The characteristic cycle and the singular support of an étale sheaf

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1 Introduction

Let \( k \) be a perfect field of characteristic \( p > 0 \). Let \( X \) be a smooth variety over \( k \) of dimension \( n \). Let \( \Lambda/\Bbb{F} \) be a finite extension with \( \ell \neq p \).

Let \( \mathcal{F} \) be a constructible complex on \( X_{\text{ét}} \) of \( \Lambda \)-modules. This means that every cohomology sheaf \( H^i(\mathcal{F}) \) is constructible for all \( i \), and vanishes for finitely many \( i \).

I want to define the singular support of the complex. This will be a union of components \( C_\alpha \) inside the cotangent bundle \( T^*X \). Since \( \dim X = n \), we have \( \dim T^*X = 2n \). Each \( C_\alpha \) will be irreducible, closed, and conical (i.e. stable under multiplication). Moreover, \( \dim C_\alpha = n \) for each \( \alpha \).

The characteristic cycle is a divisor \( \sum m_\alpha [C_\alpha] \) where \( m_\alpha \in \mathbb{Z}[1/p] \) (recall that \( p \) was the characteristic of the base field).

1.1 Analogy with microlocal analysis

There is an analogy with \( D \)-modules on a complex manifold. Under this analogy, the phenomenon of irregular singularities for \( D \)-modules corresponds to wild ramification in characteristic \( p \).

1.2 Classical examples

Suppose \( \dim X = 1 \) and \( j: U \hookrightarrow X \) is the inclusion of a dense open. Take \( \mathcal{F} = j!\mathcal{G} \) where \( \mathcal{G} \) is a locally constant sheaf on \( U \). Then the singular support is

\[
\mathcal{F} = T^*_X X \cup \bigcup_{x \in X \setminus U} T^*_x X.
\]

Then \( \mathcal{F}[-1] \) is perverse, and in general the characteristic cycle of a perverse sheaf is effective. In this case,

\[
\text{Char}(\mathcal{F}) = (-1) \left( \text{rank } \mathcal{G}[T^*_X X] + \sum_{x \in X \setminus U} \dim_{\text{tot}} \mathcal{G}[T^*_x X] \right).
\]
The integrality of the coefficient rank $\text{sw}_S$ is the Hasse-Arf theorem. Since $X$ is projective,
\[ \chi(X, \mathcal{F}) = (\text{Char}(\mathcal{F}), T_X^*X). \]
This is a reformulation of the Grothendieck-Ogg-Shafarevich formula.

2 Singular support

The definition is due to Beilinson. I need to introduce a preliminary notion.

**Definition 2.1.** Let $C \subset T_X^*X$ be a closed conical subvariety. We say that a morphism $f: X \to Y$ of smooth schemes over $k$ is $C$-transversal if $df^{-1}(C)$ is contained in the zero-section, where
\[ df: X \times_Y T^*Y \to T^*X \to T^*X. \]

**Example 2.2.** If $C = T_X^*X$, then $C$-transversal is equivalent to $f$ being smooth.

So you think of $C$-transversality as being a generalization of smoothness.

**Theorem 2.3** (Beilinson). Let $C = SS(\mathcal{F}) = \bigcup C_\alpha \subset T_X^*X$ be the smallest closed conical subset such that for every $X \leftarrow U \xrightarrow{f} \mathbb{A}^1$ with $f$ being $j^{-1}C$-transversal, and $f$ is locally acyclic relative to $j^*\mathcal{F}$ (i.e. no non-zero vanishing cycle). Then $\dim C_\alpha = \dim X$.

3 Characteristic cycle

The characteristic cycles are a generalization of the notion of “isolated singular point.” We consider a similar diagram
\[ X \leftarrow U \xrightarrow{f} \mathbb{A}^1 \]
and $u \in U$ a closed point.

**Definition 3.1.** We say that $u$ is an isolated characteristic point of $f$ if $f|_{U-\{u\}}: U-\{u\} \to \mathbb{A}^1$ is $j^*C$-transversal, where $C = \bigcup C_\alpha$ is the singular support.

We can regard $df$ as a section of $T^*U \to U$. If $u$ is an isolated characteristic point, then $df(U) \cap C_\alpha$ is an isolated intersection at $u$, and there is an intersection number $(C_\alpha, df)_{T^*U,u} \in \mathbb{N}$.

**Theorem 3.2.** There exists a unique $\mathbb{Z}[1/p]$-linear combination $\text{Char}(\mathcal{F}) = \sum m_\alpha[C_\alpha]$ such that for every $X \leftarrow U \xrightarrow{f} \mathbb{A}^1$ with isolated characteristic point $u \in U$, we have
\[ -\text{dim tot } \phi_u(\mathcal{F}, f) = (\text{Char}(\mathcal{F}), df)_{T^*U,u} \]
complex of vanishing cycles

For $\mathcal{F} = \Lambda$, we have $\text{Char}\mathcal{F} = [T_X^*X]$. This is proved by Deligne in SGA 7 XVI.

**Theorem 3.3.** If $X$ is projective, then
\[ \chi(X, \mathcal{F}) = (\text{Char}(\mathcal{F}), T_X^*X)_{T^*X}. \]
This matches up with the result in the one-dimensional case.
3 CHARACTERISTIC CYCLE

3.1 Proof ideas

For the first theorem, you choose an embedding $X \hookrightarrow \mathbb{P}^n$ and apply Beilinson’s theorem. There is a Radon transform

$$U = \text{universal family}$$

You need to prove the independence of the embedding and Milnor’s formula, which come from the stability of the total dimension of the vanishing cycle. If you have such an equality on the left hand side of the theorem, then you must have an equality on the right hand side as well.

In turn, this stability is implied by a slight generalization of the continuity of Swan conductors by Deligne/Laumon.

For the second theorem, you do an induction on dimension using hyperplane sections.