered non-behavioral explanations for the equity anomalies considered in this paper. Gomes, Kogan, and Zhang (2003) construct a model in which book-to-market serves as a proxy for the systematic risk of assets in place. Zhang (2005) demonstrates how a model with both idiosyncratic productivity and convex capital adjustment costs can generate a value premium. Cooper (2006) is similar but includes non-convex adjustment costs. Carlson, Fisher, and Gianmarino (2004) generate book-to-market effects in a model with operating leverage. Garlappi et al. (2008) and Garlappi and Yan (2011) argue that the financial distress puzzle, in particular the hump shaped relationship between return and probability of default, can be accounted for with a structural credit model featuring partial shareholder recovery.

One consistent feature in all of this work is that there is a single source of priced risk. Therefore, in explaining the value premium puzzle, for instance, the models generate higher conditional market betas of value firms than growth firms, contradicting empirical evidence. This paper differs by including a second source of priced risk, such that abnormal risk-adjusted returns can be explained by variance betas rather than market betas. In this fashion, the study is similar to Papanikolaou (2011), which includes investment shocks as a second source of priced risk to explain the value premium puzzle.

(1998), Duffie and Lando (2001), and Collin-Dufresne and Goldstein (2001). Features of these models include stochastic interest rates, endogenous default, shareholder recovery, incomplete accounting information, and mean-reverting leverage ratios. Yet, as Huang and Huang (2012) show, these models cannot jointly produce historical default probabilities and realistic credit spreads. My work is similar to Hackbarth, Miao, Morellec (2006), Chen, Collin-Dufresne, and Goldstein (2009), Bhamra, Kuehn, and Strebulaev (2010), and Chen (2010) in analyzing how business cycle variation in macroeconomic conditions impacts credit spreads. My work differs from these papers in emphasizing the role of endogenous default in resolving cross-sectional aspects of the credit spread puzzle and focusing on equity return anomalies in addition to debt pricing within a unified framework. I also develop novel tools in this paper to aid in the solution on models featuring time-varying macroeconomic risks.

3 Structural Model of the Firm

Extending Leland (1994a) and Leland (1998), I develop a continuous-time, real options model of the firm incorporating both stochastic volatility of the firm productivity process and strategic default by equityholders. The model will allow for an analysis of equity pricing as well as the full term structure of credit spreads. Table 1 defines the model’s key variables for convenient reference.

3.1 Firm Dynamics - Physical Measure

Set a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and let \(W_t\) be a Wiener process or standard Brownian motion in two dimensions under \(\mathbb{P}\), which I call the physical measure. Associated with this Brownian motion is a filtration \(\mathcal{F}_t\) satisfying the usual properties. Firms are value-maximizing and operate in a perfectly competitive environment. The productivity of a representative firm follows a stochastic process given by

\[
dX_t/X_t = \mu dt + \sqrt{Y_t} dW_t^{(1)},
\]

where \(W_t^{(1)}\) is a standard Brownian motion and \(\mu\) is the expected growth rate of productivity under the physical measure \(\mathbb{P}\).\(^6\) The variance of this process \(Y_t\) is itself stochastic and follows

\(^6\)The assumption of stochastic cash flows differs from Leland (1994) and many other models of corporate debt which directly specify a stochastic process for the value of the unlevered firm. As discussed by Goldstein et al. (2001), this assumption has several advantages with regards to the modeling of tax shields and calibration of the risk neutral drift. It is also easier to incorporate growth options into such a framework.
a mean-reverting Cox-Ingersoll-Ross (CIR) process under the physical measure given by:

\[ dY_t = \kappa_Y (\theta_Y - Y_t) + \nu_Y \sqrt{Y_t} dW_{t}^{(2)}. \] (2)

Here, \( \kappa_Y \) is the rate of mean reversion, \( \theta_Y \) is the long-run mean of variance, and \( \nu_Y \) controls the volatility of variance. The process \( W_{t}^{(2)} \) is a Brownian motion which has correlation \( \rho_Y \) with the process \( W_{t}^{(1)} \). In particular, I have that

\[
\begin{pmatrix}
W_{t}^{(1)} \\
W_{t}^{(2)}
\end{pmatrix} = 
\begin{pmatrix}
1 & 0 \\
\rho_Y & \sqrt{1 - \rho_Y^2}
\end{pmatrix}
W_t,
\] (3)

where \( W_t \) is the Wiener process defined above.

The firm begins its life as a "young (growth) firm" with initial capital \( K = K^a < 1 \) and can irreversibly expand its productive capacity to \( K = 1 \) at the discretion of the equityholders. I define firms that have exercised their growth option as "mature firms." A unit of capital costs \( \Delta \), such that exercising the growth option costs \( I = \Delta (1 - K^a) \).\(^7\) Most of the analysis in the paper will focus on mature firms.

Finally, firms pay taxes on their income at the corporate rate \( \phi \) so that the flow of after-tax profits of the unlevered firm at time \( t \) is given by

\[ (1 - \phi) X_t K_t. \] (4)

As is standard, this profit function reflects optimal choices by the firm in all other variable inputs such as labor and raw materials.

3.2 Firm Dynamics - Risk Neutral Measure

I assume the existence of a risk-neutral measure \( \mathbb{P}^* \) under which contingent claims will be priced. The dynamics of productivity and variance under the risk-neutral measure are given by

\[
dX_t = gX_t dt + \sqrt{Y_t} X_t dW_{t}^{(1)*}
\] (5)

\[
dY_t = \left( \kappa_Y (\theta_Y - Y_t) - \Gamma (Y_t) \nu_Y \sqrt{Y_t} \right) dt + \nu_Y \sqrt{Y_t} dW_{t}^{(2)*},
\] (6)

\(^7\)The choice of \( K = 1 \) is simply a normalization reflecting the scale invariance of the model. Investment occurs discretely, which allows me to solve for the contingent claims of mature and young firms recursively.
where $g > 0$ is the risk-neutral productivity growth rate and $\Gamma (Y_t)$ is defined as the market price of variance risk.\textsuperscript{8}

Since the setting is one of random, non-tradeable volatility, I am not in a complete markets framework and there exists a family of pricing measures parameterized by the market price of variance risk. As discussed in the introduction, if long-term investors see increases in volatility as deteriorations in the investment opportunity set, then the equilibrium price $\Gamma (Y_t)$ will be negative.\textsuperscript{9}

I define the asset risk premium to be $\pi_X = \mu - g$ and the (asset) variance risk premium to be $\pi_Y = \nu_Y \sqrt{\Gamma (Y_t)}$. As the latter object will be of central importance in the paper, it warrants further discussion. The variance risk premium provides the expected excess return over the risk-free rate on any security with zero asset beta and a variance beta equal to one, where these betas are respectively defined as\textsuperscript{10}

$$
\beta_X = \frac{\partial \log (\cdot)}{\partial x} \quad \text{(7)}
$$

$$
\beta_Y = \frac{\partial \log (\cdot)}{\partial y} \quad \text{(8)}
$$

For instance, it gives the expected excess return on a portfolio of delta-neutral straddles or a portfolio of variance swaps constructed to have a unit variance beta. Coval and Shumway (2001) estimate the expected excess returns on delta-neutral straddles and find them to be negative. Carr and Wu (2009) document a similar fact for variance swaps. These empirical findings are supportive of a negative variance risk premium as predicted by the ICAPM. As is standard in the literature, I parameterize this market price of variance risk as $\Gamma (Y_t) = \Gamma_0 \sqrt{Y_t}$ with $\Gamma_0 < 0$.\textsuperscript{11}

The value of the assets in place, i.e. the value of an unlevered firm, is given by the discounted present value under the risk-neutral measure of future after-tax profits generated by installed capital. Denoting this value by $U_t$ for mature firms and letting expectations

\footnotesize
\textsuperscript{8}In the baseline model, the risk-neutral growth rate, and thus asset risk premium, does not decline with volatility, consistent with empirical studies documenting the lack of a risk-return relationship in the data. See, for example, Glosten et al. (1993), Lundblad (2007), Lettau and Ludvigson (2003), Moreira and Muir (2016), and many others. Footnote 22 discusses this point further.

\textsuperscript{9}Specifically, long-term investors should view increases in market volatility as a deterioration in the investment opportunity set. If movements in the asset volatility of the representative firm are positively correlated with movements in market volatility, then fluctuations in asset volatility will carry a negative price of risk.

\textsuperscript{10}These definitions reflect the use of Itô’s lemma in calculating the required excess return of an asset under the physical measure.

\textsuperscript{11}This parameterization simply implies that the asset variance risk premium $\pi_Y = \nu_Y \sqrt{Y_t} \Gamma (Y_t)$ scales with the quantity of variance risk.
under $\mathbb{P}^*$ conditional on $\mathcal{F}_t$ be denoted by $\mathbb{E}_t^* (\cdot)$, I have

$$U_t (x) = \mathbb{E}_t^* \left[ \int_t^\infty e^{-r(s-t)} (1 - \phi) X_s ds \mid X_t = x \right] = \frac{(1 - \phi) x}{r - g}, \quad (9)$$

which is simply the Gordon growth formula given that after-tax profits grow at the rate $g$ under the risk-neutral measure. The value of assets in place for young firms, denoted by $U_t^a$, is more complicated and discussed in Appendix B.

### 3.3 Capital Structure

Firms are financed with both debt and equity issues. The firm adopts a stationary debt structure with total principal $P$ and total coupon $C$. The firm continuously rolls over debt at the fractional rate $m$. That is, at every point in time, debt with principal equal to $mP$ matures and is replaced with new debt of equal coupon, principal, and seniority to maintain stationarity. Let $p$ denote the principal on newly issued debt and $c$ the coupon. Moreover, let $p(s,t)$ and $c(s,t)$ be the principal and coupon outstanding at time $t$ for debt issued at date $s \leq t$. Since the firm retires the principal of all vintages at fractional rate $m$, the principal and coupon of each vintage declines exponentially with time:

$$p(s,t) = e^{-m(t-s)} p \quad (10)$$
$$c(s,t) = e^{-m(t-s)} c. \quad (11)$$

Integrating over all vintages at time $t$ gives the total principal and coupon of the firm:

$$P \quad = \quad \int_{-\infty}^t p(s,t) \, ds = p \int_{-\infty}^t e^{-m(t-s)} \, ds = p/m. \quad (12)$$
$$C \quad = \quad \int_{-\infty}^t c(s,t) \, ds = c \int_{-\infty}^t e^{-m(t-s)} \, ds = c/m. \quad (13)$$

Thus, the principal and coupon of newly issued debt is always equal to a fraction $m$ of the total principal and coupon in the capital structure of the firm.

Now, note that for a time $s$ vintage, the fraction of currently outstanding debt principal retired at time $t > s$ is given by $me^{-m(t-s)}$. This implies that the average maturity of debt $M$ is given by

$$M \quad = \quad \int_s^\infty t \left( me^{-m(t-s)} \right) \, dt = 1/m. \quad (14)$$

In other words, the inverse of the fractional rollover rate is a measure of the average maturity of the firm’s debt. Note that the limiting case $m = 0$ corresponds to the Leland (1994b)
model of consol debt.

Of course, the price of newly issued debt will reflect current state variables in the market. This exposes the equityholders of the firm to rollover risk. The firm will face either a cash windfall or shortfall depending on whether debt is currently priced above or below par. Additional equity must be issued if there is a cash shortfall. Since interest payments provide a tax shield, the total flow payment to equityholders of mature firms is therefore given by

\[
(1 - \phi) [X_t - C] + \tilde{d}(X_t, Y_t) - p,
\]

where \( \tilde{d}(X_t, Y_t) \) is the value of newly issued debt. The first term reflects the flow operating profits generated by the firm as well as the corporate tax rate and the tax shield, while the final two terms reflect the rollover risk faced by the equityholders.

Equityholders optimally decide when to default on their debt obligations and may issue additional equity to finance coupon payments if current cash flow is insufficient to meet their obligations. In the event of default, equityholders receive nothing and the value of their claims is zero. Debtholders receive the assets of the firm according to their vintage; however, a fraction \( \zeta \) of the total value of assets in place is lost due to bankruptcy costs.\(^{12}\)

I assume for simplicity that investment spending is financed entirely with equity. I now turn to the valuation of the firm’s contingent claims.

### 3.4 Equity Valuation

Contingent claims are priced according to the risk-neutral measure. Given equation (15), the value of equity for a mature firm is given by

\[
E(x, y) = \sup_{\tau \in \mathcal{T}} \mathbb{E}^* \left[ \int_t^\tau e^{-r(s-t)} \left\{ (1 - \phi) (X_s - C) + \tilde{d}(X_s, Y_s) - p \right\} ds \right],
\]

where \( \mathcal{T} \) is the set of \( \{ \mathcal{F}_t \} \)-stopping times. I denote the optimal stopping time, i.e. time of default, by \( \tau^B \). Intuitively, the equity value of a mature firm is simply the risk-neutral expected discounted present value of all future dividends accruing to equityholders given that the point of default is chosen optimally.

I show that the equity value of the mature firm (16) can be given as the solution to a Dirichlet/Poisson free boundary problem.

\(^{12}\)Bankruptcy costs include the direct costs of the legal proceedings, but also indirect costs such as losses of specialized knowledge and experience, reductions in trade credit, and customer dissatisfaction. In practice, the indirect costs of bankruptcy may be of an order of magnitude larger than the direct costs.
Theorem 1  The equity value of a mature firm $E(x, y)$ is the solution to

\[ \begin{align*}
(1 - \phi)(x - C) + \tilde{d}(x, y) - p + \mathcal{L}_{X,Y} E &= rE & \text{for } x > x_B(y) \\
(1 - \phi)(x - C) + \tilde{d}(x, y) - p + \mathcal{L}_{X,Y} E &\leq rE & \text{for } x, y > 0 \\
E(x, y) &= 0 & \text{for } x < x_B(y) \\
E(x, y) &\geq 0 & \text{for } x, y > 0 \\
E(x_B(y), y) &= 0 & \text{(17e)} \\
\lim_{x \to \infty} E(x, y) &= U(x) - \frac{C + mP}{r + m} - \frac{\phi C}{r} & \text{(17f)} \\
\frac{\partial E}{\partial x}|_{x=x_B(y)} &= 0 & \text{(17g)} \\
\frac{\partial E}{\partial y}|_{x=x_B(y)} &= 0, & \text{(17h)}
\end{align*} \]

where $\mathcal{L}_{X,Y}$ is the linear differential operator given by:

\[ \mathcal{L}_{X,Y} = gx \frac{\partial}{\partial x} + \frac{1}{2} y x^2 \frac{\partial^2}{\partial x^2} + \left( \kappa_Y (\theta_Y - y) - \Gamma (y) \nu_Y \sqrt{y} \right) \frac{\partial}{\partial y} + \frac{1}{2} \nu_Y^2 y \frac{\partial^2}{\partial y^2} + \rho_Y \nu_Y y \frac{\partial}{\partial x \partial y}. \]

and $x_B(y)$ is a free boundary to be determined.

Proof. See Appendix A.  ■

Here $x_B(y)$ is the point at which equityholders optimally default on their debt. Crucially, note that the boundary at which default occurs can depend on the current volatility.

The conditions in the above system are intuitive. By the multidimensional Ito’s formula, I can write equation (17a) as

\[ \frac{(1 - \phi)(x - C) + \tilde{d}(x, y) - p}{E} + \frac{E^* [dE]}{E} = r, \]

which simply says that the expected return on equity, given by the dividend yield plus the expected capital gain, must be equal to the risk-free rate under the risk-neutral measure if the firm is not in default. Equation (17b) implies that in the stop region the expected return from continuing operations must be less than or equal to the riskless rate. Otherwise, stopping would not be optimal. Equation (17e) is the usual value-matching equation which states that at the point of default the value of equity must be equal to zero. If the value, for instance, were positive then the continuation value of equity would be larger than the value of equity under default, and thus default should be optimally postponed. As $X_t \to \infty$, the probability of default in finite time approaches zero, and therefore the equity value is simply given by the present value of the assets in place $U_t$, minus the present value of future debt.
obligations \((C + mP) / (r + m)\), plus the present value of the tax shield \(\phi C / r\). This logic yields the limiting condition (17f). Finally, equations (17g) and (17h) provide the smooth-pasting conditions for the problem. These are standard for optimal stopping problems in which the stochastic process follows a regular diffusion.

