1. INTRODUCTION

This course is about the enumerative geometry of curves in Calabi-Yau threefolds.

Definition 1.0.1. A Calabi-Yau threefold is a 3-dimensional projective variety $X$ over $\mathbb{C}$ such that $K_X = \bigwedge^3 T_X^\vee \cong \mathcal{O}_X$ is trivial and $h^1(\mathcal{O}_X) = 0$.

Today we’ll explain three examples that illustrate the type of questions that we’re interested in.

1.1. Counting rational curves. Let $X = Q_5$ be a quintic hypersurface in $\mathbb{P}^4$. This is a Calabi-Yau threefold, by the adjunction formula for triviality of $K$ and the Lefschetz hyperplane theorem for triviality of $H^1(\mathcal{O}_X)$.

We are interested in the space $\text{Rat}_d(Q_5)$ of rational curves of degree $d$ in $Q_5$. We can think of such a curve as a map $F: \mathbb{P}^1 \to \mathbb{P}^4$ given by degree $d$ polynomials and factoring through $Q_5$:

$$
\begin{array}{ccc}
\mathbb{P}^1 & \to & Q_5 \\
 & F & \downarrow \\
 & \mathbb{P}^4 & 
\end{array}
$$

Now, a map $\mathbb{P}^1 \to \mathbb{P}^4$ is given by $[f_0, \ldots, f_4]$ with $f_t \in H^0(\mathcal{O}_{\mathbb{P}^1}(d)) \cong \mathbb{C}^{d+1}$, so the $F$ are parametrized by $\tilde{F} = (f_0, \ldots, f_4) \in \mathbb{C}^{5d+5}$ which do not have common zeros (which is what happens in general). The map $F$ factors through $Q_5$ if $F(f_0, \ldots, f_4) = 0 \in H^0(\mathcal{O}_{\mathbb{P}^1}(5d)) \cong \mathbb{C}^{5d+1}$ (by Riemann-Roch). We have constructed a map

$$
H^0(\mathcal{O}_{\mathbb{P}^1}(d))^{\otimes 5} \to H^0(\mathcal{O}_{\mathbb{P}^1}(5d))
$$

sending $f \mapsto F(f)$. Then $\tilde{F}^{-1}(0)$ “should” be a union of 4-dimensional cones. Note that if $f: \mathbb{P}^1 \to Q_5$, then for any $\alpha \in \text{Aut}(\mathbb{P}^1)$ we also have that $f \circ \alpha$ factors through $Q_5$. So each $f: \mathbb{P}^1 \to Q_5$ contributes to a 4-dimensional subvariety in $\tilde{F}^{-1}(0)$ (three from automorphisms and one from scaling), which leads to the following conjecture.

Conjecture 1.1.1 (Clemens). For very general $Q_5 \subset \mathbb{P}^4$, the space $\text{Rat}_d(Q_5)$ is discrete for all $d \geq 1$.

The classical method is to study the incidence scheme

$$
\Phi_d = \{(C, F) | C \subset F^{-1}(0), C \in \text{Rat}_d(\mathbb{P}^4)\}
$$

and more precisely to study its irreducibility and fibers of the obvious projection

$$
\Phi_d \xrightarrow{\pi} \text{Rat}_d(\mathbb{P}^4).
$$

Theorem 1.1.2. If $d \leq 11$, then $\Phi_d$ is irreducible. For $d \geq 12$, then $\Phi_d$ is reducible.

Irreducibility implies Clemens’ conjecture ♠♠♠ TONY: [why?], but reducibility doesn’t disprove it.
1.2. Discreteness of curves in CY 3-folds. Curves in a CY 3-fold are “expected” to be discrete. Let \( X \) be a CY 3-fold and \( C \subset X \) a smooth curve. We can form the Hilbert scheme \( \text{Hilb}_X^P \), with respect to the Hilbert polynomial

\[
P(m) = \chi(\mathcal{O}_C(m)) = n \deg H|_C + (1 - g)
\]

(there is always an ample line bundle \( H \) on \( X \) implicitly fixed). Then \( \text{Hilb}_X^P \) parametrizes curves in \( X \) with Hilbert polynomial \( P \). Assume \( g \geq 2 \) for simplicity.

We want to calculate \( \dim \mathcal{T}[C] \text{Hilb}_X^P \). A first-order deformation of a smooth subcurve \( C \) in \( X \) gives (tautologically) an abstract first-order deformation of \( C \) in moduli, so we get a map

\[
\dim \mathcal{T}[C] \text{Hilb}_X^P \rightarrow \text{Def}(C).
\]

What is the kernel? If the complex structure of \( C \) doesn’t change, then the deformation must be given by a vector field, so the kernel of this map is \( H^0(TX|_C) \). Also, there exists \( \text{Def}(C) \xrightarrow{\delta} H^1(TX|_C) \) (which requires some thinking to see!). We claim that we have an exact sequence

\[
0 \rightarrow H^0(TX|_C) \rightarrow \dim \mathcal{T}[C] \text{Hilb}_X^P \rightarrow \text{Def}(C) \xrightarrow{\delta} H^1(TX|_C)
\]

**Exercise 1.2.1.** Construct the arrows in this exact sequence, and prove that it is exact.

It is “expected” that \( \delta \) is surjective, so we can read off

\[
\dim \mathcal{T}[C] \text{Hilb}_X^P = \dim \text{Def}(C) + h^0(TX|_C) - h^1(TX|_C).
\]

By Riemann-Roch and well-known fact, this is \((3g - 3) + \deg TX|_C + \text{rank } TX|_C(1 - g)\), and \( \deg TX|_C = 0 \) since \( X \) is Calabi-Yau, so we see that this is 0.

**Remark 1.2.2.** If the curves in \( X \) are discrete, then we can enumerate the number of curves of genus \( g \) and degree \( d \) in \( X \).

This hinges on \( \delta \) being surjective. To get that to be the case, you can try to get this by varying the quintic in \( H^0(\mathbb{P}^4, \mathcal{O}(5)) \). This works for some small choices of \( g \) and \( d \), but it is hard to arrange uniformly.

1.3. Enumerating rational curves in a K3 surface.

**Definition 1.3.1.** A K3 surface is a 2-dimensional Calabi-Yau manifold.

Let \( X = S \) be a K3 surface. Pick \( L \in \text{Pic}(S) \) and \( C \in |L| \) a smooth curve. Then the adjunction formula (and triviality of \( K_S \)) tells us that

\[
g_a(C) = \frac{1}{2} L^2 + 1
\]

Applying Riemann-Roch for surfaces and using that \( T_C \otimes N_C|_S \cong \Lambda^2 TX \cong \mathcal{O}_X \), and \( \deg N_C|_X = C \cdot C = L^2 \) we also find that

\[
\dim |L| = \frac{1}{2} L^2 + 1.
\]

We expect that \{curves with at least \( k \) nodes in \( |L| \}\} has codimension \( k \), hence (expected) dimension \( \dim |L| - k \). So the number of rational nodal \( C \in |L| \), which have \( \frac{1}{2} L^2 + 1 \) nodes, should be finite. (There’s a better argument, showing that \( |L| \) must contain a rational curve, and rational curves cannot form a family.)
Theorem 1.3.2 (Chen). For general $S$, we have $\text{Pic}(S) \cong \mathbb{Z}[L]$ and rational $C \in |L|$ are nodal.

Yau-Zaslow found a way to count this finite number. For smooth $C \in |L|$ and $\mathcal{F} \to C$ an invertible sheaf, we can form the 1-dimensional sheaf $\iota_* \mathcal{F}$ on $S$.

![Figure 1.3.1. Pushforward of a line bundle from a curve in $S$.](image)

Then

$$P_{\iota_* \mathcal{F}}(n) = \chi(\iota_* \mathcal{F} \otimes L^n) = \chi(\mathcal{F} \otimes H^{\otimes n}_C) = n \deg L_C + \deg \mathcal{F} + (1 - g).$$

The point is that this is a linear polynomial. Suppose we assume that $\deg \mathcal{F} + (1 - g)$ is an odd number, and let's even assume that it is 1. Define

$$\mathcal{M}_S(P) = \{ \text{stable sheaves } \mathcal{E} \text{ of } \mathcal{O}_S\text{-modules}, \chi(\mathcal{E}(n)) = p(n) \}.$$

There are some “very bad” 1-dimensional sheaves (Figure 1.3), and the “stability” condition is intended to rule them out.

**Theorem 1.3.3.** $\mathcal{M}_S(P)$ is smooth (using that $P(0) = 1$).

There is a map $\mathcal{M}_S(P) \xrightarrow{\pi} \text{Chow}(S)$ (this is the Chow variety of curves in $S$, not the Chow group!) sending $\mathcal{E} \mapsto \text{supp}({\mathcal{E}})$. By the assumptions on the Picard group, this must land in $|L|$. Yau and Zaslow made the very interesting observation that if $C \subset L$ is a nodal curve with geometric genus $> 1$, then $e(\pi^{-1}[C]) = 0$.

**Exercise 1.3.4.** Prove this. [Hint: If $C$ is a smooth curve, then $\text{Pic}^0(C)$ is an abelian variety. Then $\text{Pic}^0(C)$ acts on $\pi^{-1}[C]$ by tensoring a vector bundle with a line bundle on $C$. Use this to show that $e(\pi^{-1}[C]) = 0$. In fact, we don't need the full abelian variety; it's enough to have a torus action.]
Therefore, \[ e(M_X(P)) = \sum_{C \in |L| \text{ rational}} e(\pi^{-1}([C])). \]

Assume that \( S \) is general. Then any rational \( C \) in \(|L|\) are necessarily nodal.

**Exercise 1.3.5.** It is an exercise to show that \( e(\pi^{-1}([C])) = 1 \) in such cases. [Hint: \( \pi^{-1}([C]) \) consists of \( \mathcal{E} \) on \( \mathbb{P}^1 \) plus identifications over two pairs of fibers, say \( P \leftrightarrow P' \) and \( Q \leftrightarrow Q' \). This is the same as giving two separate identifications, hence two \( \mathbb{P}^1 \) each with two points glued, which have Euler number 1.]

**Theorem 1.3.6.** \( e(\mathcal{M}_S(P)) \) can be calculated.

**Theorem 1.3.7** (Yau-Zaslow). If \( n_g = \# \text{Rat}_{|L|}(S) \), where \( \frac{1}{2} L^2 + 1 = g \), then

\[ \sum n_g q^g = \prod_{n=1}^{\infty} (1 - q^n)^{-24}. \]

These ideas actually come from string theory.

1.4. **Course plan.** The plan of the course is to study curves in Calabi-Yau threefolds. One could first attempt to do this by classical algebraic geometry. It turns out that this leads to a lot of tricky issues about the space of curves.

However, we shall see that if certain genericity is met (as in the above examples), then we can get good answers. The genericity amounts to saying that the counting is a topological enumeration. To achieve this in algebraic geometry, we use virtual cycles of certain moduli spaces. For this we need the moduli spaces to be proper.

Here is a sampler of the moduli spaces we’re interested in:

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**Figure 1.3.2.** A sheaf on \( S \) which is composed of the pushforward of a line bundle curve plus a skyscraper sheaf over a point. The condition of “stability” rules out these sorts of “bad” sheaves.
• Moduli of stable maps:
\[ \mathcal{M}_g(X, \alpha) = \{ C^{\text{sm}} \to X \ | \ f_*[C] = \alpha \in H_2(X, \mathbb{Z}) \} \to \overline{\mathcal{M}}_g(X, \alpha). \]

• \( C \subset X \) as subscheme
\[ \{ C^{\text{sm}} \subset X \ | \ [C] \sim \alpha \in H_2(X, \mathbb{Z}) \} \subset \text{Hilb}_X^P \]

• Another compactification of curves as subschemes: any such curve gives a structure exact sequence
\[ 0 \to \mathcal{I}_C \to \mathcal{O}_X \to \mathcal{O}_C \to 0 \]
and we can consider the data \([\mathcal{O}_X \to \mathcal{O}_C]\) in the derived category and compactify.

The first moduli leads to GW invariants, the second leads to DT invariants, and the third leads to PT invariants.

Part 1. Moduli

2. Moduli spaces

We’re going to sketch constructions of three types of moduli spaces.

• One is a moduli space of framed objects, i.e. objects equipped with an embedding in an ambient space. This includes, for instance, the Hilbert schemes. Slightly more generally, we can consider moduli of sheaves, which leads to the Quot schemes.

• Another kind of moduli space is that of unframed objects, e.g. the moduli space of curves \( \mathcal{M}_g \). This was first studied by Mumford, using the strategy of quotienting a moduli space of framed objects by automorphisms.

• The third kind of moduli spaces are the stacks, which go beyond the realm of schemes.

2.1. Moduli problems.

Example 2.1.1. Consider all quotient vector spaces \( \mathbb{C}^m \to P \) where \( \dim P = k < m \). The space of such is parametrized by \( \text{Gr}(k, \mathbb{C}^m) \). This is one of the first examples of moduli spaces.

Example 2.1.2. (Hilbert schemes) Let \( X \subset \mathbb{P}^n \) be projective and \( h \) a polynomial. Let \( H = \mathcal{O}_{\mathbb{P}^n}(n)|_X \). We consider
\[ \{ Z \subset X \ | \ P_Z = h \} \]
where \( P_Z(n) = \chi(\mathcal{O}_Z \otimes H^\otimes n) \) is the Hilbert polynomial. Usually there is no dispute about what the closed points are, but it is not necessarily clear how to define the scheme structure. Sometimes, it is appropriate to define a nonreduced scheme structure.

Grothendieck taught us that scheme structure is captured by considering not just closed points but all scheme-valued points. This leads us to consider the moduli functor
\[ \mathcal{F} : \text{Schemes} \to \text{Sets}^{\text{opp}} \]
sending
\[ S \mapsto \left\{ Z \to S \ | \ \begin{array}{l}
Z \text{ closed } \subset Z \times S \\
\text{ flat over } S \text{ for all } s \in S
\end{array} \right\}. \]
This is functorial: if \( S \to S' \), then a family over \( S' \) can be pulled back to a family over \( S \).

**Exercise 2.1.3.** Carefully write down a moduli functor for other moduli problems.

**Definition 2.1.4.** We say that \( M \) is the fine moduli space of \( \mathcal{F} \) if there exists a natural transformation

\[
\mathcal{F} \cong \text{Hom}(\cdot, M) : \text{Schemes} \to \text{Sets}^{\text{opp}}.
\]

Here \( \text{Hom}(\cdot, M)(S) = \text{Hom}(S, M) \).

In particular this implies that \( \mathcal{F}(M) = \text{Hom}(M, M) \), so there should be a family \( \mathcal{X} \to M \) corresponding to \( 1_M \in \text{Hom}(M, M) \). This \( \mathcal{X} \to M \) is called the universal family.

By the definition of \( \mathcal{X} \to 1_M \), we have that \( \mathcal{Y} \) is canonically isomorphic to the pullback of \( \mathcal{X} \) via \( \rho \).

**Example 2.1.5.** The fine moduli space for Example 2.1.1 is \( \text{Gr}(k, \mathbb{C}^m) \).

### 2.2. Quot schemes

Let \( Z \) be a closed subscheme of a projective variety \( X \) (with a given ample line bundle \( \mathcal{O}(1) \)). Then \( Z \) is defined by its ideal sheaf \( \mathcal{I}_Z \subset \mathcal{O}_X \), which fits into a “fundamental exact sequence”

\[
0 \to \mathcal{I}_Z \to \mathcal{O}_X \to \mathcal{O}_Z \to 0.
\]

The subsheaf is completely determined by the quotient sheaf, and it turns out to be better to think about quotients. So we are going to construct a moduli space for quotient sheaves \( \mathcal{O}_X \to \mathcal{F} \) with \( \chi(\mathcal{F}) = h \), where

\[
\chi(\mathcal{F}(n)) = \chi(\mathcal{F}(n)).
\]

**Definition 2.2.1.** A quotient sheaf of a fixed vector bundle \( \mathcal{E} \) over \( X \) is a surjection of sheaves \( \mathcal{E} \to \mathcal{F} \).

If \( \mathcal{E} \to \mathcal{F} \) is a surjection over \( X \), we can push it forward via a closed embedding \( \iota : X \hookrightarrow \mathbb{P}^n \). Any quotient sheaf \( \iota_* \mathcal{E} \to \mathcal{F}' \) over \( \mathbb{P}^n \) is obtained from \( \mathcal{F}' = \iota_* \mathcal{F} \). Using this fact we can punt our problems to projective space.

**Definition 2.2.2.** Fix a projective variety \( (X, \mathcal{O}(1)) \), a polynomial \( h \), and \( \mathcal{E} \) a sheaf of \( \mathcal{O}_X \)-modules. We define a moduli functor \( \text{Quot}^h_{\mathcal{E}} : \text{Schemes} \to \text{Sets}^{\text{opp}} \) sending

\[
S \mapsto \left\{ p_X^* \mathcal{E} \to \mathcal{F} \mid p_{\mathcal{F}} \text{ flat over } S \text{ for all } s \in S \right\}.
\]

**Example 2.2.3.** For \( X = \text{pt} \), we consider quotients as in Example 2.1.1.

**Example 2.2.4.** If \( X \) is a projective scheme and \( \mathcal{E} = \mathcal{O}_X \), then we are parametrizing quotients \( [\mathcal{O}_X \to \mathcal{F}] \).

**Theorem 2.2.5.** The moduli functor \( \text{Quot}^h_{\mathcal{E}} \) has a projective fine moduli space, called the Quot scheme.
Outline of proof. Let $T$ be a scheme. Suppose we have an object $[p^*_X \mathcal{E} \to \mathcal{F}] \in \text{Quot}^h_{\mathcal{E}}(T)$. We're going to apply a standard trick of twisting by a higher power of $\mathcal{O}(1)$.

First some notation: we have a line bundle $\mathcal{O}(1)$ on $X$, and we denote

$$ p^*_X \mathcal{E}(\mu) := p^*_X \mathcal{E} \otimes p^*_X \mathcal{O}(\mu). $$

Consider the exact sequence

$$ 0 \to \mathcal{H} \to p^*_X \mathcal{E} \to \mathcal{F} \to 0. $$

We can twist by $\mu$ to obtain

$$ 0 \to \mathcal{H}(\mu) \to p^*_X \mathcal{E}(\mu) \to \mathcal{F}(\mu) \to 0. $$

We can then push forward to $T$ to get a long exact sequence

$$ 0 \to p_{T*} \mathcal{H}(\mu) \to p_{T*} p^*_X \mathcal{E}(\mu) \to p_{T*} \mathcal{F}(\mu) \to R^1 p_{T*} \mathcal{H}(\mu) \to \ldots. $$

For a given $\mathcal{H}$, we have that $R^1 p_{T*} \mathcal{H}(\mu) = 0$ for large enough choice of $\mu$. That this choice can be made uniformly in $\mathcal{H}$ is highly non-obvious, and is the key ingredient in the proof. Also note that $p_{T*} p^*_X \mathcal{E}(\mu) = H^0(X, \mathcal{E}(\mu)) \otimes \mathcal{O}_T$.

