

COMPLEX MANIFOLDS

LECTURES BY JULIUS ROSS

NOTES BY TONY FENG

LENT 2014

PREFACE

These are lecture notes for a course taught in Cambridge during Lent 2014 by Julius Ross, on complex manifolds. There are likely to be errors, which are the fault of the scribe. If you find any, please let me know at tonyfeng009@gmail.com.

Contents

Preface	i
1 Complex Manifolds	1
1.1 Several complex variables	1
1.2 Complex manifolds	3
2 Almost Complex Structures	6
2.1 Linear algebra preliminaries	6
2.2 Almost complex structures	7
3 Differential Forms	10
3.1 Differential forms on Complex Manifolds	10
3.2 Dolbeault Cohomology	12
3.3 The Mittag-Leffler Problem	13
4 The $\bar{\partial}$ Poincaré Lemma	15
4.1 Holomorphic Functions on \mathbb{C}	15
4.2 Holomorphic functions of several complex variables	17
4.3 The Poincaré Lemma	18
5 Sheaves and Cohomology	21
5.1 Sheaves	21
5.2 Čech Cohomology	24
5.3 Properties of Čech cohomology	27
6 More on Several Complex Variables	31
6.1 Hartog's Theorem	31
7 Holomorphic Vector Bundles	34
7.1 Basic definitions	34
7.1.1 Bundle Constructions	35
7.2 Holomorphic Line Bundles	35

7.3	Line bundles on Projective Space	37
7.4	Ample Line Bundles	38
8	Kähler Manifolds	40
8.1	Kähler metrics	40
8.2	The Kähler Identities	44
9	Hodge Theory	52
9.1	The Hodge Decomposition	52
10	Lefschetz Theorems	57
10.1	Lefschetz, I	57
10.2	Lefschetz, II	59
11	Hermitian Vector Bundles	61
11.1	Hermitian metrics	61
11.2	The Chern connection	62
11.3	Curvature	65
11.4	Kodaira's Theorems	66
12	Example Sheet 1	68
13	Example Sheet 2	77
14	Example Sheet 3	87

Chapter 1

Complex Manifolds

1.1 Several complex variables

Identify $\mathbb{C} \cong \mathbb{R}^2$ in the standard way, $x + iy \leftrightarrow (x, y)$. Similarly, $\mathbb{C}^n \cong \mathbb{R}^{2n}$ by

$$(x_1 + y_1i, \dots, x_n + y_ni) \leftrightarrow (x_1, y_1, \dots, x_n, y_n).$$

Definition 1.1. A smooth function $f : \mathbb{C}^n \rightarrow \mathbb{C}$ is *holomorphic* if it is holomorphic in each variable.

Remark 1.2. There is a more natural definition multivariable holomorphic function (you can probably guess what it is), but it is equivalent to this one, and this is more useful for applications.

Writing $f = u + iv$, the theory of single-variable complex functions implies that being holomorphic is equivalent to satisfying the Cauchy-Riemann equations: for all j ,

$$\begin{aligned} \frac{\partial u}{\partial x_j} &= \frac{\partial v}{\partial y_j} \\ \frac{\partial u}{\partial y_j} &= -\frac{\partial v}{\partial x_j}. \end{aligned}$$

Formally define the differential operators

$$\begin{aligned} \frac{\partial}{\partial z_j} &= \frac{1}{2} \left(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right) \\ \frac{\partial}{\partial \bar{z}_j} &= \frac{1}{2} \left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right) \end{aligned}$$

Then the Cauchy-Riemann equations are equivalent to

$$\frac{\partial f}{\partial \bar{z}_j} = 0 \quad j = 1, \dots, n.$$

Again, this is probably familiar from single-variable complex analysis.

Proposition 1.3 (Maximum Principle). *Let $U \subset \mathbb{C}^n$ be open and connected. Let f be holomorphic on a region D such that $\overline{D} \subset U$. Then*

$$\max_{\overline{D}} f(z) = \max_{\partial \overline{D}} f(z)$$

and f achieves this maximum at an interior point if and only if f is constant.

Proof. For $n = 1$, this is the usual maximum principle. Apply the single-variable principle repeatedly to deduce it in higher dimensions. \square

Proposition 1.4 (Identity Principle). *Let $U \subset \mathbb{C}^n$ be open and connected and $f : U \rightarrow \mathbb{C}$ be a holomorphic function. If f vanishes on an open subset of U , then $f \equiv 0$ on U .*

Proof. This follows from the single-variable identity principle, applied repeatedly. \square

Lemma 1.5. *Let U, V be open sets in \mathbb{C}^n . A smooth map $f : U \rightarrow V$ is holomorphic if and only if df is \mathbb{C} -linear.*

Proof. Picking frames $\frac{\partial}{\partial x_k}$ and $\frac{\partial}{\partial y_k}$, we have that

$$df = \begin{pmatrix} \vdots & \vdots \\ \cdots & \frac{\partial u_j}{\partial x_k} & \frac{\partial u_j}{\partial y_k} & \cdots \\ \cdots & \frac{\partial v_j}{\partial x_k} & \frac{\partial v_j}{\partial y_k} & \cdots \\ \vdots & \vdots \end{pmatrix}.$$

Also,

$$J = \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & \ddots & \\ & & & 0 & -1 \\ & & & 1 & 0 \end{pmatrix}$$

Now, df is \mathbb{C} -linear if and only if $Jdf = (df)J$, and we see that

$$Jdf = \begin{pmatrix} \vdots & \vdots \\ \cdots & -\frac{\partial v_j}{\partial x_k} & -\frac{\partial v_j}{\partial y_k} & \cdots \\ \cdots & \frac{\partial u_j}{\partial x_k} & \frac{\partial u_j}{\partial y_k} & \cdots \\ \vdots & \vdots \end{pmatrix} \quad \text{and} \quad (df)J = \begin{pmatrix} \vdots & \vdots \\ \cdots & \frac{\partial u_j}{\partial y_k} & -\frac{\partial u_j}{\partial x_k} & \cdots \\ \cdots & \frac{\partial v_j}{\partial y_k} & -\frac{\partial v_j}{\partial x_k} & \cdots \\ \vdots & \vdots \end{pmatrix}$$

The equality of these two matrices is precisely the content of the Cauchy-Riemann equations. \square

Theorem 1.6 (Inverse function theorem). *Let U, V be open sets in \mathbb{C}^n and $f : U \rightarrow V$ a holomorphic function. Suppose that $z_0 \in U$ is such that*

$$\det J_{\mathbb{C}}(f)(z_0) \neq 0.$$

Then there exists an open subset U' containing z_0 such that $f|_{U'} : U' \rightarrow f(U')$ is a biholomorphism.

Proof. By the real inverse function theorem, there is a local smooth inverse f^{-1} . Furthermore, $df_{f^{-1}(x)}^{-1} = (df)_x^{-1}$, which is \mathbb{C} -linear because df is, so f^{-1} is holomorphic by Lemma 1.5. \square

Theorem 1.7 (Implicit function theorem). *Let $U \subset \mathbb{C}^n$ and $f : U \rightarrow \mathbb{C}^m$ a holomorphic function such that $(\frac{\partial f}{\partial z_i}(x))_{i=1}^m$ has full rank. Then there exists a holomorphic function g defined in a neighborhood of U' of x such that $(z_{m+1}, \dots, z_n) \rightarrow (g(z_{m+1}, \dots, z_n), z_{m+1}, \dots, z_n)$ is a biholomorphism onto $U' \cap f^{-1}(0)$.*

Proof. This is deduced from the (holomorphic) inverse function theorem in the usual way. Let $\mathbf{w} = (z_1, \dots, z_m)$ and $\mathbf{z} = (z_{m+1}, \dots, z_n)$. Then we define the map $\mathbb{C}^n \rightarrow \mathbb{C}^n$ by

$$(\mathbf{w}, \mathbf{z}) \mapsto (f(\mathbf{w}, \mathbf{z}), \mathbf{z}).$$

By the hypothesis, this is a diffeomorphism at x , so there exists a local holomorphic inverse. In particular, the set $(0, \mathbf{z})$ maps via this inverse to

$$(0, \mathbf{z}) \mapsto (g(\mathbf{z}), \mathbf{z}).$$

\square

1.2 Complex manifolds

Complex manifolds lie at the intersection of several different mathematical areas: several complex variables, differential geometry, and algebraic geometry. This makes their theory very rich and naturally interesting.

Let X be a smooth manifold of real dimension $2n$. We now develop the basic theory of complex manifolds, which is completely analogous to smooth manifolds except that one demands all functions to be holomorphic instead of smooth.

Definition 1.8. A *holomorphic atlas* for X is a collection of charts $(U_\alpha, \varphi_\alpha)$ where

$$\varphi_\alpha : U_\alpha \rightarrow \underbrace{V_\alpha}_{\text{open}} \subset \mathbb{R}^{2n} \cong \mathbb{C}^n$$

such that

1. $X = \bigcup_{\alpha} U_{\alpha}$, and
2. the transition functions

$$\varphi_{\alpha\beta} = \varphi_{\alpha} \circ \varphi_{\beta}^{-1}$$

are holomorphic.

Definition 1.9. Two holomorphic atlases $(U_{\alpha}, \varphi_{\alpha})$ and $(\tilde{U}_{\alpha}, \tilde{\varphi}_{\alpha})$ are said to be *equivalent* if $\varphi_{\alpha} \circ \tilde{\varphi}_{\beta}^{-1}$ is holomorphic for all α, β .

Definition 1.10. A *complex manifold* X is a smooth manifold with a choice of equivalence class of holomorphic atlases.

This is also called a “complex structure” on X . Note that if X has dimension $2n$ as a real manifold, then it has dimension n as a complex manifold.

Example 1.11. We give some examples of complex manifolds.

- \mathbb{C}^n is a complex manifold.
- $M_n(\mathbb{C}) \cong \mathbb{C}^{n^2}$ is a complex manifold. Hence $\mathrm{GL}_n(\mathbb{C})$ is a complex manifold, as it is an open subset of a complex manifold.
- Complex projective space $\mathbb{C}\mathbb{P}^n$. As a set, this consists of one-dimensional subspaces of \mathbb{C}^{n+1} . A point can be written as $[z_0, \dots, z_n]$. A holomorphic atlas is given by taking $U_i = \{z_i \neq 0\}$, and

$$\begin{aligned} \varphi_i : U_i &\rightarrow \mathbb{C}^n \\ [z_0, \dots, z_n] &\mapsto \left(\frac{z_0}{z_i}, \dots, \frac{\widehat{z}_i}{z_i}, \dots, \frac{z_n}{z_i} \right). \end{aligned}$$

The transition function on $U_i \cap U_j$ is

$$\varphi_i \circ \varphi_j^{-1}(w_0, \dots, \widehat{w}_j, \dots, w_n) = \left(\frac{w_0}{w_i}, \dots, \frac{\widehat{w}_j}{w_i}, \dots, \frac{w_n}{w_i} \right).$$

In contrast to our previous examples, $\mathbb{C}\mathbb{P}^n$ is compact.

Definition 1.12. A function $f : X \rightarrow \mathbb{C}$ is *holomorphic* if for all holomorphic charts (U, φ) , the composition

$$f \circ \varphi^{-1} : \varphi(U) \rightarrow \mathbb{C} \text{ is holomorphic.}$$

Definition 1.13. A morphism $F : X \rightarrow Y$ between complex manifolds is *holomorphic* if for all holomorphic charts (U, φ) on X and (V, ψ) on Y ,

$$\psi \circ F \circ \varphi^{-1} \text{ is holomorphic.}$$

We say that X and Y are *isomorphic* (or *biholomorphic*) complex manifolds if there exists a holomorphic function $F : X \rightarrow Y$ with holomorphic inverse.

An important theme is that there are fewer holomorphic functions than smooth functions. In the smooth theory, functions can be constructed locally and patched together using bump functions (partitions of unity). In the analytic theory, everything is much more rigid: for instance, a function is determined globally by its local behavior (the Identity Theorem). Holomorphic functions on complex manifolds are much more like regular functions on algebraic varieties.

Proposition 1.14. *Let X be compact and connected. Then any holomorphic function on X is constant.*

Proof. Since X is connected, it suffices to show that f is locally constant. Suppose otherwise. Then $\|f\|$ has a local maximum at some $x \in X$. Composing with a chart, we get a function $f \circ \varphi^{-1} : \varphi(U) \rightarrow \mathbb{C}$ assuming a local maximum in the interior. By the maximum principle, $f \circ \varphi^{-1}$ is constant. \square

Corollary 1.15. *A compact complex manifold cannot embed in \mathbb{C}^N .*

Proof. Indeed, otherwise one could obtain nonconstant holomorphic functions by pulling back holomorphic functions on \mathbb{C}^N (e.g. the coordinate functions). \square

This contrasts with the Whitney embedded theorem for smooth manifolds, which says that any real manifold embeds in a finite-dimensional Euclidean space.

Proposition 1.16. *Let X be a connected complex manifold. If $f : X \rightarrow \mathbb{C}$ is holomorphic vanishes on an open subset of X , then f is identically zero.*

Proof. Let φ be a chart around such an accumulation point, and apply the identity theorem for complex functions. \square

In particular, this prohibits a holomorphic analogue of partitions of unity. This means that it is much harder to pass from local to global, as we are accustomed to do in the theory of smooth manifolds.

Definition 1.17. Let $Y \subset X$ be a smooth submanifold of dimension $2k$. We say that Y is a *closed complex submanifold* if there exist holomorphic charts $(U_\alpha, \varphi_\alpha)$ for X where $Y \subset \bigcup U_\alpha$ and $\varphi_\alpha : U_\alpha \cap Y \rightarrow \varphi_\alpha(U_\alpha) \cap \mathbb{C}^k$ with the canonical inclusion of $\mathbb{C}^k \subset \mathbb{C}^n$ as $(z_1, \dots, z_k) \mapsto (z_1, \dots, z_k, 0, \dots, 0)$.

Definition 1.18. We say that X is *projective* if it is isomorphic to a closed complex submanifold of \mathbb{P}^N for some N .

Chapter 2

Almost Complex Structures

We might ask, to what extent can the complex structure of a manifold be captured by *linear* data? In particular, if X is a complex manifold, we should get some extra structure on TX , the tangent bundle of X . This leads to the notion of almost-complex structures.

A model case is $X = \mathbb{R}^{2n} \cong \mathbb{C}^n$, with real coordinates $x_1, y_1, \dots, x_n, y_n$. Consider the linear map $\mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$

$$(x_1, y_1, \dots, x_n, y_n) \mapsto (-y_1, x_1, \dots, -y_n, x_n).$$

Under the identification with \mathbb{C}^n , this corresponds to “multiplication by i .” Similarly on $TX = X \times \mathbb{R}^{2n}$, we have frames $\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}$ and we can define an endomorphism of the tangent bundle $J : TX \rightarrow TX$ by

$$J\left(\frac{\partial}{\partial x_j}\right) = -\frac{\partial}{\partial y_j} \quad J\left(\frac{\partial}{\partial y_j}\right) = \frac{\partial}{\partial x_j}.$$

Note that again, J^2 is -1 .

We want to investigate what extra structure we get from having this kind of endomorphism of the tangent bundle.

2.1 Linear algebra preliminaries

Let V be a real vector space.

Definition 2.1. A linear map $J : V \rightarrow V$ such that $J^2 = -1$ is called a *complex structure* on V .

On \mathbb{R}^{2n} , the endomorphism $(x_i, y_i, \dots, x_n, y_n) \mapsto (y_1, -x_1, \dots, y_n, -x_n)$ is called the *standard complex structure*.

Given a complex structure J , the relation $J^2 = -1$ implies that its eigenvalues are $\pm i$. As V is real, this means that J has no eigenspaces in V . But if we consider the complexification $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$, and any real

linear map $T : V \rightarrow V$ extends to a complex linear map $T : V_{\mathbb{C}} \rightarrow V_{\mathbb{C}}$. In particular, J extends to an endomorphism of $V_{\mathbb{C}}$ satisfying $J^2 = -1$, and it now has two eigenspaces. We let $V^{1,0}$ and $V^{0,1}$ denote the eigenspaces with eigenvalues $i, -i$.

Lemma 2.2. *With the notation above,*

1. $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$.
2. $\overline{V^{1,0}} = V^{0,1}$.

Proof. 1. This is standard linear algebra. It is worth noting that one can explicitly write this decomposition as

$$v = \underbrace{\frac{1}{2}(v - iJv)}_{\in V^{1,0}} + \underbrace{\frac{1}{2}(v + iJv)}_{\in V^{0,1}}.$$

2. Straightforward computation. Note that it is obvious from the above explicit representation. □

Let $J^* : V^* \rightarrow V^*$ be the dual map, defined by

$$J^*\alpha(v) = \alpha(J(v)) \quad \text{for all } \alpha \in V^*.$$

This is necessarily an almost-complex structure on V^* . Therefore, its complexification decomposes $V_{\mathbb{C}}^*$ into a direct sum of $+i$ and $-i$ eigenspaces.

Lemma 2.3. $(V_{\mathbb{C}})^* = \text{Hom}(V_{\mathbb{C}}, \mathbb{C}) = \text{Hom}_{\mathbb{R}}(V, \mathbb{C})$, and we have the decomposition

$$(V_{\mathbb{C}})^* = (V^*)^{1,0} \oplus (V^*)^{0,1}.$$

where

$$(V^*)^{1,0} = \{\alpha \in V_{\mathbb{C}}^* : \alpha(Jv) = i\alpha(v)\}$$

and similarly for $(V^*)^{0,1}$.

2.2 Almost complex structures

This discussion extends to real vector bundles. Given a real vector bundle V on X it makes sense to consider $V_{\mathbb{C}}$, obtained by complexifying fiber-by-fiber.

Definition 2.4. Let $V \rightarrow X$ be a real vector bundle. A bundle morphism $J : V \rightarrow V$ satisfying $J^2 = -1$ is called an *almost complex structure* on V .

Given such a bundle, we have a decomposition

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$$

where

$$J|_{V^{1,0}}(v) = iv \quad J|_{V^{0,1}}(v) = -iv.$$

This decomposition exists fiber by fiber, of course, but the point is that it varies smoothly in X . That is clear from the fact that $V^{1,0}$ can be viewed as $\ker J - i \text{id}$.

Definition 2.5. Let X be a real manifold. An *almost complex structure* on X is an almost complex structure on TX .

We now again assume that X is a complex manifold. Recall that there was a standard almost complex structure on $T\mathbb{R}^{2n}$, which we denote by J_{st} . Now, given a holomorphic chart $\varphi : U \rightarrow \mathbb{R}^{2n}$, we get a bundle map $J : TU \rightarrow TU$ by $J = D\varphi^{-1} \circ J_{st} \circ D\varphi$.

Theorem 2.6. *The J defined above is independent of the choice of holomorphic chart, and gives a well-defined almost complex structure on X .*

Proof. Suppose that φ, ψ are two different charts defined in neighborhoods of some point. We must check that

$$D\varphi^{-1} \circ J_{st} \circ D\varphi = D\psi^{-1} \circ J_{st} \circ D\psi$$

or equivalently,

$$D((\varphi \circ \psi^{-1})^{-1}) \circ J_{st} \circ D(\varphi \circ \psi^{-1}) = J_{st}.$$

Now, $\varphi \circ \psi^{-1}$ is a biholomorphism between subsets of \mathbb{C}^n , so we are done since if f holomorphic, then df commutes with J_{st} . \square

By the preceding discussion, we get a splitting

$$(TX)_{\mathbb{C}} = T^{1,0}X \oplus T^{0,1}X.$$

Definition 2.7. $T^{1,0}X$ is the *holomorphic tangent bundle* of X .

We have a similar splitting $(T^*X)_{\mathbb{C}} = (T^*X)^{1,0} \oplus (T^*X)^{0,1}$.

In terms of local coordinates, suppose that $\varphi : U \rightarrow \mathbb{R}^{2n}$ is a holomorphic chart. We say that $x_j + iy_j$ are holomorphic coordinates. By definition,

$$J\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j} \quad J\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}$$

and

$$J(dx_j) = -dy_j \quad J(dy_j) = dx_j.$$

Definition 2.8. We define

$$dz_j := dx_j + idy_j \quad d\bar{z}_j = dx_j - idy_j.$$

Then you can check that

$$J(dz_j) = J(dx_j) + iJ(dy_j) = -dy_j + idx_j = i(dx_j + idy_j) = idz_j$$

so dz_1, \dots, dz_n give a frame for $(T^*X)^{1,0}$. Similarly, $d\bar{z}_1, \dots, d\bar{z}_n$ give a frame for $(T^*X)^{0,1}$.

Definition 2.9. We define

$$\frac{\partial}{\partial z_j} = \frac{1}{2} \left(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right) \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2} \left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right)$$

These are local frames for $TX^{1,0}$ and $TX^{0,1}$, respectively.

Chapter 3

Differential Forms

3.1 Differential forms on Complex Manifolds

Recall that if X is a complex manifold, we have local holomorphic coordinates $z_j = x_j + iy_j$. We showed last time that

- dz_1, \dots, dz_n is a local frame for $(T^*X)^{(1,0)}$,
- $d\bar{z}_1, \dots, d\bar{z}_n$ is a local frame for $(T^*X)^{(0,1)}$,
- $\frac{\partial}{\partial z_j}$ is a local frame for $(TX)^{(1,0)}$,
- $\frac{\partial}{\partial \bar{z}_j}$ is a local frame for $(TX)^{(0,1)}$.

We already showed that if $f : X \rightarrow \mathbb{R}^2 \cong \mathbb{C}$ is smooth, then f is holomorphic if and only if the Cauchy-Riemann equations hold, which is equivalent to $\frac{\partial f}{\partial \bar{z}_j} = 0$ for all j .

Let $u : X \rightarrow \mathbb{R}$ be a smooth function. We have a map

$$du_x : T_x X \rightarrow T_{u(x)} \mathbb{R} \cong \mathbb{R}.$$

Abusing notation, we also let

$$du_x : T_x X \otimes \mathbb{C} \rightarrow \mathbb{C}$$

be the complexified map. If $f : X \rightarrow \mathbb{C}$ is smooth, say $f = u + iv$, then $df = du + idv$, which is a smooth section of $T^*X \otimes \mathbb{C}$.

Exercise 3.1. In a local frame, we have an equality of the expression

$$df = \sum_j \frac{\partial f}{\partial x_j} dx_j + \sum_j \frac{\partial f}{\partial y_j} dy_j = \underbrace{\sum_j \frac{\partial f}{\partial z_j} dz_j}_{\partial f} + \underbrace{\sum_j \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j}_{\bar{\partial} f}.$$

This shows that we have chosen our definitions well (or at least, we have not chosen them to be ridiculous).

Remark 3.2. The condition that f be holomorphic is equivalent to $\bar{\partial}f = 0$.

Recall that $T^*X \otimes \mathbb{C} = (T^*X)^{(1,0)} \oplus (T^*X)^{(0,1)}$. We now study what happens upon taking exterior powers. We will get summands of the following form.

Definition 3.3. Let $\bigwedge^{p,q}(T^*X) = \bigwedge^p(T^*X)^{(1,0)} \otimes \bigwedge^q(T^*X)^{(0,1)}$.

A smooth section of $\bigwedge^{p,q}(T^*X)$ is called a (p, q) -form, and (p, q) is called the “bi-degree.” Locally, it looks like

$$\sum f_{\alpha,\beta}(z) dz_{\alpha} d\bar{z}_{\beta}$$

where $|\alpha| = p$ and $|\beta| = q$, i.e. a smooth function times a form with p “holomorphic coordinates” and q “antiholomorphic coordinates.”

We then define $\mathcal{A}_{\mathbb{C}}^k(U)$ to be the space of smooth sections of $\bigwedge^k(T^*X \otimes \mathbb{C})$ over U , and $\mathcal{A}_{\mathbb{C}}^{p,q} = \bigwedge^{p,q}(T^*X)$. In particular, $\mathcal{A}_{\mathbb{C}}^0(U)$ is the set of smooth complex-valued functions on U . We will suppress the subscript \mathbb{C} when the context is clear.

Lemma 3.4. 1. *There is a natural identification*

$$\bigwedge^k(T^*X \otimes \mathbb{C}) \cong \bigoplus_{p+q=k} \bigwedge^{p,q}(T^*X)$$

and in particular,

$$\mathcal{A}_{\mathbb{C}}^k(U) = \bigoplus_{p+q=k} \mathcal{A}_{\mathbb{C}}^{p,q}(U).$$

2. *If $\alpha \in \mathcal{A}^{p,q}(U)$ and $\beta \in \mathcal{A}^{p',q'}(U)$ then*

$$\alpha \wedge \beta \in \mathcal{A}^{p+p',q+q'}(U).$$

Proof. At the level of fibers, this is all formal multilinear algebra. If $V = A \oplus B$, with corresponding representations $v_j = a_j + b_j$, then

$$v_1 \wedge \dots \wedge v_k = (a_1 + b_1) \wedge \dots \wedge (a_k + b_k) \in \bigoplus_{p+q=k} \bigwedge^p V_1 \otimes \bigwedge^q V_2.$$

At the level of vector bundles, we can establish it using local frames, or by recognizing the subspaces as kernels of the relevant operators, which are wedge powers of J and the identity. \square

What does this discussion look like in local coordinates? Choose local holomorphic coordinates z_1, \dots, z_n , write

$$dz_I = dz_{i_1} \wedge \dots \wedge dz_{i_p} \quad I = (i_1, \dots, i_p).$$

A frame for $\bigwedge^{p,q} T^*X$ is

$$dz_I \wedge d\bar{z}_J \quad |I| = p, J = |q|.$$

Example 3.5. Elements of $\mathcal{A}^{(1,0)}(\mathbb{C})$ include $z dz$ and $\bar{z} dz$. In particular, we allow non-holomorphic functions to multiply the holomorphic differentials.

3.2 Dolbeault Cohomology

Definition 3.6. Denote by $d : \mathcal{A}^k(U) \rightarrow \mathcal{A}^{k+1}(U)$ the \mathbb{C} -linear extension of the usual exterior derivative.

In addition, we can define the operators

- $\partial : \mathcal{A}^{p,q}(U) \rightarrow \mathcal{A}^{p+1,q}(U)$ which is the composition of d with projection to $\mathcal{A}^{(p+1,q)}$.
- $\bar{\partial} : \mathcal{A}^{p,q}(U) \rightarrow \mathcal{A}^{p,q+1}(U)$ which is the composition of d with projection to $\mathcal{A}^{(p,q+1)}$.

On $\mathcal{A}^{(0,0)}$, these agree with the differential operators ∂ and $\bar{\partial}$ that we already defined.

This is very concrete in locally coordinates. Locally, if $\alpha = f dz_I \wedge d\bar{z}_J \in \mathcal{A}^{p,q}(U)$, then

$$d\alpha = \underbrace{\sum_j \frac{\partial f}{\partial z_j} dz_j \wedge dz_I \wedge d\bar{z}_J}_{\partial\alpha} + \underbrace{\sum_j \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j \wedge dz_I \wedge d\bar{z}_J}_{\bar{\partial}\alpha}.$$

The following lemma is apparent from this example.

Lemma 3.7. *On $\mathcal{A}^{p,q}$, we have*

1.

$$d = \partial + \bar{\partial}.$$

2. $\partial^2 = 0, \bar{\partial}^2 = 0$, and $\partial\bar{\partial} = -\bar{\partial}\partial$.

3. If $\alpha \in \mathcal{A}^{p,q}(U)$, then

$$\partial(\alpha \wedge \beta) = \partial\alpha \wedge \beta + (-1)^{p+q}\alpha \wedge \partial\beta$$

and

$$\bar{\partial}(\alpha \wedge \beta) = \bar{\partial}\alpha \wedge \beta + (-1)^{p+q}\alpha \wedge \bar{\partial}\beta.$$

Proof. The first part follows from the above local calculation. The second follows from the fact that $d^2 = 0$. The third follows from the corresponding identity for d . \square

Remark 3.8. The operators ∂ and $\bar{\partial}$ commute with restriction because d does. This will show that they induce maps of sheaves.

Definition 3.9. The (p, q) Dolbeault cohomology group is

$$H_{\bar{\partial}}^{p,q}(X) = \frac{\ker \bar{\partial} : \mathcal{A}^{p,q} \rightarrow \mathcal{A}^{p,q+1}(X)}{\operatorname{im} \bar{\partial} : \mathcal{A}^{p,q-1} \rightarrow \mathcal{A}^{p,q}(X)}.$$

Remark 3.10. One could make the obvious analogous definition for ∂ . However, this gives no extra information, since ∂ and $\bar{\partial}$ are essentially the same (up the conjugation). This is more natural because the kernel of $\bar{\partial}$ corresponds to holomorphic forms.

Example 3.11. The convention is that negative degree groups are zero, so $H_{\bar{\partial}}^{0,0}(X)$ is the space of holomorphic functions on X .

Lemma 3.12 (Functoriality). *Let $F : X \rightarrow Y$ be holomorphic. Then the map $F^* : \mathcal{A}^k(Y) \rightarrow \mathcal{A}^k(X)$ respect the bigrading and commutes with $\bar{\partial}$, hence induces a map*

$$f^* : H_{\bar{\partial}}^{p,q}(Y) \rightarrow H_{\bar{\partial}}^{p,q}(X).$$

3.3 The Mittag-Leffler Problem

Let S be a Riemann surface (a one-dimensional complex manifold). A *principal part* at a point $x \in S$ is a Laurent series

$$\mathcal{P} = \sum_{k=1}^n a_k z^{-k}$$

where z is a holomorphic coordinate centered at x . The Mittag-Leffler problem is: given distinct points $x_1, \dots, x_n \in S$ and principal parts $\mathcal{P}_1, \dots, \mathcal{P}_n$ at these points, does there exist a meromorphic function on S , which is holomorphic away from the x_i , and has precisely these principal parts at the x_i ?

This is, in some sense, answered by Dolbeault cohomology. Clearly, locally the answer is yes, i.e. there exist neighborhoods U_j around x_j and meromorphic functions f_j on U_j with principal part \mathcal{P}_j . Consider a partition of unity (ρ_j) subordinate to the U_j , such that $\rho_j = 1$ near x_j

and $\text{supp } \rho_j \subset U_j$. Then $\sum_j \rho_j f_j$ is a function which is smooth except at the x_j , with the correct principal parts.