Similarly, letting \(E^a(x, y)\) denote the equity value of young firms, it solves the optimal stopping problem:

\[
E^a(x, y) = \sup_{\tau', \tau'' \in \mathcal{T}} \mathbb{E}_t^* \left[ \int_t^{\tau' \wedge \tau''} e^{-r(s-t)} (1 - \phi) (K_0 X_s - C) \tilde{d}(x, y) - p \right] ds + 1[\tau' < \tau''] \left( E(X_{\tau'}, Y_{\tau'}) - I \right)
\]

where \(\tau' \wedge \tau'' = \min(\tau', \tau'')\). Here \(\tau'\) is a stopping time which indicates exercising the growth option and \(\tau''\) is a stopping time which indicates default. If the firm exercises its growth option prior to default, then the value of the equity becomes \(E(X_{\tau'}, Y_{\tau'}) - I\), which gives the second term in the expression above. The equity value of a young firm can be described as the solution to an appropriate free boundary problem just as with mature firms. Basically, the limiting condition (17f) is removed and there are value-matching and smooth-pasting conditions at the boundary for exercise of the growth option. It is characterized explicitly in Appendix B.

### 3.5 Debt Valuation

As is evident from the equations above, solving the equityholders’ problem requires the value of newly issued debt. This dependence arises from the rollover risk inherent in the flow dividend to the equityholders. Let \(d(t)\) be the value at date \(t\) of debt issued at time 0 for a mature firm. According to risk-neutral pricing, the valuation is given by

\[
d(t) = \mathbb{E}_t^* \left[ \int_t^{\tau_B} \left\{ e^{-r(s-t)} e^{-ms} (c + mp) \right\} ds \right.
\]

\[
+ e^{-r(\tau_B - t)} \left( e^{-m\tau_B} p / P \right) (1 - \xi) U_{\tau_B} \] .
\]

Note that the payments to the debtholders are declining exponentially and are therefore time dependent. Furthermore, the debtholders’ claim on the assets of the firm will also depend on the point in time at which bankruptcy occurs. In general, such time-dependency would indicate that a partial differential equation involving a time derivative would need to be solved to find \(d(t)\). However, multiplying both sides of the equation by \(e^{mt}\) and noting
that \( p/P = m \) yields

\[
e^{mt}d(t) = \mathbb{E}_t^* \left[ \int_t^\tau^B \left\{ e^{-(r+m)(s-t)} (c + mp) \right\} ds \right. \\
\left. + e^{-(r+m)(x_B-t)} m (1 - \xi) U_{x_B} \right].
\] (22)

Applying Feynman-Kac, I then have the following result:

**Theorem 2** The value of the date 0 debt vintage at time \( t \) is given by \( d(t) = e^{-mt} \tilde{d}(X_t, Y_t) \) where \( \tilde{d}(X_t, Y_t) \) is the value of the newly issued debt and satisfies

\[
c + mp + \mathcal{L}_{X,Y} \tilde{d} = (r + m) \tilde{d} \quad \text{for } x > x_B(y)
\] (23a)

\[
\tilde{d}(x_B(y), y) = m (1 - \xi) U(x_B(y))
\] (23b)

\[
\lim_{x \to \infty} \tilde{d}(x, y) = \frac{c + mp}{r + m}.
\] (23c)

The value of newly issued debt is therefore the solution to a PDE that does not involve a time derivative. Rather, the exponential decline in the flow payments to the debtholders is reflected in a higher implied interest rate \( r + m \). This is the key advantage of employing the Leland (1994a) model of capital structure. To find the total value of debt at any point in time, simply integrate over the value of all outstanding debt vintages:

\[
D(t) = \int_{-\infty}^t e^{-m(t-s)} \tilde{d}(X_t, Y_t) ds = \frac{\tilde{d}(X_t, Y_t)}{m}.
\] (24)

Thus, the total value of the firm’s aggregate debt is a function only of the current productivity and volatility and does not depend on time, which is consistent with the stationarity of the overall debt structure. The debt of young firms is characterized similarly and is discussed in Appendix B.

### 4 Methodology

The solution to the model is thus a coupled system of elliptic partial differential equations, along with an unknown boundary. I utilize a novel, semi-analytic methodology based on asymptotic expansions to generate accurate approximations to the values of the contingent claims. The method operates by making assumptions on the rate of mean-reversion of the stochastic volatility process and then using either regular or singular perturbations around the appropriate parameter in the partial differential equations to recursively solve for the formal power series expansions of the debt and equity values, as well as the default boundary.
I will confine myself to first-order expansions, although higher-order terms can be calculated in a straightforward manner.

The basic principles of this method have been developed recently in the mathematical finance literature for the pricing of options under stochastic volatility. Lee (2001) develops an approach for the pricing of European options under slow-variation asymptotics of the stochastic volatility process. In a sequence of papers, authors Fouque, Papanicolaou, Sircar, and Solna have considered fast-variation asymptotics in a variety of option pricing settings (European, barrier, Asian, etc.) and have developed perturbation procedures for stochastic volatility processes with both slow and fast variation components. My baseline setting extends the small-variation asymptotics to a free boundary problem. I will further extend these methods to allow for rare disasters in the firm productivity process.

To the best of my knowledge, the use of this methodology is novel in the economics literature outside of mathematical finance, and as I will argue below, is particularly well-suited for use by economists in a variety of economic settings.

4.1 Time Scales in Volatility Modeling

The asymptotic expansions approach is based on the idea of volatility fluctuating on various time scales. A substantial number of empirical studies have validated such a concept and shown, in particular, that volatility exhibits long-run dependencies, often termed long memory. Andersen and Bollerslev (1997) and Baillie et al. (1996), for instance, introduce the Fractionally Integrated ARCH (FIGARCH) model in which autocorrelations have hyperbolic decline rather than geometric decline. Beginning with Engle and Lee (1999), a body of papers has argued that two-dimensional volatility models with both a short-run component and a long-run component show significantly better performance in matching the data. Campbell et al. (2016) document the existence of a highly persistent component to market volatility and, moreover, show that is priced in the cross-section.

I adopt this perspective in my modeling of asset variance. Specifically, in the baseline model, I argue that changes in asset volatility are driven by a highly persistent, mean-
reverting process, and that the associated shocks are priced. Whereas market volatility demonstrates fluctuations on multiple time scales, high frequency fluctuations are likely to be less important in the actual dynamics of asset volatility, since changes in asset volatility should be driven largely by fundamentals rather than market sentiment and other transient factors. More importantly, it is exactly those persistent, low frequency movements in volatility which long-run investors should care about and which should be priced in equilibrium.

Rather than directly calibrating the parameter $\kappa_Y$ to be small, though, I instead slightly modify the process for variance under the physical measure as

$$dY_t = \delta \kappa_Y (\theta_Y - Y_t) \, dt + \nu_Y \sqrt{\delta Y_t} dW_t^{(2)},$$

where $\delta > 0$ is a small parameter. Note that this is again a CIR process exactly the same as in equation (2) except that the drift term is now multiplied by $\delta$ and the diffusion term is multiplied by $\sqrt{\delta}$. This parameter directly controls the rate of mean-reversion of the process for volatility. Since I am assuming it is small, I will say that variance is *slowly mean-reverting.*

### 4.2 Asymptotic Approximation

Given this model of volatility dynamics, I can write the partial differential equations in (17a) and (23a) as:

\[
(1 - \phi) (x - C) + \tilde{d}_\delta (x, y) - p + \left( \mathcal{L}^y_x + \sqrt{\delta} \mathcal{M}^y_1 + \delta \mathcal{M}^y_2 \right) E_\delta = 0
\]

\[
c + m + \left( \mathcal{L}^y_{r+m} + \sqrt{\delta} \mathcal{M}^y_1 + \delta \mathcal{M}^y_2 \right) \tilde{d}_\delta = 0,
\]

Specifically, $\delta$ scales the spectral gap of the process for $Y_t$, or the distance between the zero eigenvalue and the first negative eigenvalue. Using eigenfunction expansions shows that the spectral gap determines the rate of mean-reversion for the process. However, the invariant or long-run distribution of $Y_t$ is

$$\Delta_Y \sim \text{Gamma} \left( \frac{2 \kappa_Y \theta_Y}{\nu^2_Y}, \frac{\nu^2_Y}{2 \kappa_Y} \right),$$

which is independent of $\delta$. This indicates that in the long run, the level of variability in the volatility of asset productivity does not depend on the parameter $\delta$. Movements in volatility are thus *slow,* but not necessarily *small.* Appendix C provides further technical details on the modeling of volatility time scales.
where $E_\delta$ and $\tilde{d}_\delta$ are the values of equity and newly issued debt respectively. The infinitesimal operators $L^y_r, M^y_1, M^y_2$ are given by

\begin{align}
L^y_r &= gx \frac{\partial}{\partial x} + \frac{1}{2} y x \frac{\partial^2}{\partial x^2} - r (\cdot) \\
M^y_1 &= \rho_Y \nu_Y y x \frac{\partial^2}{\partial x \partial y} - \Gamma (y) \nu_Y \sqrt{y} \frac{\partial}{\partial y} \\
M^y_2 &= \kappa_Y (\theta_Y - y) \frac{\partial}{\partial y} + \frac{1}{2} \nu_Y^2 y \frac{\partial^2}{\partial y^2}.
\end{align}

Here, $L^y_r$ is the time-invariant Black-Scholes operator with volatility $y$ and riskfree rate $r$, $M^y_2$ is the infinitesimal generator of the CIR process, and $M^y_1$ is an operator which accounts for correlation between the processes for asset productivity and asset volatility, as well as the Girsanov transformation between the physical and risk-neutral measures.\(^{17}\)

To solve for the contingent claims of mature firms by regular perturbation, I expand the equity value, the value of newly issued debt, and the free default boundary in powers of $\sqrt{\delta}$:

\begin{align}
E_\delta (x, y) &= E^y_0 (x) + \sqrt{\delta} E^y_1 (x) + \delta E^y_2 (x) + ... \\
\tilde{d}_\delta (x, y) &= \tilde{d}^y_0 (x) + \sqrt{\delta} \tilde{d}^y_1 (x) + \delta \tilde{d}^y_2 (x) + ... \\
x_B (y) &= x_{B,0}^y + \sqrt{\delta} x_{B,1}^y + \delta x_{B,2}^y + ...
\end{align}

I then plug these asymptotic expansions into equations (26) and (27), as well as equations (17e)-(17h) and (23b)-(23c). Taylor expansions centered around $x_{B,0} (y)$ are used to appropriately expand value-matching and smooth-pasting conditions. The system is solved by collecting terms in the powers of $\delta$ and using the method of undetermined coefficients, where here I understand the coefficients to be functions.

\(^{17}\)Note that the dynamics of volatility under the risk-neutral measure are given by:

\[ dY_t = \left( \delta \kappa (\theta - Y_t) - \Gamma (Y_t) \nu \sqrt{\delta Y_t} \right) dt + \nu \sqrt{\delta Y_t} dW_t^{(2)*} \]

for a given choice of parameter $\delta$. 

4.2.1 Principal Order Terms

I begin by computing the principal order terms in the asymptotic expansions. Collecting the order one terms for newly issued debt yields the problem:

\[
\begin{align*}
  c + mp + L_{r+m}^y \tilde{d}_0^y & = 0 \quad \text{for } x > x_{B,0}^y \quad (34a) \\
  \tilde{d}_0^y (x_{B,0}^y) & = m (1 - \xi) U (x_{B,0}^y) \quad (34b) \\
  \lim_{x \to \infty} \tilde{d}_0^y (x) & = \frac{c + mp}{r + m}. \quad (34c)
\end{align*}
\]

The principal order term is simply the value of newly issued debt under constant return variance \( y \) in the firm productivity process, given the fixed boundary \( x_{B,0}^y \). The result is thus consistent with the expression in Leland (1994a):

\[
\tilde{d}_0^y (x) = \frac{c + mp}{r + m} + \left[ m (1 - \xi) U (x_{B,0}^y) - \frac{c + mp}{r + m} \right] \left\{ \frac{x}{x_{B,0}^y} \right\}^{\gamma_1}, \quad (35)
\]

where \( \gamma_1 \) is the negative root of the following quadratic equation:

\[
g \gamma_1 + \frac{1}{2} y \gamma_1 (\gamma_1 - 1) - (r + m) = 0. \quad (36)
\]

The first term in the valuation reflects the present value of future coupon and principal payments assuming no default. The latter term takes into account the risk of default. The term \( (x/x_{B,0}^y)^{-\gamma_1} \) is akin to a probability of default. If default occurs, debtholders receive a fraction of the value of the assets of the firm but lose any future coupon and principal payments remaining. More precisely, the second component of the valuation is a perpetual digital option which pays off the term in brackets the first time the process \( X_t \) crosses the boundary \( x_{B,0}^y \).

Intuitively, the principal order term reflects the value of newly issued debt in the limiting case \( \delta = 0 \), that is a model in which volatility is fixed at its current value. This intuition carries over to computing the principal order terms for both the equity value and the default boundary. The principal order terms coincide with the equity value and default boundary.
in Leland (1994a):

\[
E_0^y (x) \ = \ U (x) - \frac{C + mP}{r + m} + \frac{\phi C}{r} + \left[ \frac{C + mP}{r + m} - (1 - \xi) U (x_{B,0}^y) \right] \left\{ \frac{x}{x_{B,0}^y} \right\}^{\gamma_1} \]

\[
- \left[ \frac{\phi C}{r} + \xi U (x_{B,0}^y) \right] \left\{ \frac{x}{x_{B,0}^y} \right\}^{\gamma_2},
\]

\( x_{B,0}^y = \left( \frac{(C + mP) \gamma_1/(r + m) - \phi C \gamma_2/r - \gamma_1 - \gamma_2}{1 - (1 - \xi) \gamma_1 - \gamma_2} \right) \frac{1}{1 - \phi}, \tag{37} \]

where \( \gamma_2 \) is the negative root to the quadratic equation:

\[
g \gamma_2 + \frac{1}{2} y \gamma_2 (\gamma_2 - 1) - r = 0. \tag{39} \]

The equity value incorporates the value of the assets in place, the present value of future debt payments, the present value tax shield, and two perpetual digital options which account for the equityholders’ default option.

4.2.2 First-Order Correction Terms

Continuing with the recursive approach, applying the parameterization \( \Gamma (Y) = \Gamma_0 \sqrt{y} \) and collecting terms of order \( \sqrt{\delta} \) yields the problems to solve for the first-order correction terms. For the value of newly issued debt, this is given by

\[
\mathcal{L}_{r+m}^y \sqrt{\delta d_1^y} = \left( A^\delta y \frac{\partial \tilde{d}_0^y}{\partial y} - B^\delta y x \frac{\partial^2 \tilde{d}_0^y}{\partial x \partial y} \right) \quad \text{for } x > x_{B,0}^y, \tag{40a} \]

\[
\sqrt{\delta d_1^y} (x_{B,0}^y) = \sqrt{\delta x_{B,1}^y} \left[ \frac{m (1 - \xi) (1 - \phi)}{r - g} - \frac{\partial \tilde{d}_0^y}{\partial x} (x_{D,0}^y) \right], \tag{40b} \]

\[
\lim_{x \to \infty} \sqrt{\delta d_1^y} (x) = 0, \tag{40c} \]

where the constants \( A^\delta, B^\delta \) are defined by

\[
A^\delta = \sqrt{\delta} \nu Y \Gamma_0, \tag{41} \]

\[
B^\delta = \sqrt{\delta} \nu Y \rho_Y. \tag{42} \]

Note that this is a one-dimensional inhomogeneous second-order boundary value problem in which \( y \) operates as a parameter. The source term is a function of comparative statics or Greeks of the debt principal order term. In particular, it is a function of the vega and the
vanna of the debt value when volatility is held constant at the level $\sqrt{y}$. Note also that the default boundary correction term $x_{B,1}^y$ appears in the boundary condition of the problem above, given by equation (40b). The problem for the first-order equity correction term takes a similar form and is given by

$$
\mathcal{L}_y \sqrt{\delta} E_1^y = \left( A^y_1 \frac{\partial E_0^y}{\partial y} - B^y_1 \frac{\partial^2 E_0^y}{\partial x \partial y} - \sqrt{\delta} \delta^y_1 \right) \quad \text{for } x > x_{B,0} 
$$

(43a)

$$
\sqrt{\delta} E_1^y (x_{B,0}) = 0 
$$

(43b)

$$
\lim_{x \to \infty} \sqrt{\delta} E_1^y (x) = 0. 
$$

(43c)

Once again, this is a one-dimensional second-order boundary value problem with a source term. Different from above, the Black-Scholes operator has for its riskfree rate $r$ instead of $r + m$ as in equation (43a). Furthermore, the source term for the problem involves the debt correction term, due to the rollover risk in the original problem, in addition to the vega and vanna of the equity value in the constant volatility case.