Now we use the (crucial!) flatness assumption. One characterization of $\mathcal{F}$ being flat over $T$ is that $\mathcal{F} \to X \times T$ is flat over $T$ if for $\mu \gg 0$, $p_{T*} \mathcal{F}(\mu)$ is a locally free sheaf of $\mathcal{O}_T$-modules. (This uses Serre’s vanishing theorem for ample line bundles plus cohomology and base change.)

Therefore, from the original quotient $[p^*_X \mathcal{E} \to \mathcal{F}] \in \text{Quot}^h(\mathcal{E})$ we get a point $[H^0(X, \mathcal{E}(\mu)) \otimes \mathcal{O}_T \to p_{T*} \mathcal{F}]$ in a certain Grassmannian, namely $\text{Gr}(H^0(X, \mathcal{E}(\mu)), h(\mu))(T)$. This globalizes to a map

$$ \xi \mapsto \psi_\mu(\xi): \text{Quot}_\mathcal{E}^h \to \text{Gr}_\mu := \text{Gr}(H^0(X, \mathcal{E}(\mu)), h(\mu)). $$

This embeds the moduli functor in a Grassmannian, and we want to show that Quot schemes are cut out as a closed subscheme of the Grassmannian.

There is a significant issue outstanding here, which is that we need to argue that we can find a single $\mu$ that works for all $\xi \in \text{Quot}_\mathcal{E}^h(T)$. For dimension 1, this is clear by Riemann-Roch. In general, one tries to slice the dimension down and use induction. We will not give the proof.

The problem is equivalent to showing that $\text{Quot}_\mathcal{E}^h(\mathcal{C})$ is “bounded” in an appropriate sense. To be precise, this should mean that it is parametrized by a finite type scheme.

Lemma 2.2.6. The map $\psi_\mu: \text{Quot}_\mathcal{E}^h \to \text{Gr}_\mu$ is injective.

(It doesn't follow immediately that $\psi_\mu$ is actually an embedding, but we'll show that later.)

Proof. Suppose that $\xi_1 = [\mathcal{E} \to \mathcal{F}_1]$ and $\xi_2 = [\mathcal{E} \to \mathcal{F}_2]$ are two elements of $\text{Quot}_\mathcal{E}^h(\mathcal{C})$ such that $\psi_\mu(\xi_1) = \psi_\mu(\xi_2)$. Then we want to show that $\xi_1 = \xi_2$.

In other words, we have that $H^0(\mathcal{F}_1(\mu)) = H^0(\mathcal{F}_2(\mu))$ as elements of $\text{Gr}_\mu$. Then we want to argue that $\mathcal{F}_1 = \mathcal{F}_2$. Well, we have the exact sequences

$$ 0 \to \mathcal{H}_1 \to \mathcal{E} \to \mathcal{F}_1 \to 0 $$

and

$$ 0 \to \mathcal{H}_2 \to \mathcal{E} \to \mathcal{F}_2 \to 0. $$
We have that $E(\mu)$ is generated by global sections for $\mu$ sufficiently large, i.e. $H^0(E(\mu)) \otimes \mathcal{O}_X \to E(\mu)$. The key point is that we can also assume for $\mu$ large that $\mathcal{H}_1(\mu)$ is generated by global sections, and that this can be arranged uniformly in $\mathcal{H}_1$. This implies that $\mathcal{H}_1(\mu)$ is the image of $H^0(\mathcal{H}_1(\mu)) \otimes \mathcal{O}_X \to H^0(E(\mu)) \otimes \mathcal{O}_X \to E(\mu)$, and similarly for $\mathcal{H}_2$.

$$\begin{array}{cccccc}
0 & \to & H^0(\mathcal{H}_1(\mu)) & \to & H^0(X, E(\mu)) & \to & H^0(\mathcal{F}_1(\mu)) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^0(\mathcal{H}_2(\mu)) & \to & H^0(X, E(\mu)) & \to & H^0(\mathcal{F}_2(\mu)) & \to & 0
\end{array}$$

So we have shown that the equality of global section implies (for large enough $\mu$) that $H^1(\mu) = H^2(\mu)$, which implies that $\mathcal{F}_1 = \mathcal{F}_2$. This shows (granting $\mu$ exists with all desired vanishing) that $\mathcal{F} \xrightarrow{\psi_\mu} \text{Hom}(-, \text{Gr}_\mu)$ is injective.

Next, we define closed subscheme $Q_\mu \subset \text{Gr}_\mu$ such that for all $\rho: T \to \text{Gr}_\mu$, $\rho$ factors through $T \xrightarrow{\rho} Q_\mu \xrightarrow{\psi_\mu} \text{Gr}_\mu$ if and only if there is a $\xi \in \text{Quot}^h_\mathcal{E}(T)$ such that $\rho = \psi_\mu(\xi)$, i.e.

$$\begin{array}{cccccc}
\text{Quot}^h_\mathcal{E} & \xrightarrow{\sim} & \text{Hom}(-, \text{Gr}_\mu) & \xrightarrow{\psi_\mu} & \text{Hom}(-, Q_\mu)
\end{array}$$

Recall that the map $\psi_\mu$ was constructed by starting with $\xi = [0 \to \mathcal{E} \to p_X^* \mathcal{E} \to \mathcal{F} \to 0] \in \text{Quot}^h_\mathcal{E}(T)$ and twisting and pushing it forward to $0 \to p_T^* \mathcal{H}(\mu) \to p_T^* p_X^* \mathcal{E}(\mu) \to p_T^* \mathcal{F}(\mu) \to 0$

The key trick is to apply this to the universal family on $\text{Gr}_\mu =: \text{Gr}$. That is a vector bundle $\mathcal{P}$ equipped with a natural surjection $0 \to \mathcal{X} \to W \otimes \mathcal{O}_{X \times \text{Gr}} \to \mathcal{P} \to 0$ where $W = H^0(X, \mathcal{E}(\mu))$. Taking global sections, we get (for large enough $\mu$) an exact sequence $0 \to K = H^0(\mathcal{X}(\mu)) \to W \to H^0(\mathcal{P}(\mu)) \to 0$.

A map $X \to \text{Gr}$ is determined by the information of a surjection of bundles over $X$ $W \otimes \mathcal{O}_X \to \mathcal{E}$, which we can suppose has kernel $\mathcal{X}$. Then for our choice of $\mu$, the image of $K \to W \otimes \mathcal{O}_X \to \mathcal{E}(\mu)$ is $\mathcal{X}(\mu) \subset \mathcal{E}(\mu)$. The point is that this recovers the subsheaf $\mathcal{X}(\mu) \subset \mathcal{E}(\mu)$.
**Proposition 2.2.7.** There exists a locally closed \( Q \subset \text{Gr}_\mu \) such that

\[
\xymatrix{
\text{Quot}_P^X \ar[r] & \text{Hom}(-, \text{Gr}_\mu) \\
\text{Hom}(-, Q) \ar[ru] &
}
\]

**Proof.** A \( T \)-point of \( \text{Quot}_P^X \to \text{Hom}(-, \text{Gr}_\mu) \) is given by a surjection \( p^*_X \mathcal{E} \to \mathcal{F} \).

We have shown how to choose a uniform \( \mu \) so that this gives a \( T \)-point of \( \text{Gr}_\mu \):

\[
W \otimes \mathcal{O}_{X \times \text{Gr}} \to p^*_X \mathcal{E}(\mu).
\]

Let \( \mathcal{H}(\mu) \) be the image of

\[
\mathcal{H} \otimes \mathcal{O}_{X \times \text{Gr}} \to W \otimes \mathcal{O}_{X \times \text{Gr}} \to p^*_X \mathcal{E}(\mu).
\]

The quotient sheaf \( P(\mu) := p^*_X \mathcal{E}(\mu)/\mathcal{H}(\mu) \) is not flat over all of \( \text{Gr} \) (as expected). What we expect is that for some closed subset \( Q \subset \text{Gr} \), all closed points \( \omega \in Q \subset \text{Gr} \) satisfy

\[
H^0(P|_{X \times \omega}(\mu)) = P(\mu) \quad \text{and} \quad H^{1>0}(P|_{X \times \omega}(\mu)) = 0.
\]

**Exercise 2.2.8.** Consider the map of sheaves

\[
\mathcal{O}_{\mathbb{A}^3} \to \mathcal{O}_{\mathbb{A}^2}
\]

given by

\[
(a_1, a_2, a_3) \mapsto (xa_1 + y^2a_2, xa_3).
\]

Check that the loci where the rank of the quotient is equal to 0, 1, 2 are locally closed subschemes.

**Lemma 2.2.9.** Let \( Z \) be a scheme. For any vector bundles \( V_1, V_2 \) and \( \varphi : \mathcal{O}_Z(V_1) \to \mathcal{O}_Z(V_2) \) and any \( r \), there exists a unique maximal locally closed \( Z_r \subset Z \) such that the cokernel of \( \mathcal{O}_Z(V_1)|_{Z_r} \to \mathcal{O}_Z(V_2)|_{Z_r} \) is a rank \( r \) locally free sheaf of \( \mathcal{O}_Z \)-modules.

This is a determinantal construction. This lemma plus a technical argument implies what we want. We're going to skip the details. The idea is that you push all calculations to the Grassmannian.

\[ \square \]

It only remains to show that \( Q \subset \text{Gr}_\mu \) is proper. This will imply that \( Q \) is projective and since \( \text{Quot}_P^X \to \text{Hom}(-, Q) \) is an equivalence. This proves that \( \text{Quot}_P^X \) has a fine moduli space.

**Proof.** We'll use the valuative criterion for properness, which says that if \( C^* = C - \{\text{pt}\} \), then a map \( C^* \to Q \) can be extended to \( C \to Q \):

\[
\xymatrix{& C^* \ar[dr] & C \ar[d] & Q \ar[dl] \\
& 0 & 0 & 0}
\]
A map $C^* \to Q$ is equivalent to an element of $\text{Quot}^p(C^*)$, which is the data of a surjection $p_X^*E \to \mathcal{F}^*$ over $X \times C^*$. The game is to extend this to a flat surjection of sheaves $p_X^*E \to \mathcal{F}$ over $X \times C$. Let $\mathcal{K}^* = \ker p_X^*E \to \mathcal{F}^*$, which fits into a short exact sequence.

$$0 \to \mathcal{K}^* \to p_X^*E \to \mathcal{F}^* \to 0.$$ Let $\mathcal{K} = \{s \in p_X^*E \mid s|_{X \times C^*} \in \mathcal{K}^*\}$. The key point is that defining this as a subsheaf of $p_X^*E$ on $X \times C$ makes it coherent and flat.

**Exercise 2.2.10.** Prove the flatness.

**Example 2.2.11.** Here is an example of what can go wrong if you extend too naïvely. If $\mathcal{F}^* = 0$, then you could take $\mathcal{O}_C \to \mathcal{O}_C \to \mathcal{O} \to 0$. This is not a flat extension.

We have finally established (modulo some details):

**Theorem 2.2.12.** $\text{Hilb}_X^p$ is represented by a projective scheme.

### 3. Moduli of sheaves via GIT

#### 3.1. Stable sheaves

Suppose $(X, H)$ is a projective scheme. Let $V$ be a holomorphic vector bundle on $X$. Then its sheaf of sections $\mathcal{O}_X(V)$ is locally free. This construction induces an equivalence

$$\{\text{Vector Bundles}/X\} \cong \{\text{locally free } \mathcal{O}_X\text{-modules}\}.$$ If $\iota: Z \hookrightarrow X$ and $\mathcal{E}$ is locally free over $Z$, then $\iota_* \mathcal{E}$ is a sheaf of $\mathcal{O}_X$-modules. We have $\mathcal{O}_X/I_Z \cong \iota_* \mathcal{O}_Z$.

**Dimension of a sheaf.** We want to quantify a notion of dimension for sheaves.

**Example 3.1.1.** $\mathcal{O}_X/I_Z$ is supported on $Z \subset X$, and so should have the same dimension as $Z$.

**Definition 3.1.2.** We say that $\mathcal{E}$ is supported on $Z$ if for all $f \in I_{Z \subset X}$, multiplication by $f$ on $\mathcal{E}$ is $0$.

$$\begin{array}{ccc}
\mathcal{E} & \times f & \mathcal{E} \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

**Example 3.1.3.** If $\mathcal{E} = k[x, y]/(x, y^2)$ then $\mathcal{E}$ is supported on $V(x, y^2)$, which has dimension $0$.

**Definition 3.1.4.** Let $\mathcal{E}$ be a sheaf and $s \in \mathcal{E}$ a non-zero section. Then we define

$$\dim s = \dim \text{Im}(\mathcal{O}_X \xrightarrow{\times s} \mathcal{E})$$

$$\quad = \dim \text{supp}(s).$$

**Example 3.1.5.** Let $X$ be a smooth variety and $\mathcal{E}$ a sheaf. The torsion subsheaf is $\mathcal{E}^\text{tor} := \{s \in \mathcal{E} \mid \dim s < \dim X\}$. The quotient $\mathcal{E}/\mathcal{E}^\text{tor}$ is torsion-free.
**Definition 3.1.6.** A sheaf $\mathcal{E}$ is of pure dimension $r$ if for any non-zero $s \in \mathcal{E}$, we have $\dim s = r$.

**Definition 3.1.7.** If $\mathcal{E}$ has pure dimension $r$, then its Poincaré polynomial is

$$P_{\mathcal{E}}(n) := \chi(\mathcal{E} \otimes H^\otimes n) = a_r n^r + \ldots$$

which has degree $r$. We also define the monic normalization

$$p_{\mathcal{E}}(n) := \frac{P_{\mathcal{E}}(n)}{a_r} = n^r + \ldots$$

**Example 3.1.8.** If $\dim X = 1$, then

$$P_{\mathcal{E}}(n) = n(\text{rank} \mathcal{E}) \deg H + \chi(\mathcal{E})$$

by Riemann-Roch, and

$$p_{\mathcal{E}}(n) = n + \frac{\chi(\mathcal{E})}{\text{rank} \mathcal{E} \deg H}.$$

We can finally give the definition of stable sheaves.

**Definition 3.1.9 (Simpson).** A $\mathcal{E}/X$ is stable if and only if it is of pure dimension $r$ and for any subsheaf $\mathcal{F} \subset \mathcal{E}$, we have

$$p_{\mathcal{F}}(n) \ll p_{\mathcal{E}}(n)$$

i.e. $p_{\mathcal{F}}(n) < p_{\mathcal{E}}(n)$ for sufficiently large $n$.

**Proposition 3.1.10.** (i) The collection of all stable sheaves which are pure of fixed Poincaré polynomial $P$ is bounded.

(ii) If $\mathcal{E}$ is stable, then $\text{Hom}(\mathcal{E}, \mathcal{E}) \cong \mathbb{C}$.

**Proof.** We give only the proof of (ii). Suppose you have $\mathcal{E} \xrightarrow{\alpha} \mathcal{E} \to \mathcal{F}$ with $\mathcal{F}$ non-zero. (Replacing $\alpha$ be $\alpha - c \text{Id}$, we can ensure that the map is not surjective.) Then there is a kernel $\mathcal{K}$, which fits into the short exact sequence

$$0 \to \mathcal{K} \to \mathcal{E} \xrightarrow{\alpha} \mathcal{E} \to \mathcal{F} \to 0.$$

Since the Hilbert polynomials are equal, we must have $\mathcal{K}$ non-zero. Since $\mathcal{E}$ is stable, $p_{\mathcal{K}} \ll p_{\mathcal{E}}$. Let $\mathcal{P}$ be the cokernel of $\mathcal{K} \to \mathcal{E}$, which fits into

$$0 \to \mathcal{K} \to \mathcal{E} \xrightarrow{\alpha} \mathcal{E} \to \mathcal{F} \to \mathcal{P} \to 0.$$

Then by the definition of stability we also have

$$p_{\mathcal{P}} \ll p_{\mathcal{E}},$$

$$P_{\mathcal{K}} + P_{\mathcal{P}} = P_{\mathcal{E}}.$$

It is an exercise to show that these equations are incompatible. □

A consequence of the first part is:

**Corollary 3.1.11.** There is a uniform $\mu_0$ such that for $\mathcal{E}$ as in the hypotheses of the proposition, $H^0(\mathcal{E}(\mu)) \otimes \mathcal{O}_X \to \mathcal{E}(\mu)$ for all $\mu \geq \mu_0$. 
We want to make a moduli space of sheaves. You always run into trouble when making a moduli space of objects with jumping automorphisms. For stable bundles we at least understand the automorphisms, which are as small as possible.

### 3.2. Moduli of stable sheaves.

We want to construct a moduli space for sheaves $\mathcal{E}$ with $P_\mathcal{E} = P$ for a fixed polynomial $P$. Pick a large $\mu$ such that $H^0(\mathcal{E}(\mu)) \otimes \mathcal{O}_X \rightarrow \mathcal{E}(\mu)$ for all stable $\mathcal{E}$. Then can try to construct a moduli space of stable sheaves by embedding it in a quot scheme via

$$[\mathcal{E}] \rightarrow [H^0(\mathcal{E}(\mu)) \otimes \mathcal{O}_X \rightarrow \mathcal{E}(\mu)].$$

Here the Quot scheme in question is $\text{Quot}_{\mathcal{O}_X}^{P(\mu)}(O_X)$.

For $N = P(\mu)$, let $O_X^{\oplus N} \rightarrow \mathcal{E}$ be the tautological surjection onto the universal bundle, corresponding to the map $\text{Gr} \rightarrow \text{Gr}$. The “map” \( \varphi \) depends on a choice of isomorphism $H^0(\mathcal{E}(\mu)) \cong \mathbb{C}^{\oplus P(\mu)}$. The ambiguity in the choice of basis is measured by $\text{GL}(N)$. In fact we can shrink the automorphisms to $\text{SL}(N)$.

Therefore, the map takes points to $\text{SL}(N)$-orbits:

$$\{\mathcal{E} | \mathcal{E} \text{ stable, } P_\mathcal{E} = P\} \xrightarrow{\text{one to SL}(N)\text{ orbit}} Q.$$

The action of $\text{SL}_n$ on $[\mathcal{O}_X^{\oplus N} \rightarrow \mathcal{E}]$ is by pre-composition. The image of $\varphi$ inside $Q$ is an $\text{SL}(N)$-invariant open subscheme. The idea is to define a moduli space of stable sheaves by taking the “quotient” with respect to the $\text{SL}(N)$-action.

### 3.3. Geometric invariant theory.

We’re going to discuss a simple version of GIT for $G = \text{SL}(n, \mathbb{C})$.

Let $V$ be a $G$-representation over $\mathbb{C}$, which we can think of in terms of a map

$$G \rightarrow \text{GL}(V).$$

Then $G$ acts on $\mathbb{P}(V) = (V - 0)/\mathbb{C}^*$. The goal is to construct a quotient scheme $\mathbb{P}V/G$, which is morally the “space of $G$-orbits”.