The smooth form $g = \sum_j \bar{\partial}(\rho_j f_j)$ satisfies $\bar{\partial}g = 0$ and is identically zero in a neighborhood of x_α . (There is actually something subtle here - f_j is not a smooth function on U_j . But since f_j is locally meromorphic, $\bar{\partial}f_j \equiv 0$ on the punctured neighborhood of x_j .) So g represents a cocycle in Dolbeault cohomology. If it were zero in cohomology, then $g = \bar{\partial}h$ and the function $f = \sum \rho_j f_j - h$ would be a solution to the Mittag-Leffler problem. So the Dolbeault cohomology groups are obstructions to Mittag-Leffler problems.

Chapter 4

The $\bar{\partial}$ Poincaré Lemma

Our goal is to show that the Dolbeault groups $H_{\bar{\partial}}^{p,q}(X)$ vanish when X is a product of discs in \mathbb{C}^n . This is essentially an analytical problem, like Poincaré's Lemma for de Rham cohomology.

4.1 Holomorphic Functions on \mathbb{C}

Definition 4.1. A *disc* in \mathbb{C} is a set $B_R = \{|z| < R\}$ for some $R \in [0, \infty]$.

Suppose that B is a bounded disc, u is a smooth complex-valued function on \bar{B} . Then Stokes' formula gives

$$\int_{\partial B} u(z) dz = \int_B du \wedge dz = \int_B \frac{\partial u}{\partial \bar{z}} d\bar{z} \wedge dz = 2i \int_B \frac{\partial u}{\partial \bar{z}} dx \wedge dy.$$

If u is holomorphic, then $\frac{\partial u}{\partial \bar{z}} = 0$ and we recover Cauchy's formula.

Theorem 4.2 (Cauchy Integral formula). *If u is smooth on \bar{B} , then*

$$u(w) = \frac{1}{2\pi i} \left(\int_{\partial B} \frac{u(z)}{z-w} dz + \int_B \frac{\partial u}{\partial \bar{z}} \frac{1}{z-w} dz d\bar{z} \right).$$

If u is holomorphic, we recover the usual Cauchy integral formula.

Proof. Apply Stokes' theorem $B \setminus B_\epsilon(w)$:

$$\int_{B \setminus B_\epsilon(w)} \frac{\frac{\partial u}{\partial \bar{z}}(z)}{z-w} d\bar{z} dz = \int_{\partial B} \frac{u(z)}{z-w} dz - \int_{\partial B_\epsilon(w)} \frac{u(z)}{z-w} dz.$$

We compute the second term by explicit parametrization, and let ϵ tend to 0:

$$\int u(w + \epsilon e^{i\theta}) i d\theta \rightarrow 2\pi i u(w).$$

□

We now give a converse to this formula, which you can think of as the $\bar{\partial}$ -lemma in one variable.

Theorem 4.3. *Let B be a bounded disc in \mathbb{C} such that $B \subset \bar{B} \subset U$, for some open set U . Let $g \in C^\infty(U)$. Then the integral*

$$u(w) = \frac{1}{2\pi i} \int_B \frac{g(z)}{z-w} dzd\bar{z}$$

is well-defined, smooth, and satisfies $\frac{\partial u}{\partial \bar{w}}(w) = g(w)$ on B .

This says that if $\alpha = g d\bar{z} \in \mathcal{A}^{0,1}(B)$ where g has compact support, then there exists u such that $\bar{\partial}u = \alpha$.

Remark 4.4. We cannot just differentiate under the integral sign here, which would give

$$\frac{\partial}{\partial \bar{w}} \frac{g(z)}{z-w} = 0,$$

since the integrand is not continuous on the domain of integration. As above, we get around this by analyzing the contribution in a small neighborhood of w .

Proof. Let ρ be a smooth bump function such that $\rho = 1$ on a neighborhood V of \bar{B} such that $\text{supp } \rho \subset U$. Then ρg is smooth.

$$\frac{1}{2\pi i} \int_B \frac{g(z)}{z-w} dzd\bar{z} = \frac{1}{2\pi i} \int_B \frac{\rho(z)g(z)}{z-w} dzd\bar{z} + \frac{1}{2\pi i} \int_B \frac{(1-\rho(z))g(z)}{z-w} dzd\bar{z}.$$

The second integrand has supported away from w , so differentiating by \bar{w} kills it. Therefore, we reduce to the case where g is smooth and compactly supported in \mathbb{C} . We then reparametrize

$$\begin{aligned} \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{g(z)}{z-w} dzd\bar{z} &= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{g(z+w)}{z} dzd\bar{z} \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \int_0^\infty g(w + re^{i\theta}) \frac{2irdrd\theta}{re^{i\theta}}. \end{aligned}$$

Written in this way, we see that the integral is well-defined and the integrand is smooth on all of \mathbb{C} , so we can differentiate under the integral sign and trace our steps backwards to deduce that

$$\frac{1}{2\pi i} \int_{\mathbb{C}} \frac{g(z)}{z-w} dzd\bar{z} = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial g(z)}{\partial \bar{z}} \frac{1}{z-w} dzd\bar{z}.$$

Then Cauchy's integral formula says that this is $g(w)$. □

4.2 Holomorphic functions of several complex variables

Definition 4.5. A polydisc in \mathbb{C}^n is a set of the form $B = B_{R_1} \times \dots \times B_{R_n}$, where the $R_i \in [0, \infty]$.

For a polydisc B , we define $\partial_0 B = \prod \partial_j B_{R_j}$.

Theorem 4.6. If B is a bounded polydisc, u is a smooth function on \bar{B} and holomorphic in B , then

$$u(w) = \frac{1}{(2\pi i)^n} \int_B \frac{u(z)}{(z_1 - w_1) \dots (z_n - w_n)} dz_1 \dots dz_n.$$

Proof. This is essentially by applying the Cauchy-formula several times. \square

Notation: If $\alpha = (\alpha_1, \dots, \alpha_n)$, with $\alpha_j \in \mathbb{N}$, then we use the usual multi-index notation

$$z^\alpha = z_1^{\alpha_1} \dots z_n^{\alpha_n}$$

$$\frac{\partial^\alpha}{\partial z^\alpha} = \frac{\partial^{\alpha_1}}{\partial z_1^{\alpha_1}} \dots \frac{\partial^{\alpha_n}}{\partial z_n^{\alpha_n}}$$

and

$$\alpha! = \alpha_1! \dots \alpha_n!.$$

Corollary 4.7. We have the Taylor series expansion

$$u(w) = \sum_{\alpha} a_{\alpha} w^{\alpha},$$

where

$$a_{\alpha} = \frac{1}{\alpha!} \frac{\partial^\alpha}{\partial z^\alpha} u|_0.$$

Proof. We write

$$\frac{1}{z_1 - w_1} = \frac{1}{z_1(1 - w_1/z_1)} = \frac{1}{z_1} \sum_{\alpha_1 \geq 0} \frac{w_1^{\alpha_1}}{z_1^{\alpha_1}} \text{ for } |w_1| \geq |z_1|.$$

So

$$\frac{1}{(z_1 - w_1) \dots (z_n - w_n)} = \frac{1}{z_1 \dots z_n} \sum_{\alpha} \frac{w^\alpha}{z^\alpha}.$$

Then

$$u(w) = \sum_{\alpha} \frac{1}{(2\pi i)^n} \left(\int_{\partial_0 B} \frac{u(z)}{z_1 \dots z_n z^\alpha} dz_1 \dots dz_n \right) w^\alpha.$$

Again, differentiating the Cauchy integral formula gives $a_{\alpha} = \frac{\partial^\alpha u}{\partial z^\alpha}|_0$. \square

4.3 The Poincaré Lemma

Theorem 4.8. *Let B be a polydisc. Assume $B \subset \bar{B} \subset U$ for some open $U \subset \mathbb{C}^n$. Suppose that $\alpha \in \mathcal{A}^{p,q}(U)$ with $\bar{\partial}\alpha = 0$. Then there exists $\beta \in \mathcal{A}^{p,q-1}(B)$ with $\alpha = \bar{\partial}\beta$ on B .*

Proof. We induct on the highest index $d\bar{z}_n$ that appears in this sum. We can write

$$\alpha = \beta \wedge d\bar{z}_n + \alpha'$$

where α' involves only $d\bar{z}_1, \dots, d\bar{z}_{n-1}$. If $\beta = \sum g_I d\bar{z}_I$.

Write

$$u_I(w_1, \dots, w_n) = \int \frac{g_I(w_1, \dots, w_{n-1}, z_n) dz_n}{z_n - w_n}.$$

Then

$$\frac{\partial u_I}{\partial w_n}(w_1, \dots, w_n) = g_I(w_1, \dots, w_n).$$

Setting

$$\delta = (-1)^{q-1} \sum_I u_I dz_I$$

we obtain

$$\begin{aligned} \bar{\partial}\delta &= \sum_I \frac{\partial u_I}{\partial z_n} d\bar{z}_I \wedge d\bar{z}_n + (-1)^{q-1} \sum_{i \neq n} \frac{\partial u_I}{\partial z_i} d\bar{z}_i \wedge d\bar{z}_I \\ &= \sum_I g_I d\bar{z}_I \wedge d\bar{z}_n + (-1)^{q-1} \sum_{i \neq n} \frac{\partial u_I}{\partial z_i} d\bar{z}_i \wedge d\bar{z}_I \end{aligned}$$

Then we can modify by the coboundary $\bar{\partial}\delta$: the cocycle $\alpha - \bar{\partial}\delta$ is closed and doesn't involve $d\bar{z}_n$. We claim that it doesn't involve $d\bar{z}_i$ for $i > n$, since α is closed, $\frac{\partial g_I}{\partial z_i} d\bar{z}_i \wedge d\bar{z}_I \wedge d\bar{z}_n = 0$ for any $i > n$.

This reduces to the case of $f d\bar{z}_1 \wedge \dots \wedge d\bar{z}_q$, and then the same argument shows that this is a coboundary. \square

Lemma 4.9. *Let B be a bounded polydisc, $B \subset \bar{B} \subset B' \subset U$ where B' is a larger bounded polydisc. Then for any multi-index α there exists a constant C_α such that for all holomorphic functions u on U ,*

$$\|\bar{\partial}^\alpha u\|_{C^0(B)} \leq C_\alpha \int_{B'} |u(z)| dz \wedge d\bar{z}.$$

Proof. This follows from the multivariate Cauchy integral formula, as in the one-variable case. \square

Corollary 4.10. *Let u_n be holomorphic on U , say $u_n \rightarrow u$ uniformly on compact sets. Then u is holomorphic.*

Proof. By the previous lemma,

$$\frac{\partial u_n}{\partial z} - \frac{\partial u_m}{\partial z}$$

converges uniformly on compact sets. Also, $\frac{\partial u_n}{\partial \bar{z}} = 0$. So u is C^1 and satisfies the Cauchy-Riemann equations, hence is holomorphic. \square

Theorem 4.11. *If B is a (possibly unbounded) polydisc in \mathbb{C}^n , then*

$$H_{\bar{\partial}}^{p,q}(B) = 0 \text{ for } q > 0.$$

In other words, if $\alpha \in \mathcal{A}^{p,q}(B)$ with $\bar{\partial}\alpha = 0$, then there exists $\beta \in \mathcal{A}^{p,q-1}(B)$ with $\bar{\partial}\beta = \alpha$.

Proof. The case $p = 0, q = 1$ is the most difficult. The others reduce easily to this one, so let's study it first.

Say $B = B_{r_1} \times \dots \times B_{r_n}$. We approximate this region by an increasing sequence of bounded polydiscs. Letting $\epsilon_i(m)$ increase to r_i as $i \rightarrow \infty$, and

$$B_m = B_{\epsilon_1} \times \dots \times B_{\epsilon_n}.$$

Then $B_1 \subset B_2 \subset \dots$ is an ascending chain of open polydiscs whose union is B .

Let α be a cocycle; we want to write it as a coboundary. We already know that we can do this after restricting to a smaller open set, so we filter our space as an increasing union of open sets, and patch together solutions. Specifically, we claim that we can find a sequence $\beta_m \in \mathcal{A}^{0,1}(B_m)$ such that

1. $\bar{\partial}\beta_m = \alpha$ on B_m , and
2. $\|\beta_{m+1} - \beta_m\|_{C^0(B_{m-1})} < 2^{-m}$.

Assuming this for now, we obtain that β_m form a Cauchy sequence, hence converge uniformly to some smooth function γ on B . Moreover, $\bar{\partial}(\beta_{m+1} - \beta_m) = 0$ on B_m , so $\beta_m - \beta_{m_0} \rightarrow \gamma - \beta_{m_0}$, and each $\beta_{m+1} - \beta_{m_0}$ is holomorphic on B_{m_0} . So $\gamma - \beta_{m_0}$ is a uniform limit of holomorphic functions, hence is holomorphic on B_{m_0} . In particular, γ is smooth on B_{m_0} , but since this is independent of m_0 we get that γ is smooth on B and $\bar{\partial}\gamma = \alpha$ on B .

Now it suffices to prove the claim. Suppose that β_1, \dots, β_m have been defined, and we want to construct β_{m+1} . By the $\bar{\partial}$ -Poincaré Lemma applied to $B_{m+1} \subset B_{m+2}$, there exists a smooth $\beta \in \mathcal{A}^{0,0}(B_{m+2})$ such that $\bar{\partial}\beta = \alpha$ on B_{m+1} . After multiplying by some smooth bump function ρ which is identically 1 on B_{m+1} and with support contained in B_{m+2} , we can assume that β is smooth on B . In order to get the norm bound, we modify β by a holomorphic function on B ; since $\beta - \beta_m$ is holomorphic on B_m , it suffices

to take a sufficiently high degree Taylor polynomial approximation to their difference.

The case for higher p, q is completed in Example Sheet 1, Question 8. \square

In fact, the same proof shows the following slightly stronger result. If $\Delta_\epsilon = \{z \in \mathbb{C} : |z| < \epsilon\}$ and $\mathring{\Delta}_\epsilon = \Delta_\epsilon - \{0\}$.

Theorem 4.12. *If $q \geq 1$ and $p \geq 0$, then*

$$H_{\bar{\partial}}^{p,q}(\Delta_{\epsilon_1} \times \dots \times \Delta_{\epsilon_R} \times \mathring{\Delta}_{\epsilon_{k+1}} \times \dots \times \mathring{\Delta}_{\epsilon_\ell}) = 0.$$

Chapter 5

Sheaves and Cohomology

5.1 Sheaves

Let X be a topological space.

Definition 5.1. A *presheaf* \mathcal{F} of groups consists of a group $\mathcal{F}(U)$ for all open subsets $U \subset X$ and restriction homomorphisms $r_{VU} : \mathcal{F}(U) \rightarrow \mathcal{F}(V)$ for each inclusion of open subset $V \subset U$.

Remark 5.2. In other words, a presheaf is a contravariant functor from the poset of open subsets of X to **Grp**.

You should think of $\mathcal{F}(U)$ as being some class (e.g. continuous, smooth, holomorphic, etc.) functions on U , and the restriction maps as being restriction of functions to a smaller domain. In keeping with this intuition, for $s \in \mathcal{F}(U)$ we write $s|_V$ for $r_{VU}(s)$.

Example 5.3. $C^0(U)$ is the group of continuous functions on U , and the restriction maps are restriction of functions.

Remark 5.4. One can similarly define a presheaf of sets, rings, vector spaces, etc. If \mathcal{O} is a pre-sheaf of rings, then one can define pre-sheaves of \mathcal{O} -modules by asking each $\mathcal{F}(U)$ to be an $\mathcal{O}(U)$ -module, in a compatible way.

Definition 5.5. A presheaf \mathcal{F} is a *sheaf* if

1. For all $s \in \mathcal{F}(U)$, if $U = \bigcup U_i$ is an open cover and $s|_{U_i} = 0$ for all i , then $s = 0$.
2. If $U = \bigcup U_i$ is an open cover and $s_i \in \mathcal{F}(U_i)$ is such that

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \text{ for all } i, j$$

then there exists $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all i .

Example. $C^0(U)$ is a sheaf. More generally, the presheaf functions satisfying some nice purely local property will be a sheaf, e.g.:

- $\mathbb{Z}(U) =$ locally constant functions $f : U \rightarrow \mathbb{Z}$,
- $\mathbb{R}(U) =$ locally constant functions $f : U \rightarrow \mathbb{R}$.

If X is a smooth manifold, we have sheaves

- $C^\infty(U) =$ smooth functions on U ,
- $\mathcal{A}^p(U) =$ smooth p -forms on U ,
- For E a vector bundle on X , $C^\infty(E)(U) =$ smooth sections of $E|_U$.

If X is a complex manifold, then we have further sheaves

- $\mathcal{O}(U) =$ holomorphic functions on U ,
- $\mathcal{O}^*(U) =$ nowhere vanishing holomorphic functions on U ,
- $\Omega^p(U) =$ holomorphic p -forms on U .

Definition 5.6. Let $U \subset \mathbb{C}^n$. A *meromorphic function* on U is a function $f : U \setminus S \rightarrow \mathbb{C}$ where $S \subset U$ is nowhere dense, such that U has an open cover $U = \bigcup U_i$ and there exists $g_i, f_i \in \mathcal{O}(U_i)$ with

$$f|_{U_i \setminus S} = \frac{f_i|_{U_i \setminus S}}{g_i|_{U_i \setminus S}}.$$

Exercise 5.7. Define a sheaf \mathcal{M} of meromorphic functions on X , and show that it is a sheaf. If X is a complex manifold, we say that f is meromorphic on X if there exists a nowhere-dense $S \subset X$ such that $f|_{X \setminus S}$ is meromorphic on some open cover by holomorphic charts.

Definition 5.8. A *morphism* $\alpha : \mathcal{F} \rightarrow \mathcal{G}$ between (pre-)sheaves on X consists of homomorphisms $\alpha_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ for all open subset $U \subset X$, such that

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\alpha_U} & \mathcal{G}(U) \\ r_{VU} \downarrow & & \downarrow r_{VU} \\ \mathcal{F}(V) & \xrightarrow{\alpha_V} & \mathcal{G}(V) \end{array}$$

Remark 5.9. If we view sheaves as functor, then this is just the notion of natural transformations, i.e. morphisms in the functor category.

Definition 5.10. We say that

$$0 \rightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \rightarrow 0$$

is *exact* if for all U , the sequence

$$0 \rightarrow \mathcal{F}(U) \xrightarrow{\alpha_U} \mathcal{G}(U) \xrightarrow{\beta_U} \mathcal{H}(U)$$

is exact and if $s \in \mathcal{H}(U)$ and $x \in U$, there exists an open neighborhood $V \subset U$ of x and $t \in G(V)$ such that $\beta_V(t) = s|_V$.

These are the notions of exactness in the functor category. In particular, this is not the same as the short exact sequence of abelian groups being exact at the level of each open set U .

Definition 5.11 (Kernels). Let $\alpha : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves. We define the kernel sheaf by

$$(\ker \alpha)(U) = \ker(\alpha_U : \mathcal{F} \rightarrow \mathcal{G}(U)).$$

Cokernels are trickier, since the naïve definition does not necessarily satisfy the sheaf axiom.

Definition 5.12 (Cokernels). If

$$0 \rightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$$

is a short exact sequence of sheaves, we define $\operatorname{coker} \alpha = \mathcal{H}$.

This does *not* imply that $\mathcal{G}(U) \rightarrow \mathcal{H}(U)$ is surjective. In general, there is a sheaf which is the cokernel of any sheaf morphism $\mathcal{F} \rightarrow \mathcal{G}$ (in the category theoretic sense), but it is not defined as the object-wise cokernel, as this is a pre-sheaf but not necessarily a sheaf. One has to take the sheafification of this naïve definition.

Example 5.13. If X is a complex manifold

$$0 \rightarrow 2\pi i\mathbb{Z} \xrightarrow{\iota} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \rightarrow 0$$

is a short exact sequence of sheaves. Given any non-vanishing holomorphic function, we can *locally* take its logarithm (but not globally). We will call this the “fundamental short exact sequence.”

Definition 5.14. A complex of sheaves

$$\mathcal{F}_1 \xrightarrow{\alpha_1} \mathcal{F}_2 \xrightarrow{\alpha_2} \dots$$

is *exact* if

$$0 \rightarrow \ker \alpha_n \rightarrow \mathcal{F}_n \rightarrow \ker \alpha_{n+1} \rightarrow 0$$

is short exact for all n .

If X is a complex manifold, we have sheaves $\mathcal{A}^{p,q}$. The $\bar{\partial}$ maps give rise to a complex of sheaves

$$\dots \xrightarrow{\bar{\partial}_{q-2}} \mathcal{A}^{p,q-1} \xrightarrow{\bar{\partial}_{q-1}} \mathcal{A}^{p,q} \xrightarrow{\bar{\partial}_q} \mathcal{A}^{p,q+1} \xrightarrow{\bar{\partial}_{q+1}} \dots$$

The $\bar{\partial}$ -Poincaré Lemma says that this is exact. Indeed, consider the sequence

$$0 \rightarrow \ker \bar{\partial}_q \rightarrow \mathcal{A}^{p,q} \rightarrow \ker \bar{\partial}_{q+1} \rightarrow 0.$$

The $\bar{\partial}$ -Poincaré Lemma said that we can *locally* find a lift of any form in $\ker \bar{\partial}_{q+1}$ (in particular, over any polydisc).

Recall that we also defined the sheaf $\Omega^p(U) = \{\sigma \in \mathcal{A}^{p,q}(U) : \bar{\partial}\sigma = 0\}$, the holomorphic $(p,0)$ forms on U . In our new language,

$$\Omega^p = \ker \bar{\partial} : \mathcal{A}^{p,0} \rightarrow \mathcal{A}^{p,1}.$$

So

$$0 \rightarrow \Omega^p \xrightarrow{\iota} \mathcal{A}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{A}^{p,1} \xrightarrow{\bar{\partial}} \dots$$

is a resolution of Ω^p . We will later see that the sheafs $\mathcal{A}^{p,q}$ are fine, hence acyclic, and can therefore be used to compute the cohomology of Ω^p .

5.2 Čech Cohomology

We consider a toy example to illustrate the idea. Let X be a topological space with open cover $X = U \cup V$. Suppose that \mathcal{F} is a sheaf on X . Say we have sections $s_U \in \mathcal{F}(U)$ and $s_V \in \mathcal{F}(V)$ - when do they come from a global section? By the sheaf axiom, this is the case if and only if $s_U|_{U \cap V} = s_V|_{U \cap V}$.

Define a map $\delta : \mathcal{F}(U) \oplus \mathcal{F}(V) \rightarrow \mathcal{F}(U \cap V)$ by sending

$$(s_U, s_V) \mapsto s_U|_{U \cap V} - s_V|_{U \cap V}.$$

Then the gluing condition is precisely $\delta(s_U, s_V) = 0$, so $\mathcal{F}(X) = \ker \delta$. So we have identified an algebraic obstruction to patching local sections into a global one.

Let X be a topological space and \mathcal{F} a sheaf on X . Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of X , indexed by a subset of \mathbb{N} (though this isn't really necessary). We introduce the notation

$$U_{\alpha_0 \dots \alpha_p} = U_{\alpha_0} \cap \dots \cap U_{\alpha_p}.$$

Define

$$C^0(\mathcal{U}, \mathcal{F}) = \prod_{\alpha} \mathcal{F}(U_\alpha)$$

and

$$C^1(\mathcal{U}, \mathcal{F}) = \prod_{\alpha < \beta} \mathcal{F}(U_{\alpha\beta}).$$

More generally,

Definition 5.15.

$$C^p(\mathcal{U}, \mathcal{F}) = \prod_{\alpha_0 < \dots < \alpha_p} \mathcal{F}(U_{\alpha_0 \alpha_1 \dots \alpha_p}).$$

By convention, for any multiindex $\alpha_0, \dots, \alpha_p$ we set

$$\sigma_{\alpha_0, \dots, \alpha_i, \alpha_{i+1}, \dots, \alpha_p} = -\sigma_{\alpha_0, \dots, \alpha_{i+1}, \alpha_i, \dots, \alpha_p}.$$

If $\alpha = (\alpha_0, \dots, \alpha_p)$, we say that $\sigma = (\sigma_\alpha) \in C^p(\mathcal{U}, \mathcal{F})$ is a p -cochain.

Definition 5.16. Define the boundary map $\delta : C^p(\mathcal{U}, \mathcal{F}) \rightarrow C^{p+1}(\mathcal{U}, \mathcal{F})$ by

$$(\delta\sigma)_{\alpha_0, \dots, \alpha_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \sigma_{\alpha_0, \dots, \widehat{\alpha_j}, \dots, \alpha_{p+1}} |_{U_{\alpha_0, \dots, \alpha_{p+1}}}.$$

Lemma 5.17. *The composition $\delta \circ \delta : C^p(\mathcal{U}, \mathcal{F}) \rightarrow C^{p+2}(\mathcal{U}, \mathcal{F})$ is the zero map.*

Proof. If $\sigma \in C^p(\mathcal{U}, \mathcal{F})$, the $\delta \circ \delta(\sigma)_{\alpha_1, \dots, \alpha_{p+2}}$ is

$$\sum_{i,j} [(-1)^i (-1)^{j-1} + (-1)^i (-1)^j] \sigma_{\alpha_1, \dots, \widehat{\alpha_i}, \dots, \widehat{\alpha_j}, \dots, \alpha_{p+2}}$$

where the two opposite signs depend on whether i or j is deleted first. \square

Example 5.18. Let $\mathcal{U} = \{U, V, W\}$ be an open cover of X . Then a $\sigma \in C^1(\mathcal{U}, \mathcal{F})$ looks like

$$\sigma = (\sigma_{UV}, \sigma_{UW}, \sigma_{VW}).$$

Then

$$\delta\sigma = \sigma_{UV} - \sigma_{UW} + \sigma_{VW}.$$

Definition 5.19. We say that $\sigma \in C^p(\mathcal{U}, \mathcal{F})$ is

- a *cocycle* if $\delta\sigma = 0$,
- a *coboundary* if $\sigma = \delta\tau$

Define

$$\check{H}^q(\mathcal{U}, \mathcal{F}) = \frac{\ker \delta : C^q(\mathcal{U}, \mathcal{F}) \rightarrow C^{q+1}(\mathcal{U}, \mathcal{F})}{\operatorname{im} \delta : C^{q-1}(\mathcal{U}, \mathcal{F}) \rightarrow C^q(\mathcal{U}, \mathcal{F})}.$$

In other words, this is the group of cocycles mod coboundaries, or the cohomology of the chain complex described earlier. Obviously, this depends on the open cover and is not our final definition of Čech cohomology.

Example 5.20. Let $X = \mathbb{P}^1$ and $\mathcal{F} = \mathcal{O}$. We pick the open cover $U = [z : 1]$ and $V = [1 : w]$. Both are isomorphic to \mathbb{C} , and their intersection is \mathbb{C}^* . So $C^0(\mathcal{U}, \mathcal{O}) = \mathcal{O}(U) \oplus \mathcal{O}(V)$ and $C^1(\mathcal{U}, \mathcal{O}) = \mathcal{O}(U \cap V)$. The map to $C^1(U, \mathcal{O})$ is

$$\delta(f + g)(z) = f(z) - g(1/z).$$

The kernel consists of $\{(f, g)\}$ where $f = g$ is constant. The image of δ consists of all holomorphic functions on \mathbb{C}^* , since all holomorphic functions on \mathbb{C}^* have Laurent series. So $\check{H}^0(\mathcal{U}, \mathcal{O}) = \mathbb{C}$ and $\check{H}^i(\mathcal{U}, \mathcal{O}) = 0$ for $i > 0$.

Definition 5.21. Given two covers \mathcal{U} and \mathcal{V} , we say that \mathcal{V} *refines* \mathcal{U} , and write $\mathcal{V} \leq \mathcal{U}$, if there exists $\alpha : \mathbb{N} \rightarrow \mathbb{N}$ such that for all β , $V_\beta \subset U_{\alpha(\beta)}$.

If $\mathcal{V} \leq \mathcal{U}$, we have natural restriction maps $\rho_{\mathcal{V}, \mathcal{U}} : C^p(\mathcal{U}, \mathcal{F}) \rightarrow C^p(\mathcal{V}, \mathcal{F})$ such that

$$\rho_{\mathcal{V}, \mathcal{U}}(\sigma_{\beta_1, \dots, \beta_p}) = \sigma_{\alpha(\beta_1), \dots, \alpha(\beta_p)}.$$

This commutes with δ , so it descends to a map on cohomology:

$$\rho_{\mathcal{V}, \mathcal{U}} : \check{H}(\mathcal{U}, \mathcal{F}) \rightarrow \check{H}(\mathcal{V}, \mathcal{F}).$$

Note that the definition of ρ depends on α . However, you can check that on the level of cohomology, it is actually independent of α .

Definition 5.22. The Čech cohomology of a sheaf \mathcal{F} on X is

$$\check{H}^p(X, \mathcal{F}) = \varinjlim_{\mathcal{U}} \check{H}^p(\mathcal{U}, \mathcal{F})$$

For X a complex manifold and $\mathcal{F} = \mathcal{O}$, this holds if each intersection is a polydisc. So the cover we used for \mathbb{P}^1 in Example 5.20 was in fact acyclic.

Example 5.23. Trivially from the definition, $\check{H}^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X)$ for *any* open cover of X . Therefore, $\check{H}^0(X, \mathcal{F}) = \mathcal{F}(X)$.