Finally, the Taylor expansion of the equity value smooth-pasting condition in $x$, equation (17g), gives the correction to the default boundary as:

$$
\sqrt{\delta} x_{B,1}^y \frac{\partial^2 E_0^y}{\partial x^2} (x_{B,0}) = -\sqrt{\delta} \frac{\partial^2 E_1^y}{\partial x} (x_{B,0}) . 
$$

(44)

Equations (40a)-(40c), (43a)-(43c), and (44) form a system of equations which can be jointly solved to determine the first-order corrections for the debt value, the equity value, and the default boundary. It is now straightforward to solve this system numerically since the differential equations are one-dimensional and the problems have fixed rather than free boundaries. The MATLAB function \textit{bvp4c}, which implements a finite elements scheme, is used to solve the boundary value problems while searching over $\sqrt{\delta} x_{B,1}^y$ to satisfy equation (44). Note that I have not used condition (17h) at all; however, it is easy to show that the approximation $E_0 (x, y) + \tilde{E}_1^y (x, y)$ is smooth across the boundary $x_{B,0} (y)$ as required.

4.2.3 Discussion and Further Applications

By making an assumption on the rate of mean-reversion of the process for volatility, this method reduces the solution of two-dimensional PDE problems with a free boundary to a recursive sequence of one-dimensional problems. The principal order terms simply reflect the contingent claims valuations and default boundary in the limiting case where volatility is fixed at its current level. The first-order correction terms are solved as a system of

$^{18}$Here, I define vega to be the comparative static with respect to variance $y$. 
equations in which the contingent claims corrections are the solutions to one-dimensional, fixed boundary problems and the source terms are functions of the comparative statics or Greeks of the principal order terms. This method is tractable and, as will be seen, provides intuitive expressions demonstrating the effects of stochastic volatility on debt and equity valuation.

A significant advantage of the methodology is that it reduces the number of parameters which need to be calibrated to generate quantitative results. Given the parametric assumption on the market price of variance risk, the only further calibration required beyond the baseline constant volatility model is that of two constants: $A^\delta = \sqrt{\delta \nu_Y \Gamma_0}$ and $B^\delta = \sqrt{5 \nu_Y \rho_Y}$. Essentially, the variance risk premium needs to be calibrated as well as a constant which has the same sign as the correlation between volatility shocks and productivity shocks. The rate of mean-reversion $\kappa_Y$ and the long-run mean of variance $\theta_Y$ do not appear at all.\footnote{This is because the operator $\mathcal{M}_2$ does not appear in the derivation of either the principal order term or the first-order correction term.}

This is especially useful in a structural model of credit incorporating stochastic volatility, as there are not good empirical estimates for the structural parameters of the asset volatility process.

Finally, I believe that this methodology is particularly well suited for use in a variety of other economic settings. The approach offers a general framework for introducing additional state variables into either deterministic or stochastic dynamic models, including but certainly not limited to volatility. First, one constructs a baseline framework in which this additional state variable is a fixed parameter and then calculates the appropriate comparative statics. If one is able to make limiting assumptions on the dynamics of this additional state variable, such as slow, small, or fast, then approximate solutions to the full model can be derived by calculating correction terms as the solutions to differential equations whose source terms are functions of the comparative statics.

For example, in the context of credit modeling, one can use this approach to calculate default probabilities, as described in Appendix A.3. As Huang and Huang (2012) point out, it is this computation in particular which has been one of the primary stumbling blocks in the construction of structural credit models which include stochastic volatility. My methodology, however, provides an effective means of overcoming this hurdle. The key, just as with the contingent claims valuations, is to find a suitable PDE characterization of the probability and then utilize a perturbation to simplify the solution of the problem. Asymptotic expansions can also be used to solve a model which includes, for instance, both stochastic volatility and rare disasters. This particular extension will be explored in a later section.
5 Equity Pricing

I begin my analysis by examining the model’s implications for equity pricing. Applying Ito’s formula, substituting in equation (17a), and taking expectations shows that the required return of a firm’s equity at time $t$ under the physical measure is given by

$$
\frac{(1 - \phi)(x - C) + \tilde{d}(x, y) - p + \mathbb{E}_t[dE^\delta]}{E^\delta} - r = \pi_X \beta_X + \pi_Y \beta_Y,
$$

where the market and variance betas $\beta_X, \beta_Y$ are respectively defined in equations (7) and (8). The excess expected return of equity over the riskfree rate is the sum of a risk premium due to its exposure to productivity risk and a risk premium due to its exposure to volatility risk. The exposures are priced according to the asset risk premium and asset variance risk premium, respectively. The first term is standard. The second term is therefore what must account for the equity anomalies I consider.

5.1 Calibration

I use a calibration procedure consistent with previous studies of structural credit models, in particular those which have documented the credit spread puzzle, since one of my goals is to understand whether exposure to volatility alone can improve model performance. The riskfree rate is set equal to 8% as in Huang and Huang (2012) and Leland (2004), equal to the historical average of Treasury rates between 1973-1998. The tax rate is equal to 15% as in Leland (2004), reflecting the corporate tax rate offset by the personal tax advantage of equity returns.\(^{20}\) I set the rate of risk-neutral productivity growth equal to 2%. This indicates that the expected return on the value of assets in place is equal to 2%, reflecting a standard payout rate of 6%. Finally, I set the asset risk premium equal to 4%, such that the asset beta is equal to approximately 0.6. This is once again consistent with Leland (2004) and only slightly less than the asset risk premia in Huang and Huang (2012). This asset risk premium generates an equity premium for the representative firm that is in line with historical averages.

Schaefer and Strebulaev (2008) provide estimates of productivity/asset volatility by credit rating. Specifically, the authors estimate asset volatility of a firm $j$ at time $t$, denoted as

\(^{20}\)Specifically, as shown in Leland (2004), given a corporate tax rate of 35%, a personal tax on bond income of 40%, and a tax rate on stock returns of 20%, the effective tax advantage of debt can be calculated as $1-(1-.35)(1-.20)/(1-.40)=.133$. 

21
\[ \sigma^2_{A,j,t}, \text{ according to} \]
\[
\bar{\sigma}^2_{A,j,t} = (1 - L_{j,t})^2 \sigma^2_{E,j,t} + L_{j,t}^2 \sigma^2_{D,j,t} + 2L_{j,t}(1 - L_{j,t}) \sigma_{ED,j,t},
\]

where \( L_{j,t} \) is the market leverage of firm \( j \) at time \( t \), \( \sigma_{E,j,t} \) is the equity volatility at time \( t \), \( \sigma_{D,j,t} \) is the debt volatility at time \( t \), and \( \sigma_{ED,j,t} \) is the covariance between debt and equity returns at time \( t \). The volatilities and covariances are calculated directly from the time series of equity and debt returns. They report that the average firm has an asset volatility of 22%. I use this volatility to price the equity of a representative firm. I set \( \rho_Y = 0 \).

For firms with leverage, I set the coupon rate equal to its historical average of 8.162%. The maturity I set at 10.5 years, equal to the average reported by Schaefer and Strebulaev (2008). I finally set distress costs equal to \( \xi = 30\% \), which generates a recovery rate of approximately 51% for the representative firm, equal to the historical average. I summarize the parameters in Table 2.

5.2 Financial Distress and Momentum

Standard structural models of the firm predict that the equities of firms close to default should earn higher returns than the equities of financially healthy firms. However, a number of authors, including Dichev (1998), Griffin and Lemmon (2002), and Campbell et al. (2008), empirically document that this is not the case. The stocks of financially distressed firms are not higher than those of financially healthy firms, and, in fact, are found to be lower in some studies. The most nuanced results are reported by Garlappi et al (2008), who measure the risk of default of a firm as the Expected Default Frequency generated by Moody’s KMV. \(^{21}\) They find that the relationship between the probability of default and returns is almost flat and slightly humped.

Figure 1 illustrates my model’s predictions for the relationship between probability of default and equity returns. Panel A plots the variance beta as a function of the 5-year probability of default, Panel B plots the equity variance risk premium as a function of the probability of default and the asset variance risk premium \( \pi_Y \), and the Panel C plots the total equity risk premium as a function of the probability of default and the asset variance risk premium. As Panel C shows, the model exactly generates the hump-shaped pattern documented by Garlappi et al. (2008) for an asset variance risk premium of close to -3\%.

As the asset variance risk premium falls, the model predicts instead that the equity return increases as the probability of default rises, in line with standard structural models of the

\(^{21}\)The authors argue that using Moody’s KMV EDF measure provides advantages relative to alternative measures due to the extensive data cleaning the Moody’s performs and the richness of their data construction.
In the model, there are two competing forces. The first is that the asset beta of the equity increases with financial distress, which tends to cause risk premia to rise with the probability of default. However, as Panel A of Figure 1 shows, the variance beta also increases. Since volatility risk carries a negative price of risk, this implies that the variance risk premium falls as the probability of default increases, as shown in Panel B, counteracting the rising premia from the first channel. For a sufficiently negative price of variance risk, this latter channel dominates, overturning the standard predictions of traditional structural models of the firm. The hump-shape feature of the relationship emerges since the variance beta decays at a faster rate than the asset beta.\textsuperscript{22}

Intuitively, the equity value of a financially distressed firm is largely comprised of the equityholders’ default option. An increase in the volatility raises the value of this default option, thereby positively impacting the equity valuation and generating a positive variance beta. Simply put, since the equityholder’s downside is capped by limited liability, while the upside potential is unlimited, an increase in uncertainty is beneficial. As the firm moves away from financial distress, the default option becomes less important in the equity valuation. Naturally, this causes the variance beta to fall. In fact, the variance beta eventually turns slightly negative due to the risk associated with having to rollover debt.\textsuperscript{23}

Finally, turning to momentum, consider forming a portfolio of financially distressed firms which is long recent winners and short recent losers. It is likely that, on average, the financial health of the winners has improved while that of the losers has deteriorated further. Then, given the hump-shaped pattern documented in Panel C of Figure 1, the equity returns of the winners should be higher than that of the losers. On the other hand, for financially healthy firms, there is little difference in either the asset beta or variance beta between winners and losers. The model is therefore consistent with the empirical evidence provided by Avramov et al. (2007) that the profits of momentum strategies are highly concentrated among a small subset of firms with low credit ratings.

\textsuperscript{22}These results are (slightly) weaker to the extent that the asset risk premium increases, i.e. risk-neutral growth rate declines, with volatility. Variance betas are smaller due to increased discounting of the future and the impact is slightly larger for distressed firms. When calibrated to match the empirical risk-return relationship, however, the results are quantitatively very similar.

\textsuperscript{23}Specifically, debt is a concave function of productivity, reflecting the fact that the probability of default is most sensitive to movements in productivity near the default barrier. Since the flow dividend accruing to equityholders depends on the market value of newly issued debt, it too is concave in firm productivity. This concavity implies that an increase in volatility decreases the expected present value of future dividends.
5.3 Book-to-Market Effects

The default option is just one embedded option in the structure of firms. Firms also have the option to expand installed capital, which the literature usually terms growth options. Growth options generically have asset betas greater than one and thus are usually considered "riskier" than assets-in-place. However, just like the default option, growth options benefit from increases in volatility. Thus firms with a greater share of growth options should have a higher variance beta than firms with few growth options. For a sufficiently negative market price of variance risk, the latter effect will dominate and firms with significant growth potential will earn lower returns than firms with little growth potential. Finally, holding productivity fixed, it is clear that firms with a greater share of growth options will have a lower book-to-market ratio than firms with few growth options. The model thus predicts that firms with low book-to-market ratios can earn lower returns than firms with high book-to-market ratios. This is exactly what the literature on the value premium finds.

To see this, I consider young firms with differing levels of initial installed capital $K^a$. As discussed previously, each young firm has the option to expand installed capital to $K = 1$. Firms with low values of $K^a$ therefore have greater growth potential than firms with high $K^a$. Assuming each firm has the same level of productivity, sorting firms by $K^a$ also sorts firms by book-to-market ratio. The quantitative results are shown in Figure 2. To isolate the impact of the growth options, I show the results for young firms with no debt. Panel A plots the variance beta as a function of the book-to-market ratio, Panel B plots the equity variance risk premium as a function of the book-to-market ratio and the asset variance risk premium $\pi_Y$, and Panel C plots the total equity risk premium as a function of the book-to-market ratio and the asset variance risk premium. As can be seen from Panel C, for a variance risk premium of approximately -3%, the model generates a sizeable spread in the equity risk premia of firms in line with the historical value premium. Once again, Panels A and B illustrate the underlying mechanisms at work. Firms with low book-to-market ratios have higher variance betas and thus more negative variance risk premia.

Finally, note that spreads in the book-to-market ratios can also be generated by differences in productivity. By the same logic as before, due to the hump-shaped relationship between default probabilities and risk premia, the model can generate a value premium among low credit rating firms even if the firms do not differ in their growth options, but instead in their productivities. The financially distressed firm with a higher productivity has a lower book-to-market ratio and a higher return since it is further away from the default boundary. Just as with momentum, this mechanism disappears when considering financially healthy firms. The model is thus consistent with the empirical results of Vassalou and Xing (2004), who document stronger book-to-market effects in highly levered, high default risk
firms. It should also be noted that Campbell et al. (2016) find that growth stocks do indeed have higher volatility betas than value firms and that the price of volatility risk is negative.

6 Debt Pricing

I now turn to the quantitative analysis of the model’s implications for debt pricing. I confine myself to analyzing the debt of mature firms. I first discuss how model is calibrated for debt of different credit ratings. The interest rate, risk-neutral growth rate, asset risk premium, and tax rate are as in Table 2. Recall that these parameters are standard and are in line with previous studies which have documented the credit spread puzzle. Table 3 details those parameters of the model which do vary by credit rating, including leverage ratios, average maturity of debt, and finally asset volatility. Target leverage ratios are from Standard & Poors (1999) and are consistent with both Huang and Huang (2012) and Leland (2004). Average maturities and average asset volatilities are from Schaefer and Strebulaev (2008).

The market price of variance risk, i.e. the constant $A^\delta$, is set to generate the target credit spread on 10-year Baa debt, but then the implied specification for the market price of variance risk is held constant for all other credit ratings and at other maturities. This implies there is only 1 degree of freedom to match the credit spreads across ratings categories and across maturities in the baseline model. I set the current productivity level $X_0 = 7.0588$ such that the current value of assets in place is equal to 100. For each credit rating, I set the total principal $P$ equal to the target leverage ratio multiplied by 100 and then solve for the coupon such that newly issued debt, and the total current value of debt, is priced at par. This will imply that the credit spread can be calculated as the coupon rate $C/P$.

This calibration procedure is standard in studies of the credit spread puzzle. The only difference is that previous studies generally calibrate the asset volatility to force the model to match historical default probabilities, whereas I set the asset volatility according to model-free empirical estimates and then ask if the model is able to jointly generate reasonable credit spreads and default rates by credit rating. The problem with the former approach, as I will demonstrate, is that the implied asset volatilities can be unreasonably high given the empirical estimates, especially at short maturities.

The only difference is that I set the distress costs $\xi = 35\%$ for Caa-rated debt so as to get a recovery rate of approximately 51%.
6.1 Credit Spreads on Intermediate/Long Maturity Debt

I begin my quantitative analysis by examining the credit spreads of 10-year and 20-year maturity debt. The second column of Table 4 reports the target historical credit spreads for 10-year maturity debt by rating category. The targets for investment grade through speculative grade debt (Aaa-Baa) are from Duffee (1998) while the targets for speculative grade through junk debt (Ba-Caa) are from Caouette, Altman, and Narayanan (1998). The questions I seek to answer are twofold. Is it possible to match historical credit spreads on intermediate to long maturity debt with a reasonable variance risk premium and can a single specification for the market price of variance risk explain credit spreads across credit ratings and maturities?