Here is the key observation. We have an embedding $\mathbb{P}V \hookrightarrow \mathbb{P}H^0(\mathcal{O}_V(\ell))'$ for $\ell > 0$, and action of $G$ on $H^0(\mathcal{O}_V(\ell))'$. The invariant functions on this space are $H^0(\mathcal{O}_V(\ell))'^G$. If $\ell$ is large, then $H^0(\mathcal{O}_V(\ell))'$ becomes large and we might hope to find enough $G$-invariant functions to find an embedding.

$$\mathbb{P}(H^0(\mathcal{O}_V(\ell))')^G \rightarrow \mathbb{P}V \rightarrow \mathbb{P}H^0(\mathcal{O}_V(\ell))'$$

The crux of the matter is then to study $G$-invariant sections in $H^0(\mathcal{O}(\ell))$.

**Definition 3.3.1.** Let $v \in V - 0$ represent $[v] \in \mathbb{P}V$. We say that

1. $[v]$ is semistable if $\overline{Gv}$ does not contain $0$,
2. $[v]$ is stable if $Gv$ is closed in $V$ and $\text{Stab}_G(v) < \infty$.
3. $[v]$ is weakly stable if $Gv$ is closed.
Lemma 3.3.2. Let $X_1, X_2 \subset V$ be two disjoint $G$-invariant closed subsets. Then there is $s \in (\text{Sym } V^\vee)^G$ such that $s|_{X_1} = 0$ and $s|_{X_2} = 1$.

Proof. Let $I(X_1)$ be the ideal of polynomials vanishing on $X_1$ and define $I(X_2)$ similarly. We know that $I(X_1) + I(X_2)$ contains the element 1, so we can find

$$1 = f_1 + f_2$$

(3.3.1)

with $f_1 \in I_1$ and $f_2 \in I_2$, which implies that $f_1|_{X_1} = 0$ and $f_1|_{X_2} = 1$, and $f_2|_{X_2} = 0$ and $f_2|_{X_1} = 1$. Then we apply a Reynolds operator

$$R: \{\text{polynomials}\} \rightarrow \{G\text{-invariant polynomials}\}.$$

At least over characteristic 0, this exists for any reductive group. (For reductive groups, GIT always amounts to careful applications of the Reynolds operators.)

Definition 3.3.3. A Reynolds operator is a $G$-equivariant map $R: W_1 \rightarrow W_1^G$ which is

1. functorial:

\[\begin{array}{ccc}
W_1 & \phi & W_2 \\
R & \downarrow & R \\
W_1^G & \rightarrow & W_2^G
\end{array}\]

2. $R(\alpha f) = \alpha R(f)$ for $G$-invariant $\alpha$.

What is the construction of the Reynolds operator? You take the maximal compact subgroup $K$ of $G$, whose tangent space complexifies to $\mathfrak{g}$. Then there is a Haar measure $\mu$ on $K$, and you can apply Weyl’s trick to:

$$R(f)(x) = \int_K f(gx) d\mu(g).$$

This is obviously $K$-invariant. It follows that it is even $G$-invariant, since this is equivalent to being $\mathfrak{g}$-invariant, and $\mathfrak{g}$ is the complexification of $\text{Lie}(K)$.

Applying this to (2) above, we obtain

$$R(1) = R(f_1) + R(f_2)$$

where $R(f_1)|_{X_1} = 0$ and $R(f_1)|_{X_2} = 1$, and $R(f_2)|_{X_2} = 0$ and $R(f_2)|_{X_1} = 1$. □

Lemma 3.3.4. Let $v \in V$ be semistable. Then there exists $\ell$ and an $s \in H^0(\mathcal{O}(\ell))^G$ such that $s(v) \neq 0$.

Let $V_\ell = H^0(\mathcal{O}(\ell))$ be the space of homogeneous polynomials of degree $\ell$ on $V$.

Proof. Take $X_1 = \overline{G \cdot v}$ and $X_2 = \{0\}$. By the definition of semistability, they are disjoint. Apply the preceding lemma to find $s \in \text{Sym}(V^\vee)^G$ such that $s|_{\overline{G \cdot v}} = 1$ and $s(0) = 0$. Write

$$s = f_1 + f_2 + \ldots + f_\ell + \ldots$$

with $f_\ell \in V_\ell^G$. So there exists $\ell$ such that $f_\ell(v) \neq 0$. □
Let $V_{ss}$ be the set of semistable points in $V - \{0\}$ and $U_{ss}$ be the image of $V_{ss}$ in $\mathbb{P} V$. Then we know that for all $[v] \in U_{ss}$, there exists $f \in V^G_t$ with $f(v) \neq 0$.

**Lemma 3.3.5.** There exists $\ell_0$ such that when $\ell_0$ divides non-zero $\ell$, for any $v \in V_{ss}$ there exists $s \in V^G_\ell$ such that $s(v) \neq 0$.

**Proof.** Take $R = \bigoplus_{\ell > 0} H^0(\mathcal{O}(\ell)) = \bigoplus V_\ell$. Then $R^G = \bigoplus_{\ell > 0} V^G_\ell$. It suffices to show that $R$ is finitely generated as an algebra over $\mathbb{C}$.

Let $S^+$ be the subset of $G$-invariant homogeneous polynomials of positive degree. We consider $S^+ \cdot \text{Sym}(V^\vee)$, the ideal generated by $G$-invariant polynomials of positive degree. It is finitely generated, say be $f_1, \ldots, f_n, g_1 \ldots, g_n$ where the $f_i$ are homogeneous and $G$-invariant.

We claim that $f_1, \ldots, f_N$ generate $R^G$. To see this, consider $h \in R^G$ with vanishing constant term. Then $h \in S^+ \mathcal{O}[V]$, and we can write

$$h = \sum_{i=1}^N f_i(g_i h_i).$$

We can assume that $\deg(g_i h_i) < \deg h$. Using that $f_i$ are homogeneous of positive degree, we have

$$h = R(h) = \sum_{i=1}^N f_i R(g_i h_i)$$

where the $R(g_i h_i)$ are $G$-invariant. The result follows by induction. \qed

Now we have constructed $U_{ss} \xrightarrow{\Phi_\ell} \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P} V}(\ell))^\vee)^G$.

We claim that the maximal domain of $\Phi_\ell$ is $U_{ss}$. Suppose $[v] \in U_{ss}$ and suppose that $\Phi_\ell$ can be extended to $[v]$. Therefore, $G v \subset s^{-1}_t(1)$ for some $s_\ell \in (H^0(\mathcal{O}_{\mathbb{P} V}(\ell))^\vee)^G$, since we can find a homogeneous polynomial taking value 1 on the image of $v$, and then by invariance it takes that value everywhere. But 0 is in the closure of the orbit by definition of not being semistable.

The morphism

$$\Phi_\ell : U_{ss} \to \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P} V}(\ell))^\vee)^G$$

factors through $\text{Proj} R^G$, where $R^G = \bigoplus_{d \geq 0} H^0(\mathcal{O}(\ell))^G$.

$U_{ss} \xrightarrow{\Phi_\ell} \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P} V}(\ell))^\vee)^G \xrightarrow{\text{Proj} R^G}$

An important issue is how we know that $U_{ss} \to U_{ss}/G$ is surjective. This had better be a property of a quotient map. It is easy to show that its image is dense, so we should check the valuative criterion for properness. This cannot be checked at the level of $U_{ss}$, since that is not proper! A “sequence” of points can go off to $\infty$ in the $G$-direction, so we somehow need to use the $G$-action to “renormalize” it.
Proposition 3.3.6. We have

(1) $\Phi_\ell$ is surjective.
(2) $\Phi_\ell$ induces a bijection from weakly stable $G$-orbits to closed points in $\text{Proj} \, R^G$.

Proof. We need to quote the semistable replacement theorem, which says:

Let $p \in C$ be a pointed smooth curve ($C$ is really the spectrum of a DVR). Let $\phi : C - p \to U_{ss}$ be a morphism. Then there exists a finite base change $f : C' \to C$ and $\sigma : C' - p' \to G$ such that

$$(\phi \circ f)^\sigma = \sigma(x) \cdot \phi(x) : C' - p \to U_{ss}$$

extends to $C' \to U_{ss}$.

$\square$

3.4. The numerical criterion. There is an effective method to characterize (semi)stable points. Again, we're going to study the special case $G = \text{SL}_n$.

Definition 3.4.1. A one-parameter subgroup is a map $\lambda : \mathbb{C}^* \to G$.

There is a basis $w_1, \ldots, w_n$ of $V$ such that $\lambda$ can be diagonalized as $w_i^{\lambda(t)} = t^{r_i} w_i$ for $r_1 \leq r_2 \leq \ldots \leq r_n$ and $\sum r_i = 0$.

Theorem 3.4.2 (Numerical criterion). For all $v \in V \setminus 0$,

(1) $v$ is stable if and only if for all one-parameter subgroups $\lambda$,

$$\lim_{t \to 0} \lambda(t) \cdot v = \infty.$$  

(2) $v$ is semistable if and only if for all $\lambda$,

$$\lim_{t \to 0} \lambda(t) \cdot v \neq 0.$$  

The point is that $\lim_{t \to 0} \lambda(t) \cdot v$ is in $\mathbb{C}^* \cdot V$ and not in $G \cdot v$ if it exists. So if $G \cdot v$ is closed (as it must be in the stable case), then this limit cannot exist. In the semistable case, it must not be 0.

Proof. Suppose that $v$ is not semistable. Then $\overline{G \cdot v}$ contains 0.

We’re going to unravel the valuative criterion for properness to see that there is a one-parameter subgroup with limit 0. The point is that if 0 is the limit of some curve, then it is the limit of the orbit of a one-parameter subgroup.

For $R$ a DVR with fraction field $K$, and $t$ a uniformizing parameter, we consider diagrams

$$\text{Spec } R \longrightarrow \overline{Gv}$$

$$\text{Spec } K \longrightarrow Gv$$

Assume that $\text{Spec } K \to Gv$ is represented by $\rho : \text{Spec } K \to G$, say

$$\rho(t) = \left( a_{ij}(t) \right) \in \text{SL}(N, K).$$
Then there exist $A_1, A_2 \in \text{SL}(R)$ such that

$$A_1 \rho A_2^{-1} = \begin{pmatrix} t^{a_1} c_1(t) \\ \vdots \\ t^{a_N} c_N(t) \end{pmatrix}$$

with the $c_i(t) \in R^\times$. We can then further modify to assume that all the $c_i$ are 1.

Suppose $\lim \rho(t) v = 0$. Since $\rho(t)V = \rho(t) A_2^{-1} A_2 v$, if we pick $\lambda$ to be the one-parameter subgroup $\lambda = A_2 \rho(t) A_2^{-1}$ (note that there is no $A_1$ in the definition) then

$$\lim_{t \to 0} \lambda(t) v = 0.$$

□

Apply the numerical criterion to $[\mathbb{C}^n \otimes V \xrightarrow{\phi} P] \in \text{Gr}(\mathbb{C}^n \otimes V, m)$. For $\lambda$ a one-parameter subgroup in $\text{SL}(n, \mathbb{C})$ we can find a basis $e_1, \ldots, e_n$ of $\mathbb{C}^n$ such that $e_i^{\lambda(t)} = t^{a_i} e_i$ for $a_1 \leq \ldots \leq a_n$. We want to show that $\lim_{t \to 0} \xi^{\lambda(t)} \neq 0$.

The homogeneous coordinates of $\xi$ in terms of a basis $u_1, \ldots, u_1$ of $V$ are

$$[\phi(e_{i_1} \otimes \omega_{j_1}) \wedge \phi(e_{i_2} \otimes \omega_{j_2}) \wedge \ldots \wedge \phi(e_{i_d} \otimes \omega_{j_d})]^{\lambda(t)} \neq 0 \in \wedge^d P \cong \mathbb{C}.$$

The one-parameter subgroup scales this by $t^{a_1 + \ldots + a_d}$. We need to verify that for all 1-parameter subgroups $\lambda$, we have $\xi^{\lambda(t)} \neq 0$. So we need to find bases $e_{i_1}, \ldots, e_{i_d}$ and $\omega_{j_1}, \ldots, \omega_{j_d}$ such that

$$\phi(e_{i_1} \otimes \omega_{j_1}) \wedge \phi(e_{i_2} \otimes \omega_{j_2}) \wedge \ldots \wedge \phi(e_{i_d} \otimes \omega_{j_d}) \neq 0$$

and $a_{i_1} + \ldots + a_{i_d} \leq 0$. If you think about this problem, you’ll realize that it the key is to consider the subgroups $\text{Im } \phi(\text{Span}\{e_{i_1}, \ldots, e_{i_d} \otimes V}) \subset P$.

**Theorem 3.4.3 (Mumford).** Let $\lambda$ be a one-parameter subgroup with $e_1, \ldots, e_n$ a choice of diagonalizing basis. Let $E_i = \text{Span}\{e_{i_1}, \ldots, e_{i_d}\}$. Let $H_i = \phi(E_i \otimes V) \subset P$. Then

$$\lim_{t \to 0} e^{\lambda(t)} = \infty$$

if and only if for any $k < n$,

$$\frac{k}{\dim H_k} < \frac{m}{\dim P}.$$

**3.5. Application to moduli of stable sheaves.** Let $\mathcal{E} \to X$ be a vector bundle of rank 2 on a smooth curve.

Given $[H^0(\mathcal{E}(\mu) \otimes \mathcal{O}_X) \to \mathcal{E}]$, we get

$$H^0(\mathcal{E}(\mu) \otimes \mathcal{O}_X(k)) \to \mathcal{E}(k)$$

Taking global sections, we get a map of vector spaces

$$[H^0(\mathcal{E}(\mu)) \otimes H^0(\mathcal{O}_X(k)) \to H^0(\mathcal{E}(k + \mu))]$$

such that the $\text{SL}(N)$ orbit of $[\mathbb{C}^n \otimes V \to P]$ is well-defined. So we want to make sense of an embedding to the quotient of $\text{Gr}$ by $\text{SL}(N)$. To do this, we need to check that the point $[\mathbb{C}^n \otimes V \to P]$ of $\text{Gr}$ is $\text{SL}(n)$-stable. Mumford’s criterion shows that this is the case.
provided that: for any basis $e_1, \ldots, e_n \in H^0(\mathcal{E}(\mu)) \cong \mathbb{C}^n$ and $E_i = \text{Span}(e_1, \ldots, e_i)$, $H_i = \phi(E_i \otimes V)$, we have for any $1 \leq k < m$

$$\frac{k}{\dim H_k} \leq \frac{n}{\dim P}.$$ 

Okay, so fix $e_1, \ldots, e_n$ and $k \in [1, n - 1]$. We need to estimate

$$h_i := \dim \Phi(\text{Span}(e_1, \ldots, e_i) \otimes V).$$

Let $\mathcal{F}_i(\mu) = \text{Span}(e_1, \ldots, e_i) \subset \Gamma(\mathcal{E}(\mu))$. The point is that

$$\text{Im} \Phi(\{e_1, \ldots, e_k\} \otimes V) \subset H^0(\mathcal{F}_i(k + \mu)) \subset H^0(\mathcal{E}(k + \mu)).$$

Since $H_i \subset H^0(\mathcal{F}(k + \mu))$, Riemann-Roch shows

$$h_i \leq (\text{Riemann-Roch number of } \mathcal{F}_i(k + \mu)) + h^1(\mathcal{F}_i(k + \mu)).$$

There are two cases.

1. If $h^1(\mathcal{F}_i(\mu)) = 0$, then we get that $i \leq h_i$ is at most $\chi(\mathcal{F}_i(\mu))$, so if $d = \deg \mathcal{O}_X(1)$ then

$$\frac{i}{\dim H_i} \leq \frac{\chi(\mathcal{F}_i(\mu))}{\dim H_i} = \frac{\chi(\mathcal{F}_i(\mu))}{\chi(\mathcal{F}_i(\mu + k))} = \frac{\mu d + \deg \mathcal{F}_i + \rank \mathcal{F}_i(1 - g)}{d(\mu + k) + \deg \mathcal{F}_i + \rank \mathcal{F}_i(1 - g)}.$$

When $\mathcal{E}$ is stable we have $\frac{\deg \mathcal{F}_i}{\rank \mathcal{F}_i} < \frac{\deg \mathcal{E}}{\rank \mathcal{E}}$, and you can check that this gives the correct inequality. We have used that for $k \gg 0$, $\dim H_i = h^0(\mathcal{F}_i(\mu + k))$.

2. If $H^1(\mathcal{F}_i(\mu)) \neq 0$ and $\rank \mathcal{E} = 2$, then this computation requires some $H^1$ inserted in. The essential case is $\rank \mathcal{F}_i = 1$. This is only possible if the degree is negative. The point is that this makes a huge gap in the inequality

$$\frac{\deg \mathcal{F}_i}{\rank \mathcal{F}_i} < \frac{\deg \mathcal{E}}{\rank \mathcal{E}}.$$

The point is that rank one subsheaves are not boundd, but the subsheaves of degree bounded from below is a bounded set.

**Theorem 3.5.1.** If $\mathcal{E}$ is stable, then for $k \gg \mu \gg 0$,

$$H^0(\mathcal{E}(\mu)) \otimes H^0(\mathcal{O}_X(k)) \longrightarrow H^0(\mathcal{E}(k + \mu))$$

is $\text{SL}(n)$-stable in the Quot scheme.
3.6. **Coarse moduli spaces.**

**Definition 3.6.1.** A scheme $M$ is a **coarse moduli space** of $F: \text{Sch} \rightarrow \text{Set}$ if there is a transformation $F \rightarrow \text{Hom}(-, M)$ such that

1. $\text{Hom}(k, M) \cong F(k)$ for algebraically closed $k$,
2. For all $N$ and $F \rightarrow \text{Hom}(-, N)$ there is a unique $\rho: M \rightarrow N$ such that the diagram

$$
\begin{array}{ccc}
F & \rightarrow & \text{Hom}(-, N) \\
\downarrow & & \downarrow \\
\text{Hom}(-, M) & \leftarrow & \\
\end{array}
$$

commutes.

The uniqueness of the coarse moduli space is clear from the fact that it possesses a universal property. The point is that in constructing moduli spaces, you sometimes have to shrink into order to get something separated. So $M$ is the space involving the least shrinkage.