Example 5.24. $\check{H}^q(X, \mathcal{A}^{r,s}) = 0$ for all $q \geq 1$. The point is that the sheaf $\mathcal{A}^{r,s}$ is a module over the *smooth* functions on X , making it much flabbier than holomorphic sheaves like \mathcal{O} . To see this, let $\sigma \in \check{H}^q(X, \mathcal{A}^{r,s})$ be represented by some $\sigma \in C^p(\mathcal{U}, \mathcal{A}^{r,s})$. We have $\delta\sigma = 0$, so let ρ_α be a partition of unity subordinate to $\{U_\alpha\}$ (assuming that the cover is locally finite). Define

$$\tau_{\alpha_0, \dots, \alpha_{q-1}} = \sum_{\beta} \underbrace{\rho_\beta \sigma_{\beta, \alpha_0, \dots, \alpha_{q-1}}}_{\text{extend by zero}}.$$

Here, since ρ_β is supported in U_β , multiplying by it allows us to extend to the complement of U_β , so that $\sigma_{\beta, \alpha_0, \dots, \alpha_{q-1}}$ is defined on $U_{\alpha_0, \dots, \alpha_{q-1}}$. As a guiding example, consider $\mathcal{U} = \{U, V, W\}$. Then by hypothesis,

$$0 = \delta\sigma = \sigma_{UV} - \sigma_{UW} + \sigma_{VW}.$$

We have $\tau_U = \sum_w \rho_w \sigma_{wU}$. Then

$$\begin{aligned} (\delta\tau)_{UV} &= \tau_V - \tau_U \\ &= \rho_U \sigma_{UV} + \rho_W \sigma_{WV} - \rho_V \sigma_{VU} - \rho_W \sigma_{WU} \\ &= \rho_U \sigma_{UV} + \rho_V \sigma_{UV} + \rho_W (\sigma_{WV} - \sigma_{WU}) \\ &= (\rho_U + \rho_V + \rho_W) \sigma_{UV}. \end{aligned}$$

Now we tackle the general case. From the definition, we have

$$\begin{aligned} (\delta\tau)_{\alpha_0, \dots, \alpha_p} &= \sum_{j=0}^p (-1)^j \tau_{\alpha_0, \dots, \widehat{\alpha}_j, \dots, \alpha_p} \\ &= \sum_{j=0}^p (-1)^j \sum_{\beta} \rho_{\beta} \sigma_{\beta \alpha_0, \dots, \widehat{\alpha}_j, \dots, \alpha_p}. \end{aligned}$$

Now, since σ is a cocycle we have

$$\begin{aligned} 0 &= (\delta\sigma)_{\beta \alpha_0 \dots \alpha_p} \\ &= \sigma_{\alpha_0 \dots \alpha_p} + \sum_{j=0}^p (-1)^{j+1} \sigma_{\beta \alpha_0, \dots, \widehat{\alpha}_j, \dots, \alpha_p}. \end{aligned}$$

Substituting this above, we find that

$$\begin{aligned} (\delta\tau)_{\alpha_0, \dots, \alpha_p} &= \sum_{\beta} \rho_{\beta} \sum_{j=0}^p \sigma_{\beta \alpha_0, \dots, \widehat{\alpha}_j, \dots, \alpha_p} \\ &= \sum_{\beta} \rho_{\beta} \sigma_{\alpha_0 \dots \alpha_p} \\ &= \sigma_{\alpha_0 \dots \alpha_p}. \end{aligned}$$

This kind of sheaf, with partitions of unity, is said to be *fine*.

5.3 Properties of Čech cohomology

Let $f : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves. This induces $f : C^p(\mathcal{U}, \mathcal{F}) \rightarrow C^p(\mathcal{U}, \mathcal{G})$ for any open cover \mathcal{U} . Furthermore, these maps commutes with δ , and hence induce a map on cohomology:

$$f_* : \check{H}^p(X, \mathcal{F}) \rightarrow \check{H}^p(X, \mathcal{G}).$$

Theorem 5.25. *Suppose that*

$$0 \rightarrow \mathcal{E} \xrightarrow{g} \mathcal{F} \xrightarrow{f} \mathcal{G} \rightarrow 0.$$

is a short exact sequence of sheaves on a manifold X . Then there exist natural maps $\delta^* : \check{H}^p(X, \mathcal{G}) \rightarrow \check{H}^{p+1}(X, \mathcal{E})$ such that there is a long exact sequence in cohomology,

$$\dots \rightarrow \check{H}^{p-1}(X, \mathcal{G}) \xrightarrow{\delta^*} \check{H}^p(X, \mathcal{E}) \xrightarrow{f_*} \check{H}^p(X, \mathcal{F}) \xrightarrow{g_*} \check{H}^p(X, \mathcal{G}) \rightarrow \dots$$

Proof. First we define δ^* . Suppose that $\sigma \in \hat{H}^p(X, \mathcal{G})$ represented by some $\sigma \in C^p(\mathcal{U}, \mathcal{G})$. The short exactness implies that there is a refinement $\mathcal{V} \leq \mathcal{U}$ such that $\rho_{\mathcal{V}\mathcal{U}}\sigma = f(\tau)$, for some $\tau \in C^p(\mathcal{V}, \mathcal{F})$. Consider $\delta\tau$:

$$f(\delta\tau) = \delta f(\tau) = \delta \rho_{\mathcal{V}\mathcal{U}}\sigma = \rho_{\mathcal{V}\mathcal{U}}\delta\sigma = 0.$$

By exactness, there exists some $u \in C^{p+1}(\mathcal{V}, \mathcal{E})$ such that $g(u) = \delta\tau$. Now $g(\delta u) = \delta g(u) = \delta^2\tau = 0$. But injectivity of g implies that $\delta u = 0$, so we can define $\delta^*[\sigma] = [u] \in \check{H}^{p+1}(X, \mathcal{E})$.

We will prove this under the assumption that there exists arbitrarily fine covers \mathcal{U} such that

$$0 \rightarrow C^p(\mathcal{U}, \mathcal{E}) \rightarrow C^p(\mathcal{U}, \mathcal{F}) \rightarrow C^p(\mathcal{U}, \mathcal{G}) \rightarrow 0$$

are exact for all p . This is an exact sequence of chain complexes, so by the Snake Lemma we obtain boundary homomorphisms $\delta^* : \check{H}^p(\mathcal{U}, \mathcal{G}) \rightarrow \check{H}^{p+1}(\mathcal{U}, \mathcal{E})$ such that

$$\dots \rightarrow \check{H}^{p-1}(\mathcal{U}, \mathcal{G}) \xrightarrow{\delta^*} \check{H}^p(\mathcal{U}, \mathcal{E}) \xrightarrow{f_*} \check{H}^p(\mathcal{U}, \mathcal{F}) \xrightarrow{g_*} \check{H}^p(\mathcal{U}, \mathcal{G}) \rightarrow \dots$$

is exact. Moreover, these commute with the restriction maps $\rho_{\mathcal{U}, \mathcal{V}}$ to finish off the proof. \square

Theorem 5.26. *Let X be a complex manifold. Then there exist natural identifications*

$$H_{\bar{\partial}}^{p,q}(X) = \check{H}^q(X, \Omega^p).$$

Recall that Ω^q denoted the sheaf of holomorphic q -forms.

Remark 5.27. We have shown that

$$0 \rightarrow \Omega^p \rightarrow \mathcal{A}^{p,0} \rightarrow \mathcal{A}^{p,1} \rightarrow \dots$$

is an acyclic resolution for the sheaf Ω^p . The derived functor cohomology then defines $H^*(X, \Omega^p)$ to be the cohomology of the complex

$$0 \rightarrow \mathcal{A}^{p,0} \rightarrow \mathcal{A}^{p,1} \rightarrow \dots$$

which is a high-level explanation of the theorem. In nice situations, the derived functor cohomology will agree with Čech cohomology. The proof we give here basically reworks the fact that in derived functor cohomology, one can take an acyclic resolution (instead of an injective resolution).

Proof. Let $\mathcal{Z}_{\bar{\partial}}^{p,q} = \ker \bar{\partial} : \mathcal{A}^{p,q} \rightarrow \mathcal{A}^{p,q+1}$. By the $\bar{\partial}$ -Lemma, we have short exact sequences of sheaves

$$0 \rightarrow \Omega^p \rightarrow \mathcal{A}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{Z}_{\bar{\partial}}^{p,1} \rightarrow 0$$

and

$$0 \rightarrow \mathcal{Z}_{\bar{\partial}}^{p,q-1} \rightarrow \mathcal{A}^{p,q-1} \xrightarrow{\bar{\partial}} \mathcal{Z}_{\bar{\partial}}^{p,q} \rightarrow 0 \text{ for all } q.$$

Of course, the content of the $\bar{\partial}$ -Lemma is the surjectivity. The long exact sequence for the first short exact sequence reads

$$\dots \rightarrow \check{H}^r(\mathcal{A}^{p,q}) \rightarrow \check{H}^r(\mathcal{Z}_{\bar{\partial}}^{p,1}) \rightarrow \hat{H}^{r+1}(\Omega^p) \rightarrow \check{H}^{r+1}(\mathcal{A}^{p,0}) \rightarrow \dots$$

but since $\mathcal{A}^{p,q}$ is fine, the middle map is an isomorphism. That shows

$$\check{H}^r(\mathcal{Z}_{\bar{\partial}}^{p,1}) \cong \hat{H}^{r+1}(\Omega^p).$$

if $r \geq 1$, and the long exact sequence for the second implies similarly that

$$\check{H}^{r+1}(\mathcal{Z}_{\bar{\partial}}^{p,q}) \cong \check{H}^r(\mathcal{Z}_{\bar{\partial}}^{p,q+1}).$$

if $q \geq 1$. This says that we can increase r by dropping q , and if we do this dropping q to 1, and then using the first row of the long exact sequence in the last step, we find:

$$\begin{aligned} \hat{H}^q(X, \Omega^p) &\cong \check{H}^{q-1}(X, \mathcal{Z}_{\bar{\partial}}^{p,1}) \\ &\cong : \\ &\cong \check{H}^1(X, \mathcal{Z}_{\bar{\partial}}^{p,q-1}) \\ &\cong \frac{H^0(\mathcal{Z}_{\bar{\partial}}^{p,q})}{\text{im } \bar{\partial} : H^0(\mathcal{A}^{p,q-1}) \rightarrow H^0(\mathcal{Z}_{\bar{\partial}}^{p,q})} \\ &\cong H_{\bar{\partial}}^{p,q}(X). \end{aligned}$$

□

Theorem 5.28. *Let X be a complex manifold. Suppose that \mathcal{U} is an open cover of X such that*

$$\check{H}^q(U_{\alpha_1} \cap \dots \cap U_{\alpha_s}, \mathcal{O}) = 0 \text{ for } q \geq 1.$$

Then $\check{H}^q(X, \mathcal{O}) = \check{H}^q(\mathcal{U}, \mathcal{O})$.

Proof. We have $\check{H}^r(U_{\alpha_1} \cap \dots \cap U_{\alpha_s}, \mathcal{O}) = H_{\bar{\partial}}^{0,r}(U_{\alpha_1} \cap \dots \cap U_{\alpha_s}) = 0$ by the Dolbeault theorem. This implies that for $r \geq 1$, we have an exact sequence

$$0 \rightarrow \mathcal{Z}_{\bar{\partial}}^{0,r-1}(U_{\alpha_1} \cap \dots \cap U_{\alpha_s}) \rightarrow \mathcal{A}^{0,r-1}(U_{\alpha_1} \cap \dots \cap U_{\alpha_s}) \rightarrow \mathcal{Z}_{\bar{\partial}}^{0,r}(U_{\alpha_1} \cap \dots \cap U_{\alpha_s}) \rightarrow 0$$

As this is true for all multi-intersections, the exactness passes to the level of cochains.

$$0 \rightarrow C^p(\mathcal{U}, \mathcal{Z}_{\bar{\partial}}^{0,r-1}) \rightarrow C^p(\mathcal{U}, \mathcal{A}^{0,r-1}) \rightarrow C^p(\mathcal{U}, \mathcal{Z}_{\bar{\partial}}^{0,r}) \rightarrow 0 \text{ is exact.}$$

This gives a long exact sequence in cohomology, with the middle terms vanishing, giving

$$\check{H}^p(\mathcal{U}, \mathcal{Z}_{\bar{\partial}}^{0,r}) \cong \check{H}^{p+1}(\mathcal{U}, \mathcal{Z}_{\bar{\partial}}^{0,r-1}).$$

We then repeat the proof of the above theorem for this open cover, which give an isomorphism to Dolbeault cohomology.

$$\check{H}^p(\mathcal{U}, \mathcal{O}) = \check{H}^p(\mathcal{U}, \mathcal{Z}_{\bar{\partial}}^{0,0}) \cong \dots \cong \check{H}^1(\mathcal{U}, \mathcal{Z}_{\bar{\partial}}^{0,p-1}).$$

As before,

$$\check{H}^1(\mathcal{U}, \mathcal{Z}_{\bar{\partial}}^{0,p-1}) \cong \frac{\check{H}^0(\mathcal{U}, \mathcal{Z}_{\bar{\partial}}^{0,p})}{\text{im } \bar{\partial}} = H_{\bar{\partial}}^{0,p}(X) = H^p(X, \mathcal{O}).$$

□

Remark 5.29. The same proof works for Ω^p . Actually, the following is true: if $\check{H}^p(U_\alpha, \mathcal{O}) = 0$ for all $p \geq 1$ (no higher intersections), then $\check{H}^1(\mathcal{U}, \mathcal{O}) \cong \check{H}^1(X, \mathcal{O})$.

Corollary 5.30. *If $\dim X = n$, then $\check{H}^q(X, \mathcal{O}) = H_{\bar{\partial}}^{0,q}(X) = 0$ if $q > n$.*

Example 5.31. $\check{H}^q(\mathbb{C}^n, \mathcal{O}) \cong H_{\bar{\partial}}^{0,q}(\mathbb{C}^n) = 0$ for all $q \geq 1$ (this is the $\bar{\partial}$ -Lemma). In addition, $\check{H}^q(\mathbb{C}^k \times (\mathbb{C}^*)^\ell, \mathcal{O}) = 0$ if $k + \ell \geq 2$, by an extension of the $\bar{\partial}$ -Lemma.

It is a fact that if X is contractible, then $\check{H}^q(X, \mathbb{Z}) = 0$ for all $q \geq 1$. Indeed, the Čech cohomology with coefficients in a constant sheaf is isomorphic to the singular cohomology with the corresponding coefficients. Recall the exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O} \rightarrow \mathcal{O}^* \rightarrow 0.$$

So then if X is contractible, its higher \mathbb{Z} -cohomology vanishes, and we get that

$$\check{H}^q(X, \mathcal{O}) \cong \check{H}^q(X, \mathcal{O}^*) \text{ for } q \geq 1.$$

Example 5.32.

$$H^p(\mathbb{P}^n, \Omega^q) = \begin{cases} \mathbb{C} & p = q \leq n \\ 0 & \text{otherwise} \end{cases}.$$

The idea of the proof is to cover \mathbb{P}^n by the standard charts and compute everything explicitly.

Chapter 6

More on Several Complex Variables

6.1 Hartog's Theorem

There are two useful results on extending holomorphic functions of several complex manifolds, both of which have been called Hartog's theorem in the literature. The intuition is that the "singularities" of a multivariable analytic function must look like an analytic hypersurface. This tells us a couple of things: if the singular locus has codimension at least 2, then it should be removable. Also, if the singular locus is compact, then in higher dimensions we also expect the function to extend. As a simple example, consider $\frac{1}{z_1}$: we get a pole along $z_1 = 0$, which defines a noncompact hypersurface.

Contrast this with the one-dimensional case. There are no non-empty (complex) codimension 2 subsets of \mathbb{C} . On the other hand, we know that there are holomorphic functions in a punctured disk that do not extend, e.g. a pole or essential singularity.

Theorem 6.1 (Hartog's Theorem 1). *Let $B_\epsilon = B_{\epsilon_1} \times \dots \times B_{\epsilon_n}$ be a polydisc in \mathbb{C}^n for $n \geq 2$. Suppose that $0 < \epsilon' < \epsilon$ for all i so that $B_{\epsilon'} \subset B_\epsilon$. Let $f : B_\epsilon - B_{\epsilon'} \rightarrow \mathbb{C}$ be holomorphic. Then there exists a unique extension $\tilde{f} : B_\epsilon \rightarrow \mathbb{C}$.*

Proof. Without loss of generality, let $\epsilon_i = 1$. For $\delta > 0$, let

$$V_i = \{z \in B_\epsilon : 1 - \delta_i < |z_i| < 1\}$$

and $V = \bigcap V_i$. So for sufficiently small δ , $V \subset B_\epsilon - B_{\epsilon'}$, hence f is holomorphic on V . We want to formalize our intuition that f cannot have a polar part, which would cut out a non-compact singular locus.

For fixed $w = (z_2, \dots, z_n)$, $|z_i| < 1$, consider the complex function of one variable

$$g_w(z_1) = f(z_1, \dots, z_n) \text{ defined on } A = \{1 - \delta < |z_1| < 1\}.$$

Therefore, this has a Laurent expansion

$$g_w(z_1) = \sum_{n=-\infty}^{\infty} a_n(w) z_1^n.$$

We claim that the $a_n(w)$ are holomorphic in w . One way to see this is to use Cauchy's integral formula

$$a_n(w) = \frac{1}{2\pi i} \int_{|\xi|=1-\frac{\delta}{2}} \frac{g_w(\xi)}{\xi^{n+1}} d\xi$$

and note that g_w is holomorphic on this region.

If we choose w so that $1 - \delta < |z_2| < 1$, then we see that g_w is holomorphic on the entire disc $|z_1| < 1$. This implies that $a_n(w) = 0$ for $n < 0$, for all such w , i.e. $1 - \delta < |z_2| < 1$. So by the identity theorem, $a_n(w) = 0$ for all $|z_2| < 1$. This implies that $f(z) = \sum_{n=0}^{\infty} a_n(w) z_1^n$ is holomorphic in z_1 over the entire polydisc (it converges absolutely since it does when $|z_2| = 1$, and the a_n are holomorphic so the maximum modulus occurs on the boundary).

□

Theorem 6.2 (Hartog's Theorem 2). *Let U be an open subset of \mathbb{C}^n for $n > 2$, and f a holomorphic function on $U - U \cap \{z_1 = z_2 = 0\}$. Then f extends to a holomorphic function on all of U .*

Proof. Let $D = \{(z_1, \dots, z_n) : |z_i| \leq r_i\} \subset U$ be a polydisc and $D' = D - D \cap \{z_1 = z_2 = 0\}$. We show that for each $z_1, \dots, z_n \in D'$, we have a Cauchy formula

$$f(z_1, \dots, z_n) = \frac{1}{(2\pi i)^n} \int_{\partial D} \frac{f(\zeta_1, \dots, \zeta_n) dz_1 \dots dz_n}{(\zeta_1 - z_1) \dots (\zeta_n - z_n)}.$$

Since the right hand side is holomorphic in D , it defines a holomorphic extension.

If $D_\epsilon = \{(\zeta_1, \dots, \zeta_n) : |z_1|, |z_2| \leq \epsilon, |\zeta_i| \leq r_i \text{ for } i > 1\}$ is a small neighborhood of (z_1, \dots, z_n) contained in D' , then Cauchy's formula implies

$$f(z_1, \dots, z_n) = \frac{1}{(2\pi i)^n} \int_{\partial D_\epsilon} \frac{f(\zeta_1, \dots, \zeta_n) dz_1 \dots dz_n}{(\zeta_1 - z_1) \dots (\zeta_n - z_n)}.$$

Therefore, we just have to show that ∂D_ϵ can be homotoped to ∂D through D' . This is where the codimension 2 hypothesis comes in.

We choose a homotopy through regions of the form

$$D(t) = \{(\zeta_1, \dots, \zeta_n) : |\zeta_i - tz_i| \leq \alpha_i(t) \quad i = 1, 2; \quad |z_i| = r_i \quad i > 2\}.$$

We require $\alpha_i(0) = r_i$ so that $D(0) = \partial D$, and $\alpha_i(1) = \epsilon_i$ so $D(1) = \partial D_\epsilon$.

In order for $D(t)$ to stay in D , we must have $\alpha_i(t) < r_i - |tz_i|$. For $D(t)$ to stay in D' , it also has to avoid (z_1, \dots, z_n) and $\{z_0 = z_1 = 0\}$. The first requirement is equivalent to $(1-t)|z_i| \neq \alpha_i(t)$ for any t . Since $r_i - |tz_i| > (1-t)|z_i|$ at $t = 0$ and $t = 1$, we can choose α so that

$$(1-t)|z_i| < \alpha(t) < r_i - |tz_i|.$$

To avoid $\{z_0 = z_1 = 0\}$, we cannot simultaneously have $\alpha_1(t) = t|z_1|$ and $\alpha_2(t) = t|z_2|$. That is fine, since we have two degrees of freedom in choosing $\alpha_1(t), \alpha_2(t)$ and we impose only one condition. (This comes down to choosing a curve $(t, \alpha_1(t), \alpha_2(t))$ in a cube that avoids the curve $(t, t|z_1|, t|z_2|)$).

□

Corollary 6.3. *If X is a complex manifold and f a holomorphic function on $X - Z$, where Z is a complex submanifold of codimension at least 2, then f extends to a holomorphic function on all of X .*

Chapter 7

Holomorphic Vector Bundles

7.1 Basic definitions

Let X be a differentiable manifold. Recall that a *complex vector bundle* E of rank r on X is a smooth manifold $E \rightarrow X$ with a smooth projection map $\pi : E \rightarrow X$ such that each fiber $E_x := \pi^{-1}(x)$ has the structure of a \mathbb{C} -vector space of dimension r , which is locally free: X has an open cover $\bigcup U_\alpha$ with trivialization $\varphi_\alpha : \pi^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{C}^r$.

If $r = 1$, we call E a complex *line bundle*.

Definition 7.1. If X is a complex manifold, we say that E is a *holomorphic vector bundle* if E is a complex manifold, π is holomorphic, and the φ_α are holomorphic.

Note that holomorphic vector bundles have stricter demands than complex vector bundles.

Recall that a vector bundle is determined by its transition functions

$$\varphi_{\alpha\beta} : \varphi_\alpha \circ \varphi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{C}^r \rightarrow (U_\alpha \cap U_\beta) \times \mathbb{C}^r.$$

Note that $\varphi_{\alpha\beta}$ can be considered as a map $U_\alpha \cap U_\beta \rightarrow \mathrm{GL}_r(\mathbb{C})$. The bundle E is holomorphic if and only if the $\varphi_{\alpha\beta}$ can be taken to be holomorphic.

From the definition, the $\varphi_{\alpha\beta}$ satisfy the *cocycle conditions*

$$\begin{aligned}\varphi_{\alpha\beta} \circ \varphi_{\beta\gamma} &= \varphi_{\alpha\gamma} \\ \varphi_{\alpha\beta} &= \varphi_{\alpha\gamma} \varphi_{\gamma\alpha}\end{aligned}$$

for all α, β, γ .

Let $\mathrm{GL}_r(\mathbb{C})$ denote the sheaf $\mathrm{GL}_r(\mathbb{C})(U) = \{g : U \rightarrow \mathrm{GL}_r(\mathbb{C}) \text{ holomorphic}\}$. (This notation is slightly confusing, since it would be also used to denote the locally constant sheaf.) If E is holomorphic, then the data $\{\varphi_{\alpha\beta}\}$ forms a cochain in $C^1(\mathcal{U}, \mathrm{GL}_r(\mathbb{C}))$. The cocycle condition says that $\delta : C^1(\mathcal{U}, \mathrm{GL}_r(\mathbb{C})) \rightarrow C^2(\mathcal{U}, \mathrm{GL}_r(\mathbb{C}))$ has $\delta\{\varphi_{\alpha\beta}\} = 0$, i.e. this cochain is actually a cocycle (hence

the name), so data of the transition functions descends to an element of $\check{H}^1(X, \mathrm{GL}_r(\mathbb{C}))$.

Note that we have technically only defined cohomology for sheaves of *abelian* groups, but we will soon specialize to $r = 1$.

Definition 7.2. Let $\pi_E : E \rightarrow X$ and $\pi_F : F \rightarrow X$ be holomorphic vector bundles. A *morphism* $f : E \rightarrow F$ is a holomorphic map such that $\pi_F f = \pi_E$ and such that the induced map $f_x : E_x \rightarrow F_x$ is \mathbb{C} -linear and $\mathrm{rank}(f_x)$ is locally constant in x .

Remark 7.3. The constancy of rank is required in order to ensure that we can take kernels and cokernels of morphisms. Be warned that this isn't how vector bundle morphisms are usually defined in differential geometry.

Definition 7.4. We write $\mathcal{O}(E)$ for the sheaf of local sections of E :

$$\mathcal{O}(E)(U) = \{s : U \rightarrow E : \pi_E \circ s = \mathrm{id}\}.$$

Example 7.5. Locally (for small enough opens) we have

$$\mathcal{O}(E)(U) \simeq \mathcal{O}(U)^r$$

since the vector bundle is trivial.

So there is an essentially surjective embedding from the category of holomorphic vector bundles to that of locally free sheaves, but this is *not* an equivalence of categories, since we imposed the constant rank condition.

7.1.1 Bundle Constructions

If E and F are holomorphic vector bundles, we can form holomorphic vector bundles $E \oplus F, E \otimes F, \bigwedge^i E, S^k E, E^*$, etc. This can be easily checked using transition functions: if E has transition functions $\varphi_{\alpha\beta}$ and F has transition functions $\psi_{\alpha\beta}$, then $E \oplus F$ has transition functions $\varphi_{\alpha\beta} \oplus \psi_{\alpha\beta}$, etc.

7.2 Holomorphic Line Bundles

Let \mathcal{O} denote the trivial line bundle $X \times \mathbb{C}$ (we are sort of abusing notation here; \mathcal{O} evokes the sheaf of sections of $X \times \mathbb{C}$). Observe that if L_1, L_2 are holomorphic line bundles, then so is $L_1 \otimes L_2$. Also, $L_1 \otimes L_1^* \cong \mathcal{O}$.

Definition 7.6. The *Picard group* $\mathrm{Pic}(X)$ is the group of holomorphic line bundles on X up to isomorphism, with the operation of tensor product.

Given a holomorphic line bundle L and an open cover $\mathcal{U} = \{U_\alpha\}$ with trivializations $\varphi_\alpha : L|_{U_\alpha} \simeq U_\alpha \times \mathbb{C}$ a biholomorphism, the collection of

transition functions for L $\{g_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}\} \in C^1(\mathcal{U}, \mathrm{GL}_1(\mathbb{C})) = C^1(\mathcal{U}, \mathcal{O}^*)$ satisfy the cocycle condition, and hence descend to an element $[g^L] \in \check{H}^1(X, \mathcal{O}^*)$.

Proposition 7.7. *The map $\Gamma: L \mapsto [g^L]$ induces an isomorphism of groups $\mathrm{Pic}(X) \cong \check{H}^1(X, \mathcal{O}^*)$.*

Proof. First we must show that this is actually well-defined. If $L \cong M$ are holomorphic line bundles, choose an open cover \mathcal{U} trivializing both L and M , so we have

$$\varphi_\alpha : L|_{U_\alpha} \simeq U_\alpha \times \mathbb{C} \quad \text{and} \quad \sigma_\alpha : M|_{U_\alpha} \simeq U_\alpha \times \mathbb{C}$$

This gives cocycles $g_{\alpha\beta}^L = \varphi_\alpha \varphi_\beta^{-1}$ and $g_{\alpha\beta}^M = \sigma_\alpha \sigma_\beta^{-1}$.

We have an isomorphism $f : L \rightarrow M$ commuting with projections to X , giving $f_\alpha : L|_{U_\alpha} \simeq M|_{U_\alpha}$. Define $h_\alpha = \sigma_\alpha f_\alpha \varphi_\alpha^{-1} : U_\alpha \times \mathbb{C} \rightarrow U_\alpha \times \mathbb{C}$. This is multiplication by some (non-vanishing) constant on each fiber, so can be regarded as a section of $\mathcal{O}^*(U_\alpha)$. Moreover,

$$\begin{aligned} (\delta h)_{\alpha\beta} &= h_\alpha h_\beta^{-1} \\ &= \sigma_\alpha f_\alpha \varphi_\alpha^{-1} (\sigma_\beta f_\beta \varphi_\beta^{-1})^{-1} \\ &= (g_{\alpha\beta}^L)^{-1} g_{\alpha\beta}^M \quad \underbrace{(f_\alpha f_\beta^{-1})}_{=1 \text{ since } f \text{ comes from a global section}} \end{aligned}$$

Clearly Γ is surjective, since any set of transition functions satisfying the cocycle condition can be patched to form a holomorphic line bundle L with those transition functions. We have to establish that it is injective (or said differently, this inverse is well-defined). Suppose $[L], [M] \in \mathrm{Pic}(X)$ such that $[g^L] = [g^M]$. That means that there exists $h = \{h_\alpha\} \in C^0(\mathcal{U}, \mathcal{O}^*)$ such that

$$h_\alpha h_\beta^{-1} = (g_{\alpha\beta}^L)^{-1} (g_{\alpha\beta}^M).$$

We can use the h_α to define an isomorphism of L and M . Let $f_\alpha : L|_{U_\alpha} \rightarrow M|_{U_\alpha}$ be $f_\alpha = \sigma_\alpha^{-1} \circ h_\alpha \circ \varphi_\alpha$. We have to verify that these glue properly, but of course that is precisely encapsulated by the above equation (details left as exercise).

That Γ is a homomorphism is evident from the definitions. \square

Recall the exact sequence of sheaves

$$0 \rightarrow 2\pi i\mathbb{Z} \xrightarrow{i} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \rightarrow 0.$$

This gives an exact sequence on cohomology,

$$\dots \rightarrow \check{H}^1(X, \mathcal{O}) \rightarrow \check{H}^1(X, \mathcal{O}^*) \xrightarrow{\delta^*} \check{H}^2(X, \mathbb{Z}) \rightarrow \dots$$

We have just made the identification $\check{H}^1(X, \mathcal{O}^*) \cong \text{Pic}(X)$.

Definition 7.8. Given $L \in \text{Pic}(X)$, the *first Chern class* of L is

$$c_1(L) = \delta^*([g^L]) \in \check{H}^2(X, \mathbb{Z}).$$

This gives a *discrete* isomorphism invariant of a line bundle. The image of $\check{H}^1(X, \mathcal{O})$ is the kernel, by exactness; this parametrizes the ambiguity of this invariant.