First, consider the performance of the model for 10-year maturity debt in which volatility is constant. The results are reported in the third and fourth columns of Table 4. As is evident, I recover the credit puzzle in this baseline model, especially for investment grade and speculative debt. For all credit ratings between Aaa-Baa, the model is never able to account for more than 30% of the target credit spread, although the performance is increasing as the rating worsens. On the other hand, the table illustrates that there is significantly less of a credit puzzle for junk debt. The baseline model performs much better at these ratings, explaining approximately 61% of the historical B credit spread and 87% of the historical Caa credit spread. This observation highlights one of the key challenges that a model needs to overcome to fully resolve the credit spread puzzle. In other words, it is important not to create a credit spread puzzle in the other direction, whereby the new model is able to better explain the credit spreads of investment grade debt, but then overpredicts the credit spreads of junk debt.

This is essentially what happens in the model incorporating stochastic volatility but in which the default boundary is specified exogenously. As can be seen from columns 5 and 6 of the table, the model is now able to explain a substantially higher proportion of the target credit spread for investment grade and speculative debt than the baseline model. However, the model significantly overpredicts the credit spreads on junk debt. The credit spread on the B-rated debt is overpredicted by 30% and the credit spread on Caa-rated debt is substantially overpredicted by 70%. This is not a particularly compelling resolution of the credit spread puzzle. Asymptotic expansions actually provide closed-form approximate solutions for the case of debt with exogenous default. These results are presented in Appendix A.4.\footnote{While the primary focus of this paper is on debt which allows for endogenous default by equityholders, this result is interesting and useful in its own right from an asset pricing perspective. There are forms of debt in which the exogenous trigger is in fact more appropriate. For instance, the debt may have certain covenants which enforce a zero net worth requirement as discussed in Leland (1994b). Alternatively, default may occur once there is insufficient cash flow to meet the debt servicing obligations and new equity cannot}
Conversely, the model incorporating stochastic volatility in which the default boundary is determined *endogenously* performs much better. The model now only slightly overpredicts the spreads on junk debt. B-rated debt is overpredicted by only 8% and Caa-rated debt by only 15%, a substantial improvement. Not only that, but the endogenous default model outperforms the exogenous default model for investment grade debt as well, especially Aaa and Aa debt. While the exogenous default model accounts for approximately 30% and 54% of the target credit spreads for Aaa and Aa debt respectively, the endogenous default model can explain 40% and 67%. The only rating category that the exogenous default model wins in is Ba-rated debt. It explains 84% of the target credit spread relative to 76% for the endogenous default model, a slight improvement. The calibration yields a parameter of $A^g = -0.2264$, which in turn implies a variance risk premium of -1.1% for an asset volatility of 22% and a premium of -1.77% for an asset volatility of 28%.

Using this value of $A^g$, I next look at the credit spreads of 20-year debt in Table 5. While I unfortunately do not have target credit spreads for junk debt at this maturity, it is clear that there once again exists a significant credit puzzle for investment grade debt. A model without stochastic volatility is unable to explain more than 25% of the historical credit spreads at any rating. Adding stochastic volatility greatly improves the pricing of investment grade debt, with the endogenous default model once again outperforming the exogenous default model for Aaa and Aa debt by a substantial amount. I also report the credit spreads for junk debt and it is apparent that the exogenous default model yet again produces significantly higher credit spreads at these ratings categories than a model with endogenous default.

Finally, given that I do not force the model to match historical default rates, it is important to see what cumulative default probabilities the model is actually generating. In particular, I need to make sure that I am not delivering higher credit spreads by simply overstating the credit risk. The targets are given by the average cumulative issuer-weighted global default rates from 1970-2007 as reported by Moody’s. The model is quite successful at matching long-maturity historical default rates for A, Baa and B rated debt as Table 6 demonstrates. It underpredicts the default probabilities in the Ba category somewhat and substantially so in the Aaa and Aa categories. Note, though, that the model-generated default probabilities for Aaa and Aa debt are much closer to reported historical rates from 1983-2007.\(^{26}\)

---

\(^{26}\)Moody’s reports a global default rate of only 0.19% for Aaa debt between the years 1983-2007 at all maturities of 8 years and above. The rates for Aa-rated debt are also substantially lower in this period than between 1970-2007.
6.2 Intuition

These quantitative results naturally lead to two essential questions. How does the inclusion of stochastic volatility increase credit spreads and why is endogenous default important in matching credit spreads across ratings categories? The answer to the first question is intuitive. The negative market price of volatility risk indicates that investors see an increase in volatility as a deterioration in the investment opportunity set. Therefore, investors require a premium in the form of higher expected returns to hold assets which do poorly when volatility increases. In general, debt is such an asset since an increase in volatility raises the probability of default which lowers the value of the claim. Consequently, the discount rates on debt should be higher in a structural credit model with stochastic volatility and a negative market price of volatility risk than in a specification in which volatility is constant. These higher discount rates then lead to lower debt prices and higher credit spreads. Indeed, Campbell et al. (2016) show that lower-rated debt has more negative variance betas and then higher rated debt.

To see this intuition more explicitly, I apply the Feynman-Kac formula to the boundary value problem defining the first-order correction for debt to derive a probabilistic interpretation. This gives:

\[
\sqrt{\delta \tilde{d}_1} (x) = E^*_t \left[ \int_t^{\tau^B (y)} e^{-(r+m)(s-t)} A^y \frac{\partial \tilde{d}_1}{\partial y} ds + e^{-(r+m)(\tau^B (y)-t)} \sqrt{\delta} \mathcal{X}_{B,1} \left\{ mU' (X_{\tau^B (y)}) - \frac{\partial \tilde{d}_1}{\partial x} (X_{\tau^B (y)}) \right\} \right],
\]

where the expectation is taken over the process \( dX_s = gX_s + \sqrt{\gamma} X_s dW_s \) with \( X_t = x \) and the stopping time defined by:

\[
\tau^B (y) = \min \left\{ s : X_s = x_{B,0}^y \right\}.
\]

In words, the correction term is an average discounted present value of comparative statics in the constant volatility model over all possible sample paths of productivity given an initial value of \( x \), taking into account the stopping time and in which the process for productivity is a geometric Brownian motion with drift \( g \) and constant volatility \( \sqrt{\gamma} \). At the stopping time, the payoff is the correction to the debt value at the default boundary. The fact that the averaging holds volatility constant reflects the assumption of a slow-moving variance process. The first term in the expression captures the intuition described above. Recall that \( A^y \) is the variance risk premium and \( \partial \tilde{d}_1 / \partial y \) is the debt vega in the constant volatility model. So
the correction term takes into account the extent to which debt in the constant volatility model covaries with variance, prices this risk according to the variance risk premium, and averages over all possible sample paths for productivity going forward. Since the vega should in general be negative for debt and the variance risk premium is negative, this constitutes a negative contribution to the correction term, raising credit spreads beyond the constant volatility baseline model.

The mechanisms underlying the improved performance of the endogenous default model are more subtle. There are two driving forces. First, while the debt vega is indeed usually negative in the constant volatility model, it is actually positive when the firm is very close to default if the barrier is chosen endogenously. This does not occur in the exogenous default model, as shown in Figure 3. When volatility increases, the option value of the equityholders increase and they respond, if able to, by postponing default. That is, \( x_{B,0}^{y} \) is a decreasing function of \( y \). Increasing volatility therefore has two effects on the value of debt. The increased riskiness raises the probability of default directly, but the shifting boundary lowers it indirectly. Near the default barrier, the latter effect dominates and the increased volatility actually increases the value of debt. This can be seen mathematically. Differentiating equations (34a)-(34c) gives the problem to solve for the debt vega of the principal order term in the endogenous default model:

\[
L_{r+m}^y \frac{\partial d_0^y}{\partial y} + \frac{1}{2} x^2 \frac{\partial^2 d_0^y}{\partial y^2} = 0 
\]

\[
\frac{\partial d_0^y}{\partial y} (x_{B,0}^{y}) = - \frac{\partial d_0^y}{\partial x} (x_{B,0}^{y}) \frac{dx_{B,0}^{y}}{dy} 
\]

\[
\lim_{x \to \infty} \frac{\partial d_0^y}{\partial y} (x) = 0 
\]

Since the value of debt is increasing at the default boundary and the boundary is decreasing with volatility, the debt vega at the boundary is positive. The result follows by continuity.

The second effect comes from the fact that in the probabilistic representation of the debt correction term, the payoff at the stopping time is positive. That is,

\[
\sqrt{\delta x_{B,1}^y} \left\{ mU' (x_{B,0}^{y}) - \frac{\partial d_0^y}{\partial x} (x_{B,0}^{y}) \right\} > 0.
\]

Intuitively, even the prospect of future movements in volatility increases the value of the equityholders’ default option, leading to a lower default boundary than in the constant volatility case. Debtholders benefit from this because it lowers the probability of default.
Technically, the term in brackets in equation (49) is negative. To see this, note that:

\[
\frac{\partial \tilde{d}_0^y}{\partial x} (x_{B,0}^y) = \frac{\partial V_0^y}{\partial x} (x_{B,0}^y) - \frac{\partial E_0^y}{\partial x} (x_{B,0}^y) \\
= \frac{\partial V_0^y}{\partial x} (x_{B,0}^y) \\
= U' (x_{B,0}^y) - \gamma_2 \left[ \xi U (x_{B,0}^y) + \frac{\phi C}{r} \right] \\
\geq U' (x_{B,0}^y),
\]

where \( V_0^y \) is the principal order term of total shareholder value as defined in Appendix A. The second inequality follows from smooth-pasting at the default boundary in the constant volatility model, the third inequality follows from differentiating the expression provided in Appendix A, and the final inequality follows from \( \gamma_2 < 0 \).

Crucially, note that these two effects are more significant in the pricing of junk debt than investment grade debt. When considering investment grade debt, default is far away and so most sample paths from the initial point will not hit the default boundary or the region near it for an extended period of time. As such, these two effects, which only occur near or at the default boundary, are heavily discounted among most sample paths. Thus they do not contribute much to the pricing of the correction term for investment grade debt, leading to similar pricing in both the endogenous and exogenous default models given a particular variance risk premium. On the other hand, it is exactly junk debt that is at risk of moving into the default region in the near future. Consequently, these effects are not discounted heavily in many of the sample paths from the initial point. Since these effects contribute positively to the value of debt, this indicates that the exogenous default model can significantly overpredict credit spreads.

It is now apparent why the endogenous default model is more successful. Since the variance risk premium is calibrated to the speculative grade Baa-rated credit spread, the implied variance risk premium is higher in the endogenous default model than the exogenous default model. However, the effects highlighted above are even stronger for junk debt than speculative grade debt. Therefore, the endogenous default model generates lower credit spreads at these ratings categories. On the other hand, since the effects are quite weak for investment grade debt and the implied variance risk premium is higher in the endogenous default model, it generates higher credit spreads at these categories as desired.
7 Short Maturity Debt and Extensions

Having quantitatively analyzed long maturity debt and discussed the mechanisms underlying the results, I now turn to a study of short maturity debt. As Table 7 illustrates, the credit spread puzzle is still present at short maturities and has familiar features. A model with constant volatility is unable to explain more than a third of the historical credit spread for Aaa-Ba debt, but accounts for a significantly greater fraction of junk credit spreads. Moreover, while the model with stochastic volatility and endogenous default certainly does improve on this baseline by a substantial amount, the performance is not as good as for long maturity debt. The model is never able to account for more than two-thirds of the historical credit spreads on investment and speculative grade debt. The performance for Aaa and Aa debt is particularly disappointing, with the model only accounting for 4.4% and 25.0% respectively of observed credit spreads.

One reason for this is that the effects of stochastic volatility are weaker at short maturities. In other words, the first-order correction term for total debt is smaller. This is because the vega of the principal order term is declining with maturity, as shown in Figure 4. Loosely stated, since productivity follows a diffusion, it can only move so far within a short period of time. Thus, increases in volatility do not significantly raise the riskiness of the firm. Since debt is not as sensitive to volatility fluctuations at short maturity, the discount rate correction in the stochastic volatility model is not as large.

This is not the whole story, however. Examining Table 8 indicates that the model is underpredicting the cumulative default probabilities of short-maturity Aaa-Baa debt. That is, the model is not only underpredicting credits spreads, but also credit risk. A key question, therefore, is how well the model could match historical prices if it more accurately reflected empirical default frequencies.

The approach of Huang and Huang (2012) and other studies to this question has been to set the asset volatility to match this moment. As discussed, though, this method is somewhat unsatisfactory since the implied asset volatility is then significantly higher than model-free empirical estimates. A more appropriate approach is to ask whether the model as currently structured is missing some element of realism which, if included, could increase the credit risk of short maturity debt. One possibility would be to include negative correlation between volatility shocks and productivity shocks, i.e. set $\rho_Y < 0$. This does not work well though, since under the assumption of slow-moving volatility, the skewness effects are weak, especially at short maturities. Instead, I consider an extension to the baseline model allowing for rare disasters in the firm productivity process.
7.1 Rare Disasters

I allow for jumps in the firm productivity process as follows:

\[
\frac{dX_t}{X_t} = \mu_t dt + \sqrt{Y_t} dW^{(1)}_t + d \left( \sum_{i=1}^{N(t)} (Q_i - 1) \right)
\]

\[
dY_t = \delta \kappa_Y (\theta - Y_t) + \nu_Y \sqrt{Y_t} dW^{(2)}_t.
\]

where \( N(t) \) is a Poisson process with rate \( \lambda > 0 \) and \( \{Q_i\} \) is a sequence of independent, identically distributed random variables which take values between zero and one. Note that a jump in this model always corresponds to a negative event, although this could easily be modified. Jumps have been included in structural modeling previously by Hilberink and Rogers (2002) and Chen and Kou (2009). My contribution is to include jumps, stochastic volatility, and endogenous default in a unified model and to describe a tractable method for solving it.

I will assume that the jump risk premium is zero to maintain my focus on increasing the credit spreads of short maturity debt.\(^{27}\) Then, the risk-neutral dynamics are given by

\[
\frac{dX_t}{X_t} = g dt + \sqrt{Y_t} dW^{(1)*}_t + d \left( \sum_{i=1}^{N(t)} (Q_i - 1) \right)
\]

\[
dY_t = \left( \kappa_Y (\theta - Y_t) - \Gamma (Y_t) \nu_Y \sqrt{Y_t} \right) dt + \nu_Y \sqrt{Y_t} dW^{(2)*}_t.
\]

In the online appendix, I provide the free boundary problem which characterizes the equity valuation. The key innovation is that the equity value must now satisfy an integro-partial differential equation, instead of a partial differential equation, to account for the jumps. The value of newly issued debt also satisfies an integro-partial differential equation.

The key assumption of the model is that the jumps are rare, i.e. \( \lambda > 0 \) is small. This allows me to utilize regular perturbation and expand the contingent claims and default boundary in powers of \( \sqrt{\delta} \) and \( \lambda \):

\[
E_{\delta,\lambda} (x, y, z) = E^y_0 (x) + \sqrt{\delta} E^y_{1,0} (x) + \lambda E^y_{0,1} (x) + \sqrt{\delta} \lambda E^y_{1,1} (x) + \cdots
\]

\[
\tilde{d}_{\delta,\lambda} (x, y, z) = \tilde{d}^y_0 (x) + \sqrt{\delta} \tilde{d}^y_{1,0} (x) + \lambda \tilde{d}^y_{0,1} (x) + \sqrt{\delta} \lambda \tilde{d}^y_{1,1} (x) + \cdots
\]

\[
x_B (y) = x^y_{B,0} + \sqrt{\delta} x^y_{B,1,0} + \lambda x^y_{B,0,1} + \cdots
\]

\(^{27}\)One can think about these jumps as being purely idiosyncratic to the firm.
As should be familiar by now, the principal order terms will reflect the model with constant volatility and no jumps. The first-order correction terms accounting for the slow-moving stochastic volatility will also be the same as before. The equity correction term accounting for the jumps is:

\[ \mathcal{L}_y^0 \lambda E_{0,1}^y = - \left( \lambda \int_0^1 [E_0^y(yx) - E_0^y(x)] f_Q(y) dy - \lambda \tilde{d}_{0,1}^{y,z} \right), \quad \text{for } x > x_{B,0}^{y,z} \]  

(57)

\[ \lambda E_{0,1}^{y,z} (x_{B,0}) = 0 \]  

(58)

\[ \lim_{x \to \infty} \lambda E_{0,1}^y (x) = 0, \]  

(59)

where \( f_Q(y) \) is the density of the jump size distribution. A similar correction term is derived for debt and, together with the corrections to the default boundary, form a system of equations.