**Theorem 3.6.2.** *The moduli of stable sheaves has a coarse moduli space.*

**Proof.** For us $F$ is the moduli functor of stable sheaves, so a point $[E] \in F(k)$ is the data of

$$[\mathcal{O}^n \cong H^0(\mathcal{E}(\mu)) \otimes \mathcal{O}_X \rightarrow \mathcal{E}(\mu)].$$

We can view this as an $\text{SL}(N)$-orbit in $\text{Quot}_{\mathcal{O}_X^n}$ which is moreover *stable*. The main issue is to construct the map

$$F \rightarrow (\text{Quot}_{\mathcal{O}_X^n})_{ss} // G \cong \text{Proj} \bigoplus_k H^0(\mathcal{O}_{\text{Quot}(k)})^G.$$

For all schemes $S$, and $\xi := [E \rightarrow X \times S] \in F(S)$ we need to construct a functorial map

$$\phi_{\xi}: S \rightarrow (\text{Quot}_{\mathcal{O}_X^n})_{ss} // G.$$

Covering $S$ by affines $S_\alpha$, we get

$$\pi_{S_\alpha}^* \pi_{S_\alpha}^* \mathcal{E}_\alpha(\mu) \rightarrow \mathcal{E}_\alpha(\mu).$$

A choice of framing $\pi_{S_\alpha}^* \pi_{S_\alpha}^* \mathcal{E}_\alpha(\mu) \cong \mathcal{O}_X^n_{X \times S_\alpha}$ defines a point in $\text{Quot}(S)$. Different choices of frames will lead to different points of the Quot scheme, but they differ by an action of $G$. Therefore, the composition map to the GIT quotient is well-defined:

$$
\begin{array}{ccc}
S_\alpha & \rightarrow & (\text{Quot}_{\mathcal{O}_X^n})_{ss} \\
\downarrow & & \downarrow \\
(\text{Quot}_{\mathcal{O}_X^n})_{ss} // G
\end{array}
$$

This allows us to define the map on the level of the $S_\alpha$, and then patch them.
In general, the stable locus is open in the semistable locus, which is compact. We define \( M_X(c_i) = \text{Quot}^P_{\mathcal{O}_X^{\oplus n}} // G \) to be the moduli of stable sheaves with fixed Poincaré polynomial (Chern class), and \( \overline{M}_X(c_i) = \text{Quot}^P_{\mathcal{O}_X^{\oplus n}} // G \) to be its compactification.

The map \( \text{Quot}_{ss} \to \text{Quot}_{ss} // G \) is a "good quotient", meaning that \( \text{Quot}_{ss} \) has a \( G \)-invariant affine cover by \( G \)-invariant open affines \( U \), so that \( U // G \) can simply be defined by \( \text{Spec} \Gamma(O_U)^G \). For \( \xi \) in the stable locus \( \text{Quot} // G \), the pre-image is a closed orbit. This fails in for points which are semistable but not stable. For that reason, \( \overline{M}_X(c_i) \) is the "moduli of \( S \)-quivariant classes of semistable sheaves". In terms of the Harder-Narasimhan filtration, there exists a unique filtration

\[
0 = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \ldots \subsetneq \mathcal{F}_n = \mathcal{E}
\]

such that \( \mathcal{F}_{k+1} / \mathcal{F}_k \) is stable and \( p_{\mathcal{F}_{k+1} / \mathcal{F}_k} \) is decreasing.

If \( \mathcal{E} \) is semistable, then there exists a filtration

\[
0 = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \ldots \subsetneq \mathcal{F}_n = \mathcal{E}
\]

such that \( \mathcal{F}_{k+1} / \mathcal{F}_k \) is stable and \( p_{\mathcal{F}_{k+1} / \mathcal{F}_k} = p_{\mathcal{E}} \). Then

\[
\text{Gr}(\mathcal{E}) := \bigoplus \mathcal{F}_{k+1} / \mathcal{F}_k
\]

is unique, and we say that \( \mathcal{E} \sim \mathcal{E}' \iff \text{Gr}(\mathcal{E}) \cong \text{Gr}(\mathcal{E}') \). \( \square \)

### 3.7. Moduli of stable maps

Let \( X \) be a projective scheme (smooth). We define

\[
\overline{M}_{g,n}(X, d) = \{ f: (C, p_1, \ldots, p_n) \to X: (*) \} / \sim
\]

where the condition \((*)\) means that

1. \( f \) is a morphism such that \( \deg f^*(\mathcal{O}_X(1)) = d \) (or more generally, that \( f_*[C] = \alpha \in H_2(X, \mathbb{Z}) \)).
2. \( (C, p_1, \ldots, p_n) \) is a pointed nodal projective curve of genus \( g \).
3. \( \# \text{Aut}(f) < \infty \), where automorphisms \( \alpha \) are commutative diagrams

\[
\begin{array}{ccc}
(C, p_1, \ldots, p_n) & \xrightarrow{\alpha} & X \\
\downarrow & & \downarrow \\
(C, p_1, \ldots, p_n)
\end{array}
\]

**Example 3.7.1.** Consider the case \( n = 0 \). Then for \( [f: C \to X] \in \overline{M}_g(X, d)(k) \), the condition that \( f \) is stable is equivalent to \( L := \omega_C \otimes f^* \mathcal{O}_X(1) \) being ample. For such \( f \), we have for \( \mu \gg 0 \)

\[
i_\mu: C \hookrightarrow \mathbb{P} H^0(L^{\otimes \mu})^\vee
\]

and thus

\[
(i_\mu, f): \mathbb{P} H^0(L^{\otimes \mu})^\vee \times X.
\]
Upon choosing a frame for $H^0(\mathcal{L}^\vee \otimes \mu)^\vee$, we have $\mathbb{P}H^0(\mathcal{L}^\vee \otimes \mu)^\vee \cong \mathbb{P}^N$. Then we can consider

$$\text{Hilb}_{\mathbb{P}^N \times X}^P \ni \left\{ \rho: C \hookrightarrow \mathbb{P}^N \times X; \rho^* \mathcal{O}_{\mathbb{P}^N}(1) \cong \omega_C \otimes \mathcal{O}_X(1) \right\} =: W.$$ 

This is defined up to a choice of frame, which should be cancelled out by the action of $\text{SL}(N + 1)$. So we would like, as before, to set

$$\overline{M}_g(X, d) := [W/G = \text{SL}(N + 1)].$$

The problem is that, unlike the situation with sheaves, it is difficult to find a $G$-invariant affine open cover. What works is to pass to an étale cover of $[W/G = \text{SL}(N + 1)]$, which gets into the territory of Deligne-Mumford stacks.

**Part 2. Deformation and obstruction theory**

**4. Obstruction theory**

**4.1. Initial examples.**

*Example 4.1.1.* Here's a toy example. Consider the equations

$$f_1 = z^2$$
$$f_2 = 2z^2 + z^3$$
$$f_3 = z$$

Let $X_i := (f_i = 0) \subset \mathbb{A}^1$. We can detect the local differences in the $X_i$ about 0 by mapping in from “thickened” neighborhoods. Consider first mapping in from “first-order neighborhoods”:

\[ \text{Spec } k[t]/t^2 \overset{\beta_1}{\longrightarrow} \text{Spec } k \overset{X}{\longrightarrow} \mathbb{A}^1 \]

Let $\beta_1: z \rightarrow t$ be a map $\text{Spec } k[t]/t^2 \rightarrow \mathbb{A}^1$. Then $\beta_1^*(f_1) = \beta_1^*(f_2) = 0$, while $\beta_1^*(f_3) \neq 0$. Therefore, $\beta_1$ factors through $X_1$ and $X_2$ but not $X_3$. These $\beta^*(f_i)$ are preliminary versions of obstruction classes.

We can continue the story to third order. Let $\beta_2$ be the lift $z \rightarrow t$.

\[ \text{Spec } k[t]/t^3 \overset{\beta_2}{\longrightarrow} \text{Spec } k[t]/t^2 \overset{\beta_1}{\longrightarrow} \text{Spec } k \overset{X}{\longrightarrow} \mathbb{A}^1 \]

Then $\beta_2^*(f_1) = t^2 \neq \beta_2^*(f_2) = 2t^2$, which tells us that $X_1$ and $X_2$ look different to third order.
Example 4.1.2. Now here’s a slightly more complicated example. Let $f = (z_1^3, z_2) \in \Gamma(O_{\mathbb{A}^2})$. We let $X := (f = 0) \subset \mathbb{A}^2$. Consider lifting to first order:

\[
\begin{array}{c}
\text{Spec } k[t]/t^2 \\
\beta_1 \\
\text{Spec } k \\
\end{array} \xrightarrow{f} \begin{array}{c}
\text{Spec } k[t]/t^3 \\
\beta_2 \\
\text{Spec } k[t]/t^2 \\
\beta_1 \\
\text{Spec } k \\
\end{array} \xrightarrow{X} \mathbb{A}^2
\]

Suppose $\beta_1$ is defined by $z_1 \mapsto t$ and $z_2 \mapsto 0$. Then $\beta_1^*(f) = 0$, i.e. the obstruction vanishes and we can extend. What about the next level?

\[
\begin{array}{c}
\text{Spec } k[t]/t^3 \\
\beta_2 \\
\text{Spec } k[t]/t^2 \\
\beta_1 \\
\text{Spec } k \\
\end{array} \xrightarrow{X} \mathbb{A}^2
\]

Suppose $\beta_2$ is defined by $z_1 \mapsto t$ and $z_2 \mapsto t^2$. Then $\beta_2$ is an extension of $\beta_1$. But $\beta_2^*(f) \in \mathbb{C}^{\otimes 2} \otimes (t^2)$ is non-zero so $\beta_2$ doesn’t factor through $X$. An obstruction class should vanish if and only if some extension is possible, and indeed one extension in this case is $z_1 \mapsto t + t^2$ and $z_2 \mapsto 0$.

The natural “home” for the obstruction class is not simply the ideal sheaf. In our example, consider the complex

\[
T_{O_{\mathbb{A}^2}} \xrightarrow{df} O_{\mathbb{A}^2} \rightarrow \mathbb{C}.
\]

Suppose we tensor with the ideal sheaf $\mathbb{C} \otimes (t^2)$. Then we get

\[
T_{O_{\mathbb{A}^2} \otimes (t^2)} \xrightarrow{df} O_{\mathbb{A}^2} \otimes (t^2) \rightarrow \mathbb{C} \otimes (t^2) \rightarrow 0
\]

The upshot is that it is not really the ideal that witnesses the obstruction, but the complex above.

4.2. The obstruction class. Given an $S$-scheme $X \to S$, suppose we can embed $X$ into a smooth $W \to S$:

\[
\begin{array}{c}
W \\
\downarrow \\
S
\end{array} \xrightarrow{X} \begin{array}{c}
W \\
\downarrow \\
S
\end{array}
\]

Then we can take $J = I_{X \subset W}$ and $L_{X/S}^{\geq -1} := [J/J^2 \to \Omega_{W/S}|_X] \in D^b(O_X)$. Note that $H^0(L_{X/S}^{\geq -1}) = \Omega_{X/S}$ (from the usual exact sequence of differentials).

Proposition 4.2.1. The complex $L_{X/S}^{\geq -1}$ is well-defined.
Proof. We need to prove independence of the embedding. Suppose we have two such $X \hookrightarrow W_1, X \hookrightarrow W_2$. We can put them together:

\[
\begin{array}{c}
W_1 \\
\downarrow \\
X \xrightarrow{\varphi} \\
\downarrow \\
W_2 \times W_2 \\
\downarrow \\
W_2
\end{array}
\]

which induces a map of complexes

\[
\begin{array}{c}
[J_1/J_1^2] \longrightarrow \Omega_{W_1/S}|_X \\
\downarrow \\
[J_{12}/J_{12}^2] \longrightarrow \Omega_{W_1 \times W_2/S}|_X.
\end{array}
\]

One can then check that this is a quasi-isomorphism, using

\[
\Omega_{W_1 \times W_2/S}|_X \cong \Omega_{W_1/S} \oplus \Omega_{W_2/S}.
\]

\[
\square
\]

We want to use this to handle intrinsic obstruction classes and lifting problems. For instance, we might want to analyze formal smoothness of $X$ at $p \in X$. For $(A, m)$ a local Artin $k$-algebra and $I$ an ideal with $mI = 0$, the formal smoothness of $X \to S$ is concerned with solving a lifting problem

\[
\begin{array}{c}
\text{Spec } A/I \\
\downarrow \\
\varphi_A/I \\
\downarrow \\
\text{Spec } A \\
\,
\end{array}
\quad
\begin{array}{c}
\varphi_A \\
\downarrow \\
\text{Spec } A/I \\
\,
\end{array}
\quad
\begin{array}{c}
X \\
\downarrow \\
\varphi \\
\downarrow \\
S
\end{array}
\]

The idea is that Spec $A$ is a little thickening of Spec $A/I$.

**Theorem 4.2.2.** There is a canonical obstruction class

\[
\text{ob}(\varphi_{A/I}, A, I) \in \text{Ext}^1_{A/I}(I_{X/S}^{\geq -1}, I)
\]

such that

1. $\text{ob}(\varphi_{A/I}, A, I) = 0$ if and only if $\varphi_A$ exists,
2. if $\varphi_A$ exists then the set of all $\varphi_A$ is an $\text{Ext}^0_{A/I}(I_{X/S}^{\geq -1}, I)$-torsor.
Proof. How to construct this class? Embed $X$ in a smooth $W/S$. Then we have some lift $\varphi$ to $W$.

This induces $\varphi^*: O_W \rightarrow A$ and we know that $J = I_{X/W}$ maps to $I$, so $J^2 \rightarrow I^2 = 0$. So $\varphi^*$ descends to a map

$$J/J^2 \varphi^* \rightarrow I.$$ 

This is part of a map $\epsilon$ of elements of $D^b(O_X)$:

$$\begin{array}{ccc}
\deg & -1 & 0 \\
I[1] := & [I \rightarrow 0] & \\
L_{X/S}^\geq := & [J/J^2 \rightarrow \Omega_{W/S}|_X] & \\
\end{array}$$

The map $\epsilon$ can be interpreted as an element in $\text{Hom}_{D^b(O_X)}(L_{X/S}^\geq, I[1]) = \text{Ext}^1(L_{X/S}^\geq, I)$. We define this to be the obstruction class $\text{ob}(\varphi_{A/I}, A, I)$ (although we have not yet shown that it is well-defined).

Let’s ignore the dependence on the embedding. What happens if we choose a different lift $\varphi: \text{Spec } A \rightarrow W$? Then we get a different class $\epsilon' \in \text{Ext}^1(L_{X/S}^\geq, I)$. We claim that there exists $\eta: \Omega_{W/S}|_X \rightarrow I$ such that

$$\epsilon - \epsilon' = \eta \circ \delta.$$ 

This will show that the two maps are homotopic in the derived category.

The two lifts $\varphi$ and $\varphi'$ coincide on the subscheme $\text{Spec } A/I$. That means that the “difference should be in the cotangent sheaf.” More precisely, Grothendieck’s definition of conormal sheaf $\Omega_W$ is as follows: for $\iota: W \rightarrow W \times W$ the diagonal embedding, we set

$$\Omega_W := \iota^*(I_\Delta/I_\Delta^2).$$

In our case, the two lifts put together give a map

$$\Phi: \text{Spec } A \rightarrow W \times W$$

which agree on $\text{Spec } A/I$, so the pullback map $\partial_{W \times W} \rightarrow A$ sends $I_\Delta$ to $I$. Since $I_\Delta^2 \rightarrow I_\Delta^2 = 0$, this descends to $I_\Delta/I_\Delta^2 = \Omega_{W/S} \rightarrow I$. This defines $\eta$.

Now we prove (1). First suppose that $\varphi_A$ exists. Then we can pick $\varphi = \varphi_A$. Then $J = 0$, so trivially $\epsilon = 0$. 
Conversely, suppose that $\epsilon = 0$. So

$$\text{Ext}^1(L_X^{-1}, I) = \text{Hom}_{D^b}(L_X^{-1}, I[1])$$

$$= H^0((L_X^{-1})^\vee \otimes I[1])$$

$$= H^1((L_X^{-1})^\vee \otimes I)$$

To compute $(L_X^{-1})^\vee$ in the derived category, we need to take a resolution $F^\bullet$ for $L_X^{-1}$:

$$[J/J^2] \longrightarrow \Omega_{W/S}|X]$$

$$\vdots$$

$$F^{-2} \longrightarrow F^{-1} \longrightarrow \Omega_{W/S}|X]$$

Then

$$(L_X^{-1})^\vee = [F^\vee_0 \rightarrow F^\vee_{-1} \rightarrow F^\vee_{-2} \ldots]$$

and we have a class $\epsilon$ in the first cohomology of

$$F^\vee_0 \otimes I \rightarrow F^\vee_{-1} \otimes I \rightarrow \ldots$$

The fact that $[\epsilon] = 0 \in H^1(F^\vee \otimes I)$, representing the homomorphism $\bar{\epsilon}: F^{-1} \rightarrow J/J^2 \rightarrow I$, tells us that $\bar{\epsilon} \in \text{Im}(d_{-1}^\vee)$, i.e. is induced from some $\eta \in \text{Hom}(\Omega_{W/S}|X], I)$. Using this $\eta$, we get a map $\varphi'$: Spec $A \rightarrow W$ factoring through $X$.

We skip (2).

4.3. Perfect obstruction theories.

**Definition 4.3.1.** We define the property $($*$$)$ as follows: $F \in D(\mathcal{O}_X)$ has property $($*$$)$ if

1. $H^{i>0}(F) = 0$,
2. $H^{-1}(F)$ and $H^0(F)$ are coherent sheaves.

If $F$ satisfies $($*$$)$, we furthermore say that $F$ is **perfect** if locally (in an appropriate topology)

$$F^* \cong [\mathcal{O}_a \rightarrow \mathcal{O}_{a+1} \rightarrow \ldots \mathcal{O}_b]$$

are locally free sheaves of $\mathcal{O}_X$-modules.

If $F$ is perfect, we furthermore say that $F$ is perfect of **perfect amplitude** $[-1,0]$ if

$$F^* \cong [\mathcal{O}_{-1} \rightarrow \mathcal{O}_0].$$

**Definition 4.3.2.** A perfect (relative) obstruction theory of $X/S$ consists of

$$\phi: E^* \rightarrow L_X^{-1} \in D(\mathcal{O}_X)$$

such that

1. $E^*$ is perfect of perfect amplitude contained in $[-1,0]$,
2. $H^0(\phi)$ is an isomorphism and $H^{-1}(\phi)$ is surjective.

**Remark 4.3.3.** A general obstruction theory replaces “perfect” with $($*$$).
What is an obstruction theory intuitively? It arises from studying a lifting problem

\[
\begin{array}{ccc}
\text{Spec } A/I & \xrightarrow{\varphi_{A/I}} & X \\
\downarrow & & \downarrow \\
\text{Spec } A & \xrightarrow{\varphi_A} & S
\end{array}
\]

Obstruction theory defines an obstruction class \( \text{ob}(\varphi_{A/I}, A, I; E) \in \text{Ext}^1(E^*, I) \) such that

1. \( \varphi_A \) exists if and only if \( \text{ob}(\varphi_{A/I}, A, I; E) = 0 \), and
2. if \( \varphi_A \) exists then the set of such is an \( \text{Ext}^0(E^*, I) \)-torsor.

Given \( \varphi: E^* \to L_{X/S}^{\geq -1} \), we define \( \text{ob}(\varphi_{A/I}, A, I; E) \) by the diagram

\[
\begin{array}{ccc}
\text{ob}(\varphi_{A/I}, A, I) & \in & \text{Ext}^1(L_{X/S}^{\geq -1}, I) \\
\downarrow & & \downarrow \\
\text{ob}(\varphi_{A/I}, A, I; E) & \in & \text{Ext}^1(E^*, I).
\end{array}
\]

We claim that (2) implies that \( \text{Ext}^0(L_{X/S}^{\leq -1}, I) \cong \text{Ext}^0(E^*, I) \). The injectivity follows from the fact that the thing is an isomorphism on \( H^0 \). Also (2) implies that the map \( \text{Ext}^1(\varphi, L) \) is injective, because \( H^1 \) is surjective. We’re skipping the proofs.