7.3 Line bundles on Projective Space

Recall that \mathbb{P}^n is the set of lines in \mathbb{C}^{n+1} . For each point $\ell \in \mathbb{P}^n$, we can associate the complex line consisting of that line. This gives a (holomorphic) line bundle on \mathbb{P}^n :

$$\begin{aligned} L &= \{(\ell, w) \in \mathbb{P}^n \times \mathbb{C}^{n+1} : w \in \ell\} \\ &= \{([z_0, \dots, z_n], (w_0, \dots, w_n)) \in \mathbb{P}^n \times \mathbb{C}^{n+1} : (w_0, \dots, w_n) = \lambda(z_0, \dots, z_n)\}. \end{aligned}$$

To see this, let's compute the transition functions. Let $U_i = \{[z_0, \dots, z_n] : z_i \neq 0\}$. Let $\varphi_i : L|_{U_i} \rightarrow U_i \times \mathbb{C}$ send $\varphi_i(\ell, w) = (\ell, w_i)$. The transition functions are

$$\varphi_{ij}(z, w) = \frac{z_i}{z_j} w \text{ on } U_i \cap U_j.$$

Definition 7.9. L is the *tautological line bundle* on \mathbb{P}^n and is denoted by $\mathcal{O}_{\mathbb{P}^n}(-1)$.

We define $\mathcal{O}_{\mathbb{P}^n}(1) = \mathcal{O}_{\mathbb{P}^n}(-1)^*$, $\mathcal{O}_{\mathbb{P}^n}(a) = \mathcal{O}_{\mathbb{P}^n}(1)^a$ if $a > 0$, and $\mathcal{O}_{\mathbb{P}^n}(a) = \mathcal{O}_{\mathbb{P}^n}(-1)^{-a}$ if $a < 0$. Notice that this is the subgroup of $\text{Pic}(\mathbb{P}^n)$ generated by the tautological line bundle.

Suppose $f : X \rightarrow Y$ is holomorphic and E is a complex vector bundle on Y . Then there is a pullback bundle $f^*E \rightarrow X$ such that $(f^*E)_x = E|_{f(x)}$. Given transition functions $g_{\alpha\beta}^E$ over $U = \{U_\alpha\}$, the bundle f^*E has transition functions $f^*g_{\alpha\beta}^E = g_{\alpha\beta}^E \circ f$ on $\{f^*(U_\alpha)\}$.

When $f = \iota : X \hookrightarrow Y$ is an inclusion map, then we write $E|_X := \iota^*E$.

Exercise 7.10. If $f : X \rightarrow Y$ is a holomorphic map, then the pullback induces a group homomorphism $f^* : \text{Pic}(Y) \rightarrow \text{Pic}(X)$.

After we study line bundles on \mathbb{P}^n , this will give useful information on any projective complex manifold.

7.4 Ample Line Bundles

Recall that if X is a complex manifold, we cannot use holomorphic functions to embed X in \mathbb{C}^n . The idea is to instead use sections of holomorphic line bundles.

Definition 7.11. X is said to be *quasiprojective* (resp. *projective*) if it is isomorphic to a (closed) submanifold of \mathbb{P}^N for some N .

For a holomorphic bundle $E \rightarrow X$, write $H^0(X, E) = \check{H}^0(X, \mathcal{O}(E))$ for the vector space of global sections of E .

We are mostly interested in the special case of holomorphic line bundles.

Definition 7.12. A *trivialization* of L over an open set $U \subset X$ is a $\xi \in \mathcal{O}^*(L)(U)$, i.e. $\xi(x) \in L_x$ is non-zero for all $x \in U$.

Given such a ξ , we define a map $\varphi : L|_U \rightarrow U \times \mathbb{C}$ by $\varphi(v) = (\pi(v), f(v))$ where $v \in L_x$ is $v = f(v)\xi(x)$ for the unique scalar $f(v)$. In this sense, a trivialization is the same as the usual notion of trivialization of bundles.

Any two trivializations ξ_α over U_α and ξ_β over U_β , the transition function is ξ_α/ξ_β over $U_\alpha \cap U_\beta$. Furthermore, any $s \in \mathcal{O}(L)(U)$ can be written uniquely as $s = f\xi$ for some $f \in \mathcal{O}(U)$ (so $\mathcal{O}(L)(U)$ is a free $\mathcal{O}(U)$ -module generated by a trivialization.)

Fix a holomorphic line bundle $\pi : L \rightarrow X$. Suppose that there are global sections $s_0, \dots, s_N \in H^0(X, L)$ and assume that $x \in X$ is such that not all $s_i(x)$ are zero. Given a trivialization ξ of U , we can write $s_i = f_i\xi$ where $f_i \in \mathcal{O}(U)$. By the non-vanishing assumption, $[f_0(x), \dots, f_N(x)] \in \mathbb{P}^N$. We claim that this is independent of ξ . Indeed, if $\tilde{\xi}$ is another trivialization then $\tilde{s}_i = g\xi$ for some $g \in \mathcal{O}(U)^*$, and this has the effect of multiplying all the coordinates by $g(x)$, which is just the same point in \mathbb{P}^N . We denote this point by $[s_0(x), \dots, s_N(x)]$, which we have now established is well-defined.

Definition 7.13. Let $V \subset H^0(X, L)$. The *base locus* of V is

$$Bs(V) := \{x \in X : s(x) = 0 \forall s \in V\}.$$

Let ϕ_V be the holomorphic map $V \setminus Bs(V) \rightarrow \mathbb{P}^N$ defined by

$$\phi_v(x) = [s_0(x), \dots, s_N(x)]$$

(note that this depends on our choice of basis. There is a way of defining it invariantly).

Such a linear subspace V is called a *linear series*. If $V = H^0(X, L)$ then we say that V is the “complete linear series” defined by L .

Even if $Bs(V) = \emptyset$, the map ϕ_V need not be an embedding. For instance, for $L = \mathcal{O}$ the induced map is constant.

Definition 7.14. We say L is *ample* if the map induced by the complete linear series $H^0(X, L^{\otimes k})$ for some $k \geq 1$ is an embedding.

This is saying that $L^{\otimes k}$ has “enough” sections so that:

1. $Bs(H^0(X, L^{\otimes k})) \neq \emptyset$.
2. The map $\phi_{L^{\otimes k}}$ is injective, i.e. for $x \neq y$ there exists $s \in H^0(X, L^{\otimes k})$ such that $s(x) = 0$ and $s(y) \neq 0$.
3. $d\phi_{L^{\otimes k}}$ to be injective.

Theorem 7.15. Let $X = \mathbb{P}^n$. There is a canonical identification for $k \geq 0$

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) \simeq \mathbb{C}[z_0, \dots, z_n]_k = \text{homogeneous polynomials of degree } k.$$

Proof. Let $s \in \mathbb{C}[z_0, \dots, z_n]_k$ and fix some $[z] \in \mathbb{P}^n$, with coordinates $z = (z_0, \dots, z_n)$. Recall that $\mathcal{O}(-1)|_{[z]} = \{\lambda z : \lambda \in \mathbb{C}\} \subset \mathbb{C}^{n+1}$. Let $\widehat{s} : \mathcal{O}(-1)|_{[z]} \rightarrow \mathbb{C}$ be defined by $\widehat{s}(\lambda z) = \lambda^k s(z_0, \dots, z_n)$. This defines a point in $(\mathcal{O}(-1)|_{[z]}^*)^{\otimes k} = \mathcal{O}(k)|_{[z]}$. Exercise: check that this defines a holomorphic global section. Clearly if $s \neq 0$ then $\widehat{s} \neq 0$.

We need to show that $s \mapsto \widehat{s}$ is surjective. Let $t \in H^0(\mathbb{P}^n, \mathcal{O}(k))$ and pick some $0 \neq s \in \mathbb{C}[z_0, \dots, z_n]_k$. Then $f := \frac{t}{s}$ is a meromorphic function on \mathbb{P}^n . Let $\pi : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n$ be the natural projection, and consider $u(z) = s(z)f([z])$. This is a *holomorphic* function on $\mathbb{C}^{n+1} - \{0\}$. By Hartog’s theorem, it extends to a holomorphic function on \mathbb{C}^{n+1} , and it is a polynomial by homogeneity, say u . Then $\widehat{u} = t$. \square

Given any line bundle $L \rightarrow X$, there is a natural multiplication $H^0(X, L^{\otimes k}) \otimes H^0(X, L^{\otimes r}) \rightarrow H^0(X, L^{\otimes(k+r)})$. The isomorphism above is compatible with the multiplicative structure, so we have an isomorphism of rings

$$\bigoplus_{k \geq 0} H^0(X, L^{\otimes k}) \simeq \mathbb{C}[z_0, \dots, z_n].$$

Chapter 8

Kähler Manifolds

8.1 Kähler metrics

We seek to put a metric on a complex manifold X that respects the complex structure.

Let V be a finite-dimensional real vector space and $J : V \rightarrow V$ an almost complex structure on V . Let $\langle \cdot, \cdot \rangle$ be an inner product on V .

Definition 8.1. We say that $\langle \cdot, \cdot \rangle$ is *compatible* with J if $\langle v, w \rangle = \langle Jv, Jw \rangle$ for all $v, w \in V$. When this holds, the *fundamental form* is

$$\omega(v, w) = \langle Jv, w \rangle.$$

Remark 8.2. Any two of $\{\langle \cdot, \cdot \rangle, J, \omega\}$ determines the third. Note that ω is an antisymmetric form, since

$$\omega(v, w) = \langle Jv, w \rangle = \langle J^2v, Jw \rangle = -\langle v, Jw \rangle = -\omega(w, v).$$

It also follows that ω is definite, since the inner product is.

Now extend these to $V_{\mathbb{C}} = V \otimes \mathbb{C}$. The inner product extends to a Hermitian inner product on $V_{\mathbb{C}}$,

$$\langle \lambda v, \mu w \rangle_{\mathbb{C}} = \lambda \bar{\mu} \langle v, w \rangle.$$

Also, ω extends \mathbb{C} -linearly to an element of $\bigwedge^2 V_{\mathbb{C}}^*$.

Lemma 8.3. *Suppose $(V, \langle \cdot, \cdot \rangle)$ is Euclidean and J is a compatible almost complex structure.*

1. *The decomposition $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$ is orthogonal with respect to $\langle \cdot, \cdot \rangle_{\mathbb{C}}$,*
2. *We have $\omega \in \Lambda^{1,1} V_{\mathbb{C}}^*$.*

Proof. 1. If $v \in V^{1,0}$ and $w \in V^{0,1}$, then

$$\langle v, w \rangle = \langle Jv, Jw \rangle = \langle iv, -iw \rangle = -\langle v, w \rangle.$$

Another way to see this is to observe that any element of $V^{1,0}$ is $v - iJv$ and any element of $V^{0,1}$ is $w + iJw$.

2. If $v, w \in V^{1,0}$ then

$$\omega(v, w) = \omega(Jv, Jw) = \omega(iv, iw) = -\omega(v, w) \implies \omega(v, w) = 0.$$

A similar calculation works for $V^{0,1}$.

□

Let X be a complex manifold. This gives an almost complex structure $J : TX \rightarrow TX$ with $J^2 = -\text{Id}$.

Definition 8.4. A Riemannian metric g on X is said to be *compatible* with the almost complex structure if for all $x \in X$, the scalar product g_x on $T_x X$ is compatible with J , i.e.

$$g_x(v, w) = g_x(J_x v, J_x w).$$

Similarly, define the fundamental form $\omega(v, w) = g(Jv, w)$. The extension of g gives a Hermitian inner product on $TX \otimes \mathbb{C}$. Suppose that we have local holomorphic coordinates z_1, \dots, z_n on X . Then dz_1, \dots, dz_n are a \mathbb{C} -basis for $(T^*X)^{1,0}$ (locally). Let

$$h_{ij} = 2\langle \partial_i, \partial_j \rangle_g.$$

Exercise 8.5. Check that

$$\omega = \frac{i}{2} \sum_{j,k} h_{jk} dz_j \wedge d\bar{z}_k.$$

Definition 8.6. We say g is a *Kähler metric* if $d\omega = 0$.

Example 8.7. On \mathbb{C}^n , with coordinates z_1, \dots, z_n , we get

$$\omega = \frac{i}{2} \sum_{i=1}^n dz_i \wedge d\bar{z}_i.$$

Example 8.8. If $\dim X = 1$ (i.e. X is a Riemann surface), then any $(1, 1)$ -form is closed.

Example 8.9 (Fubini-Study metric on \mathbb{P}^n). Let $U \subset \mathbb{P}^n$ be open and $\pi : \mathbb{C}^{n+1} \rightarrow \mathbb{P}^n$ be the natural projection map. Suppose $s : U \rightarrow \mathbb{C}^{n+1}$ is a holomorphic lift of π , i.e. $\pi s(z) = z$ for all $z \in U$. Let $\omega_{FS}|_U = \frac{i}{2\pi} \partial\bar{\partial} \log \|s\|^2$. This is $(1, 1)$ -form, but we need to check that it is well-defined, closed, and positive definite.

To show that ω_{FS} is well-defined, choose another \tilde{s} . So $\tilde{s} = fs$ for some holomorphic function f , so $|\tilde{s}|^2 = |f|^2|s|^2$. Then

$$\partial\bar{\partial}\log|\tilde{s}|^2 = \partial\bar{\partial}\log|f|^2 + \partial\bar{\partial}\log|s|^2.$$

Now,

$$\partial\bar{\partial}\log|f|^2 = \partial\bar{\partial}\log f + \partial\bar{\partial}\log\bar{f} = 0$$

since f is holomorphic. Next, note that

$$\begin{aligned} 2\omega_{FS} &= \frac{i}{2\pi}(\partial + \bar{\partial})(\bar{\partial} - \partial)\log|s|^2 \\ &= \frac{i}{2\pi}d(\bar{\partial} - \partial)\log|s|^2 \implies d\omega_{FS} = 0. \end{aligned}$$

Finally, we need to show that if

$$\omega_{FS} = \frac{1}{2} \sum h_{ij} dz_j \wedge d\bar{z}_j$$

then the (h_{ij}) is positive definite. Note that $U(n+1)$ acts on \mathbb{C}^{n+1} and on \mathbb{P}^n holomorphically, transitively, and is ω_{FS} -invariant. So it suffices to prove the positive definiteness at a particular point, e.g. $[1, 0, \dots, 0]$. This is a local calculation, left as an exercise.

Lemma 8.10. *If $f : X \rightarrow Y$ is a holomorphic embedding and w is Kähler on X , then f^*w is Kähler on X .*

Proof. The closedness follows from $df^*w = f^*dw = 0$, and the positive definiteness is obvious. \square

Corollary 8.11. *Any projective manifold is Kähler.*

Pick a local unitary frame for the hermitian metric $\varphi^1, \dots, \varphi^n$. Setting $\xi^j = \text{Re } \varphi^j$ and $\eta^j = \text{Im } \varphi^j$, we have locally

$$h = \sum_j \varphi^j \otimes \bar{\varphi}^j.$$

Then

$$g = \text{Re} \sum_j (\eta^j + i\xi_j) \otimes (\eta^j - i\xi_j) = \sum_j \eta^j \otimes \eta^j + \xi^j \otimes \xi^j$$

and

$$\omega = \frac{i}{2} \sum_j (\eta^j + i\xi_j) \wedge (\eta^j - i\xi_j) = \sum_j \eta^j \wedge \xi^j.$$

Therefore, we see that $\frac{1}{n!} \wedge \omega^n$ is a volume form on X , and we conclude that

$$\int_X \omega^n > 0.$$

when the left hand side makes sense (e.g. if X is compact).

Proposition 8.12. *If X is compact Kähler, then*

$$\dim H_{dR}^{2q}(X, \mathbb{R}) > 0.$$

Proof. Let ω be a Kähler form on X . Let $\tau = \wedge^q \omega$. Then $d\tau = 0$ as $d\omega = 0$. So $[\tau] \in H_{dR}^{2q}(X, \mathbb{R})$. Suppose for the sake of contradiction that $\tau = d\sigma$ for some σ . Then

$$\int_X \wedge^n \omega = \int_X \omega^{n-q} \wedge \tau = \int_X \omega^{n-q} \wedge d\sigma = \int_X d(\sigma \wedge \omega^{n-q}) = 0$$

(the last equality following from the fact that $d\omega = 0$) by Stokes' Theorem (we are assuming a manifold without boundary). \square

Note that this is a topological result, since the de Rham cohomology is a topological invariant (of smooth manifolds). So there is a topological obstruction to being a Kähler manifold!

Proposition 8.13. *Let ω be a $(1,1)$ -form associated to a Hermitian metric on a complex manifold X . Then $d\omega = 0$ if and only if for all $x \in X$, there exist (holomorphic) coordinates z_1, \dots, z_n centered at x so that locally*

$$\omega = \frac{i}{2} \sum h_{ij} dz_i \wedge d\bar{z}_j$$

with $h_{ij} = \delta_{ij} + O(|z|^2)$. In other words, ω defines a Kähler metric if and only if $\omega = \omega_0 + O(|z|^2)$ locally, where ω_0 is the standard form on \mathbb{C}^n .

Proof. Say $\omega = \frac{1}{2} \sum h_{ij} dz_i \wedge d\bar{z}_j$. Then

$$d\omega = \frac{1}{2} \sum \frac{\partial h_{ij}}{\partial z_k} dz_k \wedge dz_i \wedge d\bar{z}_j + \frac{1}{2} \sum \frac{\partial h_{ij}}{\partial \bar{z}_k} d\bar{z}_k \wedge dz_i \wedge d\bar{z}_j.$$

If the relation in the proposition holds, we get $\sum \frac{\partial h_{ij}}{\partial z_k} = \frac{\partial h_{ij}}{\partial \bar{z}_k} = 0$ so $d\omega = 0$.

Conversely, suppose that $d\omega = 0$. Let z_1, \dots, z_n be holomorphic coordinates at x so $\omega = \frac{1}{2} \sum_{i,j} h_{ij} dz_i \wedge d\bar{z}_j$. By a linear change of coordinates, we may assume that $h_{ij}(0) = \delta_{ij}$. The Taylor expansion will look like

$$h_{ij}(z) = \delta_{ij} + \sum_k a_{ijk} z_k + \sum_k b_{ijk} \bar{z}_k + O(|z|^2).$$

Since $h_{ij} = \overline{h_{ji}}$, we have $b_{ijk} = \overline{a_{jik}}$. Since $d\omega = 0$,

$$0 = \sum_{i,j,k} a_{ijk} dz_k \wedge dz_i \wedge d\overline{z_j} + \sum_{i,j,k} b_{ijk} d\overline{z_k} \wedge dz_i \wedge d\overline{z_j}.$$

This forces $a_{ijk} = a_{kji}$ and $b_{ijk} = b_{ikj}$.

Now let $w_j = z_j + \frac{1}{2} \sum_{i,k} a_{ijk} z_i z_k$. We claim that this will kill the linear term. Indeed,

$$\begin{aligned} dw_j &= dz_j + \frac{1}{2} \sum_{i,k} a_{ijk} (z_i dz_k + z_k dz_i) \\ d\overline{w_j} &= d\overline{z_j} + \frac{1}{2} \sum_{i,k} \overline{a_{ijk}} (\overline{z_i} d\overline{z_k} + \overline{z_k} d\overline{z_i}). \end{aligned}$$

Now we compute:

$$\begin{aligned} dw_j \wedge d\overline{w_j} &= \sum dz_j \wedge d\overline{z_j} + \frac{1}{2} \sum_{i,j,k} \overline{a_{ijk}} (\overline{z_i} dz_j \wedge d\overline{z_k} + \overline{z_k} dz_j \wedge d\overline{z_i}) \\ &\quad + \frac{1}{2} \sum_{i,j,k} a_{ijk} (z_i dz_k \wedge d\overline{z_j} + z_k dz_i \wedge d\overline{z_j}) + O(|z|^2) \\ &= \sum dz_j \wedge d\overline{z_j} + \sum_{i,j,k} a_{ijk} z_k dz_i d\overline{z_j} + \sum_{i,j,k} b_{ijk} \overline{z_k} dz_j d\overline{z_i} + O(|z|^2) \\ &= \frac{2}{i} \omega + O(|z|^2). \end{aligned}$$

□

8.2 The Kähler Identities

The Kähler condition implies strong interactions between the Riemannian geometry and complex geometry. To discuss this, we recall some operators on forms.

1. Consider (X, g) an oriented Riemannian manifold of dimension $2n$ over \mathbb{R} . There is an exterior derivative

$$d: \mathcal{A}^k \rightarrow \mathcal{A}^{k+1}$$

satisfying $d \circ d = 0$.

2. Let σ be an orientation form for X . The Hodge star $*$: $\mathcal{A}^k \rightarrow \mathcal{A}^{2n-k}$ is defined so that

$$\alpha \wedge * \beta = \langle \alpha, \beta \rangle \sigma$$

for all $\alpha \in \mathcal{A}^k$.

Set $d^* = - * d * : \mathcal{A}^k \rightarrow \mathcal{A}^{k-1}$. The Laplacian (or what is sometimes called the ‘‘Laplace-Beltrami operator’’ in this general setting) is

$$\Delta_d = d^* d + d d^* : \mathcal{A}^k \rightarrow \mathcal{A}^k.$$

3. Now suppose that X is a complex manifold of complex dimension n , and g is a Riemannian metric on X . Extend g and $*$ to the complex structure, so we get a map $* : \mathcal{A}_{\mathbb{C}}^k \rightarrow \mathcal{A}_{\mathbb{C}}^{2n-k}$. Again,

$$\alpha \wedge * \beta = g_{\mathbb{C}}(\alpha, \beta) \sigma.$$

Using the complex structure, we write $d = \partial + \bar{\partial}$ where

$$\partial : \mathcal{A}^{p,q} \rightarrow \mathcal{A}^{p+1,q} \quad \bar{\partial} : \mathcal{A}^{p,q} \rightarrow \mathcal{A}^{p,q+1}.$$

We define $\bar{\partial}^* = - * \bar{\partial} *$ and $\partial^* = - * \partial *$ and

$$\Delta_{\bar{\partial}} = \bar{\partial}^* \bar{\partial} + \bar{\partial} \bar{\partial}^* \quad \Delta_{\partial} = \partial^* \partial + \partial \partial^*.$$

If ω is a Hermitian form, let

$$L = \mathcal{A}^{p,q} \rightarrow \mathcal{A}^{p+1,q+1}$$

be $L(\alpha) = \alpha \wedge \omega$. Set

$$\Lambda = *^{-1} L * : \mathcal{A}^{p,q} \rightarrow \mathcal{A}^{p-1,q-1}.$$

Just as in Riemannian geometry, the operators ∂^* and $\bar{\partial}^*$ are constructed to be the adjoints of $\partial, \bar{\partial}$.

Lemma 8.14. *Suppose $\alpha \in \mathcal{A}^{p-1,q}$ and $\beta \in \mathcal{A}^{p,q}$. Then*

$$\langle \partial^* \alpha, \beta \rangle = \langle \alpha, \partial \beta \rangle$$

and

$$\langle \bar{\partial}^* \alpha, \beta \rangle = \langle \alpha, \bar{\partial} \beta \rangle.$$

Proof. Without loss of generality, we can prove only the first identity. Note that if $\alpha \in \mathcal{A}^k$, $* * \alpha = (-1)^{2n-k} \alpha$. Also, by Stokes’ Theorem

$$0 = \int d(\alpha \wedge * \beta) = \int \partial(\alpha \wedge * \beta) = \int \partial \alpha \wedge * \beta + (-1)^k \alpha \wedge \partial * \beta.$$

Therefore,

$$\begin{aligned}
\langle \partial\alpha, \beta \rangle &= \int \partial\alpha \wedge * \beta \\
&= (-1)^{k+1} \int \alpha \wedge \partial * \beta \\
&= (-1)^{k+1+k(2n-k)} \int \alpha \wedge ** \partial * \beta \\
&= \langle \alpha, \partial^* \beta \rangle.
\end{aligned}$$

□

Theorem 8.15. *Assume ω is Kähler. Then*

1. $\Delta_d = 2\Delta_{\bar{\partial}} = 2\Delta_{\partial}$.
2. Let $h = \sum_{k=0}^{2n} (n-k)\pi_k$ where π_k is the projection map $\mathcal{A}^n \rightarrow \mathcal{A}^k$. Then h, Λ , and L commute with Δ_d .
3. $[\Lambda, L] = h$, $[h, L] = -2L$, and $[h, \Lambda] = 2\Lambda$.

This theorem says that there are three operators on the space of harmonic forms on X : h, Λ, L which form a representation of \mathfrak{sl}_2 . The rest of the section is devoted to a proof of this theorem.

Step 1. For now, we assume that $X = \mathbb{C}^n$ and $\omega = \frac{i}{2} \sum dz_j \wedge d\bar{z}_j$ is the standard Kähler form for the Euclidean metric $g = \frac{1}{2} \sum dz_j \otimes d\bar{z}_j$.

First, we establish some notation.

Definition 8.16. For $\alpha \in \mathcal{A}^q$, $\xi \in \mathcal{A}^1$ let $\xi \vee \alpha \in \mathcal{A}^{q-1}$ be defined by

$$\langle \xi \vee \alpha, \beta \rangle = \langle \alpha, \bar{\xi} \wedge \beta \rangle \forall \beta \in \mathcal{A}^{q-1}.$$

Since the Hermitian product is non-degenerate, this specifies the operator uniquely. The point is that \vee is essentially the adjoint to \wedge .

Definition 8.17. If $\alpha \in \mathcal{A}^q$, say $\alpha = \sum_{|I|+|J|=q} \alpha_{IJ} dz_I \wedge d\bar{z}_J$ we let

$$\partial_j \alpha = \sum_{I,J} \frac{\partial \alpha_{IJ}}{\partial z_j} dz_I \wedge d\bar{z}_J \quad \partial_{\bar{j}} \alpha = \sum_{I,J} \frac{\partial \alpha_{IJ}}{\partial \bar{z}_j} dz_I \wedge d\bar{z}_J$$

Lemma 8.18. *With the notation above, $dz_j \vee dz_k = 0$ and $dz_j \vee d\bar{z}_k = g^{jk} = 2\delta_{jk}$ for all j, k .*

Proof. This is obvious from the definitions:

$$dz_j \vee dz_k := \langle dz_j, d\bar{z}_k \rangle = 0$$

and

$$dz_j \vee d\bar{z}_k := \langle dz_j, dz_k \rangle = g^{jk} = 2\delta_{jk}.$$

□

Next, we claim the following identities:

Lemma 8.19. 1. $\bar{\partial}\alpha = \sum_j dz_j \wedge \bar{\partial}_j \alpha$.

$$2. \partial_j \langle \alpha, \beta \rangle = \langle \partial_j \alpha, \beta \rangle + \langle \alpha, \bar{\partial}_j \beta \rangle.$$

$$3. \partial_j (dz_k \vee \alpha) = dz_k \vee \partial_j \alpha.$$

Proof. 1. This is obvious.

2. The important point here is that our metric standard one, so it doesn't depend on z :

$$\begin{aligned} \partial_j \langle \alpha, \beta \rangle &= \partial_j \left(\sum_I \alpha_I \bar{\beta}_I \right) \\ &= \sum_I \partial_j (\alpha_I \bar{\beta}_I) \\ &= \sum_I (\partial \alpha_I) \bar{\beta}_I + \alpha_I \bar{\partial} \beta_I \end{aligned}$$

3. Follows from pairing both sides with β and using the previous part. The point is that ∂_j commutes with $dz_k \vee$ since it commutes with its adjoint $\bar{d}z_k \wedge$. In full, gory detail:

$$\begin{aligned} \langle \partial_j (dz_k \vee \alpha), \beta \rangle &= \partial_j \langle dz_k \vee \alpha, \beta \rangle - \langle dz_k \vee \alpha, \bar{\partial}_j \beta \rangle \\ &= \partial_j \langle \alpha, d\bar{z}_k \wedge \beta \rangle - \langle \alpha, d\bar{z}_k \wedge \beta \rangle \\ &= \langle \partial_j \alpha, \bar{\partial}_j (d\bar{z}_k \wedge \beta) \rangle \\ &= \langle d\bar{z}_k \vee \partial_j \alpha, \beta \rangle. \end{aligned}$$

□

Step 2. We have the following key identity.

Proposition 8.20.

$$\bar{\partial}^* \alpha = - \sum_j dz_j \vee \partial_j \alpha.$$

Proof. Let $\alpha \in \mathcal{A}^q$ and $\beta \in \mathcal{A}^{q-1}$ with compact support. Then we have

$$\int \partial_j \langle dz_j \vee \alpha, \beta \rangle dV = 0 \text{ by Stokes' theorem and compact support.}$$

This says that $\bar{\partial}_j$ is adjoint to $(-1)^k \partial_j$. By construction,

$$\begin{aligned} \langle \bar{\partial}^* \alpha, \beta \rangle &= \langle \alpha, \bar{\partial} \beta \rangle \\ &= \sum_j \langle \alpha, d\bar{z}_j \wedge \bar{\partial}_j \beta \rangle \\ &= \sum_j \langle dz_j \vee \alpha, \bar{\partial}_j \beta \rangle \\ &= - \sum_j \langle dz_j \vee \partial_j \alpha, \beta \rangle. \end{aligned}$$

□

Step 3.

Proposition 8.21. *On \mathbb{C}^n with the standard metric, we have*

$$[\bar{\partial}^*, L] = i\partial.$$

Proof. We compute:

$$\begin{aligned} [\bar{\partial}^*, L]\alpha &= \bar{\partial}^* L\alpha - L\bar{\partial}^* \alpha \\ &= \bar{\partial}^*(\omega \wedge \alpha) - \omega \wedge \bar{\partial}^* \alpha. \end{aligned}$$

One can explicitly check that

$$dz_k \vee (\alpha \wedge \beta) = (dz_k \vee \alpha) \wedge \beta + (-1)^k \alpha \wedge (dz_k \vee \beta).$$

Therefore,

$$\begin{aligned} \bar{\partial}^*(\omega \wedge \alpha) &= - \sum_j dz_j \vee \partial_j(\omega \wedge \alpha) \\ &= - \sum_j dz_j \vee (\omega \wedge \partial_j \alpha) \\ &= - \frac{i}{2} \sum_{j,k} dz_j \vee (dz_k \wedge d\bar{z}_k \wedge \partial_j \alpha) \\ &= - \frac{i}{2} \sum_{j,k} (dz_j \vee dz_k) \wedge d\bar{z}_k \wedge \partial_j \alpha \\ &\quad + \frac{i}{2} \sum_{j,k} dz_j \wedge (dz_j \vee d\bar{z}_k) \wedge \partial_j \alpha \\ &\quad - \frac{i}{2} \sum_{j,k} dz_k \wedge d\bar{z}_k \wedge (dz_j \vee \partial_j \alpha). \end{aligned}$$

The first term vanishes by the earlier calculation. The second term can be written as $i \sum_j dz_j \wedge \partial_j \alpha = \partial \alpha$, and this is what we want to end up with. The last term will cancel out with stuff from the commutator. Indeed,

$$-\omega \wedge \bar{\partial}^* \alpha = \sum_j \omega \wedge dz^j \vee \partial_j \alpha$$

and this cancels the last term. \square

Proposition 8.22. *On any Kähler manifold X ,*

1. $[\bar{\partial}^*, L] = i\partial$
2. $[\partial^*, L] = -i\bar{\partial}$,
3. $[\Lambda, \bar{\partial}] = -i\partial^*$,
4. $[\Lambda, \partial] = i\bar{\partial}^*$.