7.1.1 Quantitative Results and Discussion

To calibrate the model, I set \( \lambda = .001 \) such that the probability of a disaster event in any given year is 1%. I assume the jump size distribution to be uniform over an interval \([Q_{\min}, Q_{\max}] \subset [0,1]\). Note that this implies the firm will lose between \( Q_{\min} \) and \( Q_{\max} \) percent of its asset value in the event of a jump. For parsimony, Table 9 displays the credit spreads for only Baa-rated debt, allowing for jump size intervals of \([.50,.75], [.25,.75], \) and \([.25,.50] \). Clearly, as the magnitudes of the jumps increase, the credit spreads rise as well.

This model also achieves the goal of proportionally increasing short-maturity credit spreads more than long-maturity credit spreads. There is a 30.4% increase in the 20-year credit spread and 30.7% increase in the 10-year credit spread as one moves from a model with no jumps to a model with \( \lambda = .001 \) and \( Q \sim U ([.25,50]) \). Credit spreads on 4-year debt increase by 45.3%, while those on 1-year debt increase by 135.7%.

The reason behind this pattern is that credit risk is increased more proportionally at shorter maturities. The intuition is that the model allows for jumps to default. Since it is difficult for pure diffusions to reach the default barrier within a short period of time, the majority of the default probability for especially short maturity debt comes from this risk. This leads to large proportional increases in credit risk as one moves beyond a pure diffusion model. However, at longer maturities it is possible to both diffuse to default as well as jump to default. Therefore, the inclusion of jump risk does not proportionally increase credit risk as much.
8 Conclusion

This paper has constructed a real-options, term structure model of the firm incorporating stochastic volatility. Shocks to variance carry a negative price of risk in the market to reflect the view of long-term investors that a persistent increase in volatility constitutes a deterioration in the investment opportunity set. The stocks of financially distressed firms are long the default option and hedge against volatility risk in the market, thus requiring lower variance risk premia than healthy firms. The model replicates the hump-shaped relationship between default probabilities and equity returns documented by Garlappi et al. (2008). The model also speaks to the risk-adjusted profitability of momentum trading strategies and explains why they are concentrated among the stocks of low credit-rating firms. The model generates a value premium and is consistent with empirical evidence that book-to-market effects are stronger among low credit rating firms.

The model further explains that firm debt is short the option to default. As a result, if volatility is stochastic and shocks are negatively priced, discount rates are higher than in a model with constant volatility. This effect lowers prices and raises credit spreads. However, it is the interaction of stochastic volatility and endogenous default which fully resolves the credit spread puzzle. In a setting with endogenous default, extremely distressed junk debt hedges volatility risk in the market and the default threshold is lower at all levels of volatility than in the model with constant volatility. This prevents the model from overpredicting the credit spreads on junk debt and also improves the performance of the model at higher ratings categories.

The paper expands on work in the mathematical finance literature by developing new tools for the solution of multidimensional optimal control and optimal stopping problems in economics. Principal order terms in the asymptotic expansion reflect contingent claims valuations in the constant volatility setting, while correction terms are the solutions to ODEs involving key comparative statics of the constant volatility model. The approach yields tractability, reduces the number of parameters which need to calibrated, and provides clean mathematical expressions to guide understanding of the mechanisms at work in the model.

Finally, the paper considers an extension to the baseline model featuring rare disasters, once again bringing to bear the tools of asymptotic expansions. Rare disasters allow the model to increase credit risk at short maturities and improve the model’s performance at matching empirical default frequencies and credit spreads at short maturities.
9 References


A Derivations and Proofs

A.1 Proof of Theorem 1

I prove Theorem 1 in a sequence of lemmas. Let
\[ \text{DIV}(x, y) = (1 - \phi)(x - C) + \bar{d}(x, y) - p \]  
(60)
denote the flow dividend which accrues to the equityholders and set:
\[ E^*(x, y) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_t^* \left[ \int_t^\tau e^{-r(s-t)} \text{DIV}(X_s, Y_s) \, ds \right]. \]  
(61)
The goal is to show that the solution \( E(x, y) \) to equations (17a)-(17h) satisfies \( E(x, y) = E^*(x, y) \) for all \( x, y > 0 \). Let \( C = \{(x, y) : x > x_B(y)\} \) denote the continuation set and \( \mathcal{D} = \{(x, y) : x \leq x_B(y)\} \) the stopping set. Finally, define the stopping time:
\[ \tau_B = \min\{t \geq 0 : (X_t, Y_t)\} \in \mathcal{D}. \]  
(62)
The first result is as follows:

**Lemma 3** The flow dividend to equityholders must be nonpositive in the stopping region; that is, \( \text{DIV}(x, y) \leq 0 \) for all \( (x, y) \in \mathcal{D} \).

**Proof.** The proof is very simple. Recalling equation (17b), we must have:
\[ \text{DIV}(x, y) + \mathcal{L}_{X,Y}E \leq rE \]
for all \((x, y) \in \mathcal{D}\). But equation (17c) says that \( E(x, y) = 0 \) for all \((x, y) \in \mathcal{D}\). Substituting this into the equation above immediately gives the desired result. ■

This result is intuitive. Since equityholders receive nothing in the event of default, it can never be optimal for them to default when they are still receiving positive dividends. Now define:
\[ Z_t = e^{-rt}E(X_t, Y_t) + \int_0^t e^{-rs} \text{DIV}(X_s, Y_s) \, ds. \]  
(63)

**Lemma 4** The process \( Z_t \) satisfies:
\[ \int_0^t e^{-rs} \text{DIV}(X_s, Y_s) \, ds \leq Z_t \leq Z_0 + M_t \]
where \( M_t \) is a continuous local martingale. The stopped process \( Z_{t \wedge \tau_B} \) satisfies:
\[ Z_{t \wedge \tau_B} = Z_0 + M_{t \wedge \tau_B}. \]  
(65)

**Proof.** The first inequality in equation (64) follows directly from the fact that \( E(x, y) \geq 0 \) for all \( x, y \). For the second inequality, note that the Itô-Doeblin formula applies to functions whose second derivatives are discontinuous on measure zero sets as long as the
first-derivatives are everywhere continuous, which itself is a consequence of smooth-pasting. Therefore, under the risk-neutral measure $\mathbb{P}^*$:

$$dZ_t = e^{-rt} \left\{ -rZ_t + \mathcal{L}_{X,Y}Z_t + DIV_t \right\} dt + \sqrt{Y_t} X_t \frac{\partial E}{\partial x} dW_t^{(1)*} + \nu Y_t \sqrt{Y_t} \frac{\partial E}{\partial y} dW_t^{(2)*} \right\}.$$ (66)

By equation (17a), this is:

$$dZ_t = e^{-rt} DIV_t I[X_t \leq x_B(Y_t)] dt + e^{-rt} \left\{ \sqrt{Y_t} X_t \frac{\partial E}{\partial x} dW_t^{(1)*} + \nu Y_t \sqrt{Y_t} \frac{\partial E}{\partial y} dW_t^{(2)*} \right\}.$$ (67)

However, by the previous lemma I know that $DIV_t \leq 0$ for all $X_t \leq x_B(Y_t)$, which implies that:

$$Z_t \leq Z_0 + M_t$$ (68)

where:

$$M_t = \int_0^t e^{-rs} \sqrt{Y_s} X_s \frac{\partial E}{\partial x} dW_s^{(1)*} + \int_0^t e^{-rs} \nu Y_s \sqrt{Y_s} \frac{\partial E}{\partial y} dW_s^{(2)*}$$ (69)

is a local continuous martingale by property of the Ito integral.

For the stopped process $Z_{t \wedge \tau^B}$, it is the case that $X_t > x_B(Y_t)$ for all $t < \tau^B$, which means that the indicator function in the drift term is equal to zero. Therefore:

$$Z_{t \wedge \tau^B} = Z_0 + M_{t \wedge \tau^B}$$ (70)

as desired. $lacksquare$

To complete the proof, I finally show:

**Lemma 5** The solution $E(x,y)$ to equations (17a)-(17h) satisfies $E(x,y) = E^*(x,y)$

**Proof.** Let $X_0 = x$ and $Y_0 = y$. Then for every stopping time $\tau$ and $n \in \mathbb{N}$:

$$\int_0^{\tau \wedge n} e^{-rs} DIV (X_s, Y_s) ds \leq E(x,y) + M_{\tau \wedge n}.$$ (71)

By the optional sampling theorem, $\mathbb{E}_0^* [M_{\tau \wedge n}] = 0$, so that:

$$\mathbb{E}_0^* \left[ \int_0^{\tau \wedge n} e^{-rs} DIV (X_s, Y_s) ds \right] \leq E(x,y).$$ (72)

Taking limits as $n \to \infty$ and applying Fatou’s lemma yields:

$$\mathbb{E}_0^* \left[ \int_0^{\tau} e^{-rs} DIV (X_s, Y_s) ds \right] \leq E(x,y).$$ (73)

Taking the supremum over all stopping times gives $E^*(x,y) \leq E(x,y)$. 

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For the other direction, by considering the stopped process $Z_t \wedge T^B$ and once again using the optional sampling theorem, I have that:

$$E \left[ e^{-r(T^B \wedge n)} E(X_t \wedge T^B, Y_t \wedge T^B) + \int_{0}^{T^B \wedge n} e^{-rs} \text{DIV}(X_s, Y_s) \, ds \right] = E(x, y). \quad (74)$$

Taking limits as $n \to \infty$ and noting that $e^{-rT^B} E(X_T, Y_T) = 0$ gives:

$$E(x, y) = \mathbb{E}_0^* \left[ \int_{0}^{T^B} e^{-rs} \text{DIV}(X_s, Y_s) \, ds \right]. \quad (75)$$

This show that $\tau^B$ is optimal and that $E(x, y) = E^*(x, y)$ for all $x, y > 0$ as desired. ■

A.2 Derivation of Equation (37)

The problem for the principal order equity term is given by:

$$\begin{align*}
(1 - \phi) (x - C) + \tilde{d}_0^y (x) - p + \mathcal{L}_r^y E_0^y &= 0 \\
E_0^y (x_{B,0}^y) &= 0 \quad \text{for } x > x_{B,0}^y \quad (76a) \\
\lim_{x \to -\infty} E_0^y (x) &= U(x) - \frac{C + mP}{r + m} + \frac{\phi C}{r} \quad (76b) \\
\frac{\partial E_0^y}{\partial x} \bigg|_{x = x_{B,0}^y} &= 0 \quad (76c)
\end{align*}$$

To solve this problem I will introduce $V_0^y$ as the solution to:

$$\begin{align*}
(1 - \phi) x + \phi C + \mathcal{L}_r^y V_0^y &= 0 \\
V_0^y (x_{B,0}^y) &= (1 - \xi) U(x_{B,0}^y) \quad (77a) \\
\lim_{x \to -\infty} V_0^y (x) &= U(x) + \frac{\phi C}{r} \quad (77b) \\
\end{align*}$$

Note that $V_0^y$ is value of debt plus equity in the case of constant volatility. It is not equal to total firm value since the government is a residual claimant on a portion of the firm’s cash flows. I now prove the following lemma:

**Lemma 6** The principal equity value term $E_0^y (x) = V_0^y (x) - \tilde{d}_0^y (x) / m$.

**Proof.** Recalling that $P = p/m$ and $C = c/m$, it is straightforward to check that:

$$\begin{align*}
V_0^y (x_{B,0}^y) - \tilde{d}_0^y (x_{B,0}^y) / m &= 0 \quad (78) \\
\lim_{x \to -\infty} V_0^y (x) - \tilde{d}_0^y (x) / m &= U(x) - \frac{C + mP}{r + m} + \frac{\phi C}{r} \quad (79)
\end{align*}$$
It therefore remains to check that \( V_0^y (x) - \tilde{d}_0^y (x) / m \) satisfies the appropriate differential equation. To this end:

\[
(1 - \phi) (x - C) + \tilde{d}_0^y (x) - p + \mathcal{L}_r^y \left( V_0^y - \tilde{d}_0^y / m \right) \\
= - C + \tilde{d}_0^y - p - (1/m) \mathcal{L}_r^y \tilde{d}_0^y \\
= - C + \tilde{d}_0^y (x) - p + (1/m) \left( c + mp - m\tilde{d}_0^y (x) \right) \\
= 0,
\]

(80)

where here I used the fact that \( c + mp + \mathcal{L}_{r+m} \tilde{d}_0^y = 0 \). This completes the proof. ■

Standard ODE techniques for the Cauchy-Euler equation give:

\[
V_0^y (x) = U (x) + \frac{\phi C}{r} \left[ \xi U (x_{B,0}^y) + \frac{\phi C}{r} \right] \left( \frac{x}{x_{B,0}^y} \right)^{\gamma_2},
\]

(81)

where \( \gamma_2 \) is the negative root of the following polynomial equation:

\[
g \gamma_2 + \frac{1}{2} y \gamma_2 (\gamma_2 - 1) - r = 0.
\]

(82)

The expression for \( \tilde{d}_0^y (x) \) is provided in equation (35). Computing \( V_0^y (x) - \tilde{d}_0^y (x) / m \) then gives equation (37). Differentiating this equation with respect to \( x \) and evaluating at \( x_{B,0}^y \) to solve the smooth-pasting condition gives the endogenous default boundary (38).

### A.3 Cumulative Default Probabilities

To calculate cumulative default probabilities under stochastic volatility, I find a suitable PDE characterization of the probability and then utilize asymptotic expansions to simplify the solution of the problem. Letting \( u (l, x, y) \) denote the cumulative survival probability within \( l \) years, by the backwards Kolmogorov equation:

\[
\left( - \frac{\partial}{\partial l} + \mathcal{L}_0 + \sqrt{\delta \nu \rho y x} \frac{\partial^2}{\partial x \partial y} + \delta M_2^y \right) u = 0 \tag{83a}
\]

\[
u (l, x_B (y), y) = 0 \tag{83b}
\]

\[
\lim_{x \to -\infty} u (l, x, y) = 1 \tag{83c}
\]

\[
u (0, x, y) = 1, \tag{83d}
\]

where \( x_B (y) \) is the optimal default boundary from the equityholders’ problem.\(^{28}\) Importantly, note that the cumulative survival probabilities are calculated with respect to the dynamics of \( X_t \) and \( Y_t \) under the physical measure rather than the risk-neutral measure.

\(^{28}\) Recall that \( \mathcal{L}_0 \) is the Black-Scholes operator with a riskfree rate set equal to zero. Cumulative default probabilities are simply given by \( 1 - u (l, x, y) \).
Now expand this survival probability in powers of $\sqrt{\delta}$:

$$u(l, x, y) = u_0^y(l, x) + \sqrt{\delta} u_1^y(l, x) + \delta u_2^y(l, x) + \cdots,$$  \hspace{1cm} (84)

substitute into equations (83a)-(83d), and expand the boundary conditions using Taylor expansions. Once again, the principal order term reflects the cumulative survival probability in the limiting case where volatility is fixed at the current level. I do not reproduce the expression here, but it can be looked up in any standard textbook treatment on the hitting times of geometric Brownian motion.

The correction term is an inhomogeneous partial differential equation with $y$ as a parameter:

$$\left(-\frac{\partial}{\partial l} + \mathcal{L}_0\right) \sqrt{\delta} u_1^y(l, x, y) = -B^y_{yx} \frac{\partial^2 u_0^y}{\partial x \partial y}$$  \hspace{1cm} (85a)

$$\sqrt{\delta} u_1^y(l, x, y) = -\sqrt{\delta} x_{B,0}^y \frac{\partial u_0^y}{\partial x} (l, x_{B,0}^y)$$  \hspace{1cm} (85b)

$$\lim_{x \to \infty} \sqrt{\delta} u_1^y(l, x, y) = 0$$  \hspace{1cm} (85c)

$$\sqrt{\delta} u_1^y(0, x, y) = 0.$$  \hspace{1cm} (85d)

As expected, the source term is a function of a comparative statics of the survival probability in the constant volatility case; however, contrary to the expressions above, the vega of the principal order term and the variance risk premium do not appear in equation (85a). The correction term can be calculated in MATLAB using the function pdepe, which implements a finite-difference scheme.