5. Construction of virtual cycles

5.1. The normal cone. Suppose we have a vector bundle \( E \to W \) of rank \( m \) over a smooth variety of dimension \( n \). Suppose \( X \subset W \) is a subvariety defined by \( s = 0 \), where \( s \) is a section of \( W \). Then

1. \( \dim_x X \geq n - m \) for all \( x \in X \).
2. If \( \dim X = n - m \), then \( [X] = c_{\text{top}}(E) \in A_{n-m} W \).
3. Even if \( \dim X > n - m \), we can define \( c_{\text{top}}(E, s) \in A_{n-m} X \) and we set

\[
[X]_{\text{vir}} := c_{\text{top}}(E, s) \in A_{n-m} X
\]

viewing \( n - m \) is the expected dimension of \( X \). This \( c_{\text{top}}(E, s) \) is defined via MacPherson’s deformation to the normal cone.

To explain this, consider the section \( t^{-1} \cdot s: W \to E \) as \( t \to 0 \). More precisely, consider the limiting behavior of the graph \( \Gamma_{t^{-1} s} \). Then

\[
\lim_{t \to 0} \Gamma_{t^{-1} s} =: N_{X/W} \subset E|_X.
\]

Explicitly \( N_{X/W} \) is the fiber over \( 0 \in \mathbb{A}^1 \) of

\[
\bigcup_{t \neq 0} \Gamma_{t^{-1} s} \subset E \times \mathbb{A}^1.
\]

Then we define

\[
c_{\text{top}}(E, s) := 0^1_{E|_X}[N_{X/W}] \subset A_{s} X.
\]

The point is that in general, you can take a cycle in the vector bundle and move it by rational equivalence to something vertical, and then intersect with the 0 section.
Let $J := I_{X \subset W}$. Then we have a complex $[J/J^2 \to \Omega_W|_X] \in D^b(X)$. This admits a map from $[\mathcal{E}^\vee|_X \xrightarrow{d s} \Omega_W|_X]$, which is the identity on $\Omega_W|_X$ and the map $\phi$. We claim that this map in the derived category

$$L^{-1}_{X/S} := [J/J^2 \to \Omega_W|_X]$$

$$E^\bullet := [\mathcal{E}^\vee|_X \xrightarrow{d s} \Omega_W|_X]$$

is a perfect obstruction theory.

Note that another way of describing the normal cone is as $C_{X/W} = \text{Spec}_X \bigoplus_{n \geq 0} J^n/J^{n+1}$.

There is an obvious surjection

$$\text{Sym}^\bullet(J/J^2) \to \bigoplus_{n \geq 0} J^n/J^{n+1}$$

inducing a closed embedding

$$C_{X/W} \hookrightarrow \text{Spec}_X \text{Sym}^\bullet(J/J^2).$$

If I have an obstruction theory

$$\mathcal{E}^\bullet := [\mathcal{E}_{-1} \to \mathcal{E}_0] \xrightarrow{\phi} L^{-1}_{X/S}$$

given by

$$\mathcal{E}_{-1} \xrightarrow{} J/J^2$$

$$\downarrow$$

$$\mathcal{E}_0 \xrightarrow{} \Omega_W|_X$$

then we get an embedding

$$\text{Spec}_X \text{Sym}^\bullet(J/J^2) \hookrightarrow \text{Spec}_X \text{Sym}^\bullet \mathcal{E}_{-1} := E_{-1}$$

This depends on $X \hookrightarrow W$ and $\phi$. We want to get rid of these dependencies.

Go back to the definition of $L^{-1}_{X/S}$. We have

$$\text{Spec}_X \text{Sym}^\bullet(J/J^2) \hookrightarrow \text{Spec}_X \text{Sym}^\bullet \mathcal{E}_{-1} =: E_{-1}$$

We can think of $T_{W/S|X}$ as a group over $X$, which acts on $\text{Sym}^\bullet(J/J^2)$. So we can take the quotient as an Artin stack over $S$, which fits into maps

$$[C_{X/W}/T_{W/S|X}] \hookrightarrow [\text{Spec Sym}(J/J^2)/T_{W/S|X}] \hookrightarrow [E_{-1}^\vee/T_{W/S|X}] \quad (5.1.1)$$

**Definition 5.1.1.** We define

$$[E_{-1}^\vee/T_{W/S|X}] = h^1/h^0((E^\bullet)^\vee).$$

It depends only on $E^\bullet$.

We define $[\text{Spec Sym}(J/J^2)/T_{W/S|X}]$ to be the intrinsic normal sheaf $\mathcal{N}_{X/S}$. 


Finally, we define $[C_X/W/T_{W/S}]_X$ to be the *intrinsic normal cone* $\mathcal{C}_{X/S}$.

So diagram (3) expresses inclusions "cone stack" into "abelian cone stack" into "vector bundle stack".

Suppose $F_\bullet \in D(\mathcal{O}_X)$ is of class $(\ast)$, i.e. $h^{>0}(F_\bullet) = 0$ and $h^{-1}(F_\bullet)$ and $h^0(F_\bullet)$ are coherent. What is $h^1/h^0(F_\bullet)$? Take a resolution

$$F \cong [\ldots F_{-2} \rightarrow F_{-1} \rightarrow F_0]$$

where the $F_i$ are locally free. Then

$$\mathcal{F}^\vee = [\mathcal{F}_0^\vee \rightarrow \mathcal{F}_{-1}^\vee \rightarrow \ldots]$$

and we get $\mathcal{F}_{-1}^\vee \subset \mathcal{F}_{-1}^\vee$.

**Definition 5.1.2.** We define

$$h^1/h^0(F_\bullet) = [\text{Spec}_X \text{Sym}^* \ker d_{-1}^\vee / F_0^\vee].$$

The point is that $\mathcal{N}_{X/S}$ is canonically isomorphic to $h^1/h^0((L_{X/S})^\vee)$. If $E_\bullet$ is perfect with perfect amplitude, then $h^1/h^0(E_\bullet)^\vee = [E_{-1}^\vee / E_0^\vee]$.

**Lemma 5.1.3.** The stack $h^1/h^0(F_\bullet)$ is well-defined and has property $(\ast)$.

5.2. **Crash course on derived categories.** We write $\text{Sh}(\mathcal{O}_X)$ for the category of sheaves of $\mathcal{O}_X$-modules. This is an *abelian category*, meaning that there the familiar operations of kernels, cokernels, direct sums.

From $\text{Sh}(\mathcal{O}_X)$ we can construct $C(\mathcal{O}_X)$, the category of *complexes* valued in $\text{Sh}(\mathcal{O}_X)$, whose objects are complexes of $\mathcal{O}_X$-modules

$$
\ldots \rightarrow \mathcal{E}_i \xrightarrow{d_i} \mathcal{E}_{i+1} \xrightarrow{d_{i+1}} \mathcal{E}_{i+2} \rightarrow \ldots
$$

and morphisms are morphisms of complexes.

**Shifts.** Given a complex

$$
\text{deg} & -1 & 0 \\
E_\bullet & \mathcal{E}_{-1} & \mathcal{E}_0
$$

we can form a shifted complex

$$
\text{deg} & -2 & -1 & 0 \\
E_\bullet[1] & \mathcal{E}_{-1} & \mathcal{E}_0
$$

**Definition 5.2.1.** A *triangulated category* is category (plus some extra structure) equipped with a collection of distinguished triangles

$$A_\bullet \rightarrow B_\bullet \rightarrow C_\bullet \xrightarrow{+1} A_\bullet[1]$$
satisfying certain axioms. (For the details, we recommend Kashiwara’s “Derived categories and sheaves”.) The most important one is that one gets a long exact sequence in cohomology

$$\cdots \to H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C^\bullet) \to H^{i+1}(A^\bullet) \to \cdots$$

Intuitively, these play the role of short exact sequences of sheaves.

Example 5.2.2. For \(f: A^\bullet \to B^\bullet\) in \(C(O_X)\), we get a distinguished triangle

$$A^\bullet \to B^\bullet \to MC(f) \to A^\bullet[1]$$

The mapping cone \(MC(f)\) is the total complex formed from the double complex

$$\begin{array}{ccc}
\cdots & \to & A_{i-1} \\
\downarrow & & \downarrow \\
\cdots & \to & B_{i-1}
\end{array}$$

as

$$\begin{array}{ccc}
\cdots & \to & A_i \\
\downarrow & & \downarrow \\
\cdots & \to & B_i
\end{array}$$

(check your sign connections).

Definition 5.2.3. We can form the homotopy category \(K(O_X)\), whose objects are the same as those of \(C(O_X)\) but morphisms are quotiented out by the equivalence relation of homotopy:

$$f_1 \sim f_2: A^\bullet \Rightarrow B^\bullet$$

if there exists \(\eta: A_{i+1} \to B^i\) such that \(f_1 - f_2 = \eta \circ \delta + \delta \circ \eta\).

\[
\begin{array}{ccc}
A_i & \xrightarrow{\delta} & A_{i+1} \\
\downarrow & & \downarrow \\
B_i & \xrightarrow{\delta} & B_{i+1}
\end{array}
\]

Note that this implies

$$H^*(f_1) = H^*(f_2): H^*(A^\bullet) \to H^*(B^\bullet).$$

Definition 5.2.4. Let \(N \subset K(O_X)\) be the objects \(A^\bullet\) with \(H(A^\bullet) = 0\). The derived category \(D(O_X)\) is \(K(O_X)/N\).

Given \(A \in D(O_X)\) and \(B \in D(O_X)\), a morphism \(\varphi: A \to B\) in \(D(O_X)\) is the same as a diagram

\[
\begin{array}{ccc}
A^\bullet & \xrightarrow{\text{quasi-isom}} & B^\bullet \\
\downarrow & & \downarrow \\
E^\bullet & \xrightarrow{\text{quasi-isom}} & \cdots
\end{array}
\]

(A quasi-isomorphism is a map \(f: A^\bullet \to B^\bullet\) such that \(H^i(f): H^i(A) \cong H^i(B)\) is an isomorphism.)
Given $E^* = [E_{-1} \to E_0] \in D(O_X)$ with $E_i$ locally free, we can form the stack
\[ h^1/h^0(E') = [E'_{-1}/E'_0], \]
We claim that this is well-defined as an Artin stack, i.e. independent of the presentation.

**Proof.** We work locally. If $E = [E_{-1} \to E_0] \cong [E'_{-1} \to E'_0]$, then by definition we can dominate both complexes by quasi-isomorphisms
\[
\begin{array}{ccc}
[F_{-1} \to F_0] & \text{quasi-isom} & [E'_{-1} \to E'_0] \\
[E_{-1} \to E_{-1}] & \text{quasi-isom} & [E'_{-1} \to E'_0]
\end{array}
\]
We claim that $F_{-1}$ can be chosen to be locally free. First we make $F_{-1} \to E_{-1}$ surjective. We just take the direct sum with something that surjects onto the cokernel of $F_{-1} \to E_{-1}$:
\[
\begin{array}{ccc}
A \oplus F_{-1} & \longrightarrow & A \oplus F_0 \\
\downarrow & & \downarrow \\
E_{-1} & \longrightarrow & E_0
\end{array}
\]
Similarly we can also make $F_0 \to E_0$ surjective.
So we have reduced to assuming that both maps $F_{-1} \to E_{-1}$ and $F_0 \to E_0$ are surjective.
\[
\begin{array}{ccc}
F_{-1} & \longrightarrow & F_0 \\
\downarrow & & \downarrow \\
E_{-1} & \longrightarrow & E_0
\end{array}
\]
Since we are working locally, we can split these surjections. So we can assume $F_{-1} = A_{-1} \oplus E_{-1}$ and $F_0 = A_0 \oplus E_0$, with locally free kernel $B_0 \oplus E_0$.
\[
\begin{array}{ccc}
B_0 \oplus E_0 & \longrightarrow & F_{-1} \\
\downarrow & & \downarrow \\
F_{-1} & \longrightarrow & A_0 \oplus B_0 \\
\downarrow & & \downarrow \\
E_{-1} & \longrightarrow & E_0
\end{array}
\]
Then we replace $F_{-1}$ with the fibered product. This shows that
\[ h^1/h^0([E_{-1} \to E_0]) = h^1/h^0([F_{-1} \to F_0]). \]
\[
\square
\]
Consider $C_X \subset h^1/h^0(L_{X}^{\leq -1})^\vee$. What is the geometry of $C_X|_x$ for $x \in X$?
Example 5.2.5. Suppose that $X$ has surface singularities. Suppose it is inside a three-dimensional $W$.

Suppose $X \subset W_0 \subset W$. Then $C_{X/W}|_X \cong C_{X/W_0}|_X \oplus N_{W_0/W}/X$. There is an action of $T_{W_0}|_X$ on $C_{X/W_0}|_X$ and of $T_W|_X$ on $C_{X/W_0}|_X$, and the quotients are canonically identified:

$$\frac{C_{X/W}|_X}{T_W|_X} \cong \frac{C_{X/W_0}}{T_{W_0}|_X}.$$

5.3. The cotangent complex. Let $A \to B$ be a map of rings. View this as a map of schemes

$$\text{Spec } B \longrightarrow X \longrightarrow \text{Spec } A \longrightarrow S$$

We construct a canonical resolution $P_\bullet \to B$ as follows:

$$\ldots A[A[B]] \xrightarrow{\varepsilon} A[B] \xrightarrow{\varepsilon} B$$

Elements of $A[B]$ are $\sum a_{i_1 \ldots i_k}^{n_1 \ldots n_k} [\alpha_{i_1}]^{n_1} \ldots [\alpha_{i_k}]^{n_k}$. The map $\varepsilon$ is by deleting the brackets.

Elements of $A[A[B]]$ are sums of things with two layers of brackets. The two maps are determined by which layer of bracket to delete.

This forms a simplicial complex, $\Omega_{P_k/A}$.

Definition 5.3.1. We define the cotangent complex to be the resolution

$$L_X := \ldots \to \Omega_{P_k/A} \otimes_{P_k, \varepsilon} B \to \Omega_{P_{k-1}/A} \otimes_{P_{k-1}, \varepsilon} B \to \ldots$$

Theorem 5.3.2. Suppose that $\pi: X \to S$ is a relative DM morphism. Then there exists $L_{X/S} \in D(\mathcal{O}_X)$ such that

1. $H^{>0}(L_{X/S}) = 0$,
2. $H^{-1}(L_{X/S})$ and $H^0(L_{X/S})$ are coherent,
3. $H^0(L_{X/S}) = \Omega_{X/S},$
4. $L_{X/S}^{-1}$ is the truncated cotangent complex.

Also

$$\pi^* L_S \to L_X \to L_{X/S} \xrightarrow{+1}$$

is distinguished triangle.
5.4. **Virtual cycles.** Suppose \( X \to S \) is a DM stack. We can form \( L_{X/S} \). Suppose we have an étale covering \( U \to S \). We can find an \( S \)-embedding \( U \hookrightarrow W \) into some smooth \( W \to S \). Then we can form the intrinsic normal cone

\[
C_{U}^{\text{intr}} = [C_{U/W}/T_{W|U}] \subset h^1/h^0(L_{U/S}^{\geq -1})^\vee.
\]

Such a collection (by the canonical nature of the intrinsic normal cone \( C^{\text{intr}} \)) descends to

\[
C_{X/S}^{\text{intr}} \subset h^1/h^0(L_{X/S}^{\geq -1})^\vee = h^1/h^0(L_{X/S}^\vee).
\]

For what follows, it is useful to recall the construction of the top chern class earlier, for \( X \subset W \) cut out by a section \( s \) of a vector bundle \( E \).

Assume that there is a perfect obstruction theory \( E \xrightarrow{\phi_X} L_{X/S} \). Then we have an embedding

\[
h^1/h^0(L_{X/S}^{\geq -1})^\vee \hookrightarrow h^1/h^0(E^\vee).
\]

Suppose \( E^\bullet \simeq [E_{-1} \to E_0] \in C(\mathcal{O}_X) \), where \( E_i \) are locally free. Then we get

\[
[C_{X/S}^{\text{intr}}] \hookrightarrow h^1/h^0(E^\vee) = [E^\vee_{-1}/E^\vee_0].
\]

The map

\[
\tilde{\phi} : E^\vee_1 \to h^1/h^0(E^\vee)
\]

is flat, so we have a flat pullback \( \tilde{\phi}^*[C_{X/S}^{\text{intr}}] \in A_s(E^\vee_1) \). Then we define

\[
[X^{\text{intr}}]_{\phi_X} = 0^1_{E^\vee_1} [\tilde{\phi}^*[C_{X/S}^{\text{intr}}]] \in A_{\text{rank } E_{-1} - \text{rank } E_0}(X).
\]

**Example 5.4.1.** Let \( Y \) be a smooth projective scheme. Consider

\[
X = \overline{M}_{g,n}(Y, \alpha) \quad \xrightarrow{\quad} \quad S = \overline{M}_{g,n}
\]

We also have a universal curve ‘\( \mathcal{C} \)’ over \( X \):

\[
\pi \quad \xrightarrow{\quad} \quad \mathcal{C} \quad \xrightarrow{f} \quad Y
\]

**Theorem 5.4.2** (Illusie). *There is an obstruction theory*

\[
(R\pi_*f^*T_Y)^\vee \to L_{X/S}.
\]

**Lemma 5.4.3.** \( R\pi_*f^*T_Y \) is quasi-isomorphic to \( [\mathcal{E}_0 \to \mathcal{E}_1] \in D(\mathcal{O}_X) \) with \( \mathcal{E}_0, \mathcal{E}_1 \) locally free.
6. Deformation theory

6.1. Classical approach to obstruction theory. Let \( \mathcal{E} \) be a locally free sheaf on \( X \).

- Def\( \mathcal{E} \) is the space of 1st-order deformations of \( \mathcal{E} \).
- \( \text{ob}(\mathcal{E}) \) is the space of obstructions to deforming \( \mathcal{E} \).

A first-order deformation looks like an extension from \( \mathcal{E}_{A/I} \) on \( X \times \text{Spec} \, A/I \) to some \( \mathcal{E} \) on \( X \times \text{Spec} \, A \), where \( mI = 0 \).

\[
\begin{array}{c}
\mathcal{E}_0 \ar{r} & \mathcal{E}_0' \ar{r} & \mathcal{E}_{A/I} \ar{r} & \mathcal{E}_A
\end{array}
\]

\[
\begin{array}{c}
X' \ar{r} & X \times \text{Spec} \, A/I' \ar{r} & X \times \text{Spec} \, A
\end{array}
\]

Definition 6.1.1. We define the obstruction space \( \text{Ob}(\mathcal{E}_0) = H^2(X, \mathcal{E}_0 \otimes \mathcal{E}_0) = \text{Ext}^2(\mathcal{E}_0, \mathcal{E}_0) \).