Proof. For (1), recall that as ω is Kähler, around any x there are holomorphic coordinates z_1, \dots, z_n in which $\omega = \omega_0 + o(|z|^2)$, where ω_0 is the standard form. Since $[\bar{\partial}^*, L]$ only involves first order derivatives in the metric coefficients, the calculation for \mathbb{C}^n shows that this is still true.

Then (2) follows from conjugating (noting that ω is a *real* $(1,1)$ -form), and (3) and (4) from taking adjoints. \square

Proposition 8.23. *On any Kähler manifold (X, ω) we have*

$$\Delta_d = 2\Delta_\partial = 2\Delta_{\bar{\partial}}.$$

Proof. We claim that

$$\begin{aligned} \bar{\partial}^* \partial + \partial \bar{\partial}^* &= 0 \\ \partial^* \bar{\partial} + \bar{\partial} \partial^* &= 0. \end{aligned}$$

This is just formal: the first identity is

$$\begin{aligned} \bar{\partial}^* \partial + \partial \bar{\partial}^* &= -i[L^*, \partial] \partial - i\partial[L^*, \partial] \\ &= -iL^* \partial \partial + i\partial L^* \partial - i\partial L^* \partial + i\partial \partial L^* \\ &= 0 \text{ since } \partial \partial = 0. \end{aligned}$$

The other one follows by conjugation.

Now we claim that $\Delta_d = \Delta_\partial + \Delta_{\bar{\partial}}$. Write

$$\Delta_d = dd^* + d^*d = (\partial + \bar{\partial})(\partial^* + \bar{\partial}^*) + (\partial^* + \bar{\partial}^*)(\partial + \bar{\partial})$$

and expand and collect terms, canceling the cross-terms.

Finally, one computes that $\Delta_\partial = \Delta_{\bar{\partial}}$ in a similar fashion: the right hand side is

$$-i\bar{\partial}[L^*, \partial] - i[L^*, \partial]\bar{\partial} = -i\bar{\partial}L^*\partial + i\bar{\partial}\partial L^* - iL^*\partial\bar{\partial} + i\partial L^*\bar{\partial}$$

and the right hand side is

$$i\partial[L^*, \bar{\partial}] + i[L^*, \bar{\partial}]\partial = i\partial L^*\bar{\partial} - i\partial\bar{\partial}L^* + iL^*\bar{\partial}\partial - i\bar{\partial}L^*\partial.$$

These are equal since $\partial\bar{\partial} = -\bar{\partial}\partial$. \square

Now, we are ready to prove the remaining Kähler identities.

Theorem 8.24. *Let (X, ω) be Kähler. Define $L : \mathcal{A}^* \rightarrow \mathcal{A}^*$ by $L(\alpha) = \alpha \wedge \omega$, and $\Lambda = L^*$. Let $\pi_k : \mathcal{A}^* \rightarrow \mathcal{A}^k$ be the projection, and define*

$$h = \sum_{p=0}^{2n} (n-k)\pi_k.$$

Then

1. *Then h, Λ , and L commute with Δ_d .*
2. *$[\Lambda, L] = h$, $[h, L] = -2L$, and $[h, \Lambda] = 2\Lambda$.*

Proof. First we study the commutators with h . By linearity, suffices to check the identities on $\alpha \in \mathcal{A}^{p,q}$ where $p+q=k$. Then it is clear that

$$[h, \Delta_\partial]\alpha = (n-k)\Delta_\partial\alpha - \Delta_\partial(n-k)\alpha = 0.$$

Also,

$$\begin{aligned} [h, L]\alpha &= hL\alpha - Lh\alpha \\ &= (n - (k+2))L\alpha - L(n-k)\alpha \\ &= -2L\alpha. \end{aligned}$$

Taking the adjoint (and noting that $h^* = h$ since $\pi_k^* = \pi_k$) we obtain the third commutation relation.

We show that $[L, \Delta_d] = 0$. (Note that this is equivalent to showing that ω is harmonic). Using $d(\omega \wedge \alpha) = \omega \wedge d\alpha$, we have $[L, d] = 0$. Then

$[L, \partial] = 0$ and $[L, \bar{\partial}] = 0$. Therefore,

$$\begin{aligned} [L, \Delta_\partial] &= [L, \partial\partial^* + \partial^*\partial] \\ &= \partial[L, \partial^*] + [L, \partial^*]\partial \\ &= -i(\partial\bar{\partial} + \bar{\partial}\partial) \\ &= 0. \end{aligned}$$

Taking adjoints, we get that Λ commutes with Δ_d (since Δ_d is self-adjoint).

The only thing left to see is that $[\Lambda, L] = h$. We can recast this as saying that if $\alpha \in \mathcal{A}^{p,q}$ then

$$[\Lambda, L]\alpha = (n - p - q)\alpha.$$

This has no derivatives, so it certainly follows if it is shown on (\mathbb{C}^n, ω_0) , which can be checked explicitly. In the case $n = 1$, we have $\Lambda(g(z)dzd\bar{z}) = g(z)$, and the commutator identity follows in that case. In general, write

$$L = \sum_k L_k \text{ where } L_k\alpha = dz_k \wedge d\bar{z}_k \wedge \alpha$$

and $\Lambda = \sum_k \Lambda_k$, where $\Lambda_k = L_k^*$ just removes $dz_k \wedge d\bar{z}_k$ if it exists and kills the form otherwise (we are ignoring issues with constants). Furthermore, it is clear that $[L_k, \Lambda_j] = 0$ if $j \neq k$, so this reduces to the one-variable case. \square

Chapter 9

Hodge Theory

9.1 The Hodge Decomposition

Our aim is to use the Kähler identities to get info on $H_{\bar{\partial}}^{p,q}(X)$. We do this indirectly, by considering the *harmonic* forms.

Definition 9.1. Given an oriented Riemannian manifold (X, g) , we define the space of *harmonic forms* of degree k to be

$$\mathcal{H}^k(X, g) = \{\alpha \in \mathcal{A}^k(X) : \Delta_d \alpha = 0\}.$$

Remark 9.2. On \mathbb{R}^n with the standard Riemannian metric, Δ_d reduces to the usual Laplace operator, so functions lying in the kernel of Δ_d are harmonic functions in the usual sense.

Lemma 9.3. *We have*

$$\Delta_{\bar{\partial}} \alpha = 0 \iff \bar{\partial} \alpha = \bar{\partial}^* \alpha = 0.$$

Proof. The direction \Leftarrow is obvious since $\Delta_{\bar{\partial}} = \bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial}$. For the other direction, if $\Delta_{\bar{\partial}} \alpha = 0$ then

$$0 = (\Delta_{\bar{\partial}} \alpha, \alpha) = (\bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial}, \alpha) = \|\bar{\partial}^* \alpha\|^2 + \|\bar{\partial} \alpha\|^2.$$

□

If (X, g) is Kähler, then the Kähler identities imply that $\mathcal{H}^k(X, g)$ is also the set of forms annihilated by $\Delta_{\bar{\partial}}$ (or Δ_{∂} , since they are the same). Let

$$\mathcal{H}_{\bar{\partial}}^{p,q}(X, g) = \{\alpha \in \mathcal{A}^{p,q}(X) : \Delta_{\bar{\partial}} \alpha = 0\}.$$

Recall that for a compact oriented Riemannian manifold (X, g) , we have the following Hodge decomposition:

Theorem 9.4 (Hodge Decomposition for Riemannian manifolds). *If (X, g) is a compact Riemannian manifold, then there is a natural orthogonal decomposition*

$$\mathcal{A}^k(X) \cong \mathcal{H}^k(X) \oplus d\mathcal{A}^{k-1}(X) \oplus d^*\mathcal{A}^{k-1}(X).$$

Note that another way to write this is

$$\mathcal{A}^k(X) \cong \mathcal{H}^k(X) \oplus \Delta_d(\mathcal{A}^k(X))$$

since $\Delta_d\mathcal{A}^k(X) = dd^*\mathcal{A}^k(X) \oplus d^*d\mathcal{A}^k(X) = d\mathcal{A}^{k-1}(X) \oplus d^*\mathcal{A}^{k+1}(X)$ by the above.

If (X, g) is in addition Kähler (keeping the compactness hypothesis), then we have an inner product

$$(\alpha, \beta) = \int_X \alpha \wedge *\beta = \int_X g_{\mathbb{C}}(\alpha, \beta)dV.$$

This is obviously positive definite, so it induces a norm on $|\mathcal{A}^k(X)|$. (It might be worth warning that the forms are not complete with respect to this norm.) The analogous Hodge decomposition for complex Kähler manifolds is:

Theorem 9.5 (Hodge Decomposition for Kähler manifolds). *If (X, g) is a compact Kähler manifold, then there are natural orthogonal decompositions*

$$\begin{aligned} \mathcal{A}^{p,q}(X) &\cong \mathcal{H}^{p,q}(X) \oplus \partial\mathcal{A}^{p-1,q}(X) \oplus \partial^*\mathcal{A}^{p+1,q}(X) \\ &\cong \mathcal{H}^{p,q}(X) \oplus \bar{\partial}\mathcal{A}^{p,q-1}(X) \oplus \bar{\partial}^*\mathcal{A}^{p,q+1}(X). \end{aligned}$$

Moreover, the spaces $\mathcal{H}^{p,q}(X)$ are finite dimensional.

Again, this could be written as

$$\begin{aligned} \mathcal{A}^{p,q}(X) &\cong \mathcal{H}^{p,q}(X) \oplus \Delta_{\partial}(\mathcal{A}^{p,q}(X)) \\ &\cong \mathcal{H}^{p,q}(X) \oplus \Delta_{\bar{\partial}}(\mathcal{A}^{p,q}(X)) \end{aligned}$$

As mentioned just above we have $\Delta_d = 2\Delta_{\partial} = 2\Delta_{\bar{\partial}}$, so we may identify $\mathcal{H}^{p,q} = \mathcal{H}_{\partial}^{p,q} = \mathcal{H}_{\bar{\partial}}^{p,q}$.

Corollary 9.6. *The map $\mathcal{H}_{\bar{\partial}}^{p,q}(X) \rightarrow H_{\bar{\partial}}^{p,q}(X)$ is an isomorphism, i.e. each class in $H^{p,q}(X)$ is represented by a unique harmonic form.*

Proof. Lemma 9.3 shows that the map $\alpha \mapsto [\alpha]$ from $\mathcal{H}^{p,q} \rightarrow H_{\bar{\partial}}^{p,q}$ is well-defined.

Now let's show surjectivity. Let $\alpha \in \mathcal{A}^{p,q}(X)$ with $\bar{\partial}\alpha = 0$. By the Hodge Decomposition Theorem, we may write

$$\alpha = \beta_1 + \bar{\partial}\beta_2 + \bar{\partial}^*\beta_3 \text{ where } \beta_1 \text{ is harmonic.}$$

Applying $\bar{\partial}$ to both sides, we see that $0 = \bar{\partial}\bar{\partial}^*\beta_3 \implies \bar{\partial}^*\beta_3 = 0$ (by a similar argument as before, consider $(\bar{\partial}\bar{\partial}^*\beta_3, \beta_3) = 0$). Therefore, $[\alpha] = [\beta_1]$ in $H_{\bar{\partial}}^{p,q}(X)$.

Now for injectivity, suppose that $\alpha \in \mathcal{H}^{p,q}(X, g)$ and $[\alpha] = 0$ in $H_{\bar{\partial}}^{p,q}(X)$. Then $\alpha = \bar{\partial}\beta$ for some β , so $0 = \bar{\partial}^*\bar{\partial}\beta$ implies $\bar{\partial}\beta = 0$. \square

By the same argument in the Riemannian case, we have

Corollary 9.7. *The map $\mathcal{H}^k(X) \rightarrow H_{dR}^k(X)$ is an isomorphism, i.e. each class in $H_{dR}^k(X)$ is represented by a unique harmonic form.*

Some operations on $\mathcal{H}^{p,q}(X, g)$

1. Conjugation $\alpha \mapsto \bar{\alpha}$ sends harmonic forms to harmonic forms since $\bar{\partial}\bar{\alpha} = \bar{\partial}\alpha = 0$, hence induces an isomorphism

$$H_{\bar{\partial}}^{p,q}(X, g) \cong H_{\bar{\partial}}^{q,p}(X, g).$$

Note that we are really using the Kähler identities here, since a priori we would end up in H_{∂} .

2. The Hodge star $\alpha \mapsto *\alpha$ sends harmonic forms to harmonic forms since $\partial^* * \alpha = \pm * \partial\alpha = 0$, hence gives an isomorphism

$$\mathcal{H}_{\bar{\partial}}^{p,q}(X, g) \cong \mathcal{H}_{\bar{\partial}}^{n-q, n-p}(X, g).$$

3. Serre duality: consider the pairing

$$\mathcal{H}_{\bar{\partial}}^{p,q}(X, g) \times \mathcal{H}_{\bar{\partial}}^{n-p, n-q}(X, g) \rightarrow \mathbb{C}$$

where $(\alpha, \beta) \mapsto \int_X \alpha \wedge \beta$. If $\alpha \neq 0$, then $(\alpha, *\bar{\alpha})$ pairs to $\int \alpha \wedge *\bar{\alpha} > 0$, giving an isomorphism

$$\mathcal{H}_{\bar{\partial}}^{p,q}(X, g) \cong \mathcal{H}_{\bar{\partial}}^{n-p, n-q}(X, g)^*.$$

Since the harmonic forms are canonically identified with Dolbeault cohomology (by above), we get a similar pairing on the Dolbeault cohomology.

4. There is a Lefschetz operator

$$L : \mathcal{A}^{p,q}(X) \rightarrow \mathcal{A}^{p+1, q+1}(X)$$

sending $L(\alpha) = \alpha \wedge \omega$. So as $[L, \Delta_{\bar{\partial}}] = 0$ we get an induced map

$$L : \mathcal{H}_{\bar{\partial}}^{p,q}(X, g) \rightarrow \mathcal{H}_{\bar{\partial}}^{p+1, q+1}(X, g).$$

Define $h^{p,q} = \dim H_{\bar{\partial}}^{p,q}(X)$ on any complex manifold X . These are certainly finite if X is compact Kähler (since the harmonic forms are finite dimensional). On a compact Kähler X , the *Hodge diamond* is the array

$$\begin{array}{ccccccc}
 & & & & h^{0,0} & & \\
 & & & & & & \\
 & & & & h^{0,1} & & h^{1,0} \\
 & & & & & & \\
 & & & h^{0,2} & & h^{1,1} & & h^{2,0} \\
 & & & & & & \\
 & h^{0,3} & & h^{1,2} & & h^{2,1} & & h^{3,0} \\
 & & & \vdots & & \vdots & & \\
 & & & h^{n-1,n} & & h^{n,n-1} & & \\
 & & & & & & & \\
 & & & & & & & h^{n,n}
 \end{array}$$

Note that the rows are symmetric by conjugation, the columns are symmetric by the Hodge star, and their composition is the symmetry of Serre duality.

Theorem 9.8. *Let (X, g) be compact Kähler. Then there exists a decomposition*

$$H^k(X, \mathbb{C}) \cong \bigoplus_{p+q=k} H_{\bar{\partial}}^{p,q}(X)$$

independent of the chosen Kähler structure.

This is in accordance with what we got earlier, but the point is that this doesn't depend on the metric.

Proof. The decomposition is induced by the Hodge decomposition

$$H_{dR}^k(X, \mathbb{C}) \cong \mathcal{H}_{\bar{\partial}}^k(X, \mathbb{C}) \cong \bigoplus_{p+q=k} \mathcal{H}_{\bar{\partial}}^{p,q}(X, g) \cong \bigoplus_{p+q=k} H_{\bar{\partial}}^{p,q}(X).$$

We must show that the composition is independent of the choice of g . Since we know that the composition is actually an isomorphism, it suffices to show that if g_1, g_2 are two Kähler metrics, and $\alpha_1 \in \mathcal{H}^{p,q}(X, g_1)$, $\alpha_2 \in \mathcal{H}^{p,q}(X, g_2)$ such that $[\alpha_1] = [\alpha_2] \in H_{\bar{\partial}}^{p,q}(X)$, then $[\alpha_1] = [\alpha_2]$ in $H_{dR}^k(X, \mathbb{C})$.

To see this, suppose $[\alpha_1] = [\alpha_2]$ in $H_{\bar{\partial}}^{p,q}(X)$, so $\alpha_1 = \alpha_2 + \bar{\partial}\gamma$ for some γ . So $d(\bar{\partial}\gamma) = d(\alpha_1 - \alpha_2) = 0$ as $\Delta_d(\alpha_1 - \alpha_2) = 0$. Moreover, $\bar{\partial}\gamma$ is orthogonal to $\mathcal{H}^k(X, g)$ by the (Kähler) Hodge decomposition theorem, so $\bar{\partial}\gamma \in d(\mathcal{A}_{\mathbb{C}}^{k-1}(X))$ by the (Riemannian) Hodge decomposition theorem. That implies $[\alpha_1] - [\alpha_2]$ in $H_{dR}^k(X, \mathbb{C})$. \square

Chapter 10

Lefschetz Theorems

10.1 Lefschetz, I

Throughout this chapter, X is compact Kähler. The basic question is: how much of H^2 can we see from line bundles on X ? To make this more precise, recall that from the fundamental exact sequence we had

$$\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}).$$

Since $\mathbb{Z} \subset \mathbb{C}$, we have $H^2(X, \mathbb{Z}) \subset H^2(X, \mathbb{C})$. Last time, we proved that if X is compact Kähler then we have a Hodge decomposition

$$H^2(X, \mathbb{C}) \cong H_{\bar{\partial}}^{2,0}(X) \oplus H_{\bar{\partial}}^{1,1}(X) \oplus H_{\bar{\partial}}^{0,2}(X).$$

Definition 10.1. Let $H^{1,1}(X, \mathbb{Z}) = H^2(X, \mathbb{Z}) \cap H_{\bar{\partial}}^{1,1}(X)$.

This is the image of the map $H^2(X, \mathbb{Z}) \hookrightarrow H^2(X, \mathbb{C})$ in $H_{\bar{\partial}}^{1,1}(X)$. The Lefschetz theorem on $(1,1)$ classes identifies the image of $c_1 : \mathrm{Pic}(X) \rightarrow H^2(X, \mathbb{Z})$ with $H^{1,1}(X, \mathbb{Z})$.

The inclusion $\mathbb{C} \subset \mathcal{O}_X$ induces

$$f_1 : H^2(X, \mathbb{C}) \rightarrow H^2(X, \mathcal{O}_X) \underbrace{\cong}_{\text{Dolbeault}} H_{\bar{\partial}}^{0,2}(X).$$

Here and throughout, we are identifying $H^*(X, \mathbb{C})$ with $H_{dR}^*(X, \mathbb{C})$. We also have a projection $f_2 : H^2(X, \mathbb{C}) \rightarrow H_{\bar{\partial}}^{0,2}(X)$ by Hodge decomposition.

Lemma 10.2. *With the notation above, $f_1 = f_2$.*

Proof. Recall the complexes of sheaves we used to prove the Dolbeault theorem:

$$\begin{array}{ccccccc} \mathbb{C} & \longrightarrow & \mathcal{A}^0 & \longrightarrow & \mathcal{A}^1 & \longrightarrow & \mathcal{A}^2 \\ \downarrow & & \downarrow & & \downarrow \pi_1 & & \downarrow \pi_2 \\ \mathcal{O}_X & \longrightarrow & \mathcal{A}^0 & \longrightarrow & \mathcal{A}^{0,1} & \longrightarrow & \mathcal{A}^{0,2} \end{array}$$

By chasing through the Dolbeault isomorphism, the map $f_1 : H^2(X, \mathbb{C}) \rightarrow H_{\bar{\partial}}^{0,2}(X)$ can be given as follows. Let $[\alpha] \in H^2(X, \mathbb{C})$ be represented by some $\alpha \in \mathcal{A}_{\mathbb{C}}^2(X)$, then $f_1([\alpha]) = [\pi_2(\alpha)]$ in $H_{\bar{\partial}}^{0,2}(X)$. ♠♠♠ Tony: [The rest of the argument is a pedantic way of saying that the result is automatic from construction of the Hodge Decomposition.]

We have to show that $f_2[\alpha] = [\pi_2(\alpha)]$. To see this, we can take α to be the (unique) harmonic representative with respect to some Kähler metric g . Then $\pi_2(\alpha)$ is the (unique) harmonic representative of $[\pi_2(\alpha)]$ in $H_{\pi_1}^{0,2}(X)$. On the other hand, $\pi_2(\alpha)$ also represents $f_2([\alpha])$ by our construction of the Hodge decomposition in Theorem 9.8, so we conclude that $p([\alpha]) = [\pi_2(\alpha)]$. \square

Theorem 10.3. *The image of the first Chern class $H^1(X, \mathcal{O}^*) \hookrightarrow H^2(X, \mathbb{C})$ is $H^{(1,1)}(X, \mathbb{Z})$.*

Proof. Let $\alpha \in H^2(X, \mathbb{Z}) \subset H^2(X, \mathbb{C})$. Say $\alpha = \alpha^{2,0} + \alpha^{1,1} + \alpha^{0,2}$ where $\alpha^{p,q} \in H_{\bar{\partial}}^{p,q}(X)$. As $\alpha \in H^2(X, \mathbb{Z})$ is real, we have $\alpha^{2,0} = \overline{\alpha^{0,2}}$. So $\alpha \in H_{\bar{\partial}}^{1,1} \iff \alpha^{2,0} = 0$.

By the preceding lemma, $[\alpha^{2,0}]$ is precisely the class obtained by from the composition

$$\begin{array}{ccccc}
 \text{Pic}(X) & \longrightarrow & H^2(X, \mathbb{Z}) & \xrightarrow{f} & H^2(X, \mathcal{O}) \cong H_{\bar{\partial}}^{0,2}(X) \\
 & & \searrow & & \nearrow \beta \mapsto \beta^{0,2} \\
 & & & H^2(X, \mathbb{C}) &
 \end{array}$$

So $\alpha^{0,2} = 0 \iff \alpha \in \ker f$, which is what the exactness tells us. \square

Let X be a complex manifold. The Jacobian of X is $\text{Pic}^0(X) = \ker c_1 = \{L \in \text{Pic}(X) : c_1(L) = 0\}$. So understanding $\text{Pic}(X)$ comes down to understanding the discrete group $H^{1,1}(X, \mathbb{Z})$ and $\text{Pic}^0(X)$. Note that $\text{Pic}(X)^0(X) = H^1(X, \mathcal{O})/H^1(X, \mathbb{Z})$ from the long exact sequence. If X is compact Kähler then one can show that $H^1(X, \mathbb{Z})$ is a full-rank lattice in $H^1(X, \mathcal{O})$. So it is not hard to then get that $\text{Pic}^0(X)$ is a complex manifold (in fact, a complex torus).

Remark 10.4. It is a fact that given a smooth hypersurface $D \subset X$ one can associate a line bundle $\mathcal{O}(D)$ with a unique $s \in H^0(\mathcal{O}(D))$ such that $D = \{s = 0\}$. If X is projective then one can show that $H^{1,1}(X, \mathbb{Z})$ is spanned over \mathbb{Q} by $c_1(\mathcal{O}_D)$ for such divisors.

Definition 10.5. Let $Z \subset X$ be a complex submanifold of complex dimension $n - p$. We define $[Z] \in H^{p,p}(X, \mathbb{C})$ by requiring

$$\int_X \alpha \wedge [Z] = \int_Z \alpha|_Z \quad \forall \alpha \in H^{2n-2p}(X, \mathbb{Q}).$$

Naïvely, one could hope that all classes in $H^{p,p}(X, \mathbb{C}) \cap H^*(X, \mathbb{Z})$ are linear combinations of the classes corresponding to submanifolds $[Z]$. We saw above that this is actually true for $H^{1,1}$, but it cannot be true in general. The Hodge conjecture (a Millenium Prize Problem) predicts that this is so for *rational* classes, i.e. any class in $H^{p,p}(X) \cap H^*(X, \mathbb{Z})$ is a \mathbb{Q} -linear combination of classes of the form $[Z]$ for $Z \subset X$.

10.2 Lefschetz, II

Recall that if X is Kähler and ω is the Kähler form, we considered the operator $L(\alpha) = \alpha \wedge \omega$. By the Kähler identities we have $[L, \Delta] = 0$ so this induces

$$L : \mathcal{H}^{p,q}(X, \omega) \rightarrow \mathcal{H}^{p+1,q+1}(X, \omega)$$

hence also

$$L : H_{\bar{\partial}}^{p,q}(X) \rightarrow H_{\bar{\partial}}^{p+1,q+1}(X)$$

and

$$L : H^k(X, \mathbb{C}) \rightarrow H^{k+2}(X, \mathbb{C}).$$

Theorem 10.6 (Hard Lefschetz). *Let X be compact Kähler of complex dimension n . Then*

$$L^k : H^{n-k}(X, \mathbb{C}) \rightarrow H^{n+k}(X, \mathbb{C})$$

is an isomorphism.

This result (and improvements) follow from the general theory about finite-dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$ and the Kähler identities. For completeness, we give a discussion of representations of \mathfrak{sl}_2 . As a vector space, this consists of traceless 2×2 complex matrices. A standard basis is

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and these satisfy the commutation relations

$$[X, Y] = H, \quad [H, X] = 2X, \quad [H, Y] = -2Y.$$

A *representation* of \mathfrak{sl}_2 is a Lie algebra homomorphism $\mathfrak{sl}_2 \rightarrow \mathfrak{gl}(V)$. We say that ρ is *irreducible* if it has no proper subrepresentations. It is a fact that any finite dimensional representation of \mathfrak{sl}_2 is semisimple, i.e. a direct sum of irreducibles.

In any representation, H has eigenspaces V_λ called *weight spaces*. Observe that if $v \in V_\lambda$, then $Xv \in V_{\lambda+2}$ and $Yv \in V_{\lambda-2}$. We say that $v \in V$ is a *highest weight vector* if it's an eigenvector for H and $Xv = 0$.

Lemma 10.7. *If V is an irreducible finite dimensional representation of \mathfrak{sl}_2 , then $V \cong V_n \oplus V_{n-2} \oplus \dots \oplus V_{-n}$ where $XV_\lambda = V_{\lambda+2}$ and $YV_\lambda = V_{\lambda-2}$.*

Proof. Let $v \in V_\lambda$ be a highest weight vector with weight λ (it is true but not trivial that such a vector exists). We claim that $\{v, Yv, Y^2v, \dots\}$ spans V . The elements are linearly independent, since they are eigenvectors with distinct eigenvalues: we can inductively show that $H(Y^k v) = (\lambda - 2k)v$. You can check that the subspace V' generated by this set is subrepresentation: it is obviously preserved by Y, H and the result for X follows by using the commutation relations.

If V is finite dimensional, then for some n we have $Y^n v \neq 0$ but $Y^{n+1} v = 0$. We calculate

$$\begin{aligned} Xv &= 0 \\ XYv &= YXv + Hv = \lambda v \\ XY^2v &= (\lambda + \lambda - 2)Yv \\ &\vdots \\ XY^n v &= YXY^{n-1}v + HY^{n-1}v = (n\lambda - n^2 + n)Y^{n-1}v. \end{aligned}$$

Therefore, $\lambda = n$. □

In particular, a byproduct of the proof is that $X^m : V_{-m} \rightarrow V_m$ is an isomorphism. If V is not an irreducible representation, then it is a direct sum of irreducibles of this form, so X^m is still an isomorphism.

Now, let's return to studying a compact Kähler manifold (X, ω) . The Kähler identities show that $H = h, X = L, Y = \Lambda$ give a representation on $\mathcal{H}^*(X, g)$. The eigenspace of h with eigenvalue $n - p$ is $\mathcal{H}^p(X, g) \cong H^p(X, \mathbb{C})$. So the preceding discussion proves the Hard Lefschetz theorem.

Chapter 11

Hermitian Vector Bundles

11.1 Hermitian metrics

Let $E \rightarrow X$ be a complex vector bundle over a smooth manifold X . We define

$$\mathcal{A}^k(E)(U) = \mathcal{A}^k(U) \otimes C^\infty(E)(U) \text{ (the sheaf of smooth sections of } E\text{).}$$

If X is a complex manifold, then we get a splitting

$$\mathcal{A}^k(E) = \bigoplus_{p+q=n} \mathcal{A}^{p,q}(E)$$

just from the usual splitting on \mathcal{A}^k .

Definition 11.1. A *Hermitian metric* on E is a smoothly varying Hermitian metric on the fibers E_x , for $x \in X$.

If e_1, \dots, e_r is a local frame for E , then $[h_{ij} = h(e_i, e_j)]$ is a Hermitian matrix varying smoothly with x . We call (E, h) a *Hermitian vector bundle*. It is convenient to think of h as giving an isomorphism $h : E \simeq E^*$. (Note that we have to choose the conjugate complex structure on E^* , or allow this to be conjugate linear (so not a morphism of complex vector bundles).)

Exercise 11.2. If E, F are given Hermitian metrics, then the bundles $E \oplus F, E \otimes F, E^*, \bigwedge^i E$ etc have naturally induced Hermitian metrics.

Now assume that E is a *holomorphic* vector bundle.