A.4 Debt Valuation with Exogenous Default Boundary

Let the exogenous boundary be given by $\pi_B$. A perturbation approach can be used in a similar fashion as above to determine the approximate value of debt in the stochastic volatility model. As should be familiar by now, the principal order term reflects the constant volatility case and is simply given by equation (35) with $x_{B,0}^y$ replaced by $\pi_B$. It turns out that in the case of $\rho_y = 0$, a relatively simple explicit expression can be derived for the first-order correction term.

**Theorem 7** If the default boundary is set exogenously at $\pi_B$ and the correlation between productivity shocks and volatility shocks is equal to zero, then the first-order correction term in the asymptotic expansion of newly issued debt is given by

$$\sqrt{\delta} d_1^y(x) = \frac{\ln (x/\pi_B)}{2 \left(g + \gamma_1 y - \frac{1}{2} y^2 \right)} A^y \left[ \frac{\partial \tilde{d}_0^y}{\partial y} + \frac{1}{2 \left(g + \gamma_1 y - \frac{1}{2} y^2 \right)^2} y x^2 \frac{\partial^2 \tilde{d}_0^y}{\partial y^2} \right],$$  \hspace{1cm} (86)

where $\gamma_1$ is the negative root of equation (36).

Thus, the first-order correction term involves both the vega and gamma of the value of newly issued debt in the constant volatility model. These are provided explicitly in
Appendix A. In fact, an explicit expression can be computed for the case of $\rho_Y \neq 0$ as well, but it and its derivation are particularly cumbersome. Moreover, it is not needed in my analysis and I therefore omit it. The method of derivation, however, follow closely to the proof of the above result. I begin with a preliminary useful lemma.

**Lemma 8** The following identity holds:

\[
\mathcal{L}^{y}_{r+m} \left\{ \frac{1}{g + \gamma_1 y - \frac{1}{2} y} \ln \left( \frac{x}{\overline{\tau}_B} \right) \left( \frac{x}{\overline{\tau}_B} \right)^{\gamma_1} \right\} = \left( \frac{x}{\overline{\tau}_B} \right)^{\gamma_1}
\]  

(87)

**Proof.** By direct computation, I show that:

\[
\mathcal{L}^{y}_{r+m} \left\{ \frac{1}{g + \gamma_1 y - \frac{1}{2} y} \ln \left( \frac{x}{\overline{\tau}_B} \right) \left( \frac{x}{\overline{\tau}_B} \right)^{\gamma_1} \right\} = \frac{1}{g + \gamma_1 y - \frac{1}{2} y} \left\{ g \left( \frac{x}{\overline{\tau}_B} \right)^{\gamma_1} \left[ 1 + \gamma_1 \ln \left( \frac{x}{\overline{\tau}_B} \right) \right] 
+ \frac{1}{2} y \left( \frac{x}{\overline{\tau}_B} \right)^{\gamma_1} \left[ 2 \gamma_1 - 1 + \gamma_1 \left( \gamma_1 - 1 \right) \ln \left( \frac{x}{\overline{\tau}_B} \right) \right] \right\} 
- \left( r + m \right) \left( \frac{x}{\overline{\tau}_B} \right)^{\gamma_1} \ln \left( \frac{x}{\overline{\tau}_B} \right) 
\]

\[
= \left( \frac{x}{\overline{\tau}_B} \right)^{\gamma_1} \left[ \frac{g + \gamma_1 y - \frac{1}{2} y}{g + \gamma_1 y - \frac{1}{2} y} + \ln \left( \frac{x}{\overline{\tau}_B} \right) g \gamma_1 + \frac{1}{2} g \gamma_1 (\gamma_1 - 1) - \left( r + m \right) \left( \frac{x}{\overline{\tau}_B} \right)^{\gamma_1} \right] 
\]

\[
= \left( \frac{x}{\overline{\tau}_B} \right)^{\gamma_1}
\]

due the definition of $\gamma_1$. ■

I next compute an explicit expression for the gamma and vega of the principal order term, that is the vega in the constant volatility model with exogenous default boundary.

**Lemma 9** The gamma of the debt principal order term in the exogenous default model is given by:

\[
\frac{\partial^2 \tilde{d}_0}{\partial x^2} = \left[ m \left( 1 - \xi \right) U \left( \overline{\tau}_B \right) - \frac{c + mp}{r + m} \right] \gamma_1 (\gamma_1 - 1) \left\{ \frac{x}{\overline{\tau}_B} \right\}^{\gamma_1 - 2}.
\]  

(88)

The vega of the principal order term is given by:

\[
\frac{\partial \tilde{d}_0}{\partial y} = - \left[ m \left( 1 - \xi \right) U \left( \overline{\tau}_B \right) - \frac{c + mp}{r + m} \right] \frac{\gamma_1 (\gamma_1 - 1)}{2 \left( g + \gamma_1 y - \frac{1}{2} y \right)} \ln \left( \frac{x}{\overline{\tau}_B} \right) \left( \frac{x}{\overline{\tau}_B} \right)^{\gamma_1}.
\]  

(89)

**Proof.** The gamma of the principal order term can be calculated directly by differentiating equation (35) with respect to $x$. By the symmetry of partial derivatives, If I differentiate the ODE and boundary condition for $\tilde{d}_0$ with respect to $y$, I get the following problem to solve for vega:

\[
\mathcal{L}^{y}_{r+m} \frac{\partial \tilde{d}_0}{\partial y} + \frac{1}{2} x^2 \frac{\partial^2 \tilde{d}_0}{\partial y^2} = 0
\]  

(90a)

\[
\frac{\partial \tilde{d}_0}{\partial y} \left( \overline{\tau}_B \right) = 0
\]  

(90b)
Since the gamma remains bounded as $x \to \bar{x}_B$, equation (89) clearly satisfies the boundary condition. It remains to check that it satisfies the differential equation. This is a consequence of the lemma above. Define:

$$A(\bar{x}_B) = m (1 - \xi) U(\bar{x}_B) - \frac{c + mp}{r + m}.$$  \hspace{1cm} (91)

Then I can show:

$$\mathcal{L}^y_{r+m} \left\{ -A(\bar{x}_B) \frac{\gamma_1 (\gamma_1 - 1)}{2 \left(g + \gamma_1 y - \frac{1}{2} y\right)} \ln \left( \frac{x}{\bar{x}_B} \right) \left( \frac{x}{\bar{x}_B} \right)^\gamma_1 \right\}$$

$$= -\frac{1}{2} A(\bar{x}_B) \gamma_1 (\gamma_1 - 1) \mathcal{L}^y_{r+m} \frac{1}{g + \gamma_1 y - \frac{1}{2} y} \ln \left( \frac{x}{\bar{x}_B} \right) \left( \frac{x}{\bar{x}_B} \right)^\gamma_1$$

$$= -\frac{1}{2} A(\bar{x}_B) \gamma_1 (\gamma_1 - 1) \left( \frac{x}{\bar{x}_B} \right)^\gamma_1$$

$$= -\frac{1}{2} x^2 \frac{\partial^2 \bar{y}}{\partial y^2},$$  \hspace{1cm} (92)

as desired. \hspace{1cm} □

Note that I could have simply calculated the vega of the principal order term directly through differentiation, recognizing that:

$$\frac{\partial}{\partial y} \left( \frac{x}{\bar{x}_B} \right)^\gamma_1 = \left( \frac{x}{\bar{x}_B} \right)^\gamma_1 \ln \left( \frac{x}{\bar{x}_B} \right) \frac{d\gamma_1}{dy}$$

$$= -\frac{1}{2} \gamma_1 (\gamma_1 - 1) \frac{1}{g + \gamma_1 y - \frac{1}{2} y} \ln \left( \frac{x}{\bar{x}_B} \right) \left( \frac{x}{\bar{x}_B} \right)^\gamma_1,$$  \hspace{1cm} (94)

where $d\gamma_1/\partial y$ is computed through implicit differentiation of equation (36). However, the proof given above illustrates the usefulness of the first lemma. I can now prove the theorem above.

**Proof.** I will break the calculation into parts. First, note that:

$$\mathcal{L}^y_{r+m} \frac{\ln \left( \frac{x}{\bar{x}_B} \right)}{2 \left(g + \gamma_1 y - \frac{1}{2} y\right)} A^\delta y \frac{\partial \bar{y}}{\partial y}$$

$$= -\frac{1}{2} A(\bar{x}_B) \frac{\gamma_1 (\gamma_1 - 1)}{2 \left(g + \gamma_1 y - \frac{1}{2} y\right)} A^\delta y \mathcal{L}^y_{r+m} \ln \left( \frac{x}{\bar{x}_B} \right)^2 \left( \frac{x}{\bar{x}_B} \right)^\gamma_1$$  \hspace{1cm} (95)
Then:

\[
\mathcal{L}^y_{r+m} \ln \left( \frac{x}{\bar{x}_B} \right)^2 \left( \frac{x}{\bar{x}_B} \right)^{\gamma_1} = g \left( \frac{x}{\bar{x}_B} \right)^{\gamma_1} \left[ 2 \ln \left( \frac{x}{\bar{x}_B} \right) + \gamma_1 \ln \left( \frac{x}{\bar{x}_B} \right)^2 \right] \\
+ \frac{1}{2} y \left( \frac{x}{\bar{x}_B} \right)^{\gamma_1} \left[ 2 + 2 (2 \gamma_1 - 1) \ln \left( \frac{x}{\bar{x}_B} \right) + \gamma_1 (\gamma_1 - 1) \ln \left( \frac{x}{\bar{x}_B} \right)^2 \right] \\
- (r + m) \ln \left( \frac{x}{\bar{x}_B} \right)^2 \left( \frac{x}{\bar{x}_B} \right)^{\gamma_1} \\
= y \left( \frac{x}{\bar{x}_B} \right)^{\gamma_1} + 2 \ln \left( \frac{x}{\bar{x}_B} \right) \left( \frac{x}{\bar{x}_B} \right)^{\gamma_1} \left( g + \gamma_1 y - \frac{1}{2} y \right) \\
+ \ln \left( \frac{x}{\bar{x}_B} \right)^2 \mathcal{L}^y_{r+m} \left( \frac{x}{\bar{x}_B} \right)^{\gamma_1} \\
= y \left( \frac{x}{\bar{x}_B} \right)^{\gamma_1} + 2 \ln \left( \frac{x}{\bar{x}_B} \right) \left( \frac{x}{\bar{x}_B} \right)^{\gamma_1} \left( g + \gamma_1 y - \frac{1}{2} y \right) 
\]  

(96)

Therefore:

\[
\mathcal{L}^y_{r+m} \frac{\ln \left( \frac{x}{\bar{x}_B} \right)}{2 \left( g + \gamma_1 y - \frac{1}{2} y \right)} A^\delta y \frac{\partial \tilde{y}^y_0}{\partial y} \\
= - \frac{1}{2} A^\delta y A \left( \bar{x}_B \right) \frac{\gamma_1 (\gamma_1 - 1)}{2} \left( g + \gamma_1 y - \frac{1}{2} y \right)^2 y \left( \frac{x}{\bar{x}_B} \right)^{\gamma_1} \\
- A^\delta y A \left( \bar{x}_B \right) \frac{\gamma_1 (\gamma_1 - 1)}{2} g \frac{1}{g + \gamma_1 y - \frac{1}{2} y} \ln \left( \frac{x}{\bar{x}_B} \right) \left( \frac{x}{\bar{x}_B} \right)^{\gamma_1} \\
= - \frac{1}{2} A \left( \bar{x}_B \right) \frac{\gamma_1 (\gamma_1 - 1)}{2} A^\delta y \frac{1}{g + \gamma_1 y - \frac{1}{2} y} \ln \left( \frac{x}{\bar{x}_B} \right) \left( \frac{x}{\bar{x}_B} \right)^{\gamma_1} A^\delta y \frac{\partial \tilde{y}^y_0}{\partial y} 
\]  

(97)

Finally, I show that:

\[
\mathcal{L}^y_{r+m} \frac{\ln \left( \frac{x}{\bar{x}_B} \right)}{2 \left( g + \gamma_1 y - \frac{1}{2} y \right)^3} A^\delta y \frac{1}{2} y x^2 \frac{\partial^2 \tilde{y}^y_0}{\partial y^2} \\
= \frac{1}{2} A^\delta y A \left( \bar{x}_B \right) \frac{\gamma_1 (\gamma_1 - 1)}{2} \left( g + \gamma_1 y - \frac{1}{2} y \right)^2 y \mathcal{L}^y_{r+m} \frac{1}{g + \gamma_1 y - \frac{1}{2} y} \ln \left( \frac{x}{\bar{x}_B} \right) \left( \frac{x}{\bar{x}_B} \right)^{\gamma_1} \\
= \frac{1}{2} A^\delta y A \left( \bar{x}_B \right) \frac{\gamma_1 (\gamma_1 - 1)}{2} \left( g + \gamma_1 y - \frac{1}{2} y \right)^2 y \left( \frac{x}{\bar{x}_B} \right)^{\gamma_1} 
\]  

(98)

This completes the proof since:

\[
\mathcal{L}^y_{r+m} \frac{\ln \left( \frac{x}{\bar{x}_B} \right)}{2 \left( g + \gamma_1 y - \frac{1}{2} y \right)} A^\delta y \left( \frac{\partial \tilde{y}^y_0}{\partial y} + \frac{1}{2 \left( g + \gamma_1 y - \frac{1}{2} y \right)^2 y x^2 \frac{\partial^2 \tilde{y}^y_0}{\partial y^2}} \right) = A^\delta y \frac{\partial \tilde{y}^y_0}{\partial y} 
\]  

(100)
B Contingent Claims of Young (Growth) Firms

In this appendix, I provide the problems to solve for the equity and debt values of growth firms and provide the details of how to solve for them by regular perturbation.

B.1 Equity Valuation of Growth Firms

The equity valuation of the growth firm is given by the following optimal stopping problem under the risk-neutral measure:

\[
E_\delta^a(x, y) = \sup_{\tau', \tau'' \in \mathcal{T}} \mathbb{E}_t^* \left[ \int_{\tau'}^{\tau''} e^{-r(s-t)} \left\{ (1 - \phi) (K_a X_s - C) + \tilde{d}_\delta^a(x, y) - p \right\} ds \right] + \mathbb{I} [\tau' < \tau''] (E_\delta (X_{\tau'}, Y_{\tau'}) - I),
\]

where \( \tau' \) is the stopping time which denotes investment and \( \tau'' \) is the stopping time which denotes default. Upon defaulting, equityholders again receive nothing. At the investment threshold however, the equity value equals to the equity value a mature firm minus the cost of investment. Recall that by assumption, the cost of investment must be borne by equityholders, i.e., the firm cannot issue new debt to finance the purchase of additional capital.

By the same logic used to prove Theorem 1, we can characterize the equity valuation as a free boundary problem.

**Theorem 10** The equity value of a growth firm \( E_\delta^a(x, y) \) is the solution to:

\[
DIV^a_\delta(x, y) + \left( \mathcal{L}^y + \sqrt{\delta M_1^y} + \delta M_2^y \right) E_\delta^a = 0 \quad \text{for } x \in (x_B^a(y), x_I^a(y))
\]

\[
E_\delta^a(x_B^a(y), y) = 0 \quad (102a)
\]

\[
E_\delta^a(x_I^a(y), y) = E_\delta(x_I^a(y), y) - I \quad (102b)
\]

\[
\frac{\partial E_\delta^a}{\partial x} \bigg|_{x = x_B^a(y), \delta} = 0 \quad (102c)
\]

\[
\frac{\partial E_\delta^a}{\partial y} \bigg|_{x = x_B^a(y), \delta} = 0 
\]

\[
\frac{\partial E_\delta^a}{\partial x} \bigg|_{x = x_I^a(y), \delta} = \frac{\partial E_\delta}{\partial x} \bigg|_{x = x_I^a(y), \delta} \quad (102d)
\]

\[
\frac{\partial E_\delta^a}{\partial y} \bigg|_{x = x_I^a(y), \delta} = \frac{\partial E_\delta}{\partial y} \bigg|_{x = x_I^a(y), \delta} \quad (102e)
\]

\[
\frac{\partial E_\delta^a}{\partial x} \bigg|_{x = x_I^a(y), \delta} = \frac{\partial E_\delta}{\partial x} \bigg|_{x = x_I^a(y), \delta} \quad (102f)
\]

\[
\frac{\partial E_\delta^a}{\partial y} \bigg|_{x = x_I^a(y), \delta} = \frac{\partial E_\delta}{\partial y} \bigg|_{x = x_I^a(y), \delta} \quad (102g)
\]

where

\[
DIV^a_\delta(x, y) = (1 - \phi) (K_a x - C) + \tilde{d}_\delta^a(x, y) - p
\]

and \( x_B^a(y) \) and \( x_I^a(y) \) are free boundaries to be determined.