Theorem 6.1.2. There is an obstruction class \( \text{ob}(\mathcal{E}_{A/I}, AI) \in \text{Ob}(\mathcal{E}_0) \otimes_k I \) such that an extension exists if and only if \( \text{ob}(\mathcal{E}_{A/I}, AI) \) vanishes.

Furthermore, if Pic\( X \) is smooth then we can take \( \text{Ob}(\mathcal{E}_0) = \text{Ext}^2(\mathcal{E}_0, \mathcal{E}_0) \otimes_\mathbb{Z} \) (corresponding to the traceless subsheaf).

Proof. We cover \( X \) by open affines \( X_\alpha \) trivializing \( \mathcal{E}_\alpha := \mathcal{E}|_{X_\alpha \times \text{Spec} \, A/I} \).

\[
\begin{array}{c}
\mathcal{E} \ar{r} & \mathcal{E}_\alpha
\end{array}
\]

\[
\begin{array}{c}
X \times \text{Spec} \, A/I \ar{r} & X_\alpha \times \text{Spec} \, A/I
\end{array}
\]

We begin by choosing arbitrary extensions of \( \mathcal{E}_\alpha \) to \( \tilde{\mathcal{E}}_\alpha \) over \( X_\alpha \times \text{Spec} \, A \). We also choose arbitrary extensions of the transition functions \( f_{\beta \alpha} : \mathcal{E}_\alpha \to \mathcal{E}_\beta \), which satisfy the cocycle condition, to \( \tilde{f}_{\beta \alpha} \in H^0(X_{\alpha \beta} \times \text{Spec} \, A, \tilde{\mathcal{E}}_\alpha \otimes \tilde{\mathcal{E}}_\beta) \). We know that

\[
\tilde{f}_{\alpha \beta} \circ \tilde{f}_{\gamma \beta} \circ \tilde{f}_{\beta \alpha} \in H^0(X_{\alpha \beta \gamma} \times \text{Spec} \, A, \tilde{\mathcal{E}}_\alpha \otimes \tilde{\mathcal{E}}_\gamma \otimes \tilde{\mathcal{E}}_\beta)
\]

is the identity modulo \( I \), so \( \tilde{f}_{\alpha \beta} \circ \tilde{f}_{\gamma \beta} \circ \tilde{f}_{\beta \alpha} \in H^0(X_{\alpha \beta \gamma}, \tilde{\mathcal{E}}_\alpha \otimes \tilde{\mathcal{E}}_\beta \otimes \tilde{\mathcal{E}}_\gamma) \). But

\[
H^0(X_{\alpha \beta \gamma}, \tilde{\mathcal{E}}_\alpha \otimes \tilde{\mathcal{E}}_\beta \otimes \tilde{\mathcal{E}}_\gamma) = H^0(X_{\alpha \beta \gamma}, \text{Hom}(\mathcal{E}_\alpha, \mathcal{E}_\alpha) \otimes A).
\]

Using that \( I^2 = 0 \), you check that \( \delta \tilde{f} \) is a closed 2-cycle in \( C^2([X_\alpha], \text{Hom}(\mathcal{E}, \mathcal{E}) \otimes A I) \). We define

\[
\text{ob}(\mathcal{E}_{A/I}, A, I) := [\delta \tilde{f}] \in H^2(X, \text{Hom}(\mathcal{E}, \mathcal{E}) \otimes I).
\]

Exercise 6.1.3. Prove that \( \text{ob}(\mathcal{E}_{A/I}, A, I) = 0 \) if and only if the choices \( \tilde{f}_{\alpha \beta} \) can be modified to satisfy the cocycle condition.

Furthermore, if Pic\( X \) is smooth then

\[
\text{ob}(\mathcal{E}_{A/I}, A, I) \in \ker(\text{Ext}^2(\mathcal{E}_0, \mathcal{E}_0) \otimes I \overset{\text{tr}}{\longrightarrow} H^2(X, \mathcal{O}_X) \otimes I)
\]

Why? The formation of the obstruction class is compatible with taking determinants, so we have a map
\[
\operatorname{Ext}^2_{X \times A/I}(\mathcal{E}, \mathcal{E}) \otimes I \xrightarrow{\text{trace}} \operatorname{Ext}^2_{X \times A/I}(\det \mathcal{E}, \det \mathcal{E}) \otimes I
\]
\[
\text{ob}(\mathcal{E}_A/I, A, I) \to \text{ob}(\det \mathcal{E}_A/I, A, I)
\]
The smoothness of \(\text{Pic}(X)\) implies that there are no obstructions to deforming line bundles, so that \(\operatorname{Ext}^2_{X \times A/I}(\det \mathcal{E}, \det \mathcal{E}) \otimes I = 0\).

\[\square\]

Remark 6.1.4. More generally, if \(X\) is smooth then one can still form a determinant in the Grothendieck group by taking a resolution of \(\mathcal{E}\) by vector bundles
\[
0 \to \mathcal{F}_n \to \ldots \to \mathcal{F}_0 \to \mathcal{E} \to 0
\]
and defining \(\det \mathcal{E} = \bigotimes (\det \mathcal{F}_i)^{(-1)^i}\).

Remark 6.1.5. The same sort of argument shows that
\[
\operatorname{Def}(\mathcal{E}) \cong H^1(X, \operatorname{End}(\mathcal{E})).
\]

Theorem 6.1.6. With choices of identification \(\operatorname{Def}(\mathcal{E}_0) \cong \mathbb{C}^n\) and \(\operatorname{Ob}(\mathcal{E}_0) \in \operatorname{Ext}^2_{X}(\mathcal{E}_0, \mathcal{E}_0) \cong \mathbb{C}^m\), there exists \(f \in \mathbb{C}[[z_1, \ldots, z_n]]^{\otimes m}\) such that \((f = 0)\) is the germ of the moduli of sheaves at \([\mathcal{E}_0]\).

Locally the tangent space to the moduli space looks like \(\operatorname{Def}(\mathcal{E}_0)\), and the theorem is telling us that the moduli space itself is cut out by equation “coming from” the obstruction space.

6.2. Modern obstruction theory. In modern terms, an obstruction theory is a morphism \(\phi_X : E^* \to I_X^*\) in the derived category such that
- \(H^0(\phi_X)\) is an isomorphism and
- \(H^1(\phi_X)\) is surjective.

Let \(X\) be a smooth projective variety, \(c\) a fixed total Chern class, and \(\mathcal{M}\) the moduli space of stable sheaves with Chern class \(c\). This is quasi-projective.

Remark 6.2.1. \(\mathcal{M}\) might not have a universal family, i.e. it may only be a coarse moduli space. This is basically coming from the fact that even stable sheaves have nontrivial automorphisms.

However, we can always cover \(\mathcal{M}\) by étale maps \(U \to \mathcal{M}\) such that there is a tautological family \(\mathcal{E} \to X \times U\). We can then consider the pushforward to \(\mathcal{M}\):

\[
\begin{array}{ccc}
\mathcal{E} & \to & X \times U \\
\pi_U & & \downarrow \\
& U &
\end{array}
\]

Atiyah class. We explain the construction of the Atiyah class. Let \(Y\) be a scheme and \(\mathcal{E}\) a sheaf of \(\mathcal{O}_Y\)-modules. In the classical version, with \(Y\) smooth, you consider
\[
0 \to I_\Delta/I_\Delta^2 \to \mathcal{O}_{Y \times Y}/I_\Delta^2 \to \mathcal{O}_{Y \times Y}/I_\Delta \to 0
\]
We know that $I_{\Delta}/I_{\Delta}^2 = i_{\Delta*}\Omega_Y$ and $\partial_{Y\times Y}/I_{\Delta} = i_*\partial_{\Delta}$, so we can rewrite the above sequence as

$$0 \to i_{\Delta*}\Omega_Y \to \partial_{Y\times Y}/I_{\Delta}^2 \to i_*\partial_{\Delta} \to 0$$

So the class of this extension is an element of

$$\text{Ext}^1(i_{\Delta*}\partial_Y, i_{\Delta*}\Omega_Y) = \text{Hom}(i_{\Delta*}\partial_Y, i_{\Delta*}\Omega_Y[1])$$
i.e. a map (in the derived category) $i_{\Delta*}\partial_Y \to i_{\Delta*}\Omega_Y[1]$. If $p_1, p_2$ are the two projection maps $Y \times Y \to Y$, then this induces

$$p_{1*}(p_2^*E \otimes i_{\Delta*}\partial_Y) \to p_{1*}(p_2^*E \otimes i_{\Delta*}\Omega_Y[1])$$

This is a map $E \to E \otimes \Omega_Y[1]$, which defines a class $A(E) \in \text{Ext}^1(E, E \otimes \Omega_Y)$, which is the Atiyah class. (See an article “...Atiyah class...” by Huybrechts and Thomas.)

**Theorem 6.2.2.** The Atiyah class induces an obstruction theory

$$\phi: \pi_{U*}(\text{Hom}(\mathcal{E}, \mathcal{E}_0)[1]) \to L_{U}^{\geq -1}.$$

**Proof.** We apply the preceding discussion to $Y = X \times U$. We upgrade everything by replacing $\Omega$ by the cotangent complex (since we are dealing with potentially singular $Y$). Embed $Y \leftarrow W$ for some smooth $W$ and let $J = I_{Y \subset W}$. Using the conormal exact sequence

$$\Omega_{W/Y} = (J/J^2)|_Y \to \Omega_{W}|_Y \to \Omega_Y \to 0$$

and

$$0 \to i_*\Omega_Y \to \partial_{Y\times Y} \to i_*\partial_{\Delta Y} \to 0$$

we obtain (after some calculation)

$$i_*\partial_{\Delta Y} \cong [i_{\Delta*}(J/J^2) \to I_{\Delta W}|_{Y \times Y} \to \partial_{Y\times Y}].$$

Then we get

$$i_{\Delta*}\Omega_{Y}|_{Y \times Y} \cong [i_{\Delta*}(J/J^2) \to I_{\Delta W}|_{Y \times Y} \to \partial_{Y\times Y}].$$

We can string these together into a map in the derived category

$$
\begin{array}{ccc}
[i_{\Delta*}(J/J^2)] & \to & I_{\Delta W}|_{Y \times Y} \\
\downarrow & & \downarrow \\
[i_{\Delta*}(J/J^2)] & \to & I_{\Delta W}/I_{\Delta W}^2|_{Y \times Y} = i_{\Delta W*}\Omega_{W}|_{Y \times Y} \\
\end{array}
$$

Then we get

$$p_{1*}((\partial_{\Delta Y} \to i_{\Delta*}\Omega_{Y}|_{Y \times Y}^{-1}[1]) \otimes p_1^*\mathcal{E}) \in \text{Ext}_Y^1(\mathcal{E}, \mathcal{E} \otimes \Omega_{Y}^{\geq -1}).$$

Apply this to $Y = X \times U$, over which we have a tautological family $\mathcal{E}$. Then we get an Atiyah class $A(\mathcal{E}) \in \text{Ext}^1_{X \times U}(\mathcal{E}, \mathcal{E} \otimes \Omega_{X \times U}^{\geq -1})$. We have

$$\Omega_{X \times U}^{\geq -1} = \pi_{U*}\Omega_X \otimes \pi_{U*}\Omega_{Y}^{\geq -1}.$$ 

From the projection we get a map

$$\text{Ext}^1_{X \times U}(\mathcal{E}, \mathcal{E} \otimes \Omega_{X \times U}^{\geq -1}) \to \text{Ext}^1_{X \times U}(\mathcal{E}, \mathcal{E} \otimes \pi_{U*}\Omega_{Y}^{\geq -1}) = \text{Ext}^1_{X \times U}(\text{Hom}(\mathcal{E}, \mathcal{E}), \pi_{U*}\Omega_{Y}^{\geq -1}).$$
Then applying Verdier duality gives
\[ \text{Ext}^1_{X \times U}(\text{Hom}(\mathcal{E}, \mathcal{E}), \pi_\ast \mathcal{L}^{-1}) \cong \text{Ext}^{-2}_U(\pi_U(\text{Hom}(\mathcal{E}, \mathcal{E}) \otimes \pi_\ast \omega_X), \mathcal{L}^{-1}_U). \]

Tracing through the isomorphisms, the Atiyah class corresponds to a class \( \overline{\mathcal{A}}(\mathcal{E}) \) in \( (\text{Hom}(\mathcal{E}, \mathcal{E}) \otimes \pi_\ast \omega_X, \mathcal{L}^{-1}_U) \), which corresponds to a map
\[ \pi_U(\text{Hom}(\mathcal{E}, \mathcal{E}) \otimes \pi_\ast \omega_X)[2] \rightarrow \mathcal{L}^{-1}_U. \]

6.3. Calabi-Yau threefolds.

**Theorem 6.3.1.** If \( X \) is a Calabi-Yau threefold and \( c \) is a total Chern class, then \( M := M_X(c)^{13} \) is projective and we have a virtual cycle \([M]^{\text{vir}} \in A_{\text{vir. dim. } M}(M)\)

where \[
\text{vir. dim. } M = \dim T_{[\mathcal{E}]} M = \dim \text{Ext}^1(E, E)_0 - \dim \text{Ext}^2(E, E)_0 = 0
\]
by Serre duality. The Donaldson-Thomas invariant is \( \deg M^{\text{vir}} \).

Let \( X \) be a smooth projective variety. Let \( M = M_{g, n}(X, \alpha) \) for \( \alpha \in H_2(X, \mathbb{Z}) \). We have a universal curve
\[ \mathcal{E} \xrightarrow{f} X \]
\[ \mathcal{M} \ni [f : C \rightarrow X] \]
\[ M_{g, n} \ni [C] \]
We then get a relative obstruction theory (according to Illusie)
\[ (R\pi_\ast(f^\ast \Omega_X^1))^\vee \rightarrow \mathcal{L}_{M_{g, n}(X, \alpha)/M_{g, n}} \]
In this case the virtual dimension is
\[ \text{vir. dim. } M = n + (3 - \dim X)(g - 1) + \int \alpha c_1(T_X). \]

Given a perfect obstruction theory \( E^\bullet \rightarrow \mathcal{L}^{-1}_X \), we have the intrinsic normal cone \( C_X \subset h^1/h^0((E^\bullet)^\vee) \).

**Definition 6.3.2.** We define \([M]^{\text{vir}} = 0 \left[ C_X \right] \in A_*(X). \)
Part 3. Gromov-Witten Theory

7. Constructing Gromov-Witten invariants

7.1. Summary of virtual cycles. The picture is always that we have \( X \subset W \) cut out by a section \( s \) of some vector bundle \( E \to W \). We have a normal cone \( C_{X/W} \subset E|_X \) and we define a virtual class

\[
[X]_{vir} = 0_{E}^{!}[C_{X/W}] = c_{\text{top}}(E, s) \in A_{\dim W - \rank E}(X).
\]

In this situation, we have an obstruction theory

\[
E^* := \left[ E|_X \to \Omega_W|_X \right]
\]

\[
L_{X}^{\geq -1} = \left[ J/J^2 \to \Omega_W|_X \right]
\]

We pretend that \( E^* \) is a two-term complex \([E_{-1} \to E_0]\), which one should imagine as being \([T_{W}|_X \sigma \to E|_X]\). In particular, for \( x \in X \) we imagine \( H^0(E^*|_x) = T_x X \) and \( H^1(E^*|_x) = \text{Ob}(x) \).

The map \((L_{X}^{\geq -1})^\vee \to (E^*)^\vee\) induces a map of the intrinsic normal cone

\[
C_X \leftarrow h^1/h^0(E^*) = [E_{-1}^\vee/E_0^\vee].
\]

We define \([X]_{vir} = 0_{E_1}^{!}[C_{X,E_1}]\). Then

\[
\dim[X]_{vir} = \dim C_X + \rank E_0 - \rank E_{-1}.
\]

To figure out \( \dim C_X \), we need to recall the construction of the intrinsic normal cone. The construction is by embedding \( X \hookrightarrow W \) and forming \([C_{X/W}/TW|_X]\). So \( \dim C_{X/W} = \dim W \), hence \( \dim C_X = \dim W - \dim W = 0.\)

7.2. Moduli of stable maps. We consider

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{f} & X \\
\downarrow & & \downarrow \\
M_g(X, \alpha) & \ni [f: C \to X] & \\
\downarrow & & \downarrow \\
M_{g,n} & \ni [C] & 
\end{array}
\]

We have a relative obstruction theory

\[
(R_{\pi_*}(f^*\Omega_X))^{\vee} \to \Omega_{M_{g,n}(X,\alpha)/M_{g,n}}
\]

It may be more intuitive to dualize

\[
T_{M_{g,n}(X,\alpha)/M_{g,n}} \to R_{\pi_*}(f^*\Omega_X^{\vee}) \sim [E_0 \to E_1]
\]
Consider the intrinsic normal cone $C_{\overline{M}_g(X)/M_g} \hookrightarrow h^1/h^0(T_{\overline{M}_g(X)/M_g})$, which fits into a diagram

$$
\begin{array}{c}
C_{\overline{M}_g(X)/M_g} \\
\downarrow \\
h^1/h^0(R\pi_* f^* T_X) = [E_1/E_0] \\
\end{array}
$$

To calculate the dimension of the normal cone, choose smooth embedding $\overline{M}_g(X) \rightarrow W$ over $M_g$. (This $M_g$ is really an Artin stack since automorphisms may become infinite after forgetting the map, but that’s okay because we are just doing a local calculation.) Then we have

$$C_{\overline{M}_g(X)/M_g} = C_{\overline{M}_g(X)/W/TW/M_g}$$

(We should think of $M_g$ as being a Deligne-Mumford stack, even though it’s technically an Artin stack.) Then

$$\dim[\overline{M}_g(X, \alpha)]^{\text{vir}} = \dim M_g + \dim E_0 - \dim E_1$$

$$= \dim M_g + \chi(f^* T_X)$$

$$= \dim M_g + \dim X(1 - g) + \deg f^* T_X$$

$$= \dim M_g + \dim X(1 - g) + \alpha \cdot c_1(T_X)$$

$$= 3g - 1 + \dim X(1 - g) + c_1(T_X) \cdot \alpha$$

$$= (3 - \dim X)(g - 1) + c_1(T_X) \cdot \alpha.$$ 

\[\text{Tony: [still not sure if the second equality is BS or not]}\]

Conclusion: the dimension of the virtual cycle is

$$\dim[\overline{M}_g(X, \alpha)]^{\text{vir}} = (3 - \dim X)(g - 1) + c_1(T_X) \cdot \alpha. \quad (7.2.1)$$

7.3. Example: the quintic threefold. Let $X = (x_1^5 + \ldots + x_5^5 = 0) \subset \mathbb{P}^4$ be a quintic threefold. We view it as being cut out by a section $s \in H^0(\mathbb{P}^4, \mathcal{O}(4))$. By adjunction we see that $K_X \cong \mathcal{O}_X$, and by the standard Chern class calculation we see that $c_1(T_X) = 0$. Putting also $\dim X = 3$ into the dimension formula (7.4.1), we see that

$$[\overline{M}_g(X, d)]^{\text{vir}} \in A_0(\overline{M}_g(X, d)).$$

By composing with $X \hookrightarrow \mathbb{P}^4$, we have a canonical map

$$\overline{M}_g(X, d) \hookrightarrow \overline{M}_g(\mathbb{P}^4, d).$$
Over $\overline{M}_g(\mathbb{P}^4, d)$ we have a universal curve $\mathcal{C}$, which admits a map $f$ to $\mathbb{P}^4$.