Proposition 11.3. *There is a natural map \mathbb{C} -linear map $\bar{\partial} = \bar{\partial}_E : \mathcal{A}^{p,q}(E) \rightarrow \mathcal{A}^{p,q+1}(E)$ satisfying*

$$\bar{\partial}_E(\alpha \otimes s) = \bar{\partial}\alpha \otimes s + \alpha \otimes \bar{\partial}_E s \text{ for all } \alpha \in \mathcal{A}^{p,q}(U), s \in C^\infty(E)(U).$$

Proof. In a local holomorphic frame (e_1, \dots, e_n) , we define

$$\bar{\partial}_E(\alpha \otimes e_i) = \bar{\partial}\alpha \otimes e_i.$$

To see that this is well-defined, let $e'_i = \psi^{ij}e_j$ be a new choice of holomorphic frame (we are using Einstein summation convention), so that the ψ_{ij} are holomorphic functions. Then

$$\bar{\partial}_E(\alpha \otimes \psi^{ij}e_j) = \bar{\partial}_E(\psi^{ij}\alpha \otimes e_j) = \psi^{ij}\bar{\partial}\alpha \otimes e_j = \bar{\partial}\alpha \otimes \psi^{ij}e_j.$$

□

11.2 The Chern connection

Definition 11.4. A *connection* on a complex vector bundle is a sheaf map $D : \mathcal{A}^0(E) \rightarrow \mathcal{A}^1(E)$ such that

$$D(fs) = df \otimes s + fDs \text{ for } f \in C^\infty(U), s \in \mathcal{A}^0(E)(U).$$

If e_1, \dots, e_n is a local frame for E then this gives a connection matrix

$$De_i = \sum_j \Theta_{ij}e_j$$

where $\Theta = (\Theta_{ij})$ is a matrix of 1-forms.

Definition 11.5. Let E be a holomorphic vector bundle and D be a connection on E . Then we can write $D = D' + D''$, where

$$D' : \mathcal{A}^0(E) \rightarrow \mathcal{A}^{1,0}(E) \quad D'' : \mathcal{A}^0(E) \rightarrow \mathcal{A}^{0,1}(E).$$

We say that D is *compatible* with the holomorphic structure on E if $D'' = \bar{\partial}_E : \mathcal{A}^0(E) \rightarrow \mathcal{A}^{0,1}(E)$.

Here is an alternate characterization which is useful for local analysis.

Proposition 11.6. *Let E be a holomorphic vector bundle and D a connection on E . Then D is compatible with the holomorphic structure on E if and only if in every local holomorphic frame (s_1, \dots, s_n) , the connection coefficients $D = (\Theta_{ij})$ are all $(1, 0)$ forms.*

Proof. The $(0, 1)$ part of $Ds_j = \sum_{ij} \Theta_{ij}s_j$ vanishes since s_j is holomorphic. Since vanishing can be checked locally, this shows that if D is compatible with the holomorphic structure then the $(0, 1)$ part of D must vanish in every holomorphic frame.

Conversely, if that is the case then for any local holomorphic frame (s_1, \dots, s_n) over U and $\alpha_j \in C^\infty(U)$, we have

$$D(\alpha_j s_j) = d\alpha_j \otimes s_j + \alpha_j Ds_j.$$

Projecting to the $(0, 1)$ -part shows that

$$D''(\alpha_j s_j) = \bar{\partial}\alpha_j \otimes s_j,$$

which is the same as the local description of $\bar{\partial}_E$. \square

Definition 11.7. If (E, h) is a Hermitian vector bundle, a connection D on E is *compatible* with h if

$$d(\alpha, \beta)_h = (D\alpha, \beta)_h + (\alpha, D\beta)_h \quad \text{for all } \alpha, \beta \in \mathcal{A}^0(E).$$

Again, there is a local characterization of this condition.

Proposition 11.8. *Let (E, h) be a Hermitian vector bundle. A connection D on E is compatible with h if and only if for every unitary frame (e_1, \dots, e_n) , the connection coefficients $D = (\Theta_{ij})$ form a skew-Hermitian matrix, i.e. $\Theta_{ij} = -\overline{\Theta_{ji}}$.*

Proof. If (e_1, \dots, e_n) is a unitary frame, then $(e_i, e_j)_h = \delta_{ij}$. Therefore,

$$0 = d(e_i, e_j) = (De_i, e_j) + (e_i, De_j) = \left(\sum_k \Theta_{ik} e_k, e_j\right) + \left(e_i, \sum_k \Theta_{jk} e_k\right) = \Theta_{ij} + \overline{\Theta_{ji}}.$$

Conversely, suppose that (Θ_{ij}) is skew-Hermitian in any unitary frame. Since the equality of forms can be checked locally, it suffices to prove that for any $\alpha, \beta \in \mathcal{A}^0(U)$ we have

$$d(\alpha, \beta) = (D\alpha, \beta) + (\alpha, D\beta).$$

By the above computation, this is satisfied when α, β are among $\{e_1, \dots, e_n\}$. Both sides are linear in the arguments, so it suffices to show that they have the same $C^\infty(U)$ dependence, i.e. if the above identity holds then

$$d(f\alpha, \beta) = (D(f\alpha), \beta)_h + (f\alpha, D\beta)$$

The left hand side expands out as

$$\begin{aligned} d(f\alpha, \beta) &= df \otimes (\alpha, \beta) + fd(\alpha, \beta) \\ &= df \otimes (\alpha, \beta) + f(D\alpha, \beta) + f(\alpha, D\beta). \end{aligned}$$

The right hand side expands as

$$\begin{aligned} (D(f\alpha), \beta) + (f\alpha, D\beta)_h &= (df \otimes \alpha, \beta) + (fD\alpha, \beta) + (f\alpha, D\beta) \\ &= df \otimes (\alpha, \beta) + f(D\alpha, \beta) + f(\alpha, D\beta). \end{aligned}$$

□

There are many connections, but a unique one which is compatible with a holomorphic and Hermitian structure (which is the Levi-Civita connection in the real case). This is called the *metric connection* or *Chern connection*.

Proposition 11.9. *If (E, h) is a holomorphic, Hermitian vector bundle then there exists a unique connection D compatible with the holomorphic structure and with h .*

Proof. Let's start with uniqueness. Let e_1, \dots, e_n be a local frame and $h_{ij} = h(e_i, e_j)$. Write

$$De_i = \sum \Theta_{ij} e_j.$$

Then

$$\begin{aligned} dh_{ij} &= d(e_i, e_j)_h \\ &= \left(\sum_k \Theta_{ik} e_k, e_j \right) + \left(e_i, \sum_k \Theta_{jk} e_k \right) \\ &= \sum_k \Theta_{ik} h_{k,j} + \sum_k \bar{\Theta}_{jk} h_{i,k} \end{aligned}$$

As D is compatible with the complex structure, the Θ_{ij} are $(1, 0)$ forms. So

$$\partial h_{ij} = \sum_k \Theta_{ik} h_{kj} \quad \text{and} \quad \bar{\partial} h_{ij} = \sum_k \bar{\Theta}_{jk} h_{ik}.$$

Therefore, $\Theta = (\partial h)h^{-1}$. This gives uniqueness. On the other hand, this formula satisfies the conditions necessary, which also gives existence. □

A connection D extends to

$$D : \mathcal{A}^p(E) \rightarrow \mathcal{A}^{p+1}(E)$$

by

$$D(\alpha \otimes s) = d\alpha \otimes s + (-1)^p \alpha \wedge Ds \quad \alpha \in \mathcal{A}^p(U), s \in \mathcal{O}(E).$$

11.3 Curvature

Definition 11.10. The *curvature* of D is the map

$$F_D : D^2 : \mathcal{A}^0(E) \rightarrow \mathcal{A}^2(E).$$

Lemma 11.11. F_D is linear over C^∞ .

Proof. We have

$$\begin{aligned} F_D(fs) &= D^2(fs) = D(df \otimes s + fDs) \\ &= d^2f \otimes s - df \otimes Ds + df \otimes Ds + f \otimes D^2s. \end{aligned}$$

The tricky sign swap in the second term comes from the definition of how we extend D . Since $d^2 = 0$, this is exactly what we want. \square

Therefore, this map F_D looks locally like multiplication by a matrix of two-forms:

Corollary 11.12. F_D is induced by an element $F_D \in \mathcal{A}^2(\text{End}(E))$.

Given a local frame e_1, \dots, e_r , the *connection matrix* is

$$\Theta e_i = \sum_j \Theta_{ij} e_j$$

where the Θ_{ij} are one-forms. Given any local section $s = \sum s_i e_i$,

$$Ds = \sum ds_i \otimes e_i + \sum s_i \Theta_{ij} e_j.$$

We can write this as $D = d + \Theta$.

In this notation, the curvature is

$$\begin{aligned} D^2s &= (d + \Theta)(d + \Theta)s \\ &= (d + \Theta)(ds + \Theta s) \\ &= d^2s + (d\Theta)s - \Theta(ds) + \Theta ds + \Theta\Theta s \\ &= (d\Theta + \Theta \wedge \Theta)s. \end{aligned}$$

Lemma 11.13. 1. If (E, h) is hermitian and D is compatible with h , then $h(F_D s_i, s_j) + h(s_i, F_D s_j) = 0$.

2. If E is holomorphic and D is compatible with the holomorphic structure, then F_D has no $(0, 2)$ -part, i.e. $F_D \in \mathcal{A}^{2,0}(\text{End}(E)) \oplus \mathcal{A}^{1,1}(\text{End}(E, E))$.

3. If D is the metric connection on holomorphic (E, h) then F_D is a skew-hermitian form in $\mathcal{A}^{1,1}(\text{End}(E))$ (i.e. the identity in 1 holds).

Proof. 1. Since the statement is local, we can assume that $E = X \times \mathbb{C}^r$ and pick a *unitary* local frame e_1, \dots, e_r . Therefore, we can write $D = d + \Theta$ where $\Theta^* = -\Theta$ as D is compatible with h .

We have $(d\Theta + \Theta \wedge \Theta)^* = d\Theta^* - \Theta^* \wedge \Theta^*$ (the $-$ sign comes from flipping the factors in the wedge), and this is

$$\begin{aligned} &= -d\Theta - (-\Theta \wedge -\Theta) \\ &= -d\Theta - \Theta \wedge \Theta \\ &= -F_D \end{aligned}$$

Here's another way to see this. We have $dd(s_i, s_j)_h = d(Ds_i, s_j)_h + d(s_i, Ds_j)_h = 0$. On the other hand,

$$dh(Ds_i, s_j) = h(DDs_i, s_j) - h(Ds_i, Ds_j)$$

(the sign coming again from the tricky sign swap) and

$$dh(s_i, Ds_j) = h(Ds_i, Ds_j) + h(s_i, DDs_j).$$

2. We have $D : \mathcal{A}^k(E) \rightarrow \mathcal{A}^{k+1}(E)$ which restricts to $D : \mathcal{A}^{p,q}(E) \rightarrow \mathcal{A}^{p+1,q}(E) \oplus \mathcal{A}^{p,q+1}(E)$. In the notation from before, $D = D' \oplus D''$ where $D'' = \bar{\partial}_E$ as D is compatible with the holomorphic structure, so

$$\begin{aligned} D^2 &= (D' + D'')^2 \\ &= (D')^2 + D'\bar{\partial}_E + \bar{\partial}_E + \bar{\partial}_E^2 \end{aligned}$$

so we see that the $(0, 2)$ part vanishes.

Alternatively, if $D = d + \Theta$ and D is compatible with the holomorphic structure, then Θ has type $(1, 0)$. So

$$\begin{aligned} F_D &= d\Theta + \Theta \wedge \Theta \\ &= \underbrace{\partial\Theta}_{2,0} + \underbrace{\bar{\partial}\Theta}_{1,1} + \underbrace{\Theta \wedge \Theta}_{2,0} \end{aligned}$$

3. Follows from the first two parts. □

11.4 Kodaira's Theorems

We write F_n for the curvature of the Chern connection on a holomorphic, Hermitian vector bundle.

Example 11.14. If L is a holomorphic, hermitian *line* bundle with metric h , we have

$$F_D \in \mathcal{A}^{1,1}(\text{End}(E)) \text{ skew-hermitian}$$

hence is a real $(1, 1)$ -form. If z is a local unitary frame for L , then $h(z) = \|z\|^2$. In this local frame, $\Theta = \partial \log h$ and $F = \partial \bar{\partial} \log h$.

In particular, if $X = \mathbb{P}^n$ and $L = \mathcal{O}_{\mathbb{P}^n}(1)$, we have a natural hermitian metric on $L^* = \mathcal{O}_{\mathbb{P}^n}(-1)$ since the total space minus zero section is $\mathbb{C}^{n+1} - 0$, and we can pull back the standard metric on \mathbb{C}^{n+1} . This gives a Fubini-Study Hermitian metric on L^* , hence on L . By definition, the Fubini-Study Kähler metric on \mathbb{P}^n is

$$\omega_{FS} = \frac{i}{2\pi} \partial \bar{\partial} \log(1 + \|w\|^2) \text{ on } \{[1 : w]\}.$$

We have just seen that this coincides with $\frac{i}{2\pi} F|_{h_{FS}}$.

We have just seen an example of a metric whose curvature is a Kähler form. This is important enough to make it into a definition.

Definition 11.15. We say a holomorphic line bundle L is *positive* if there exists a Hermitian metric h such that $\frac{i}{2\pi} F_h$ is a Kähler form, i.e.

$$\frac{iF_h}{2\pi} = \sum h_{ij} dz_i d\bar{z}_j$$

where h_{ij} are positive-definite.

If $f : X \rightarrow Y$ is holomorphic, and E is holomorphic and hermitian on Y , then $F^*E \rightarrow X$ is holomorphic and f^*h is Hermitian on F^*E , and $F_{f^*h} = f^*F_h$.

So if $f : X \rightarrow \mathbb{P}^n$ is holomorphic, then $L := f^*\mathcal{O}_{\mathbb{P}^n}(1)$ for some holomorphic L , then f^*h_{FS} is a positive Hermitian metric on L so L is positive. In particular, we can take k th roots to deduce that any ample divisor is positive.

Theorem 11.16 (Kodaira Embedding). *L is ample if and only if L is positive.*

Theorem 11.17 (Kodaira Vanishing). *Let L be a positive line bundle on a compact Kähler manifold of complex dimension n . Then*

$$H^q(X, \Omega^p \otimes L) = 0 \text{ if } p + q > n.$$

Chapter 12

Example Sheet 1

1. Let U be an open subset of $\mathbb{C}^m = \mathbb{R}^{2m}$ and $f : U \rightarrow \mathbb{C}^n = \mathbb{R}^{2n}$ be a smooth function. The complex Jacobian of f is defined to be the matrix

$$J_{\mathbb{C}}(f) = \left(\frac{\partial f_i}{\partial z_j} \right)$$

where $z_j = x_j + iy_j$ are the standard coordinates on U . With the standard coordinates $w_j = u_j + iv_j$ on \mathbb{C}^n , compute the matrix of $df : T\mathbb{R}^m \rightarrow T\mathbb{R}^{2n}$. Find also the matrix of the induced map

$$df_{\mathbb{C}} : T\mathbb{R}^{2m} \otimes \mathbb{C} \rightarrow T\mathbb{R}^{2n} \otimes \mathbb{C}$$

in terms of the frames $\left\{ \frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_j} \right\}$ and $\left\{ \frac{\partial}{\partial w_j}, \frac{\partial}{\partial \bar{w}_j} \right\}$ and related this to $J_{\mathbb{C}}$ when f is holomorphic. Conclude that $\det df = |\det J_{\mathbb{C}}|^2$. Use this to prove that a complex structure induces a natural orientation on the underlying manifold.

As a real function, f is $(u_1, v_1, u_2, v_2, \dots, u_n, v_n)$. Therefore,

$$df = \begin{pmatrix} \vdots & \vdots & & \\ \cdots & \frac{\partial u_j}{\partial x_k} & \frac{\partial u_j}{\partial y_k} & \cdots \\ \cdots & \frac{\partial v_j}{\partial x_k} & \frac{\partial v_j}{\partial y_k} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

In terms of the frames $\left\{ \frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_j} \right\}$ and $\left\{ \frac{\partial}{\partial w_j}, \frac{\partial}{\partial \bar{w}_j} \right\}$, $J_{\mathbb{C}}$ has matrix

$$df_{\mathbb{C}} = \begin{pmatrix} \vdots & \vdots & & \\ \cdots & \frac{\partial f_j}{\partial z_k} & \frac{\partial f_j}{\partial \bar{z}_k} & \cdots \\ \cdots & \frac{\partial f_j}{\partial z_k} & \frac{\partial f_j}{\partial \bar{z}_k} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

If f is holomorphic, then $\frac{\partial f_j}{\partial \bar{z}_k} = \frac{\partial \bar{f}_j}{\partial z_k} = 0$ and $\frac{\partial \bar{f}_j}{\partial \bar{z}_k} = \overline{\frac{\partial f_j}{\partial z_k}}$. In particular, if $n = m$ then

$$\det df = |\det J_{\mathbb{C}}|^2.$$

Here is another, more conceptual perspective on the isomorphism $df_{\mathbb{C}} \sim J_{\mathbb{C}} \oplus \overline{J_{\mathbb{C}}}$. Let $V \cong \mathbb{R}^{2m}$ be a real vector space. As we have seen, a complex structure on V is the same thing as a map $J : V \rightarrow V$ satisfying $J^2 = -1$. (Given such a map, we can define a linear action of \mathbb{C} on V by having i act as multiplication by J .) We denote the associated complex vector space by (V, J) . Complexifying, we have a map $J_{\mathbb{C}} : V_{\mathbb{C}} \rightarrow V_{\mathbb{C}}$ satisfying $J_{\mathbb{C}}^2 = -1$ and we denote by $V^{1,0}$ the $+i$ eigenspace.

Lemma 12.1. *There is a natural isomorphism $(V, J) \rightarrow V^{1,0}$.*

Proof. The map can be described explicitly as $v \mapsto \frac{v - iJv}{2}$. Then

$$(a + Jb)v \mapsto \frac{(a + Jb)v - iJ(a + Jb)v}{2} = (a + ib) \frac{v - iJv}{2}.$$

□

This describes $V_{\mathbb{C}} = V^{1,0} + V^{0,1}$ as $(V, J) \oplus \overline{(V, J)}$.

Lemma 12.2. *The conjugation map $z \otimes v \mapsto \bar{z} \otimes v : \mathbb{C} \otimes V \rightarrow \mathbb{C} \otimes V$ is an \mathbb{R} -linear involution of $V_{\mathbb{C}}$ interchanging $V^{1,0}$ and $V^{0,1}$.*

Proof. Clear from our explicit description of $V^{1,0}$ as $v - iJv$. □

Now, if $T : (V, J) \rightarrow (V', J')$ is a \mathbb{C} -linear map, then we get also get an \mathbb{R} -linear map $T_{\mathbb{R}} : V \rightarrow V'$. The complexification $T_{\mathbb{C}} : V_{\mathbb{C}} \rightarrow V'_{\mathbb{C}}$ is the direct sum of T and \overline{T} , by the previous two lemmas. In particular, if $(V', J') = (V, J)$ then we have $\det T_{\mathbb{C}} = |\det T|^2$.

Back to the setting of complex manifolds, TU is endowed with a complex structure by the complex structure on \mathbb{C}^m . If f is holomorphic, then df is \mathbb{C} -linear (see next question), and $J_{\mathbb{C}}$ is just the map induced on $T^{1,0}U$ since $\{\frac{\partial}{\partial z_j}\}$ is a frame for $T^{1,0}$. So the result follows from our preceding discussion.

In particular, we see that the Jacobians of the transition functions for any frame $\{\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_j}\}$ respecting the natural decomposition of $TX \otimes \mathbb{C}$ have positive determinant, implying that a complex manifold is orientable.

2. Included as Lemma 1.5 and Theorem 1.6.

3. Show that a closed complex submanifold X of a complex manifold Y is naturally a complex manifold and the inclusion $\iota : X \rightarrow Y$ is holomorphic.

By hypothesis, there exist open sets $\{U_\alpha\}$ in Y covering X , and an atlas $\{\varphi_\alpha : U_\alpha \cong V_\alpha \subset \mathbb{C}^n\}$ such that $\varphi_\alpha|_{U_\alpha \cap X} \cong V_\alpha \cap \{z_n = 0\}$, the latter being an open subset of \mathbb{C}^{n-1} . The transition functions are holomorphic, since a holomorphic map to V landing in V' is the same as a holomorphic map to V' .

4. Let $f : \mathbb{C}^n \rightarrow \mathbb{C}$ be holomorphic and $0 \in \mathbb{C}$ be a regular value (i.e. the complex Jacobian $J_{\mathbb{C}}$ is surjective at any point in $f^{-1}(0)$). Show that $Z := f^{-1}(0)$ is a complex submanifold of \mathbb{C}^n .

(ii) Let F be a homogeneous polynomial in $n+1$ variables. Show that the set

$$Z = \{[z_0, \dots, z_n] \in \mathbb{P}^n : F(z_0, \dots, z_n) = 0\}$$

is well defined. By considering the map $f : \mathbb{C}^{n+1} - \{0\} \rightarrow \mathbb{C}$ induced by F , show that if 0 is a regular value of f then Z is a complex submanifold of \mathbb{P}^n .

(i) For any point $p \in Z$, the implicit function theorem 1.7 implies that an open neighborhood (in Z) of p can be described as $(z_1, \dots, z_{n-1}, \phi(z_1, \dots, z_{n-1}))$ where ϕ is holomorphic. The chart

$$(z_1, \dots, z_n) \mapsto (z_1, \dots, z_{n-1}, z_n - \phi(z_1, \dots, z_{n-1}))$$

sends a neighborhood of p to the subset $z_n = 0$ of an open subset of \mathbb{C}^n , exhibiting Z as a closed submanifold in this chart.

(ii) The set is well-defined by homogeneity. The previous part applied to $f : \mathbb{C}^{n+1} - \{0\}$ shows that if 0 is a regular value of f then $Z(f) \subset \mathbb{C}^{n+1}$ is a complex submanifold of \mathbb{C}^{n+1} , so the same is true after intersecting with the standard open (say) $\tilde{U}_i = \{z_i \neq 0\}$, and we can choose coordinates for $Z(f)$ that look like $(z_0, \dots, \underbrace{t}_i, \dots, z_n = 0)$.

Let U_i be the image of this open in \mathbb{P}^n . Then $Z(F)$ is described in U_i by $(z_0, \dots, 1, \dots, z_n = 0)$.

5. Prove that a closed submanifold $D \subset \mathbb{C}^n$ of complex dimension $n - 1$ is given by $D = Z(f)$ for some holomorphic $f : \mathbb{C}^n \rightarrow \mathbb{C}$.

By hypothesis, there is an open cover $\{U_\alpha\}$ of \mathbb{C}^n such that in each open set, D is cut out by a local equation $f_\alpha \in \mathcal{O}(U_\alpha)$. (This is still true if $D \cap U_\alpha = \emptyset$, since we can take f_α to be non-vanishing).

An important and subtle point is that we can choose f_α to not be divisible by the square of any non-unit in $\mathcal{O}(U_\alpha)$, possibly after refining our cover. (This is related to the fact that the logarithm cannot

be defined in an open neighborhood of the origin, or else we could take $f_\alpha^{1/n}$ for each n .) Otherwise, f_α would be divisible by arbitrarily high powers of non-units. Consider the germ of f_α in $p \in D$. Any holomorphic germ $g_p \in \mathcal{O}_p$ is a unit in a neighborhood of p if it doesn't vanish at p . If it does vanish at p , then its power series has no constant term, so g_p^n has terms all of order at least n .

Remark 12.3. More generally, the Auslander-Buchsbaum theorem guarantees that a regular local ring is a UFD, so we can apply this analysis to any smooth variety.

Now, on $U_\alpha \cap U_\beta$ the hypersurface D is cut out by squarefree equations f_α and f_β , so

$$g_{\alpha\beta} := \frac{f_\alpha}{f_\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta).$$

We can then view $(g_{\alpha\beta}) \in C^1(\mathcal{U}_\alpha, \mathcal{O}^*)$. This is a cocycle, since

$$g_{\alpha\beta} g_{\beta\delta} g_{\delta\alpha} = \frac{f_\alpha f_\beta f_\delta}{f_\beta f_\delta f_\alpha} = 1.$$

Therefore, it descends to a cohomology class $\check{g} \in \check{H}^1(\mathbb{C}^n, \mathcal{O}^*)$. If this is the 0 class, then \check{g} is a coboundary, so (after possibly refining the cover again) there is an open cover where

$$g_{\alpha\beta} = \frac{h_\alpha}{h_\beta} \quad h_\alpha \in \mathcal{O}^*(U_\alpha) \forall \alpha.$$

Now we claim that $\frac{f_\alpha}{h_\alpha}$ defines a global section cutting out D (since h_α is nonvanishing, it cuts out the same local subset as f_α , so the only thing to check is that it is well-defined). Indeed,

$$\frac{f_\alpha}{h_\alpha} = \frac{g_{\alpha\beta} f_\beta}{g_{\alpha\beta} h_\beta} = \frac{f_\beta}{h_\beta}.$$

Finally, we show that the class \check{g} is 0 by showing that the entire cohomology group $\check{H}^1(\mathbb{C}^n, \mathcal{O}^*)$ vanishes. By the exponential short exact sequence of sheaves, we have a long exact sequence

$$\dots \rightarrow \check{H}^1(\mathbb{C}^n, \mathcal{O}) \rightarrow \check{H}^1(\mathbb{C}^n, \mathcal{O}^*) \rightarrow \check{H}^2(\mathbb{C}^n, \mathbb{Z}) \rightarrow 0$$

By the Dolbeault theorem, $\check{H}^1(\mathbb{C}^n, \mathcal{O}) \cong H^{0,1}(\mathbb{C}^n) = 0$ by the $\bar{\partial}$ -Poincaré Lemma. Also, $\check{H}^2(\mathbb{C}^n, \mathbb{Z}) = H^2(\mathbb{C}^n, \mathbb{Z}) = 0$. Therefore, the middle group vanishes as well.

6. Let $f : X \rightarrow Y$ be a holomorphic map between complex manifolds. Prove that if α is a (p, q) -form on Y then $f^*\alpha$ is a (p, q) form on X .

Using this, show that f induces a homomorphism

$$f^* : H_{\bar{\partial}}^{p,q}(Y) \rightarrow H_{\bar{\partial}}^{p,q}(X)$$

given by

$$f^*[\alpha] = [f^*\alpha].$$

There are two ways to see this: using the invariant definitions, or using local coordinates. From the invariant perspective, note that the pullback is defined by

$$f^*\alpha(X) = \alpha(df(X)) \text{ for } X \in T_{\mathbb{C}}^k X.$$

Therefore, it suffices to show that if $X \in T^{p,q}X$, then $df(X) \in T^{p,q}X$. Since $T_{\mathbb{C}}^{p,q} = \bigwedge^p T^{1,0} \otimes \bigwedge^q T^{0,1}$, it suffices to establish this in the special case $(1, 0)$ and $(0, 1)$, and since those spaces are conjugate it suffices to show that df preserves $T^{1,0}$. Since $T^{1,0}$ is the $+i$ eigenspace of J , this follows from the fact that df commutes with J because f is holomorphic (Lemma 1.5).

In local coordinates (w_1, \dots, w_n) on Y and (z_1, \dots, z_m) on X , we have

$$f^*dw_j = d(f_j) = \sum_k \frac{\partial f_j}{\partial z_k} dz_k + \frac{\partial f_j}{\partial \bar{z}_k} d\bar{z}_k$$

but the second term vanishes because f is holomorphic.

To show that f descends to a map on Dolbeault cohomology, we must show that f commutes with $\bar{\partial}$, since this implies f^* preserves cocycles and coboundaries. Since f^* preserves the bigrading by the previous part, this follows from the fact in the smooth case that f^* commutes with d ; recall that this can be checked on functions and 1-forms.

7. Let \mathcal{F} be a sheaf on a topological space X . Define the stalk of \mathcal{F} at $x \in X$ by

$$\mathcal{F}_x = \varprojlim_{U \ni x} \mathcal{F}(U).$$

Observe that any sheaf morphism $\alpha : \mathcal{F} \rightarrow \mathcal{G}$ induces a morphism $\alpha_x : \mathcal{F}_x \rightarrow \mathcal{G}_x$. Prove that

$$0 \rightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \rightarrow 0$$

is short exact if and only if

$$0 \rightarrow \mathcal{F}_x \xrightarrow{\alpha_x} \mathcal{G}_x \xrightarrow{\beta_x} \mathcal{H}_x \rightarrow 0$$

is short exact.

Lemma 12.4. $\mathcal{F}(U) \hookrightarrow \prod_{x \in U} \mathcal{F}_x$.

Proof. If $s \in \mathcal{F}(U)$ maps to 0, then around each $x \in U$ there is a V_x such that $s|_{V_x} = 0$. By the identity axiom, $s = 0$. \square

This shows that if $\mathcal{F}_x \xrightarrow{\alpha_x} \mathcal{G}_x$ is injective for all x , then so is $\mathcal{F} \xrightarrow{\alpha} \mathcal{G}$. Conversely, if $f_x \in \mathcal{F}_x$ maps to 0 in \mathcal{G}_x , then this is witnessed by representatives in some V , so if $\mathcal{F}(U) \hookrightarrow \mathcal{G}(U)$ for all U then $f_x = 0$.

The equivalence of surjectivity is obvious from our definition of a surjective map of sheaves.

Finally, we consider exactness at the middle term. First suppose that the sequence of sheaves is short exact. If $g \in \mathcal{G}_x$ maps to 0 in \mathcal{H}_x , then there is a representative $g \in \mathcal{G}(U)$ for g_x mapping to 0 in $\mathcal{H}(U)$, so it is the image of $f \in \mathcal{F}(U)$, hence $g_x = \alpha_x(f_x)$.

Conversely, suppose the sequence of stalks is exact for all x . If $g_x \in \mathcal{G}(U)$ maps to 0 in $\mathcal{H}(U)$, then $\beta_x(g_x) = 0$ for all $x \in U$. Therefore, $g_x = \alpha_x(f_x)$ for $f_x \in \mathcal{F}_x$. For each x , there is some V_x with a lift $\tilde{f}_x \in \mathcal{F}(V_x)$ restricting to f_x . Let $\tilde{g}_x = \alpha(\tilde{f}_x)$. We know that the stalk of \tilde{g}_x at x agrees with g_x , so by shrinking V_x further if necessary we may assume that $\tilde{g}_x = g|_{V_x}$.