**Proof.** The proof follows the same logic as the proof of Theorem 1. ■
The free boundary problem is similar to the one specified for the equity value of mature firms. At the default boundary, the equity value must be equal to zero and the derivatives must be continuous. However, now there is no limiting condition as \( x \to \infty \). Instead, there are additional value-matching and smooth-pasting conditions. At the investment threshold, the equity value must be equal to the equity of a mature firm minus the cost of investment. Once again, optimality in conjunction with the nature of a regular diffusion requires that the derivatives of the valuation be continuous across this barrier. Finally, the dividend to the equityholders now reflects the lower level of capital and the valuation of newly issued debt for young firms.

### B.2 Debt Valuation of Growth Firms

Let \( I \) denote the optimal stopping time for investment and \( D \) the optimal stopping time for default:

\[
I = \min \left\{ t : (X_t, Y_t) = (x_{I,a}^a, Y_t) \right\}
\]

\[
D = \min \left\{ t : (X_t, Y_t) = (x_{D,a}^a, Y_t) \right\}
\]

Existing debt is simply rolled over when the firm invests in capital. Therefore, the value of a vintage of debt at time \( t \) issued at date 0 is given by:

\[
d_a^a (t) = \mathbb{E}_t^s \left[ \int_t^{\tau_I \wedge \tau_D} e^{-r(s-t)} e^{-ms} (c + mp) \right. \left. ds + \mathbb{1}_{\tau^D < \tau^I} e^{-r(\tau^D - t)} \left( e^{-m\tau_D} p/P \right) (1 - \xi) U_{\tau_D} \right.
\]

\[
+ \mathbb{1}_{\tau^I < \tau^D} e^{-r(\tau^I - t)} d_\delta (\tau^I) \right]
\]

Upon default, debtholders receive a fraction of the value of the assets in place (minus bankruptcy costs) according to their vintage. At the investment threshold, the debt value equals the value of debt of identical vintage in a mature firm.

Noting that \( p/P = m \) and \( d_\delta (\tau^I) = e^{-m\tau^I} d_\delta (X_{\tau^I}, Y_{\tau^I}) \), it follows that:

\[
e^{mt} d_a^a (t) = \mathbb{E}_t^s \left[ \int_t^{\tau_I \wedge \tau_D} e^{-(r+m)(s-t)} (c + mp) ds + \mathbb{1}_{\tau^D < \tau^I} e^{-(r+m)(\tau^D - t)} m (1 - \xi) U_{\tau_D}^a \right.
\]

\[
+ \mathbb{1}_{\tau^I < \tau^D} e^{-(r+m)(\tau^I - t)} d_\delta (X_{\tau^I}, Y_{\tau^I}) \right]
\]

where \( U_{\tau^I}^a \) is the value of the unlevered young firm.\(^{29}\) By Feynman-Kac the following theorem holds.

**Theorem 11** The value of the date 0 debt vintage at time \( t \) for a young firm is given by

\(^{29}\)The value of the unlevered firm is also found through the use asymptotic expansions, using the same procedure described here and setting the principal equal to zero.
\[ d^a(t) = e^{-mt} \tilde{d}^a(X_t, Y_t) \] where \( \tilde{d}^a(X_t, Y_t) \) is the value of the newly issued debt and satisfies:

\[
c + mp + \left( \mathcal{L}_{r+m}^y + \sqrt{\delta} \mathcal{M}_1^y + \delta \mathcal{M}_2^y \right) \tilde{d}_0^y = 0 \quad \text{for} \quad x \in (x_{B,0}^a(y), x_{I,0}^a(y)) \tag{108a}
\]

\[
\tilde{d}_0^y(x_B^a(y), y) = m(1 - \xi) U^a(x_B^a(y)) \tag{108b}
\]

\[
\tilde{d}_0^y(x_I^a(y), y) = \tilde{d}_0^y(x_I^a(y), y) \tag{108c}
\]

The total value of debt \( D^a = \tilde{d}^a/m \).

I now expand the contingent claims and default boundaries in powers of \( \sqrt{\delta} \):

\[
E_0^y(x, y) = E_0^{a,y}(x) + \sqrt{\delta} E_1^{a,y}(x) + \delta E_2^{a,y}(x) + \ldots \tag{109a}
\]

\[
\tilde{d}_0^y(x_B^a(y), y) = \tilde{d}_0^{a,y}(x_B^a(y)) + \sqrt{\delta} \tilde{d}_1^{a,y}(x_B^a(y)) + \delta \tilde{d}_2^{a,y}(x_B^a(y)) + \ldots \tag{109b}
\]

\[
x_B^a(y) = x_{B,0}^{a,y} + \sqrt{\delta} x_{B,1}^{a,y} + \delta x_{B,2}^{a,y} + \ldots \tag{109c}
\]

These are substituted into the problems above. Boundary conditions are expanded with Taylor series. Finally, contingent claims of mature firms are also written using asymptotic expansions.

### B.3 Principal Order Terms

The principal order terms reflect the valuations and default boundary in the constant volatility case. The problem for debt is:

\[
c + mp + \mathcal{L}_{r+m}^y \tilde{d}_0^y = 0 \quad \text{for} \quad x \in (x_{B,0}^{a,y}, x_{I,0}^{a,y}) \tag{110a}
\]

\[
\tilde{d}_0^{a,y}(x_{B,0}^{a,y}) = m(1 - \xi) U^a(x_{B,0}^{a,y}) \tag{110b}
\]

\[
\tilde{d}_0^{a,y}(x_{I,0}^{a,y}) = \tilde{d}_0^y(x_{I,0}^{a,y}) \tag{110c}
\]

and the problem for equity is:

\[
(1 - \phi) (x K_a - C) + \tilde{d}_0^{a,y}(x) - p + \mathcal{L}_r^y E_0^{a,y} = 0 \quad \text{or} \quad x \in (x_{B,0}^{a,y}, x_{I,0}^{a,y}) \tag{111a}
\]

\[
E_0^{a,y}(x_{B,0}^{a,y}) = 0 \tag{111b}
\]

\[
E_0^{a,y}(x_{I,0}^{a,y}) = E_0^y(x_{I,0}^{a,y}) - I \tag{111c}
\]

\[
\frac{\partial E_0^{a,y}}{\partial x} \Big|_{x=x_{B,0}^{a,y}} = 0 \tag{111d}
\]

\[
\frac{\partial E_0^{a,y}}{\partial x} \Big|_{x=x_{I,0}^{a,y}} = \frac{\partial E_0^y}{\partial x} \big|_{x=x_{I,0}^{a,y}} \tag{111e}
\]

I follow an indirect approach to solve for \( E_0^{a,y} \) as in appendix A.2. I introduce the sum of debt and equity values \( V_0^{a,y} \), which is the solution to:

\[
(1 - \phi) K_a x + \phi C + \mathcal{L}_r^y V_0^{a,y} = 0 \quad \text{for} \quad x \in (x_{B,0}^{a,y}, x_{I,0}^{a,y}) \tag{112a}
\]

\[
V_0^{a,y}(x_{B,0}^{y}) = (1 - \xi) U^a(x_{B,0}^{y}) \tag{112b}
\]

\[
V_0^{a,y}(x_{I,0}^{y}) = V_0^y(x_{I,0}^{y}) \tag{112c}
\]
Then $E_{0}^{a,y} = V_{0}^{a,y} - \tilde{d}_{0}^{a,y} / m$ and the default/investment boundaries are found by applying the two smooth-pasting conditions.

### B.4 First-Order Correction Terms

Finally, the first-order corrections once again reflect comparative statics in the constant volatility case as well as boundary corrections. The first-order correction for debt is:

$$
L_{r+m}^{y} \sqrt{\delta d_{1}^{a,y}} = \left( A^{y} \frac{\partial \tilde{d}_{0}^{a,y}}{\partial y} - B^{y} x \frac{\partial^{2} \tilde{d}_{0}^{a,y}}{\partial x \partial y} \right) \quad \text{for } x \in (x_{B,0}^{a,y}, x_{I,0}^{a,y}) \quad (113a)
$$

$$
\sqrt{\delta d_{1}^{a,y}} (x_{B,0}^{a,y}) = \sqrt{\delta} x^{a,y}_{B,1} \left[ m \left( 1 - \xi \right) \left( 1 - \phi \right) \frac{r - g}{r} - \frac{\partial \tilde{d}_{0}^{a,y}}{\partial x} (x_{B,0}^{a,y}) \right] \quad (113b)
$$

$$
\sqrt{\delta d_{1}^{a,y}} (x_{I,0}^{a,y}) = \sqrt{\delta} x^{a,y}_{I,1} \left[ \frac{\partial \tilde{d}_{0}^{y}}{\partial x} (x_{I,0}^{a,y}) - \frac{\partial \tilde{d}_{0}^{a,y}}{\partial x} (x_{I,0}^{a,y}) \right] + \sqrt{\delta d_{1}^{y}} (x_{I,0}^{a,y}) \quad (113c)
$$

The first-order correction for equity is:

$$
L_{r}^{y} \sqrt{\delta E_{1}^{a,y}} = \left( A^{y} \frac{\partial E_{0}^{a,y}}{\partial y} - B^{y} x \frac{\partial^{2} E_{0}^{a,y}}{\partial x \partial y} - \sqrt{\delta d_{1}^{a,y}} \right) \quad \text{for } x \in (x_{B,0}^{a,y}, x_{I,0}^{a,y}) \quad (114a)
$$

$$
\sqrt{\delta E_{1}^{a,y}} (x_{B,0}^{a,y}) = 0 \quad (114b)
$$

$$
\sqrt{\delta E_{1}^{a,y}} (x_{I,0}^{a,y}) = \sqrt{\delta} E_{1}^{a,y} (x_{I,0}^{a,y}) \quad (114c)
$$

Note that I used the smooth-pasting condition of the principal order term at the default and investment boundaries to derive the final two equations. Finally, the corrections in the default/investment barriers must satisfy:

$$
\sqrt{\delta} x^{a,y}_{B,1} \frac{\partial^{2} E_{0}^{a,y}}{\partial x^{2}} (x_{B,0}^{a,y}) = -\sqrt{\delta} \frac{\partial E_{1}^{a,y}}{\partial x} (x_{B,0}^{a,y}) \quad (115)
$$

$$
\sqrt{\delta} x^{a,y}_{I,1} \frac{\partial^{2} E_{0}^{a,y}}{\partial x^{2}} (x_{I,0}^{a,y}) = -\sqrt{\delta} \frac{\partial E_{1}^{a,y}}{\partial x} (x_{I,0}^{a,y}) \quad (116)
$$

All of this together solves a system of equations which can be solved numerically in MATLAB.

### C Volatility Time Scales

This technical appendix elaborates on the discussion of volatility time scales in section 4.1 and specifically on the role of $\delta$ in controlling the rate of mean-reversion. The discussion follows closely the textbook treatment provided in Fouque, Papanicolaou, Sircar, and Solna (2011). I define the infinitesimal generator of a time-homogenous, ergodic Markov process $Y_{t}$ to be:

$$
L h(y) = \lim_{t \to 0} \frac{P_{t} h(y) - h(y)}{t}, \quad (117)
$$
where:

\[ \mathbf{P}_t h (y) = \mathbb{E} [h (Y_t) \mid Y_0 = y]. \quad (118) \]

For example, the infinitesimal generator of the Cox-Ingersoll-Ross process of equation (2) is given by:

\[ \mathcal{M}_2^y = (\theta Y - y) \frac{\partial}{\partial y} + \frac{1}{2} \nu^2 y \frac{\partial^2}{\partial y^2}. \quad (119) \]

In general, the infinitesimal generator of a regular diffusion can be found by considering Ito’s formula and the backwards Kolmogorov equation. To find the invariant distribution of the process \( Y_t \), which exists by ergodicity, I look for a distribution \( \Lambda \) for \( Y_0 \) which satisfies for any bounded \( h \):

\[ \frac{d}{dt} \int \mathbb{E} [h (Y_t) \mid Y_0 = y] d\Lambda (y) = 0, \quad (120) \]

where the integral is taken of the state space on which the Markov process is defined. By the backward Kolmogorov equation for a time-homogenous Markov process:

\[ \frac{d}{dt} \mathbf{P}_t h (y) = \mathcal{L} \mathbf{P}_t g (y), \quad (121) \]

it follows that:

\[ \frac{d}{dt} \int \mathbb{E} [h (Y_t) \mid Y_0 = y] d\Lambda (y) = \int \frac{d}{dt} \mathbf{P}_t h (y) d\Lambda (y) = \int \mathcal{L} \mathbf{P}_t h (y) d\Lambda (y) = \int \mathbf{P}_t h (y) \mathcal{L}^* d\Lambda (y), \quad (122) \]

where \( \mathcal{L}^* \) is the adjoint operator of \( \mathcal{L} \) defined uniquely by:

\[ \int \alpha (y) \mathcal{L} \beta (y) dy = \int \beta (y) \mathcal{L}^* \alpha (y) dy \quad (123) \]

for test functions \( \alpha, \beta \). The invariant distribution, therefore, is the solution to the equation:

\[ \mathcal{L}^* \Lambda = 0, \quad (124) \]

since the relation above must hold for all \( h \). I denote integration with respect to the invariant distribution by \( \langle h \rangle \) and suppose that the invariant distribution has a mean.

Now define the process \( Y_t^\delta \) according the infinitesimal generator \( \delta \mathcal{L} \). For a CIR process, this procedure then gives a new process which is given explicitly in equation (25). The adjoint of the operator \( \delta \mathcal{L} \) is clearly given by \( \delta \mathcal{L}^* \). Therefore, it is immediately clear that the invariant distribution of the process \( Y_t^\delta \) is independent of the choice of \( \delta \), as described in section 4.1. This indicates that, in the long-run, the choice of \( \delta \) does not affect the degree of variability in the process \( Y_t^\delta \).

Now I suppose that the process \( Y_t \) is reversible, or that the operator \( \mathcal{L} \) has a discrete spectrum and that zero is an isolated eigenvalue. This allows the formation of an orthonormal
basis of $L^2(\Lambda)$ and to consider the eigenfunction expansion of a function $h(y)$ by:

$$h(y) = \sum_{n=0}^{\infty} d_n \psi_n(y) \quad (125)$$

where each $\psi_n$ satisfies:

$$\mathcal{L}\psi_n = \lambda_n \psi_n \quad (126)$$

and where $0 = \lambda_0 > \lambda_1 > \cdots$. Each constant satisfies $d_n = \langle h \psi_n \rangle$ and in particular $d_0 = \langle h \rangle$, since the eigenfunction associated with the zero eigenvalue is simply $\psi_0 = 1$. Next consider the backwards Kolmogorov equation:

$$\frac{d}{dt} P_t h(y) = \mathcal{L} P_t h(y) \quad (127)$$

and look for a solution of the form:

$$P_t h(y) = \sum_{n=0}^{\infty} z_n(t) \psi_n(y) \quad (128)$$

Substituting this expression into the backwards Kolmogorov equation and using equation (126) gives an ODE for each $z_n(t)$:

$$z'_n(t) = \lambda_n z_n(t) \quad (129)$$

with initial condition $z_n(0) = d_n$. Solving, this implies that:

$$P_t h(y) = \sum_{n=0}^{\infty} c_n e^{\lambda_n t} \psi_n(y) \quad (130)$$

From this, it is possible to show that:

$$|P_t g(y) - \langle h \rangle| \leq C e^{\lambda_1 t},$$

for some constant $C$. In words, the spectral gap, defined as the magnitude of the first negative eigenvalue, controls the rate of mean reversion of the process to the invariant distribution.