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{f} & \mathbb{P}^4 \\
\downarrow \pi & & \\
\overline{M}_g(\mathbb{P}^4, d) & & 
\end{array}$$

We can pull back $\mathcal{O}(5)$ on $\mathbb{P}^4$ to $\mathcal{C}$ and then push forward to $\overline{M}_g(\mathbb{P}^4, d)$. This gives a bundle $\pi_* f^* \mathcal{O} \mathbb{P}^4(5)$ on $\overline{M}_g(\mathbb{P}^4)$. Also, the section $s$ pulls back to a global section $f^* s \in H^0(\mathcal{C}, f^* \mathcal{O}(5))$. In these terms, a point $[u : C \to \mathbb{P}^4] \in \overline{M}_g(\mathbb{P}^4, d)$ comes from $\overline{M}_g(X, d)$, i.e. $f$ factors through $X$, if and only if $u^* s = 0$. So $\overline{M}_g(X, d)$ is “cut out” in $\overline{M}_g(X, d)$ by $f^* s$.

The key point is that when $g = 0$, the stack $\overline{M}_g(\mathbb{P}^4, d)$ is smooth and $\pi_* f^* \mathcal{O}(5)$ is locally free.

In this case $g = 0$ we have

$$\dim \pi_* f^* \mathcal{O}(5) = \dim \overline{M}_g(\mathbb{P}^4, d) = 5d + 1.$$ 

Kontsevich defined

$$[\overline{M}_0(X, d)]^{vir} := \mathcal{E}_{5d+1}(\pi_* f^* \mathcal{O}(5), f^* s) \in A_0(\overline{M}_0(X, d)).$$

As far as we know, it is an open question whether the virtual cycle $[\overline{M}_0(X, d)]^{vir}$ defined using the tangent complex $\mathbb{T}_{\overline{M}_0(X, d)/M_0} \to R\pi_* f^* T_X$ agrees with Kontsevich’s construction.

8. K3 surfaces

8.1. The period map for K3 surfaces. Let $S$ be a K3 surface. (It is a deep theorem that all such are diffeomorphic.) An example is a quartic surface in $\mathbb{P}^3$. Then

$$H^2(S, \mathbb{Z}) \cong U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$$

where

$$E_8 = \begin{pmatrix}
2 & -1 \\
-1 & 2 & -1 \\
& -1 & 2 & -1 \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
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& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
\end{pmatrix}$$

and $E_8(-1) = -E_8$. In particular, we have rank $H^2(S, \mathbb{Z}) = 22$.

We have the Hodge decomposition

$$H^2(S, \mathbb{C}) = H^{2,0}(S) \oplus H^{1,1}(S) \oplus H^{0,2}(S).$$

By the Dolbeault Theorem, $H^{2,0}(S) \cong H^0(S, K_S) \cong \mathbb{C}$ since $K_S$ is trivial. That tells us that the Hodge numbers are $h^{2,0} = h^{0,2} = 1$ and $h^{1,1} = 20$.

Let $V = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$ and consider

$$V_C = V \otimes_{\mathbb{Z}} \mathbb{C}.$$ 

Pick an isomorphism $V \cong H^2(S, \mathbb{Z})$. Then we have the line

$$\mathbb{C} \omega_S \subset H^2(S, \mathbb{C}) \subset V_C.$$
We consider the corresponding point $[\omega_S] \in \mathbb{P}V_C$. We know that

1. $(\omega_S, \omega_S) = 0$ since $\omega_S \sim \omega_S$ is a $(4,0)$ form, and
2. $(\omega_S, \overline{\omega_S}) > 0$.

**Definition 8.1.1.** We define the period domain $M \subset \mathbb{P}V_C \cong \mathbb{P}^{21}_C$ to be

$$M = \left\{ C\mu \mid (\mu, \overline{\mu}) = 0 \right\}.$$ 

In particular, the first condition imposes one complex condition and the second is open, so $\dim M = 20$.

Suppose that we have a family $X \to \Delta$ of K3 surfaces, and $X_0 = S$

$$X \supset X_0 = S \quad \Delta \ni 0.$$ 

Then there is a canonical isomorphism $H^2(X_t, \mathbb{Z}) \cong H^2(S, \mathbb{Z})$, since $X$ deformation retracts to $S$. This induces

$$H^2(X_t, \mathbb{C}) \cong H^2(S, \mathbb{C}) = V_C.$$ 

The line $[\omega_{X_t}]$ defines a period map $\Delta \to \mathbb{P}V_C$ factoring through $M$:

**Theorem 8.1.2 (Local Torelli theorem).** *For any K3 surface $X_0$, there is a family $X \to \Delta^{20}$ such that the induced period map

$$X \quad \Delta^{20} \quad M$$

$\sigma : \Delta^{20} \to M$ is a local biholomorphism onto an open subset of $M$.*

A cohomology class $\beta \in H^2(X, \mathbb{Z})$ then $\beta = c_1(L)$ if and only if $\beta \in H^{1,1}(X, \mathbb{Z}) := H^{1,1}(X, \mathbb{R}) \cap H^2(X, \mathbb{Z})$.

Since the generator of the 2,0 class is $\omega_S$, the algebraic classes in $S$ are precisely $\omega_S^1 \cap H^2(S, \mathbb{Z})$ (since $H^2(S, \mathbb{Z})$ is a real subspace, any class of it in $\omega_S^1$ automatically lies in $\omega_S^{-1}$).

Therefore, if $X_t$ is a deformation of $S$, then $\beta \in H^{1,1}(S, \mathbb{Z})$ ceases to be $(1,1)$ to first order if

$$\frac{d}{dt} (\omega_t, \beta)_{t=0} \neq 0.$$ 

Pick $\beta \in H^{1,1}(S, \mathbb{Z})$ so $\beta^{PD.} \in H_2(S, \mathbb{Z}) = H_2(X, \mathbb{Z})$ such that $\beta$ ceases to be $(1,1)$ to first order when deforming $t \in \Delta$. 
**Proposition 8.1.3.** For such a choice of \( \beta \), the map \( \overline{M}_g(X_0, \beta) \hookrightarrow \overline{M}_g(X, \beta) \) is an isomorphism.

**Proof.** The two moduli problems are the same, since any map from a proper algebraic curve \( C \to X \) is contracted by the composition to \( \Delta \), hence lies in a fiber. By the choice of \( \beta \), the only fiber admitting algebraic classes is \( X_0 \), even to first order. \( \square \)

8.2. **Fundamental cycles of K3 surfaces.** We calculated that

\[
\dim[\overline{M}_g(X, \alpha)]^{\text{vir}} = (3 - \dim X)(g - 1) + c_1(T_X) \cdot \alpha
\]

which is 0 for Calabi-Yau threefolds.

For K3 surfaces, \( K_X \cong \mathcal{O}_X \) and the formula tells us that

\[
\dim[\overline{M}_g(X, \alpha)]^{\text{vir}} = g - 1.
\]

**Theorem 8.2.1** (Folklore). For \( X \) a K3 surface and any \( \alpha \),

\[
[\overline{M}_g(X, \alpha)]^{\text{vir}} = 0.
\]

**Proof.** The key idea is that the virtual cycle \( [\overline{M}_g(X, \alpha)]^{\text{vir}} \) is “deformation invariant”, i.e. invariant under deformations of \((X, \alpha)\). In other words, if we have a disk \( S \) with a base-point 0, and a deformation

\[
\begin{array}{ccc}
X & \hookrightarrow & X' \\
\downarrow & & \downarrow \\
0 & \hookrightarrow & S
\end{array}
\]

For \( \alpha \in H_2(X, \mathbb{Z}) \hookrightarrow H_2(X', \mathbb{Z}) \), we get a map

\[
\overline{M}_g(X', \alpha) \to S
\]

because any map \([u : C \to X]\) is necessarily contracted by the composition with projection to \( S \), hence factors through the fiber over some point in \( S \). So there exists \([Z] \in \overline{M}_g(X', \alpha) \to \overline{M}_g(X, \alpha) \).
Figure 8.2.1. The surface $X_0$ inside $X$. Any proper curve $C$ mapping to $X$ must land in $X_0$.

$A_s(\overline{M}_g(\mathcal{X}, \alpha))$ such that $[\overline{M}_g(X_s, \alpha)]^{\text{vir}} = s^! [Z]$. Here we are using a general Gysin map:

$$
\begin{array}{ccc}
W \times_Y Y' & \longrightarrow & W \\
\downarrow & & \downarrow \\
Y' & \longrightarrow & Y
\end{array}
$$

If $X$ is K3 and $\alpha \in H_2(X, \mathbb{Z})$. If $\overline{M}_g(X, \alpha) = \emptyset$ then obviously the virtual cycle is 0. If $\overline{M}_g(X, \alpha)$ is non-empty then the Poincaré dual class $\alpha^{PD}$ lies in $H^{1,1}(X, \mathbb{R}) \cap H^2(X, \mathbb{Z})$, since it is algebraic. We can use the period map to deform the K3 to become non-algebraic, so that $\alpha^{PD} \notin X^{1,1}(X, \mathbb{R}) \cap H^2(X, \mathbb{Z})$.

We now change notation. Suppose $X \rightarrow S$ is a family of projective smooth K3 surfaces over an affine curve $S$. Then $K_X$ is trivial, so $X$ is a (local) Calabi-Yau threefold. We have $X_0$ the fiber over $0 \in S$.

For $0 \in S$ such that (1) $\alpha^{PD} \in H^{1,1}(X_0, \mathbb{R})$; we want to deform so that nearby $\alpha$ is not algebraic, evidenced by not being a $(1,1)$ class, so we also ask that (2) for any $s \neq 0 \in S$, $\alpha \notin H^1(X_s, \mathbb{R})$. In fact we will demand even more strongly that $\alpha \notin H^{1,1}(X_s, \mathbb{R})$ to the first order deformation of $X_s$. This implies $\overline{M}_g(X_0, \alpha) = \overline{M}_g(X, \alpha)$.

Obstruction theory furnishes a map $T_{\overline{M}_g(X, \alpha)/M_g} \rightarrow R\pi_* f^* T_X$, and similarly for $X_0$. By functorially, these are compatible:

$$
\begin{array}{ccc}
T_{\overline{M}_g(X_0, \alpha)/M_g} & \longrightarrow & T_{\overline{M}_g(X, \alpha)/M_g} \\
\downarrow & & \downarrow \\
R\pi_* f^* T_{X_0} & \longrightarrow & R\pi_* f^* T_X
\end{array}
$$
The bottom map comes from part of the short exact sequence

\[ 0 \to T_{X_0} \to T_X|_{X_0} \to N_{X_0/X} \cong \mathcal{O}_{X_0} \to 0 \]

which gives a distinguished triangle

\[ R\pi_+ f^* T_{X_0} \to R\pi_+ f^* T_X \to R\pi_+ f^* \mathcal{O}_{X_0} \to +1 \]

We pretend that this is a map of *two-term complexes* \( R\pi_+ f^* T_{X_0} = [E_0 \to E_1] \) and \( R\pi_+ f^* T_X = [F_0 \to F_1] \) and \( R\pi_+ f^* \mathcal{O}_{X_0} = [\pi_* \mathcal{O}_C \to R^1 \pi_* \mathcal{O}_C] \) where *C* is the universal curve over \( \overline{M}_g(X_0, \alpha) \), which splits

\[
\begin{array}{ccc}
E_0 & \longrightarrow & F_0 \\
\downarrow & & \downarrow \\
E_1 & \longrightarrow & F_1 \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
\]

Since the two moduli problems are rigged to be *the same* for \( X_0 \) and \( X \), the complexes \( E^* \) and \( F^* \) are quasi-isomorphic. Let us pretend that also \( E_0 \sim F_0 \). Then we can form a “boundary map”

\[ \pi_* \mathcal{O}_C \hookrightarrow E_1 \to F_1 \to R^1 \pi_* \mathcal{O}_C. \]

We have the normal cones \( C_{\overline{M}_g(X_0, \alpha), E_1} \subset E_1 \) and \( C_{\overline{M}_g(X, \alpha), F_1} \subset F_1 \). So we can define the virtual cycles

\[
[M_g(X_0, \alpha)]^{vir} = 0^1_{E_1} [C_{\overline{M}_g(X_0, \alpha), E_1}]
\]

\[
[M_g(X, \alpha)]^{vir} = 0^1_{F_1} [C_{\overline{M}_g(X, \alpha), F_1}]
\]

But

\[
[C_{\overline{M}_g(X, \alpha), E_1}] \cong [C_{\overline{M}_g(X, \alpha), F_1}]
\]

By this isomorphism, the image of

\[ \pi_* \mathcal{O}_C \to \mathcal{O}_{\overline{M}_g(X_0, \alpha)} \to E_1 \]

doesn’t meet the cone \( [C_{\overline{M}_g(X, \alpha), E_1}] \), so \( 0^1_{[C_{\overline{M}_g(X, \alpha), E_1}]} \) is trivial.

In the proof we found that

\[ 0^1_{F_1} [C_{\overline{M}_g(X, \alpha), F_1}] = 0^1_{E_1/\theta} [C_{\overline{M}_g(X, \alpha), E_1}] \cap c_{top}(R^1 \pi_* \mathcal{O}). \]

**Definition 8.2.2.** We define \( 0^1_{E_1/\theta} [C_{\overline{M}_g(X, \alpha), E_1}] \) to be the *reduced Gromov-Witten invariant* of K3 surfaces. We also define \( c_{top}(R^1 \pi_* \mathcal{O}) \) to be \((-1)^{g} \lambda_{g} \).

8.3. **Modifying the obstruction theory.** We found above that K3 surfaces have vanishing virtual class, which is bad. In this section we’ll show how to modify to get an interesting fundamental class (we already touched on this last time in defining the reduced Gromov-Witten invariant). The key point is to examine the difference between the moduli problems for the K3 surface and a small deformation of it, which is a “local Calabi-Yau threefold”.
**Example 8.3.1.** Suppose $X_0 \to E$ is a $(-2)$-curve and $\beta = [E]$, so $N_{E/X_0} \cong \mathcal{O}_E(-2)$. Then $N_{E/X} \cong \mathcal{O}_E(-1) \oplus \mathcal{O}_E(-1)$. Indeed, this follows by considering the rank and degree and the fact that there can be no deformations of $E$ in $X$ (the latter observation ruling out the existence of a summand of non-negative degree).

Consider $[u : E \cong \mathbb{P}^1 \to X_0] \in \overline{M}_0(X_0, \beta)$. We claim that

$$T_{[u]}\overline{M}_0(X_0, \beta) = \text{coker}(H^0(T_{\mathbb{P}^1}) \to H^0(u^*T_{X_0}))$$

Why? The obstruction theory was

$$(R\pi_*f^*T_X)^\vee \to \mathcal{L}_{\overline{M}_g(X, \beta)/M_g}.$$ 

where the maps are described in the diagram

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{f} & X \\
\downarrow \pi & & \\
\overline{M}_g(X, \beta) & & \\
\end{array}$$

with $\mathcal{C}$ the universal curve over $\overline{M}_g(X, \beta)$. Dualizing, we can think of this in terms of the tangent complex

$$T_{\overline{M}_g(X, \beta)/M_g} \to R\pi_*f^*T_X.$$ 

By the assumptions on the obstruction theory this is an *isomorphism* in degree 0, so for any closed point $[u] \in \overline{M}_g(X, \beta)$ we have

$$H^0(T_{\overline{M}_g(X, \beta)/M_g}[u]) = T_{[u]}(\overline{M}_g(X, \beta)/M_g) \sim H^0(R\pi_*f^*T_X) = H^0(u^*T_X).$$

This is the relative tangent space. The full tangent space $T_{[u]}\overline{M}_0(X_0, \beta)$ is actually *smaller* because $M_0$ is an Artin stack of dimension $-3$, and it’s the cokernel of the deformations of $\mathbb{P}^1$, which are three-dimensional. This justifies our claim

$$T_{[u]}\overline{M}_0(X_0, \beta) = \text{coker}(H^0(T_{\mathbb{P}^1}) \to H^0(u^*T_{X_0})).$$

We then need to take a cokernel ($M_g$ has dimension $-3$, because we’re dealing with Artin stacks).

♠♠♠ **TONY:** [still to BS-y. There should be a short exact sequence of cotangent complexes that expresses this clearly...] From the sequence

$$0 \to T_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(2) \to u^*T_{X_0} \to \mathcal{O}_{\mathbb{P}^1}(-2) \to 0.$$ 

we deduce that $u^*T_{X_0} \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$. Therefore, $\mathbb{C}$ implies that the tangent space $T_{[u]}\overline{M}_0(X_0, \beta)$ vanishes.

Arguing similarly as above, the obstruction spaces are

$$\text{Ob}_{[u]} \overline{M}_0(X_0, \beta) = H^1(u^*T_{X_0})$$

$$= H^1(\mathcal{O}(2) \oplus \mathcal{O}(-2))$$

$$\cong \mathbb{C}.$$ 

Therefore we see that the virtual dimension of $\overline{M}_0(X_0, \beta)$ is

$$\text{vir. dim} \overline{M}_0(X_0, \beta) = 0 - 1 = -1.$$
However, 

\[ \text{Ob}_{[u]} \overline{M}_0(X, \beta) = H^1(u^* T_X) \]

\[ = H^1(\mathcal{O}(2) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1)) \]

\[ = 0. \]

Therefore, 

\[ \text{vir. dim} \overline{M}_0(X, \beta) = 0 - 0 = 0 \]

which is good. This is baffling because the moduli problem \( \overline{M}_0(X_0, \beta) \) and \( \overline{M}_0(X, \beta) \) are equivalent. This is the kind of issue that we want to address.