We claim that the sections \tilde{f}_x are compatible. Indeed, for each $z \in V_x$ we have $\alpha(\tilde{f}_x)_z = g_z$, so for each $z \in V_x \cap V_y$ we see that $\alpha(\tilde{f}_x)_z = \alpha(\tilde{f}_y)_z$. Since α_x is injective, \tilde{f}_x and \tilde{f}_y agree on all stalks, and then Lemma 12.4 implies that $\tilde{f}_x = \tilde{f}_y$. Now the sheaf axiom glues all the \tilde{f}_x to a section $f \in \mathcal{F}(U)$. We must have $\alpha(U)(f) = g$, since equality holds after mapping to each stalk and by Lemma 12.4 again.

Remark 12.5. The argument above may seem rather involved, but it is guided by the simple intuition that a section of a sheaf is a “bundle of compatible stalks.” Having produced a bundle of stalks f_x for the candidate section f , we have to show that they are compatible, which is true because their images under the injective map α are compatible.

8. Complete the proof of the $\bar{\partial}$ -Poincaré Lemma 4.11 (in the remaining cases $q > 1$).

We proceed by induction on q , having established the base case $q = 1$. We suppose that $\bar{\partial}\alpha = 0$. As before, we filter B by an increasing sequence of bounded polydiscs B_1, B_2, \dots such that $\bigcup B_i = B$.

- (a) We claim that there exist smooth p, q forms β_1, β_2, \dots such that $\bar{\partial}\beta_m|_{B_m} = \alpha$.

Indeed, since α is $\bar{\partial}$ -closed Lemma 4.3 guarantees that there exists β_m defined on B_{m+1} such that $\bar{\partial}\beta_m|_{B_m} = \alpha$. Then we can multiply β_m by a smooth bump function which is $\equiv 1$ on B_m and supported in B_m to extend it to all of B .

(b) Next, we claim that we may arrange that $\beta_m|_{B_{m-1}} = \beta_{m+1}|_{B_{m-1}}$. Indeed, we have $\bar{\partial}(\beta_{m+1} - \beta_m)|_{B_m} = 0$, so the induction hypothesis furnishes us with some γ on B_m such that $\beta_{m+1} - \beta_m = \bar{\partial}\gamma$ on B_m . Multiplying by a bump function $\equiv 1$ on B_{m-1} and supported in B_m , we may extend γ to B , so $\beta_{m+1} - \gamma$ is defined on B and restricts to β_m on B_{m-1} .

(c) Obviously, the β_m converge to β satisfying $\bar{\partial}\beta = \gamma$.

9. (i) Let $X = \mathbb{C}^*$. Consider the open cover $U_0 = \mathbb{C} - \mathbb{R}^+ \times \{0\}$ and $U_2 = \mathbb{C} - \mathbb{R}_- \times \{0\}$. Compute $\check{H}^q(\mathcal{U}, \mathbb{Z})$.

$C^0(\mathcal{U}, \mathbb{Z})$ consists of pairs (a, b) where $a \in \mathbb{Z}(U_0) = \mathbb{Z}$ and $b \in \mathbb{Z}(U_1) = \mathbb{Z}$. The boundary map is $(a, b) \mapsto (a - b, a - b)$, so the cocycles are the pairs where $a = b$, i.e. $\check{H}^0(\mathcal{U}, \mathbb{Z}) \cong \mathbb{Z}$.

$C^1(\mathcal{U}, \mathbb{Z})$ consists of an element of $\mathbb{Z}(U_0 \cap U_1) = \mathbb{Z} \oplus \mathbb{Z}$. The coboundaries are the diagonal, and the cocycles are everything since there are no higher intersections, so $\check{H}^1(\mathcal{U}, \mathbb{Z}) \cong \mathbb{Z}$.

All higher cohomology groups are automatically 0.

(ii) Let $M = \mathbb{C}^2 - 0$. Consider the cover $\mathcal{U} = \{U_0, U_1\}$ consisting of $U_0 = \mathbb{C} \times \mathbb{C}^*$ and $U_1 = \mathbb{C}^* \times \mathbb{C}$. Show that $\check{H}^1(M, \mathcal{O})$ is infinite-dimensional and $\check{H}^q(M, \mathcal{O})$ is trivial for $q > 1$.

Since $U_0 \cap U_1 = \mathbb{C}^* \times \mathbb{C}^*$, Example 5.31 shows that this cover actually computes Čech cohomology. So it is obvious that $\check{H}^q(M, \mathcal{O})$ vanishes for $q \geq 2$.

To compute $\check{H}^1(\mathcal{U}, \mathcal{O})$, observe that the 0-cochains $\mathcal{C}^0(\mathcal{U}, \mathcal{O})$ consist of (f, g) where $f_0 \in \mathcal{O}(U_0)^*$ and $f_1 \in \mathcal{O}(U_1)^*$. The 1-cochains consist of $h \in \mathcal{O}(U_0 \cap U_1)^*$, and note that $U_0 \cap U_1 = \mathbb{C}^*$. Therefore, $\check{H}^1(\mathcal{U}, \mathcal{O})$ consists of holomorphic functions on $\mathbb{C}^* \times \mathbb{C}^*$ modulo functions that can be written as $f - g$, where f is holomorphic on $\mathbb{C} \times \mathbb{C}^*$ and g is holomorphic on $\mathbb{C}^* \times \mathbb{C}$.

We claim that $\frac{1}{(z_0 z_1)^k}$ are independent elements of the cokernel. The independence will follow if we can show that they are non-zero, since we can take a linear combination and multiply through by $(z_0 z_1)^{k-1}$, where k is the largest exponent appearing in the linear combination.

To see that they are non-zero suppose that

$$f_0 - f_1 = \frac{1}{(z_0 z_1)^k}$$

with f_0, f_1 as above. Then $(z_0 z_1)^k f_0 = 1 + (z_0 z_1)^k f_1$. The left hand side is holomorphic on $\mathbb{C} \times \mathbb{C}^*$ and the right hand side is holomorphic on $\mathbb{C}^* \times \mathbb{C}$, so they both extend to holomorphic functions on all of M . But then Hartog's theorem implies that they extend to holomorphic functions on all of \mathbb{C}^2 , say g_0 and g_1 .

Now, $\frac{g_0}{(z_0 z_1)^k} = f_0$ is holomorphic on $\mathbb{C} \times \mathbb{C}^*$, so $\frac{g_0}{z_0}$ is holomorphic on $\mathbb{C} \times \mathbb{C}^*$. It is also holomorphic on $\mathbb{C}^* \times \mathbb{C}$ since z_0 is nonvanishing there, so it *also* extends to a global entire function, say h_0 . Similarly, $\frac{g_1}{z_1} = h_1$ is entire. So we have

$$\frac{h_0}{z_1^k} - \frac{h_1}{z_0^k} = 1 \implies z_0^k h_0 - z_1^k h_1 = 1.$$

Setting $z_0 = 0$, we see that $z_1 h_1(0, z_1) = 1$, where $h_1(0, z_1)$ is entire on \mathbb{C}^1 . This is clearly impossible.

10. Let $X = \mathbb{P}^1$ and P, Q be distinct points of X . Let $\mathcal{O}_X(-P-Q)$ denote the sheaf of holomorphic functions vanishing at both P and Q . Show that there is a short exact sequence of sheaves

$$0 \rightarrow \mathcal{O}_X(-P-Q) \rightarrow \mathcal{O}_X \rightarrow \mathbb{C}_P \oplus \mathbb{C}_Q \rightarrow 0$$

where the sheaf on the right should be carefully defined. For any sheaf \mathcal{F} write $\Gamma(\mathcal{F}) = \mathcal{F}(X)$ for the space of sections on all of X . Prove that the map $\Gamma(\mathcal{O}_X) \rightarrow \Gamma(X, \mathbb{C}_P \oplus \mathbb{C}_Q)$ is not surjective and that $\check{H}^1(\mathcal{O}_X(-P-Q)) \neq 0$.

The sheaf \mathbb{C}_P is a skyscraper sheaf at P . This can be characterized in a few different ways. It is the unique sheaf whose stalk at P is \mathbb{C} and all other stalks vanish. Alternatively,

$$\mathbb{C}_P(U) = \begin{cases} \mathbb{C} & P \in U \\ 0 & \text{otherwise} \end{cases}$$

with restriction maps being identity or 0.

To check exactness, we can check exactness at each stalk. At a stalk at $R \neq P, Q$, the sequence reads

$$0 \rightarrow \mathcal{O}_X(-P-Q)_R \rightarrow (\mathcal{O}_X)_R \rightarrow 0 \rightarrow 0$$

so we have to show that $\mathcal{O}_X(-P-Q)_R \cong (\mathcal{O}_X)_R$. This is obvious, since the open sets containing R but not P or Q is cofinal among open sets containing R .

Without loss of generality, we check exactness of the stalks at P only. Then the sequence reads

$$0 \rightarrow \mathcal{O}_X(-P-Q)_P \rightarrow (\mathcal{O}_X)_P \rightarrow \mathbb{C} \rightarrow 0$$

For the same reason as before, $\mathcal{O}_X(-P-Q)_P \cong \mathcal{O}_X(-P)_P$, and substituting this above makes the exactness obvious again.

Note that $H^0(X, \mathcal{O}_X) = \mathbb{C}$ and $H^0(X, \mathbb{C}_P \oplus \mathbb{C}_Q) \cong \mathbb{C} \oplus \mathbb{C}$, so the map is obviously not surjective. Its cokernel injects into $H^1(\mathcal{O}_X(-P-Q))$, so the latter group is non-zero.

Chapter 13

Example Sheet 2

1. Let X be a differentiable manifold. We let \mathbb{R} denote the sheaf of locally constant real-valued functions, and \mathcal{A}^p the sheaf of p -forms on X .

(i) Show that there is an exact sequence of sheaves

$$0 \rightarrow \mathbb{R} \rightarrow \mathcal{A}^0 \rightarrow \mathcal{A}^1 \rightarrow \dots$$

(ii) Show that $\check{H}^q(X, \mathcal{A}^p) = 0$ for $q \geq 1$ and all $p \geq 0$.

(iii) Using induced long exact sequences, as in lectures, prove that $\check{H}^p(X, \mathbb{R}) = H_{dR}^p(M)$.

(i) The exactness can be checked locally, where it is a consequence of Poincaré's lemma.

(ii) This is immediate from the fact that \mathcal{A}^p is fine.

(iii) One can give a direct argument, completely analogous to what we did for Dolbeault cohomology. Instead, we will show more generally what we claimed there: if

$$0 \rightarrow \mathcal{F} \rightarrow I_0 \xrightarrow{d_1} I_1 \rightarrow \dots$$

is a resolution of \mathcal{F} by acyclic sheaves, then $H^p(\mathcal{F}) = \frac{\ker d_{p+1}}{\operatorname{im} d_p}$.

Indeed, we have short exact sequences of sheaves

$$0 \rightarrow Z_{i-1} \rightarrow I_{i-1} \xrightarrow{d_i} Z_i \rightarrow 0$$

for each $i > 0$. By the long exact sequence in cohomology,

$$H^{j+1}(Z_{i-1}) \cong H^j(Z_i) \text{ for } j \geq 1.$$

Therefore,

$$\frac{Z_p(X)}{d_p I_{p-1}(X)} = H^1(Z_{p-1}) \cong H^2(Z_{p-2}) \cong \dots \cong H^{p-1}(Z_1) = H^p(\mathcal{F}).$$

2. Let X be a complex manifold, and denote the induced almost complex structure by $J : TX \rightarrow TX$. Explain how multiplication by i gives an \mathbb{R} -linear map

$$\tilde{J} : (T^*X)^{0,1} \rightarrow (T^*X)^{0,1}$$

such that $\tilde{J}^2 = -\text{Id}$. Prove that there is a natural identification

$$T^*X \cong (T^*X)^{1,0}$$

taking J to \tilde{J} .

This was already done in Example Sheet 1, Question 1.

(ii) Suppose now that g is a Riemannian metric on X compatible with J and let $g_{\mathbb{C}}$ be the extension to $TX \otimes \mathbb{C}$ given by

$$g_{\mathbb{C}}(\lambda v, \mu w) = \lambda \bar{\mu} g(v, w) \quad \lambda, \mu \in \mathbb{C}; v, w \in T_x X.$$

Show that under the isomorphism in part (i), we have

$$g = 2g_{\mathbb{C}} \text{ on } (T^*X)^{1,0}.$$

The explicit isomorphism is $v \mapsto \frac{v - iJv}{2}$. We compute

$$\begin{aligned} g_{\mathbb{C}}\left(\frac{v - iJv}{2}, \frac{w - iJw}{2}\right) &= \frac{1}{4} (g(v, w) - g(v, iJw) - g(iJv, w) + g(iJv, iJw)) \\ &= \frac{1}{2} g(v, w) \end{aligned}$$

(iii) Now suppose we have local coordinates z_1, \dots, z_n on X so that dz_1, \dots, dz_n are a frame for $(T^*X)^{(1,0)}$. Show that if $h_{ij} = 2g_{\mathbb{C}}(dz_i, dz_j)$ then $h = (h_{ij})$ is a hermitian matrix, and the associated fundamental form is

$$\omega = \frac{i}{2} \sum_{i,j} h_{ij} dz_i \wedge d\bar{z}_j.$$

With $z_j = x_j + iy_j$, we compute

$$\begin{aligned}\omega(dz_i, d\bar{z}_j) &= g_{\mathbb{C}}(Jdz_i, dz_j) \\ &= ig_{\mathbb{C}}(dz_i, dz_j) \\ &= \frac{i}{2}h_{ij}.\end{aligned}$$

That establishes the result.

(iv) Deduce that any hermitian metric is given by a unique such form ω with h_{ij} hermitian (and that this metric is Kähler if and only if ω is closed).

The computation in (iii) shows that h can be recovered from ω .

3. Let $U_0 = \{[1, z] \in \mathbb{P}^n : z = (z_1, \dots, z_n) \in \mathbb{C}^n\}$. Show that the Fubini-Study form on \mathbb{P}^n can be written locally on U_0 as

$$\omega_{FS} = \frac{i}{2\pi} \left(\frac{\sum_j dz_j \wedge d\bar{z}_j}{1 + \|z\|^2} - \frac{\sum_j \bar{z}_j dz_j \wedge \sum_j z_j d\bar{z}_j}{(1 + \|z\|^2)^2} \right).$$

Deduce that ω_{FS} defines a Kähler metric on \mathbb{P}^n .

It will be useful to record that

$$\bar{\partial}(1 + \|z\|^2) = \sum_j z_j d\bar{z}_j \quad \partial(1 + \|z\|^2) = \sum_j \bar{z}_j dz_j.$$

Now, $\frac{2\pi}{i}\omega_{FS} = \partial\bar{\partial}\log(1 + \|z\|^2)$, which is

$$\begin{aligned}\partial\bar{\partial}\log(1 + \|z\|^2) &= \partial \left(\frac{\sum_j z_j d\bar{z}_j}{1 + \|z\|^2} \right) \\ &= \frac{\sum_j dz_j \wedge d\bar{z}_j}{1 + \|z\|^2} - \frac{\sum_j \bar{z}_j dz_j \wedge \sum_j z_j d\bar{z}_j}{(1 + \|z\|^2)^2}\end{aligned}$$

We already showed that ω_{FS} is well-defined and closed, so it suffices to show that it is positive-definite. From the preceding computation, we see that $h_{ij} = (1 + \sum |w_i|^2)\delta_{ij} - \bar{w}_i w_j$. Therefore, if in terms of the standard Hermitian inner product on \mathbb{C}^n we have

$$u^t(h_{ij})\bar{u} = \|u\|^2 + \|w\|^2\|u\|^2 - u^t\bar{w}w^t u.$$

By Cauchy-Schwarz, the last term is $\leq \|u\|^2 \cdot \|w\|^2$, so we are done.

4. Let $\mathbb{P}^{n-1} \subset \mathbb{P}^n$ be the standard inclusion. Show that the restriction of the Fubini-Study metric on \mathbb{P}^n gives the Fubini-Study metric on \mathbb{P}^{n-1} .

We interpret the standard inclusion to be $(z_1, \dots, z_{n-1}) \mapsto (z_1, \dots, z_{n-1}, 0)$. (We could take any of the other inclusions where $z_j = 0$ as well.) Then

clearly $\partial|_{\mathbb{P}^n}$ restricts to $\partial|_{\mathbb{P}^{n-1}}$ and likewise for $\bar{\partial}$, and also the norm on \mathbb{P}^n restricts to the norm on \mathbb{P}^{n-1} , so the result follows.

5. Show that any holomorphic line bundle on a disc $\Delta \subset \mathbb{C}$ is trivial. Deduce that any holomorphic line bundle on \mathbb{P}^1 is of the form $\mathcal{O}(n)$ for some integer n .

Remark 13.1. It is true more generally that any line bundle on \mathbb{P}^n is isomorphic to $\mathcal{O}(m)$. The easiest way to see this is to prove that

- (a) $H^i(\mathbb{P}^n, \mathcal{O}) = 0$ for $i > 0$. This can be shown directly using Čech cohomology on the standard open cover.
- (b) The long exact sequence of the exponential map shows that $H^1(\mathbb{P}^1, \mathcal{O}^*) = \mathbb{Z}$.
- (c) The (first) Chern class $c_1 : H^1(\mathbb{P}^n, \mathcal{O}^*) \rightarrow H^2(\mathbb{P}^n, \mathbb{Z})$ takes $\mathcal{O}(1)$ to 1. This is the hard part, partly because the Chern class receives a new definition every time it is mentioned. One justification is that the first Chern class of a complex line bundle is the Euler class of the underlying real vector bundle, which is Poincaré dual to the zero set of a transverse section. Since a section of $\mathcal{O}(1)$ is a linear polynomial, its zero locus is the class of a hyperplane in \mathbb{P}^n , which generates. Therefore, $c_1(\mathcal{O}(1))$ is a generator of $H^2(\mathbb{P}^n, \mathbb{Z}) \cong \mathbb{Z}$.

We will present an ad hoc argument that works for \mathbb{P}^1 . First, the Dolbeault theorem implies $H^q(\Delta, \mathcal{O}) \cong H^{0,q}(\Delta) = 0$ for $q > 0$ by the $\bar{\partial}$ -Poincaré Lemma. By the long exact sequence associated to the short exact exponential sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O} \rightarrow \mathcal{O}^* \rightarrow 0$, we have $H^1(\Delta, \mathcal{O}^*) \cong H^2(X, \mathbb{Z}) = 0$. This shows that any complex line bundle on Δ is trivial.

Take a cover of \mathbb{P}^1 by two discs $U_0, U_1 \cong \Delta$ whose intersection is an annulus V . Let $\mathcal{U} = \{U_0, U_1\}$.

Lemma 13.2. $\check{H}^1(\mathcal{U}, \mathcal{O}) = 0$.

Remark 13.3. Our cover is good, so in fact this implies $\check{H}^1(\mathbb{P}^1, \mathcal{O}) = 0$.

Proof. This boils down to the assertion that if $f \in \mathcal{O}(V)$, we can write $f = g_0 - g_1$ where $g_0 \in \mathcal{O}(U_0)$ and $g_1 \in \mathcal{O}(U_1)$. By Cauchy's formula,

$$f(w) = \frac{1}{2\pi i} \int_{\partial V} \frac{f(z) dz}{z - w} = \underbrace{\frac{1}{2\pi i} \int_{\partial U_0} \frac{f(z) dz}{z - w}}_{g_0} - \underbrace{\frac{1}{2\pi i} \int_{\partial U_1} \frac{f(z) dz}{z - w}}_{g_1}.$$

□

Now, let L be a line bundle on \mathbb{P}^1 . By the first part, L is trivial over U_0 and U_1 , so let s_0 and s_1 be trivializations over those two open sets. We have a transition function $g(z) = \frac{s_0}{s_1} \in \mathcal{O}^*(U_0 \cap U_1)$. Let d be the winding number of g , so $h(z) = z^{-d}g(z)$ has a winding number zero. Therefore, $\log h(z)$ is well-defined and holomorphic (rigorous argument: the integral of $d \log(z^{-d}g(z))$ over any closed loop vanishes).

By the Lemma, we may write $\log h(z) = h_0(z) - h_1(z)$ where $h_j \in \mathcal{O}(U_j)$. Then $z^{-d}g(z) = h(z) = e^{h_0(z)}e^{-h_1(z)}$. Since $e^{-h_j(z)}$ is non-vanishing and holomorphic on U_j , we can define new trivializations $\tilde{s}_j(z) := s_j(z)e^{-h_j(z)}$. Then

$$\tilde{s}_0(z) = s_0(z)e^{-h_0(z)} = g(z)s_1(z)z^d g^{-1}(z)e^{-h_1(z)} = z^d \tilde{s}_1(z)$$

showing that L is isomorphic to $\mathcal{O}(d)$.

6. Let Δ_d and Δ_∂ denote the d and ∂ -Laplacians, respectively, on \mathbb{C}^n with the standard metric. Show that $\Delta_\partial(f) = -\frac{1}{2} \sum_j \partial^2 f / \partial x_j^2 + \partial^2 f / \partial y_j^2$. Verify directly that $\Delta_d = 2\Delta_\partial$.

Let $dV = dz_1 \wedge d\bar{z}_1 \wedge \dots \wedge dz_n \wedge d\bar{z}_n$ be the volume form on \mathbb{C}^n . We denote by \widehat{dz}_i the form with only dz_i missing, e.g.

$$\widehat{dz}_2 = dz_1 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \wedge \dots \wedge dz_n \wedge d\bar{z}_n.$$

Observe that $*dz_i = \widehat{dz}_i$ and $*d\bar{z}_i = -\widehat{d\bar{z}_i}$.

Since ∂^* decreases degrees, $\Delta_{\partial}f = \partial * f = - * \partial * \partial f$.

$$\begin{aligned} \Delta_{\partial}f &= - * \partial * \sum_j \frac{\partial f}{\partial z_j} dz_j \\ &= - * \partial \sum_j \frac{\partial f}{\partial z_j} \widehat{dz}_j \\ &= - * \sum_{i,j} \frac{\partial^2 f}{\partial z_i \partial z_j} dz_i \wedge \widehat{dz}_j \\ &= - * \sum_j \frac{\partial^2 f}{\partial z_j^2} dV \\ &= - \sum_j \frac{\partial^2 f}{\partial z_j^2} \\ &= -\frac{1}{4} \sum_j \frac{\partial}{\partial z_j} \left(\frac{\partial f}{\partial x_j} - i \frac{\partial f}{\partial y_j} \right) \\ &= -\frac{1}{4} \sum_j \frac{\partial^2 f}{\partial x_j^2} - 2i \frac{\partial^2 f}{\partial x_j \partial y_j} + \frac{\partial^2 f}{\partial y_j^2} \end{aligned}$$

Note that the Cauchy-Riemann equations imply

$$\frac{\partial f}{\partial x_j} = \frac{\partial u}{\partial x_j} + i \frac{\partial v}{\partial x_j} = \frac{\partial v}{\partial y_j} - i \frac{\partial u}{\partial y_j} = -i \frac{\partial f}{\partial y_j},$$

so the above is

$$-\frac{1}{2} \sum_j \frac{\partial^2 f}{\partial x_j^2} + \frac{\partial^2 f}{\partial y_j^2}$$

as desired.

7. Prove that any complex manifold admits a Hermitian structure.

Let $\mathcal{U} = \{(U_\alpha, \varphi_\alpha)\}$ be a holomorphic atlas for a complex manifold X . By pulling back the standard Hermitian metric on \mathbb{C}^n via φ_α , we obtain a Hermitian metric h_α on φ_α . Let $\{\rho_\alpha\}$ be a partition of unity subordinate to U_α . Then $\sum \rho_\alpha h_\alpha$ is a Hermitian metric on X .

Remark 13.4. The Hopf surface is the quotient $(\mathbb{C}^2 - \{0\})/\mathbb{Z}$ where $k \cdot (z, w) = (\lambda^k z, \lambda^k w)$ for some $\lambda \in (0, 1)$ is evidently diffeomorphic to $S^1 \times S^3$, and does not have a Kähler structure since it is compact with vanishing H^2 . We show that it is indeed a complex manifold in Example Sheet 3, Question 4.

8. Let X be a complex manifold. Show that $T^{1,0}X$ is naturally a holomorphic vector bundle which we denote by T_X . Given a submanifold

$Y \subset X$ prove that there is a natural inclusion $T_Y \subset TX|_Y$, which has quotient \mathcal{N}_Y which is a holomorphic bundle on Y .

In local coordinates (z_1, \dots, z_n) on $U_\alpha \subset X$, a frame for $T^{1,0}X|_{U_\alpha}$ is $\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}$. Suppose that $(z_1, \dots, z_n)^T = \psi(z'_1, \dots, z'_n)$, where ψ is a holomorphic transition function. The induced transition function on $T^{1,0}X$ is

$$\frac{\partial}{\partial z_j} \mapsto \sum_i \frac{\partial z'_i}{\partial z_j} \frac{\partial}{\partial z'_i} = (\psi_{ij})^{-1} \frac{\partial}{\partial z'_j}$$

and $(\psi_{ij})^{-1}$ is holomorphic by the inverse function theorem.

We have a natural inclusion $TY \hookrightarrow TX|_Y$, which induces $T_{\mathbb{C}}Y \hookrightarrow T_{\mathbb{C}}X|_Y$. Since the inclusion is holomorphic, its differential commutes with the almost complex structure J so we have an inclusion of $+i$ eigenspaces $T^{1,0}Y \hookrightarrow T^{1,0}X|_Y$.

Finally, in U_α we may suppose that in our local coordinates (z_1, \dots, z_n) $Y \cap U_\alpha$ is cut out by $z_n = 0$, so a local frame for $TY|_{U_\alpha}$ is $\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_{n-1}}$ is a frame for TY . In another frame (z'_1, \dots, z'_n) for U_β where $Y \cap U_\beta$ is cut out by $z'_n = 0$, we have

$$\frac{\partial}{\partial z_n} \mapsto \sum_i \frac{\partial z'_i}{\partial z_n} \frac{\partial}{\partial z'_i} = \frac{\partial z'_n}{\partial z_n} \frac{\partial}{\partial z'_n} + \dots$$

so the transition function for N_Y is the holomorphic function $\frac{\partial z'_n}{\partial z_n}$.

9. Let L_1 and L_2 be line bundles on a complex manifold X . Suppose that there exists a complex submanifold Y of codimension at least 2 such that L_1 is isomorphic to L_2 on $X \setminus Y$. Prove that $L_1 \cong L_2$.

A vector bundle isomorphism is an isomorphism of the total spaces (as complex manifolds) over the identity on the base, restricting to a linear map on the fibers.

Let $U = X \setminus Y$ and $f : L_1|_U \rightarrow L_2|_U$ be an isomorphism with inverse $g : L_2|_U \rightarrow L_1|_U$. Since $L_1|_U$ has codimension 2 in L_1 , this extends to a holomorphic map $\tilde{f} : L_1 \rightarrow L_2$ by Hartog's Theorem 6.2. Similarly, g extends to a holomorphic map $\tilde{g} : L_2|_U \rightarrow L_1|_U$. Since $g \circ f = \text{Id}|_{L_1|_U}$ and vice versa, the same holds for the holomorphic extension by the identity principle. So we have established that f extends to an isomorphism of total spaces.

Let π_i denote the projection maps from $L_i \rightarrow X$. Then we have

$$\pi_1 = \pi_2 \circ \tilde{f} \text{ on } L_1|_U$$

hence $\pi_1 = \pi_2 \circ \tilde{f}$ by the identity principle again. Therefore, \tilde{f} sends fibers to fibers.

The only thing left to show is that \tilde{f} restricts to a linear map on each fiber. This can be checked locally, so we may assume that L_1, L_2 are trivial and \tilde{f} is of the form

$$(x, z) \mapsto (x, \varphi(z)).$$

Consider two maps $L_1 \times \mathbb{C} \rightarrow L_2$ defined by

$$\begin{aligned} (x, z, \lambda) &\mapsto (x, \lambda\varphi(z)), \\ (x, z, \lambda) &\mapsto (x, \varphi(\lambda z)). \end{aligned}$$

Both are holomorphic, and they agree on a subset of codimension 2 (namely $U \times \mathbb{C} \times \mathbb{C}$), hence they agree everywhere by the identity principle. That establishes linearity, and we are done.

Remark 13.5. A different argument using transition functions was suggested in class. It was stated slightly incorrectly, but we can make it work as follows.

Let $\{U_\alpha\}$ be an open cover of X trivializing both L_1 and L_2 . Then $\{U \cap U_\alpha\}$ is an open cover of U , and since $L_1 \cong L_2$ over U we may assume that the transition functions for L_1, L_2 are actually equal. That is, for each U_α we have bundle isomorphisms

$$\begin{aligned} (\pi_1, f_\alpha \circ \pi_1) &: L_1|_{U \cap U_\alpha} \cong (U \cap U_\alpha) \times \mathbb{C} \\ (\pi_2, g_\alpha \circ \pi_2) &: L_2|_{U \cap U_\alpha} \cong (U \cap U_\alpha) \times \mathbb{C}. \end{aligned}$$

Moreover, $f_\alpha = \tau_{\alpha\beta} f_\beta$ and $g_\alpha = \tau_{\alpha\beta} g_\beta$ on $U_\alpha \cap U_\beta$.

Since f_α and f_β are defined on a codimension-2 subset, they extend to \tilde{f}_α and \tilde{f}_β which are non-vanishing since the extension of an isomorphism is an isomorphism (by the previous argument). Similarly, $\tau_{\alpha\beta}$ extends to $\tilde{\tau}_{\alpha\beta}$ satisfying

$$\tilde{f}_\alpha = \tilde{\tau}_{\alpha\beta} \tilde{f}_\beta$$

by the identity axiom. This shows that in fact L_1 and L_2 have the same transition functions over all of X .

10. (i) Let $K_X = \det(T_X^*)$, which is a holomorphic line bundle called the canonical bundle of X . Show that if $Y \subset X$ is a complex submanifold, then K_Y is isomorphic to $K_X|_Y \otimes \det N_Y$.