Finally, it is trivial to see that the eigenfunctions/eigenvalues of the infinitesimal generator $\delta \mathcal{L}$ are given by:

$$\mathcal{L} \delta \psi_n = \lambda_n \delta \psi_n. \quad (132)$$

That is, the spectrum of $\delta \mathcal{L}$ is simply a scaling of the spectrum of $\mathcal{L}$ according to the parameter $\delta$. But this implies that the spectral gap of the process $Y^\delta_t$ is proportional to $\delta$ and, therefore, that $\delta$ controls the rate of mean-reversion of the process.
Table 1: Variables in Structural Model of the Firm

<table>
<thead>
<tr>
<th>Variable</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_t$</td>
<td>Asset productivity</td>
</tr>
<tr>
<td>$Y_t$</td>
<td>Asset variance</td>
</tr>
<tr>
<td>$U^a, U$</td>
<td>Value of assets in place</td>
</tr>
<tr>
<td>$E^a, E$</td>
<td>Equity values</td>
</tr>
<tr>
<td>$\tilde{d}, \tilde{d}$</td>
<td>Value of newly issued debt</td>
</tr>
<tr>
<td>$D^a, D$</td>
<td>Total debt values</td>
</tr>
<tr>
<td>$V^a, V$</td>
<td>Total debt value plus equity value</td>
</tr>
<tr>
<td>$u, u$</td>
<td>Cumulative survival probabilities</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>Riskfree rate</td>
</tr>
<tr>
<td>$\mu, \gamma$</td>
<td>Expected productivity growth rates</td>
</tr>
<tr>
<td>$\kappa_Y$</td>
<td>CIR rate of mean-reversion</td>
</tr>
<tr>
<td>$\theta_Y$</td>
<td>Long-run variance</td>
</tr>
<tr>
<td>$\nu_Y$</td>
<td>Volatility of variance</td>
</tr>
<tr>
<td>$\rho_Y$</td>
<td>Correlation between productivity and variance</td>
</tr>
<tr>
<td>$K^a$</td>
<td>Capital stock of young firms</td>
</tr>
<tr>
<td>$I$</td>
<td>Cost of investment</td>
</tr>
<tr>
<td>$\phi$</td>
<td>Corporate tax rate</td>
</tr>
<tr>
<td>$\xi$</td>
<td>Bankruptcy costs</td>
</tr>
<tr>
<td>$P$</td>
<td>Total principal</td>
</tr>
<tr>
<td>$C$</td>
<td>Total coupon</td>
</tr>
<tr>
<td>$m$</td>
<td>Rollover rate of debt</td>
</tr>
<tr>
<td>$\pi_X$</td>
<td>Asset risk premium</td>
</tr>
<tr>
<td>$\Gamma(Y_t)$</td>
<td>Market price of variance risk</td>
</tr>
</tbody>
</table>

Note: This table defines the key variables, parameters, and notation used throughout the paper. A superscript $a$ refers to young firms. The parametric specification for the market price of volatility risk is given by $\Gamma(Y_t) = \Gamma_0 \sqrt{\gamma_t}$.

Table 2: Calibration of Model Parameters

<table>
<thead>
<tr>
<th>$r$</th>
<th>$\phi$</th>
<th>$g$</th>
<th>$\pi_X$</th>
<th>$\bar{\sigma}_A$</th>
<th>$\xi$</th>
<th>$\rho_Y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>.08</td>
<td>.15</td>
<td>.02</td>
<td>.04</td>
<td>.22</td>
<td>.30</td>
<td>0</td>
</tr>
</tbody>
</table>

Note: This table provides the calibrated values of the interest rate, the tax rate, the risk-neutral growth rate, the asset risk premium, the average asset volatility, the fractional bankruptcy costs, and the correlation between productivity shocks and volatility shocks.
Table 3: Calibration of Model Parameters by Credit Rating

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Aaa</th>
<th>Aa</th>
<th>A</th>
<th>Baa</th>
<th>Ba</th>
<th>B</th>
<th>Caa</th>
</tr>
</thead>
<tbody>
<tr>
<td>Target Leverage</td>
<td>13.1%</td>
<td>21.1%</td>
<td>32.0%</td>
<td>43.3%</td>
<td>53.5%</td>
<td>65.7%</td>
<td>80.0%</td>
</tr>
<tr>
<td>Avg. Asset Vol.</td>
<td>22%</td>
<td>22%</td>
<td>22%</td>
<td>22%</td>
<td>23%</td>
<td>28%</td>
<td>28%</td>
</tr>
<tr>
<td>Avg. Maturity (yr.)</td>
<td>10.16</td>
<td>9.45</td>
<td>10.13</td>
<td>9.14</td>
<td>7.11</td>
<td>7.39</td>
<td>2.48</td>
</tr>
</tbody>
</table>

Note: This table provides calibration values for the target leverage ratio, average asset volatility, and average maturity of debt by credit rating.

Table 4: Credit Spreads on 10-Year Maturity Debt (bps)

<table>
<thead>
<tr>
<th>Credit Rating</th>
<th>Target</th>
<th>Baseline</th>
<th>Exog. Default</th>
<th>Endog. Default</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>% Explained</td>
<td>% Explained</td>
<td>% Explained</td>
<td>% Explained</td>
</tr>
<tr>
<td>Aaa</td>
<td>47</td>
<td>2</td>
<td>4.3%</td>
<td>14</td>
</tr>
<tr>
<td>Aa</td>
<td>69</td>
<td>6</td>
<td>8.7%</td>
<td>37</td>
</tr>
<tr>
<td>A</td>
<td>96</td>
<td>19</td>
<td>19.8%</td>
<td>84</td>
</tr>
<tr>
<td>Baa</td>
<td>150</td>
<td>43</td>
<td>28.7%</td>
<td>150</td>
</tr>
<tr>
<td>Ba</td>
<td>310</td>
<td>93</td>
<td>30.0%</td>
<td>260</td>
</tr>
<tr>
<td>B</td>
<td>470</td>
<td>286</td>
<td>60.9%</td>
<td>607</td>
</tr>
<tr>
<td>Caa</td>
<td>765</td>
<td>663</td>
<td>86.7%</td>
<td>1307</td>
</tr>
</tbody>
</table>

Note: This table provides target and model-generated 10-year credit spreads by ratings category. Calibration parameters are provided in Tables 2 and 3. There is zero correlation between productivity shocks and variance shocks. The variance risk premium is set to match the 10-year historical Baa credit spread. Historical target credit spreads for Aaa-Baa debt are from Duffee (1998) while those for lower ratings categories are from Caouette, Altman, and Narayanan (1998). The baseline model holds volatility constant. The exogenous default model incorporates stochastic volatility, but sets the default barrier to be equal to that of the baseline model. The endogenous default model is the full stochastic volatility model.
Table 5: Credit Spreads on 20-Year Maturity Debt (bps)

<table>
<thead>
<tr>
<th></th>
<th>Target</th>
<th>Baseline</th>
<th>Exog. Default</th>
<th>Endog. Default</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Model</td>
<td>% Explained</td>
<td>Model</td>
<td>% Explained</td>
</tr>
<tr>
<td>Aaa</td>
<td>59</td>
<td>3</td>
<td>5.1%</td>
<td>28</td>
</tr>
<tr>
<td>Aa</td>
<td>87</td>
<td>8</td>
<td>9.2%</td>
<td>63</td>
</tr>
<tr>
<td>A</td>
<td>117</td>
<td>22</td>
<td>18.8%</td>
<td>121</td>
</tr>
<tr>
<td>Baa</td>
<td>198</td>
<td>46</td>
<td>23.2%</td>
<td>202</td>
</tr>
<tr>
<td>Ba</td>
<td>N/A</td>
<td>91</td>
<td>N/A</td>
<td>326</td>
</tr>
<tr>
<td>B</td>
<td>N/A</td>
<td>258</td>
<td>N/A</td>
<td>692</td>
</tr>
<tr>
<td>Caa</td>
<td>N/A</td>
<td>535</td>
<td>N/A</td>
<td>1256</td>
</tr>
</tbody>
</table>

Note: This table provides target and model-generated 20-year credit spreads by ratings category. Calibration parameters are provided in Tables 2 and 3. There is zero correlation between productivity shocks and variance shocks. The variance risk premium is set to match the 10-year historical Baa credit spread. Historical target credit spreads for Aaa-Baa debt are from Duffee (1998) while those for lower ratings categories are from Caouette, Altman, and Narayanan (1998). The baseline model holds volatility constant. The exogenous default model incorporates stochastic volatility, but sets the default barrier to be equal to that of the baseline model. The endogenous default model is the full stochastic volatility model.

Table 6: Cumulative Default Probabilities - Long Maturity

<table>
<thead>
<tr>
<th></th>
<th>10yr</th>
<th>15yr</th>
<th>20yr</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Target</td>
<td>Model</td>
<td>Target</td>
</tr>
<tr>
<td>Aaa</td>
<td>0.53%</td>
<td>0.01%</td>
<td>1.00%</td>
</tr>
<tr>
<td>Aa</td>
<td>0.52%</td>
<td>0.15%</td>
<td>1.09%</td>
</tr>
<tr>
<td>A</td>
<td>1.31%</td>
<td>1.08%</td>
<td>2.40%</td>
</tr>
<tr>
<td>Baa</td>
<td>4.35%</td>
<td>4.16%</td>
<td>7.60%</td>
</tr>
<tr>
<td>Ba</td>
<td>18.43%</td>
<td>13.40%</td>
<td>27.53%</td>
</tr>
<tr>
<td>B</td>
<td>40.92%</td>
<td>38.33%</td>
<td>50.21%</td>
</tr>
</tbody>
</table>

Note: This table reports historical and model-generated cumulative default probabilities of firms within 10, 15, and 20 years by ratings category. Target expected default frequencies are the average cumulative issuer-weighted global default rates from 1970-2007 as reported by Moody’s. Calibration parameters are provided in Tables 2 and 3. There is zero correlation between productivity shocks and variance shocks.
Table 7: Credit Spreads on 4-Year Maturity Debt (bps)

<table>
<thead>
<tr>
<th></th>
<th>Target</th>
<th>Baseline</th>
<th>Exog. Default</th>
<th>Endog. Default</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Model</td>
<td>% Explained</td>
<td>Model</td>
<td>% Explained</td>
</tr>
<tr>
<td>Aaa</td>
<td>46</td>
<td>0.5</td>
<td>1.1%</td>
<td>2</td>
</tr>
<tr>
<td>Aa</td>
<td>56</td>
<td>3</td>
<td>5.4%</td>
<td>13</td>
</tr>
<tr>
<td>A</td>
<td>87</td>
<td>12</td>
<td>13.8%</td>
<td>38</td>
</tr>
<tr>
<td>Baa</td>
<td>149</td>
<td>36</td>
<td>24.2%</td>
<td>91</td>
</tr>
<tr>
<td>Ba</td>
<td>310</td>
<td>94</td>
<td>30.3%</td>
<td>189</td>
</tr>
<tr>
<td>B</td>
<td>470</td>
<td>344</td>
<td>73.2%</td>
<td>541</td>
</tr>
<tr>
<td>Caa</td>
<td>N/A</td>
<td>1072</td>
<td>N/A</td>
<td>1570</td>
</tr>
</tbody>
</table>

Note: This table provides target and model-generated 4-year credit spreads by ratings category. Calibration parameters are provided in Tables 2 and 3. There is zero correlation between productivity shocks and variance shocks. The variance risk premium is set to match the 10-year historical Baa credit spread. Historical target credit spreads for Aaa-Baa debt are from Duffee (1998) while those for lower ratings categories are from Caouette, Altman, and Narayanan (1998). The baseline model holds volatility constant. The exogenous default model incorporates stochastic volatility, but sets the default barrier to be equal to that of the baseline model. The endogenous default model is the full stochastic volatility model.

Table 8: Cumulative Default Probabilities - Short Maturity

<table>
<thead>
<tr>
<th></th>
<th>2yr Target</th>
<th>Model</th>
<th>4yr Target</th>
<th>Model</th>
<th>6yr Target</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aaa</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.03%</td>
<td>0.00%</td>
<td>0.17%</td>
<td>0.00%</td>
</tr>
<tr>
<td>Aa</td>
<td>0.02%</td>
<td>0.00%</td>
<td>0.11%</td>
<td>0.00%</td>
<td>0.26%</td>
<td>0.01%</td>
</tr>
<tr>
<td>A</td>
<td>0.09%</td>
<td>0.00%</td>
<td>0.34%</td>
<td>0.03%</td>
<td>0.61%</td>
<td>0.21%</td>
</tr>
<tr>
<td>Baa</td>
<td>0.48%</td>
<td>0.01%</td>
<td>1.36%</td>
<td>0.43%</td>
<td>2.32%</td>
<td>1.48%</td>
</tr>
<tr>
<td>Ba</td>
<td>3.02%</td>
<td>0.53%</td>
<td>7.65%</td>
<td>3.70%</td>
<td>11.77%</td>
<td>7.42%</td>
</tr>
<tr>
<td>B</td>
<td>10.20%</td>
<td>8.33%</td>
<td>20.33%</td>
<td>20.39%</td>
<td>28.74%</td>
<td>28.48%</td>
</tr>
</tbody>
</table>

Note: This table reports historical and model-generated cumulative default probabilities of firms within 2, 4, and 6 years by ratings category. Target expected default frequencies are the average cumulative issuer-weighted global default rates from 1970-2007 as reported by Moody’s. Calibration parameters are provided in Tables 2 and 3. There is zero correlation between productivity shocks and variance shocks.
Table 9: Baa-Rated Credit Spreads with Disaster Risk

<table>
<thead>
<tr>
<th></th>
<th>Target</th>
<th>( \lambda = 0 )</th>
<th>( \lambda = .001 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>[.50,.75]</td>
<td>[.25,.75]</td>
</tr>
<tr>
<td>20yr</td>
<td>195</td>
<td>194</td>
<td>216</td>
</tr>
<tr>
<td>10yr</td>
<td>150</td>
<td>150</td>
<td>165</td>
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<tr>
<td>4yr</td>
<td>149</td>
<td>95</td>
<td>109</td>
</tr>
<tr>
<td>1yr</td>
<td>N/A</td>
<td>28</td>
<td>37</td>
</tr>
</tbody>
</table>

Note: This table provides target and model-generated Baa-rated credit spreads at 20yr, 10yr, 4yr, 1yr maturities in a model with rare disasters. The parameter \( \lambda \) controls the arrival rate of the disasters and the intervals reflect the uniform distribution of the jump sizes. Other calibration parameters are provided in Tables 2 and 3. \( A^d = -.2264 \) and \( B^d = 0 \). Historical target credit spreads are from Duffee (1998).
Figure 1: Relationship between Financial Distress and Equity Returns

Note: Panel A plots the variance beta as a function of the 5-year probability of default, Panel B plots the equity variance risk premium as a function of the probability of default and the asset variance risk premium, and Panel C plots the total equity risk premium as a function of the probability of default and the asset variance risk premium. The principal equals 35 and the coupon rate is 8.162%. The other calibration parameters are provided in Table2.
Figure 2: Relationship between Book-to-Market and Equity Returns

Note: Panel A plots the variance beta as a function of the book-to-market ratio, Panel B plots the equity variance risk premium as a function of the book-to-market ratio and the asset variance risk premium, and Panel C plots the total equity risk premium as a function of the book-to-market ratio and the asset variance risk premium. The firm has no debt and the per-unit cost of capital is equal to 30. The productivity level is $x=2$. The other calibration parameters are provided in Table2.
Figure 3: Debt Vegas: Endogenous and Exogenous Default

Note: This figure illustrates how debt vegas vary with productivity in the constant volatility model under endogenous and exogenous default assumptions. Firms are mature. The riskless rate, corporate tax rate, risk-neutral rate of productivity growth, and asset risk premium are set according to Table 2. Total principal equals 43.3 and the coupon rate equals 8.168%. Current volatility of the productivity process equals 22%.

Figure 4: Total Debt Vegas and Average Maturity

Note: This figure illustrates how total debt vegas in the constant volatility model vary with the average debt maturity and productivity of the firm. Firms are mature. The riskless rate, corporate tax rate, risk-neutral rate of productivity growth, and asset risk premium are set according to Table 2. Total principal equals 43.3 and the coupon rate equals 8.168%. Current volatility of the productivity process equals 22%.