Let \( W = \overline{M}_g(X_0, \beta) = \overline{M}_g(X, \beta) \) where \( X \) is a deformation of \( X_0 \) over some sufficiently small \( \Delta \). We have

\[ '\mathcal{C}' \xrightarrow{f} X_0 \subset X \]

\[ \pi \]

\[ W \]

We get two obstruction theories

\[ \mathbb{L}_{W/M_g} \rightarrow \mathbb{L}_{W/M_g} \]

\[ (R \pi_* f^* T_X)^{\vee} \rightarrow (R \pi_* f^* T_{X_0})^{\vee} \]

From the short exact sequence

\[ 0 \rightarrow T_{X_0} \rightarrow T_X|_{X_0} \rightarrow N_{X_0/X_0} = \mathcal{O}_X \rightarrow 0 \]

we get the distinguished triangle

\[ R \pi_* f^* T_{X_0} \rightarrow R \pi_* f^* T_X \rightarrow R \pi_* \mathcal{O}_\mathcal{C} \rightarrow +1 \]

What we want to do is to find some way of “shrinking” the dimension of the obstruction space of \( \overline{M}_g(X_0, \beta) \). This obstruction space was by definition \( (R^1 \pi_* f^* T_{X_0})^{\vee} \). We have a map of complexes

\[ R \pi_* f^* T_{X_0} \rightarrow \tau_{\geq 1} R \pi_* f^* T_{X_0} = R^1 \pi_* f^* T_{X_0} [-1] \]  

(8.3.2)

Contraction with the symplectic form on \( X_0 \) gives

\[ f^* T_{X_0} \sim f^* \Omega_{X_0} \rightarrow \omega_{\mathcal{C}/W} \]

which allows us to continue the sequence (6) as

\[ R \pi_* f^* T_{X_0} \rightarrow \tau_{\geq 1} R \pi_* f^* T_{X_0} = R^1 \pi_* f^* T_{X_0} [-1] = R^1 \pi_* \Omega_{\mathcal{C}/W} = \mathcal{O}_W [-1] \]

with the last equality following from Serre duality.

Define the complex \( MC \) to be the mapping cone of \( R \pi_* f^* T_{X_0} \rightarrow \mathcal{O}_W [-1] \), so that there is a distinguished triangle

\[ MC \rightarrow R \pi_* f^* T_{X_0} \rightarrow \mathcal{O}_W [-1] \rightarrow +1 \]

(8.3.3)
which dualizes to
\[ \mathcal{O}_W[1] \to (R\pi_*f^*T_{X_0})^\vee \to (MC)^\vee +1 \]

We have two intrinsic normal cone classes
\[ C_{\overline{M}_g(X,\beta)/M_g} \subset h^1/h^0(R\pi_*f^*T_X) \]
\[ C_{\overline{M}_g(X_0,\beta)/M_g} \subset h^1/h^0(MC) \]

They fit into the short exact sequence obtained from (7):
\[
\begin{array}{ccccccc}
0 & \to & h^1/h^0(M_C) & \to & h^1/h^0(R\pi_*f^*T_X) & \to & R^1\pi_*\mathcal{O}_C & \to & 0
\end{array}
\]
\[ C_{\overline{M}_g(X_0,\beta)/M_g} \to \cong C_{\overline{M}_g(X,\beta)/M_g} \]

As indicated by the diagram, the cones are isomorphic. You can basically think of them as sub-bundles. Then
\[ 0_{h^1/h^0(MC)}[C_W/M_g] = 0_{h^1/h^0(R\pi_*f^*T_X)}[C_W/M_g] \cap c_{top}(R^1\pi_*\mathcal{O}_C) = (\pi_*\mathcal{O}_C/W)^\vee. \]

Letting \( \lambda_g = c_g(\pi_*\mathcal{O}_C/W) \) be the top Chern class appearing above, we obtain which we define to be the reduced Gromov-Witten invariant for \( X_0 \):
\[ ([\overline{M}_g(X_0,\beta)]^{vir})_{\text{red}} := (-1)^g \lambda_g \cap [\overline{M}_g] = ([\overline{M}_g(X_0,\beta)]^{vir})_{\text{red}} \in A_0(\overline{M}_g(X_0,\beta)). \]

We need to justify some claims we made above. We claimed a distinguished triangle
\[ (R\pi_*f^*T_{X_0})^\vee \to MC^\vee \to (R^1\pi_*\mathcal{O}_C)[1]. \]

Is this right? Are the shifts right? ♠♠♠ TONY: [...] Mapping from the left should be \( R^1\pi_*\mathcal{O}_C \). We need to compare this distinguished triangle with
\[ (R\pi_*f^*T_X)^\vee \to (R\pi_*f^*T_{X_0})^\vee \to R^1\pi_*\mathcal{O}_C +1. \]

Fit in
\[
\begin{array}{ccccccc}
(R\pi_*f^*T_X)^\vee & \to & (MC)^\vee & \to & (\ast) & \to & (R\pi_*f^*T_{X_0})^\vee
\end{array}
\]
\[
\begin{array}{ccccccc}
(R\pi_*f^*T_X)^\vee & \to & (R^1\pi_*\mathcal{O}_C)^\vee & \to & R^1\pi_*\mathcal{O}_C[1] & \to & (\ast) +1
\end{array}
\]

We have to prove that \( (\ast) = R^1\pi_*\mathcal{O}_C[1] \). This statement is hard to pin down in the literature. The key is to consider for \( [u : C \to X_0] \in \overline{M}_g(X_0,\beta) \) the symplectic pairing
\[ u^*T_{X_0} \to u^*\Omega_{X_0} \to \omega_C \]
and the tangent sequence
\[ 0 \to u^*T_{X_0} \to u^*T_X|_{X_0} \to N_{X/X_0} \cong \mathcal{O}_{X_0} \to 0 \]
fit into a long exact sequence
\[ 0 \to H^0(u^* T_{X_0}) \xrightarrow{\sim} H^0(u^* T_X) \to H^0(\mathcal{O}_C) = H^1(\omega_C) \]
\[ \to H^1(u^* T_{X_0}) \to H^1(u^* T_X) \to H^1(\mathcal{O}_C) \]
and we want to cancel out \( H^0(\mathcal{O}_C) \cong \mathbb{C} \). We need to prove that \( H^1(MC) \to H^1(u^* T_{X_0}) \to H^1(u^* T_X) \) is an isomorphism. Then we can see that \( H^0(*) = 0 \) and \( H^1 \) is what we claim.

Consider the short exact sequence induced by \( X \to \Delta \):
\[ 0 \to T_{X_0} \to T_X|_{X_0} \to \mathcal{O}_{X_0} \to 0 \]
which is given by the Kodaira-Spencer class \( KS \in \text{Ext}^1_{X_0}(\mathcal{O}_{X_0}, T_{X_0}) \).

On the other hand, we have a period map \( \sigma : \Delta \to \mathbb{P}V_C \). Then \( \sigma(0) \in \text{Hom}(H^{2,0}(X_0), H^{1,1}(X_0) \oplus H^{0,2}(X_0)) \). We have to relate these two classes. That is what underlies the desired identifications. And the answer is that the Kodaira-Spencer class \( KS \in H^1(T_{X_0}) \) induces a map
\[ H^{2,0}(X_0) = H^0(X_0, K_{X_0}) \xrightarrow{\text{KS}} H^1(\Omega_{X_0}) = H^{1,1}(X_0). \]
which is precisely the differential of the period map.

9. The Yau-Zaslow Theorem

9.1. Setup. Let \( S \) be a projective K3 surface and \( \beta_2 \in H_2(X, \mathbb{Z}) \) an algebraic class. We consider \( M_0(S, \beta) \).

**Theorem 9.1.1** (Chen). Let \( \ell > 0 \) and consider
\[ M_\ell = \left\{ (X, H) \mid X = \text{K3 surface, } H \subset X \text{ ample divisor} \atop \text{H}^2 = 2\ell c_1(H) \text{ primitive} \right\}. \]
There exists a dense open subset \( U \subset M_\ell \) such that for all \( (X, H) \in U \) any rational \( C \in |H| \) is nodal.

(Henceforth we denote by \( L \) a general line bundle and by \( H \) an ample one.)

If \( S \) is a K3 surface and \( L \in \text{Pic}(S) \) with \( L^2 = 2\ell \), then by Riemann-Roch for surfaces we have
\[ \chi(L) = \frac{1}{2}(L^2 + L \cdot K_S) + \chi(\mathcal{O}_S) = \frac{1}{2} L^2 + 2. \]
Assume \( |L| \neq \emptyset \), i.e. \( L \) is effective. Then \( h^1(L) = 0 \) and \( h^2(L) = 0 \) by Serre-duality and the triviality of \( K_S \). This implies that
\[ \dim |L| = h^0(L) - 1 = \frac{1}{2} L^2 + 1. \]
If \( C \in |L| \) is general then \( C \) is smooth, so
\[ K_C = K_S \otimes N_{C/S}|_C = \mathcal{O}_S(C)|_C \]
so
\[ 2g(C) - 2 = \deg K_C = C \cdot C = L^2 \]
and we find that \( g(C) = \frac{1}{2} L^2 + 1 \). Heuristically, having a node is a codimension one condition, so we expect that there should exist finitely many rational \( C \in |L| \).
Remark 9.1.2. The Riemann-Roch calculation shows that there does not exist a family of rational curves in a K3 surface, since the canonical line bundle is trivial. We have to argue that there always exists some rational curve.

Let \( C = |L| \) be a nodal rational curve. Let \( \beta = [L] \) and assume that \( \beta \) is primitive, and also assume that \( \text{Pic}(S) = \mathbb{Z}[L] \). (These conditions will be satisfied for general \( S \).) Consider the moduli space \( \overline{M}_0(S, \beta) \) and \([u : C \to S] \in \overline{M}_0(S, \beta)\). Consider the map from the normalization

\[ C^{\text{nor}} \to C \to S. \]

The normalization is a union of \( \mathbb{P}^1 \)s, so the image of the normalization is several copies of \( L \) by the assumptions. But by primitivity the normalization must be a single \( \mathbb{P}^1 \). We claim that \([u] \in \overline{M}_0(S, \beta)\) is open and closed.

**Proof.** We can verify the openness if we show that \( \text{Def}(u) = 0 \). Well,

\[ \text{Def}(u) = \text{coker} \left( H^0(T_{\mathbb{P}^1}) \xrightarrow{u_*} H^0(u^*T_S) \right) . \]

By the assumption of nodal singularities the map \( T_{\mathbb{P}^1} \xrightarrow{u_*} u^*T_S \) is an injective map of bundles (i.e. the embedding \( u \) separates tangent vectors), so the cokernel is locally free. By degree considerations, the cokernel must be \( \mathcal{O}(-2) \). Therefore, we have a short exact sequence

\[ 0 \to T_{\mathbb{P}^1} \xrightarrow{u_*} u^*T_S \to \mathcal{O}(-2) \to 0. \]

Thus we must have a splitting \( u^*T_S \cong \mathcal{O}(2) \oplus \mathcal{O}(-2) \). This shows that the cokernel of \( u_* \) vanishes.

What about the obstructions? We have

\[ \text{Ob}[u] = H^1(u^*T_S) = H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) \cong \mathbb{C} \]

while the reduced obstruction group vanishes. So \([u]\) contributes 1 to \( \deg(\overline{M}_0(S, \beta))_{\text{red}} \).

Adding in Chen’s theorem that rational curves are all nodal, we find: for general \((X, H) \in M_\ell\),

\[ \deg(\overline{M}_0(S, \beta))_{\text{red}} = \#\{\text{rational curves}\} \]

\( \square \)

9.2. **Counting via moduli of sheaves.** So, how do we count nodal curves? The idea of Yau-Zaslow is to consider the moduli space \( M_S(c) \) of stable sheaves \( E \) with

- \( \text{rank } E = 0 \),
- \( c(E) = c_1(E) = [L] = \beta \) and
- \( \chi(E) = 1 \).

**Theorem 9.2.1** (Mukai, Gieseker,...). \( M_S(c) \) is projective and smooth.

**Proof.** Suppose \([E] \in M_S(C)\). Then

\[ T_{[E]} M_S(C) = \text{Ext}^1_S(E, E)_0 = \text{Ext}^1_S(E, E) \]

with the last equality following from \( H^1(\mathcal{O}_S) = 0 \). Also

\[ \text{Ob}_{[E]} M_S(C) = \text{Ext}^2_S(E, E)_0 \cong \text{Ext}^0_S(E, E)_0 = 0 \]
Hilbert-Chow map. We are going to define a morphism
\[ M_S(c) \to |L| \]
This is called the Hilbert-Chow map. Also, \( E \) has a resolution of length at most 3
\[ 0 \to F_3 \to F_2 \to F_1 \to E \to 0. \]
Using this we can define \( \det E := \det F_1 \otimes (\det F_2)^\vee \otimes \det F_3 \in \text{Pic}(S) \). We know that \( E|_{S^0} = 0 \) for some \( S^0 \subset S \) open and dense. Then \( \det E|_{S^0} \cong \mathcal{O}_{S^0} \) has a section. We claim that it can be extended to \( S \). By Hartog's theorem, it suffices to extend over codimension-one points.

**Exercise 9.2.2.** Convince yourself that this is possible. (Consider for instance what happens when \( E = \mathcal{O}_C \).)

From the section \( \sigma_E : \mathcal{O}_S \to \det E \),
we obtain a locus \( \sigma_E^{-1}(0) \) contained in the support of \( \mathcal{E} \). We define the Hilbert-Chow map to be \( E \mapsto \sigma_E^{-1}(0) \).

**Theorem 9.2.3 (Yau-Zaslow).** We have
\[ \# \{ \text{rational curves in } |L| \} = e(M_S(c)). \]

9.3. Some background/ingredients.

**Theorem 9.3.1 (Mukai).** \( M_S(c) \) is a smooth, projective, symplectic variety.

To define the symplectic structure we need to define a pairing
\[ \text{Ext}^1_S(E, E) \times \text{Ext}^1_S(E, E) \to \mathbb{C} \]
which globalizes to a holomorphic non-degenerate form
\[ T M_S(c) \times T M_S(c) \to \mathcal{O}_{M_S(c)} \]
since \( \text{Ext}^1_S(E, E) = T^1_{[E]} M_S(c) \). This is obtained through the cup product
\[ \text{Ext}^1_S(E, E) \times \text{Ext}^1_S(E, E) \to \text{Ext}^2_S(E, E) \xrightarrow{\text{tr}} H^2(\mathcal{O}_S) = \mathbb{C}. \]

**Proposition 9.3.2.** \( M_S(c) \) is birational to \( S^{[g]} \), the Hilbert scheme of length \( g \) 0-dimensional subschemes of \( S \). (This is a symplectic resolution of \( S^{(g)} \).)

**Proof.** We'll show how to define the map. Take \([E] \in M_S(C)\). We have the Hilbert-Chow map
\[ [E] \mapsto \sigma_E^{-1}(0) \in |L|. \]
Every curve in \( |L| \) is irreducible and reduced. Call \( C_E := \sigma_E^{-1}(0) \). We also know \( \chi(E) = 1 \).
For general \( E \) we can arrange \( h^0(E) = 1 \), and this map takes \( E \) to \( s^{-1}(0) \) for \( s \) a non-zero section in \( H^0(C_E, E) \).

♠♠♠ TONY: [this is just the self-intersection of the support of \( E \)... suspicious]
Theorem 9.3.3 (Bayteyev-Kontsevich). Suppose $M_S(c)$ and $S^{[g]}$ are smooth, projective, Calabi-Yau birational. Then $b_i(M_S(c)) = b_i(S^{[g]})$ for all $i$, so in particular $e(M_S(c)) = e(S^{[g]})$.

\[ \text{\color{red}{TONY: [hypothesis doesn’t make sense. One is a resolution of another...]} } \]

Theorem 9.3.4 (Goettsche). We have the identity of formal $q$-series:

\[ \frac{q}{\Delta(q)} = \sum_{g \geq 0} e(S^{[g]})q^g \quad \text{where} \quad \Delta(q) = q \prod (1 - q^n)^{24}. \]

9.4. Proof of Yau-Zaslow. Consider

\[ M_S(c) \twoheadrightarrow [E] \]

\[ \mu \downarrow \quad \downarrow \]

\[ |L| \mapsto \sigma^{-1}_E(0) \]

We have $\sigma^{-1}_E(0) = \text{supp } E$. then

\[ E = \iota_* E' \]

where $E'$ is a sheaf on $\text{supp } E$. \[ \text{\color{red}{TONY: [doesn’t seem right...]} } \]

If $C \in |L|$ is rational, then $e(\mu^{-1}([C])) = 1$.

If $C \in |L|$ such that $g(C^{\text{nor}}) \geq 1$, then $e(\mu^{-1}(C)) = 0$. Then, by a cut-and-paste argument we have

\[ e(M_S(c)) = e(\bigcup \mu^{-1}(C)) = \# \{ \text{C rational } \subset |L| \}. \]

Let $[E] \in M_S(c)$ and $C = \text{supp } E$. Then $E'$ is a rank 1 sheaf of $O_C$-modules such that $E = \iota_* E'$. In other words, $\mu^{-1}(C)$ is the space of rank one torsion-free sheaves on $C$ with $\chi = 1$.

Now comes an argument of Beauville.

Theorem 9.4.1 (Beauville). If $g(C^{\text{nor}}) \geq 1$, then for all $p$ there is a free $\mathbb{Z}/p$ action on $\mu^{-1}([C])$. (Thus $e(\mu^{-1}(C)) = 0$.)

Proof. If $\mathcal{L}$ is a rank one torsion sheaf on $C$, consider the automorphism sheaf

\[ \text{Aut}_{\mathcal{O}_C}(\mathcal{L}) \]

Since $\text{Aut}(\mathcal{L})$ is a finitely generated $\mathcal{O}_C$-module, there exists $C' \rightarrow C$ such that $\text{Aut}(\mathcal{L}) = \pi_* \mathcal{O}_{C'}$, and an invertible sheaf $\mathcal{L}'$ on $C'$ such that $\mathcal{L} = \pi_* \mathcal{L}'$. Indeed, we can take $C' = \text{Spec } C(\text{Aut}(\mathcal{L}))$. This $C'$ will be intermediate between $C^{\text{nor}}$ and $C$.

If $g(C^{\text{nor}}) \geq 1$ then we can find a $p$-torsion element $[N]$ in $\text{Pic}^0(C)$ such that $N \otimes_{\mathcal{O}_C} \mathcal{O}_{C^{\text{nor}}} \not\cong \mathcal{O}_{C^{\text{nor}}}$. We use this to define a $\mathbb{Z}/p$ action on rank one sheaves on $C$ by tensoring. That this action is free is guaranteed by the following lemma, which concludes the proof. \[ \square \]

Lemma 9.4.2. Let $N$ be an invertible sheaf on $C$ such that $L \otimes N \cong L$. Then $\pi^* N \cong \mathcal{O}_{C'}$.

Example 9.4.3. Example: over the nodal plane curve $xy = 0$, two distinct sheaves with full support $k[x,y]/(xy)$ and $k[x,y]/(y) \oplus y k[x,y]/(x)$.
Proof. The isomorphism gives a non-vanishing section of the sheaf $\text{Hom}_{\mathcal{O}_C}(L, L \otimes N) = \text{Hom}_{\mathcal{O}_C}(L, L) \otimes N$. By the definition of $C'$ this is the same as $\pi_\ast \mathcal{O}_{C'} \otimes \mathcal{O}_{\mathcal{O}_C} N$. By the projection formula, this is $\pi_\ast \pi^\ast N$. So the isomorphism gives a (non-vanishing) global section

$$H^0(\text{Hom}(L, L \otimes N)) = H^0(\pi_\ast \pi^\ast N) = H^0(\pi^\ast N) \implies \pi^\ast N = \mathcal{O}_{\mathcal{O}_C}.$$

$\square$