We have a short exact sequence of vector bundles

$$0 \rightarrow T_Y \rightarrow T_X \rightarrow \mathcal{N}_Y \rightarrow 0.$$

Whenever one has a short exact sequence of vector spaces,

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

then one gets an isomorphism by taking top exterior powers

$$\det A \otimes \det C \cong \det B$$

defined on pure tensors by

$$(u_1 \wedge \dots \wedge u_a) \otimes (g(b_1) \wedge \dots \wedge g(b_c)) \mapsto f(u_1) \wedge \dots \wedge f(u_a) \wedge b_1 \wedge \dots \wedge b_c.$$

This is well-defined because all lifts of b_1, \dots, b_c differ by elements in the image of A , which will be killed after wedging with its top form. By naturality, we can take top exterior powers of our short exact sequence of vector bundles to deduce

$$\det TX|_Y = \det T_Y \otimes \det \mathcal{N}_Y.$$

Dualizing, we get

$$K_X|_Y \cong K_Y \otimes \det \mathcal{N}_Y^*,$$

and tensoring with \mathcal{N}_Y gives the result.

(ii) Suppose that L is a holomorphic line bundle on X and $0 \neq s \in H^0(X, L)$. Suppose further that $Y = \{x \in X : s(x) = 0\}$ is a complex submanifold and assume that ds does not vanish along Y (that is, if s is locally given by a holomorphic function f then df does not vanish on Y). Prove a relation between N_Y and $L|_Y$, and use this to find a formula for K_Y in terms of L and K_X .

The statement is geometrically intuitive: locally Y looks like the subset $s = 0$, so the line spanned by s is complementary to TY in TX . Led by this intuition, we guess that $N_Y \cong L|_Y$, which we prove locally.

Suppose that L is trivialized over U_α and U_β by sections x_α and x_β , with transition function $x_\alpha = g_{\alpha\beta}x_\beta$. Let the restrictions of s to U_α and U_β be $s_\alpha = f_\alpha x_\alpha$ and $s_\beta = f_\beta x_\beta$, respectively. Then

$$f_\alpha x_\alpha = f_\beta x_\beta = \frac{f_\beta}{g_{\alpha\beta}}(g_{\alpha\beta}x_\beta) \implies g_{\alpha\beta}f_\alpha = f_\beta. \quad (13.1)$$

Since $ds|_Y \neq 0$, we have that f_α is not divisible by the square of any non-unit in any stalk, since if we had $(f_\alpha)_x = ch_x^2$ then $d(f_\alpha)_x$ would be divisible by h_x , hence vanish at x . Therefore, f_α is a chart coordinate, so we can find local charts for $Y \subset X$ of the form $(z_1, \dots, z_{n-1}, f_\alpha)$ on U_α (after possibly shrinking U_α).

In these local coordinates, a frame for $TX|_Y$ is $\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_{n-1}}, \frac{\partial}{\partial f_\alpha}$ and the first $n - 1$ sections form a frame for TY , so a frame for $N_Y|_Y$ is $\frac{\partial}{\partial f_\alpha}$. The transition function relating these coordinates to coordinates $(z'_1, \dots, z'_{n-1}, f_\beta)$ on U_β is

$$(z_1, \dots, z_{n-1}, f_\alpha) = (z'_1, \dots, z'_{n-1}, g_{\alpha\beta}(z'_1, \dots, z'_n, 0) f_\beta).$$

Therefore, the transition function for $N_Y|_Y$ is given by (using (13.1))

$$\frac{\partial}{\partial f_\alpha} = \frac{\partial f_\beta}{\partial f_\alpha} \frac{\partial}{\partial f_\beta} + \dots = g_{\alpha\beta}(z'_1, \dots, z'_n, 0) \frac{\partial}{\partial f_\beta} + \dots$$

These are precisely the transition functions for $L|_Y$.

Chapter 14

Example Sheet 3

1. *Prove that a Kähler form is harmonic.*

One way is simply to use the Kähler identities. Recall from Theorem 8.24 that $[\Lambda_d, L] = 0$. Applying this operator to the 1, we find that $\Delta_d(\omega) - L\Delta_d 1 = 0$, and the second term obviously vanishes so $\Delta_d(\omega) = 0$.

You can also do this by explicit computation in local coordinates. Let $\alpha_1, \beta_1, \dots, \alpha_n, \beta_n$ be an orthonormal basis for T^*X such that $J(\alpha_i) = \beta_i$. This can be constructed inductively, like what one does to obtain a symplectic basis: first pick a unit vector α_1 , and set $\beta_1 = J(\alpha_1)$. By compatibility,

$$g(\beta_1, \beta_1) = g(J\alpha_1, J\alpha_1) = g(-\alpha_1, -\alpha_1) = 1.$$

Also, $g(\alpha_1, \beta_1) = g(\alpha_1, J\alpha_1) = g(J\alpha_1, -\alpha_1) = -g(\alpha_1, J\alpha_1)$ shows orthogonality. The basis is then constructed by induction.

In this local frame, we have

$$\omega = \sum \alpha_i \wedge \beta_i.$$

By definition, ω is harmonic if $(dd^* + d^*d)\omega = 0$. Since ω is Kähler, $d\omega = 0$. Therefore, we need to check that $d^*\omega$ is closed. But $d^* = *d*$ and we can explicitly see that

$$*(\alpha_j \wedge \beta_j) = \alpha_1 \wedge \beta_1 \wedge \dots \wedge \widehat{\alpha}_j \wedge \widehat{\beta}_j \wedge \dots \wedge \alpha_n \wedge \beta_n.$$

It is clear that this is true up to sign, but since $(\alpha_1, \beta_1, \dots, \alpha_n, \beta_n)$ is an oriented coframe the sign $+1$ is also clear. In particular, we see that $*\omega = \frac{1}{(n-1)!}\omega^{n-1}$, so $d(*\omega) = 0$.

2. *Prove that $H^{p,q}(\mathbb{P}^n) = 0$ except when $p = q \leq n$, in which case it is one-dimensional. Use this to find the Picard group of \mathbb{P}^n .*

Recall the Hodge decomposition

$$H^*(X, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X)$$

We know the (singular) cohomology of projective space: $H^*(X, \mathbb{C}) \cong \mathbb{C}[z]/z^n$, where $|z| = 2$. Since $h^{p,q} = h^{q,p}$, this is only possible if the only non-zero groups are in bi-degree (p, p) , and these all have dimension one.

3. Find an exact sequence involving the holomorphic tangent bundle of \mathbb{P}^n , the trivial bundle and the tautological bundle. Use this to show that

$$K_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(-n-1).$$

The exact sequence is

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{O}_{\mathbb{P}^n}(1)^{n+1} \rightarrow T_{\mathbb{P}^n} \rightarrow 0.$$

Before explaining where this comes from, it may be helpful and clarifying to consider the question for the tangent bundle rather than the holomorphic tangent bundle.

In the *topological* category, one in fact has a splitting $T\mathbb{P}^n \oplus \underline{\mathbb{C}} \cong L^{-n-1}$, where $\underline{\mathbb{C}}$ is the trivial bundle and L is the tautological bundle. Indeed, recall that an open chart for $\mathbb{P}^n(V)$ in the neighborhood of a particular line ℓ is $\text{Hom}_{\mathbb{C}}(\ell, V/\ell)$. Complexifying and taking the $(1, 0)$ part returns this same space. Choosing a Hermitian metric on \mathbb{P}^n , this is isomorphic to $\text{Hom}(\ell, \ell^\perp)$, so we can identify the tangent bundle as $\{(\ell, \varphi) : \varphi \in \text{Hom}(\ell, \ell^\perp)\}$. On the other hand, the trivial bundle is $\{(\ell, \varphi) : \varphi \in \text{Hom}(\ell, \ell)\}$. Taking the direct sum, we obtain

$$\{(\ell, \varphi) : \varphi \in \text{Hom}(\ell, \ell^\perp \oplus \ell) = \text{Hom}(\ell, V)\}.$$

Since the tautological bundle is $\{(\ell, v) : v \in \ell\}$, this is visibly the dual of L^{n+1} . Dualizing, we find that $T^*\mathbb{P}^n \oplus \underline{\mathbb{C}} \cong L^{n+1}$, which shows that the canonical bundle is $\mathcal{O}(-n-1)$.

This doesn't work in the holomorphic (or algebraic) category, since we do not have the vector bundle ℓ^\perp (there is no *holomorphic* splitting of the bundle morphism $V \rightarrow V/\ell$). However, one still has an "Euler exact sequence" of sheaves whose dual is

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{O}_{\mathbb{P}^n}(1)^{n+1} \rightarrow T_{\mathbb{P}^n} \rightarrow 0.$$

The meaning is as follows. If we think of \mathbb{P}^n as the quotient of $X = \mathbb{C}^{n+1} - 0$ by \mathbb{C}^* , then the vector fields on X are freely generated by $\frac{\partial}{\partial x_i}$, so a general vector field on X can be written as

$$V(x) = \sum v_i(x_0, \dots, x_n) \frac{\partial}{\partial x_i}.$$

The vector fields on \mathbb{P}^n should correspond to “ \mathbb{C}^* -equivariant” vector fields on X . In other words, $v_i(tx) \frac{\partial}{\partial tx_i} = v_i(x) \frac{\partial}{\partial x_i}$, and from this it is clear that the v_i must be linear functions on X , which is the same as sections of $\mathcal{O}(1)$ over \mathbb{P}^n . That furnishes a map $\mathcal{O}(1)^{n+1} \rightarrow T\mathbb{P}^n$. What’s the kernel? Recall Euler’s formula for a homogeneous polynomial F of degree d :

$$dF(x_0, \dots, x_n) = \sum x_i \frac{\partial F}{\partial x_i}.$$

Applying this to a rational function $\frac{F}{G}$ on \mathbb{P}^n , we see that (x_0, \dots, x_n) maps to the trivial derivation. In fact, the kernel is freely generated by this section. We have now defined the maps in a purported short exact sequence

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}(1)^{n+1} \rightarrow T_{\mathbb{P}^n} \rightarrow 0.$$

To check exactness, we can work locally. Without loss of generality, we consider the standard open set U_0 with coordinates $(z_1, \dots, z_n) = (\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})$. The tangent bundle is trivialized over this open set with local frame $\partial_i := \frac{\partial}{\partial z_i}$.

To study the map $\mathcal{O}(1)^{n+1} \rightarrow T_{\mathbb{P}^n}$, we relate ∂_i and $\frac{\partial}{\partial x_i}$, observe that for $i > 1$ we have

$$\frac{\partial}{\partial x_i} f \left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0} \right) = \frac{1}{x_0} \partial_i f(z_1, \dots, z_n)$$

and for $i = 0$,

$$\frac{\partial}{\partial x_0} f \left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0} \right) = \sum_{i=1}^n -\frac{x_i}{x_0^2} \partial_i f(z_1, \dots, z_n).$$

Therefore, the map $\mathcal{O}(1)^{n+1} \rightarrow T_{\mathbb{P}^n}$ sends

$$(f_0, \dots, f_n) \mapsto \sum f_i \frac{\partial}{\partial x_i} = \sum_i \left(\frac{f_i}{x_0} - \frac{f_0 x_i}{x_0^2} \right) \partial_i.$$

From this, we visibly see that the kernel is the free submodule generated by $(1, z_1, \dots, z_n)$, which is the image of the restriction $\mathcal{O} \rightarrow \mathcal{O}(1)^{n+1}$.

Remark 14.1. Yet another point of view is that there is a canonical identification $T_\ell \mathbb{P}(V) \cong \text{Hom}(\ell, V/\ell)$ (even in the algebraic category). If you have seen deformation theory, a nice point of view is that $\mathbb{P}(V)$ represents inclusions of a line bundle into the trivial bundle tensored with V , so the tangent space to $\mathbb{P}(V)$ at a line consists of the space of first-order deformations of that line in V , i.e. flat families

$$\begin{array}{ccc} \ell & \longleftarrow & \mathcal{V} \\ \downarrow & & \downarrow \\ \text{Spec } k & \longleftarrow & \text{Spec } k[\epsilon]/\epsilon^2 \end{array}$$

and such things are described by a homomorphisms $\text{Hom}(\ell, V/\ell)$ (this is just a little commutative algebra; the quotienting corresponds to the flatness condition).

Using this, we see that $T_\ell \mathbb{P}(V) \cong L_\ell^* \otimes V_\ell/L_\ell$. Tensoring with L_ℓ , we obtain an exact sequence

$$0 \rightarrow \mathcal{O}_\ell \rightarrow V \otimes L_\ell \rightarrow T_\ell \mathbb{P}(V) \otimes L_\ell \rightarrow 0.$$

This gives isomorphisms on the fibers of the exact sequence we want. To deduce the global exactness in the algebraic category, we use the fact that the sheaves are locally free and apply Nakayama's Lemma.

4. *The k -th Betti number of a smooth manifold is $b_k = \dim_{\mathbb{R}} H^k(M, \mathbb{R})$. Show that the odd betti numbers b_{2k+1} of a compact Kähler manifold are even.*

Note that $b_k = \dim_{\mathbb{C}} H^k(M, \mathbb{C})$. By the Hodge Decomposition Theorem,

$$H^k(M, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(X, \mathbb{C}).$$

Using $h^{p,q} = h^{q,p}$ on a Kähler manifold, we deduce the result.

The Hopf surface (discussed in Example Sheet 2, Question 7) is a complex manifold homeomorphic to $S^1 \times S^3$, which fails this property, so it cannot be given a Kähler structure.

5. *Calculate the dimensions of the cohomology groups $H^i(\mathbb{P}^n, \mathcal{O}(r))$ for all values of n, i, r .*

You can compute this directly with Čech cohomology, although it is pretty involved. The advantage of that approach is that it illuminates Serre duality on \mathbb{P}^n , which is then used to deduce it for projective

varieties. However, it is relatively easy to obtain the dimensions directly.

We claim that the only non-zero cohomology groups are $H^0(\mathbb{P}^n, \mathcal{O}(k))$ for $k \geq 0$ and their Serre duals, namely $H^n(\mathbb{P}^n, \mathcal{O}(-n-1-k))$. We'll prove this by induction, so let's first establish it for \mathbb{P}^1 by direct computation. Choosing the usual cover by U_0 and U_1 , we see that the restriction maps for $\mathcal{O}(-k)$ are $(f(z), g(1/z)) \rightarrow (z^k f(z), g(1/z))$, so the cokernel is spanned by z, z^2, \dots, z^{k-1} . This verifies our claim when $n = 1$.

In general, we have a short exact of sheaves associated to the inclusion $\mathbb{P}^{n-1} \rightarrow \mathbb{P}^n$:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^n} \rightarrow \iota_* \mathcal{O}_{\mathbb{P}^{n-1}} \rightarrow 0.$$

Twisting by k , we deduce

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(k-1) \rightarrow \mathcal{O}_{\mathbb{P}^n}(k) \rightarrow \iota_* \mathcal{O}_{\mathbb{P}^{n-1}}(k) \rightarrow 0.$$

We claim that $H^i(\iota_* \mathcal{O}_{\mathbb{P}^{n-1}}(k)) \cong H^i(\mathcal{O}_{\mathbb{P}^{n-1}}(k))$. This generally phenomenon is most naturally phrased in algebro-geometric terms, where it is clear from the Čech complex that Čech cohomology is preserved by pushforward under affine morphisms. The result can be seen in this case because the standard open cover of \mathbb{P}^n , whose intersections are all cohomologically trivial, restricts to the standard open cover of \mathbb{P}^{n-1} . With this in hand, the long exact sequence in cohomology reads

$$\dots \rightarrow H^{i-1}(\mathbb{P}^{n-1}, \mathcal{O}(k)) \rightarrow H^i(\mathbb{P}^n, \mathcal{O}(k)) \rightarrow H^i(\mathbb{P}^n, \mathcal{O}(k-1)) \rightarrow \dots$$

Assuming our claim on \mathbb{P}^{n-1} to be established, we see that for $1 < i < n$, the claim is automatic by induction on k on \mathbb{P}^n . When $i = 1$, it is clear from the explicit description of $H^0(\mathcal{O}(k))$ that the map $H^0(\mathbb{P}^n, \mathcal{O}(k)) \rightarrow H^0(\mathbb{P}^{n-1}, \mathcal{O}(k))$ is surjective. Inducting on k , the “base case” is established by Kodaira vanishing or explicit computation; the rest is straightforward.

6. Let X be the Hopf manifold $\mathbb{C}^d - 0 / \sim$, where $(w_1, \dots, w_d) \sim (z_1, \dots, z_d)$ iff $w_j = 2^s z_j$ for all j for some $s \in \mathbb{Z}$. For $d > 1$, show that X is a complex manifold and $h^0(X, \Omega_X^r) = 0$ for all $1 \leq r \leq d$.

A quotient of a manifold by a proper, free group action is again a manifold ♠♠♠ Tony: [explain this?], and it is clear that \mathbb{Z} acts properly on $\mathbb{C}^{n+1} - 0$ since every compact subset gets squished arbitrarily small or blown up arbitrarily large.

If $H^0(X, \Omega^r) \neq 0$ for some $r > 0$, let $\omega \in H^0(X, \Omega^r)$. Pulling back via the natural projection map $\pi : \mathbb{C}^{n+1} - 0 \rightarrow X$, we obtain a holomorphic differential $\pi^*\omega$ on \mathbb{C}^{n+1} which is *invariant* under the action of \mathbb{C}^* . We can write

$$\pi^*\omega(z_0, \dots, z_n) = f(z_0, \dots, z_n)dz_I.$$

By Hartog's extension theorem, f extends to a holomorphic function on all of \mathbb{C}^* . The invariance $\pi^*\omega(\vec{z}) = \pi^*\omega(2^j\vec{z})$ implies that

$$f(2^{-j}z_0, \dots, 2^{-j}z_n) = 2^j f(z_0, \dots, z_n).$$

Taking the limit as $j \rightarrow \infty$ contradicts the fact that f extends continuously to 0.

7. Suppose that L is a holomorphic line bundle. Suppose that $s \in H^0(X, L)$ is non-zero and $Y = \{s = 0\}$ is a complex submanifold with $ds \neq 0$ along Y . Show that the sheaf $\mathcal{I}_Y \subset \mathcal{O}_X$ consisting of functions vanishing along Y can be identified with the sheaf of sections of the dual bundle L^* . Use this to explain the short exact sequence

$$0 \rightarrow L^{\otimes(k-1)} \rightarrow L^{\otimes k} \rightarrow L^{\otimes k}|_Y \rightarrow 0.$$

Suppose now that L is positive and E is a holomorphic line bundle. For all $i > 0$, show that $H^i(X, E \otimes L^{\otimes k}) = 0$ for $k \gg 0$. Assuming the Kodaira embedding theorem, and Bertini's theorem that a general hyperplane section of a complex projective variety is a complex submanifold, show that $h^0(X, E \otimes L^{\otimes r})$ is a polynomial in r for $r \gg 0$. What is this polynomial when $X = \mathbb{P}^n$ and $L = \mathcal{O}_{\mathbb{P}^n}(1)$ and $E = \mathcal{O}_X$?

We can check that $\mathcal{I}_Y \cong L^*$ by examining the transition functions. Suppose that L is trivialized over U_α and U_β by sections x_α and x_β , with transition function $x_\alpha = g_{\alpha\beta}x_\beta$. Let the restrictions of s to U_α and U_β be $s_\alpha = f_\alpha x_\alpha$ and $s_\beta = f_\beta x_\beta$, respectively. Since $ds|_Y \neq 0$, we have that f_α is not divisible by the square of any non-unit in any stalk, since if we had $(f_\alpha)_x = ch_x^2$ then $d(f_\alpha)_x$ would be divisible by h_x , hence vanish at x . Therefore, \mathcal{I}_Y is trivialized over U_α and U_β by f_α and f_β , and the identity

$$f_\alpha x_\alpha = f_\beta x_\beta = \frac{f_\beta}{g_{\alpha\beta}}(g_{\alpha\beta}x_\alpha) \implies f_\alpha = \frac{f_\beta}{g_{\alpha\beta}}.$$

shows that the transition function for \mathcal{I}_Y is the inverse of that for L , hence $\mathcal{I}_Y \cong L^*$.

Therefore, we have the short exact sequence of a closed submanifold

$$0 \rightarrow \mathcal{I}_Y \cong L^* \rightarrow \mathcal{O}_X \rightarrow \iota_* \mathcal{O}_Y \rightarrow 0.$$

Tensoring with $L^{\otimes k}$ (which is exact because $L^{\otimes k}$ is a line bundle), we obtain

$$0 \rightarrow L^{\otimes(k-1)} \rightarrow L^{\otimes k} \rightarrow L^{\otimes k}|_Y \rightarrow 0.$$

Here $L^{\otimes k}|_Y = \iota^* \mathcal{O}_Y \otimes L_{\otimes k} = \iota^* (\mathcal{O}_Y \otimes \iota^* L^{\otimes k})$.

For the second part, by Kodaira vanishing it suffices to show that $E \otimes L^k$ is positive. Take any hermitian metric g on E , and a positive metric h on L . Then $g \otimes h^{\otimes k}$ is a metric on $E \otimes L^{\otimes k}$, and the corresponding curvature form is $F_g + kF_h$. If X is compact, then since F_h is positive-definite we can ensure that $F_g + kF_h$ is positive definite for $k \gg 0$. ♠♠♠ Tony: [the compactness isn't in the hypothesis ... how to do without it?]

Next, we use the exact sequence

$$0 \rightarrow L^{\otimes(k-1)} \rightarrow L^{\otimes k} \rightarrow L^{\otimes k}|_Y \rightarrow 0.$$

Twisting by E and taking Euler characteristics gives

$$\chi(E \otimes L^{\otimes k}) - \chi(E \otimes L^{\otimes(k-1)}) = \chi(E \otimes L^{\otimes k}|_Y).$$

Since L is positive, the previous part implies that the higher cohomology vanishes for $k \gg 0$. Now the result follows from induction on $\dim X$, provided that we can show it for the zero-dimensional case, which is obvious. Note that we are using a fact about polynomials: if $f(n) - f(n - 1)$ is a polynomial, then so is f .

8. If $V \subset X$ is the inclusion of a codimension-one complex manifold, show that there are exact sequences of sheaves (and define the sheaves involved)

$$0 \rightarrow \Omega_X^p(-V) \rightarrow \Omega_X^p \rightarrow \Omega_X^p|_V \rightarrow 0$$

and

$$0 \rightarrow \Omega_V^{p-1}(-V) \rightarrow \Omega_X^p|_V \rightarrow (\Omega_V^p) \rightarrow 0.$$

There is some obvious abuse of notation here, since the same notation $\Omega_X^p|_V$ is used to denote a sheaf on X and a vector bundle on V .

We have the short exact sequence associated to a closed hypersurface.

$$0 \rightarrow \mathcal{O}_X(-V) \rightarrow \mathcal{O}_X \rightarrow \iota_* \mathcal{O}_V \rightarrow 0.$$

Tensoring with Ω_X^p is exact since Ω_X^p is locally free (it is the sheaf of sections of a vector bundle).

$$0 \rightarrow \Omega_X^p(-V) \rightarrow \Omega_X^p \rightarrow \Omega_X^p|_V \rightarrow 0.$$

where $\Omega_X^p|_V = \iota_* \mathcal{O}_V \otimes \Omega_X^p \cong \iota_*(\iota^* \Omega_X^p)$.

For the second exact sequence, we dualize the sequence from Example Sheet 2, Question 10 to obtain

$$0 \rightarrow \mathcal{N}_{V/X}^* \cong \mathcal{O}_V(-V) \rightarrow \Omega_X|_V \rightarrow \Omega_V \rightarrow 0$$

where now $\Omega_X|_V = \iota^* \Omega_X$. Note that $\mathcal{N}_{V/X}^* \cong L^*|_V \cong \mathcal{O}_V(-V)$. We take $p - 1$ wedge powers to obtain the exact sequence requested in the problem

$$0 \rightarrow \mathcal{O}_V(-V) \otimes \Omega_V^{p-1} \rightarrow \Omega_X \rightarrow \Omega_V^p \rightarrow 0.$$

Where does this come from? In general, given an exact sequence of vector spaces

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

where A is one-dimensional we obtain by taking exterior powers

$$0 \rightarrow A \otimes \bigwedge^{p-1} B \xrightarrow{\tilde{f}} \bigwedge^p B \xrightarrow{\tilde{g}} \bigwedge^p C \rightarrow 0.$$

Here $\tilde{g} = \bigwedge^p g$, and $\tilde{f}(a, g(b_1) \wedge \dots \wedge g(b_{p-1})) = a \wedge b_1 \wedge \dots \wedge b_{p-1}$. The point is that this is well-defined because any two pre-images of $g(b_1)$ differ by an element of A , which is killed off by the wedge with a . (More generally, it's clear that we can do this as long as $p > \dim A$.) By comparing dimensions, we see that this is indeed exact.

9. Let M be a smooth manifold and E a complex vector bundle on M . For ψ a complex 1-form on M , we consider $d\psi$ as an alternating 2-form via the natural identification. For complex vector fields X, Y , show that

$$2d\psi(X, Y) = X\psi(Y) - Y\psi(X) - \psi([X, Y]).$$

Suppose that $D : \mathcal{A}(E) \rightarrow \mathcal{A}^1(E)$ is a connection on E , with $R = D^2 : \mathcal{A}(E) \rightarrow \mathcal{A}^2(E)$ the curvature of D . With the 2-form part of $R \in \mathcal{A}(\text{Hom}(E, E))$ considered as an alternating form, show that

$$2R(X, Y) = [D_X, D_Y] - D_{[X, Y]}.$$

It suffices to check the equality locally. By linearity of both sides in ψ and X, Y , it suffices to check the result when X, Y , and ψ are all pure tensors. Without loss of generality (by conjugating), we may assume $X = X^i \partial_i$ and $Y = Y^j \partial_j$. Then

$$[X, Y] = X^i (\partial_i Y^j) \partial_j - Y^j (\partial_j X^i) \partial_i$$

If ψ is type $(0, 1)$ then both sides are zero, so we may assume that $\psi = f(z) dz_k$ and only $\partial \psi$ contributes to the left hand side. Both sides also vanish unless $k = i$ or $k = j$, so we assume that $k = j$ without loss of generality.

$$\begin{aligned} d\psi(X, Y) &= \sum_{\ell} \frac{\partial f}{\partial z_{\ell}} dz_{\ell} \wedge dz_j(X, Y) \\ &= \frac{\partial f}{\partial z_i} dz_i \wedge dz_j(X, Y) \\ &= \frac{1}{2} (X^i Y^j) \partial_i f. \end{aligned}$$

The right hand side is

$$\begin{aligned} X\psi(Y) - Y\psi(X) - \psi([X, Y]) &= X^i \partial_i (f Y_j) - \psi(X^i (\partial_i Y^j) \partial_j) \\ &= X^i (\partial_i f) Y^j + X^i f (\partial_i Y^j) - f X^i \partial_i Y^j \\ &= \partial_i f (X^i Y^j). \end{aligned}$$

The second part is a painful explicit computation. To ease notation, we adopt Einstein summation convention. We can check equality locally, so let (e^1, \dots, e^n) be a local frame for E . In this local frame, we have connection coefficients $D = (\Theta_{ij})$. We check that both sides are equal as sections of $\text{Hom}(E, E)$. Applying the right hand side to e_i , we find that

$$\begin{aligned} ([D_X, D_Y] - D_{[X, Y]})e^i &= D_X D_Y e^i - D_Y D_X e^i - D_{[X, Y]} e^i \\ &= D_X (\Theta_{ij}(Y) e^j) - D_Y (\Theta_{ij}(X) e^j) - \Theta_{ij}([X, Y]) e^j \\ &= X d(\Theta_{ij}(Y)) e^j + \Theta_{ij}(Y) \Theta_{jk}(X) e^k \\ &\quad - Y d(\Theta_{ij}(X)) e^j + \Theta_{ij}(X) \Theta_{jk}(Y) e^k \\ &\quad - \Theta_{ij}([X, Y]) e^j \\ &= X d(\Theta_{ij}(Y)) e^j - Y d(\Theta_{ij}(X)) e^j \\ &\quad + \Theta \wedge \Theta(X, Y) e^j - \Theta_{ij}([X, Y]) e^j. \end{aligned}$$

On the other hand, $R_D = d\Theta + \Theta \wedge \Theta$, and by the first part of the problem we have

$$2d\Theta(X, Y) = Xd(\Theta) - Yd\Theta - \Theta([X, Y])$$

so the two sides are equal, as desired.

10. Let D be a connection on a complex vector bundle E . We define the dual connection D^* on E^* by specifying that for local sections σ of E^* and s of E , we have the identity

$$(D^*\sigma)(s) = d(\sigma(s)) - \sigma(Ds).$$

Check that D^* is a connection

Given a hermitian metric on E , we define a dual hermitian metric on E^* by specifying that the dual frame to any unitary frame is unitary. If D is a hermitian connection on a hermitian vector bundle E , check that D^* is a hermitian connection on E^* .

Suppose now that E is a hermitian vector bundle over a complex manifold and D is the associated Chern connection on E . Show that D^* is the associated Chern connection on E^* .

By definition,

$$\begin{aligned} D^*(f\sigma)(s) &= d(f\sigma(s)) - f\sigma(Ds) \\ &= \sigma(s)df + fd(\sigma(s)) - f\sigma(Ds). \end{aligned}$$

We have to check that this agrees with the Leibniz rule, which says

$$\begin{aligned} D^*(f\sigma)(s) &= \sigma(s)df + fD^*\sigma(s) \\ &= \sigma(s)df + fd(\sigma(s)) - f\sigma(Ds). \end{aligned}$$

Indeed, they do agree.

Next, we suppose (E, h) is hermitian. By Proposition 11.8, it suffices to check that the coefficients of D^* are skew-hermitian in any local unitary frame. If (e_1, \dots, e_n) is a local unitary frame for E , then Proposition 11.8 tells us that (Θ_{ij}) is skew-hermitian. We wish to find the coefficients of the dual connection, so write

$$D^*e_i^* = \sum \Theta_{ij}^*e_j^*.$$

To find Θ_{ij}^* , we evaluate on e_j :

$$D^*e_i^*(e_j) = \Theta_{ij}^*.$$

By definition, the left hand side is

$$d(e_i^*(e_j)) - e_i^*(De_j) = 0 - e_i^*\left(\sum \Theta_{jk}e_k\right) = \Theta_{ji}.$$

We have thus found that $\Theta_{ij}^* = \Theta_{ji}$, i.e. the coefficient matrix of the dual is the transpose. Now it is obvious that (Θ_{ij}^*) is skew-hermitian if (Θ_{ij}) is.

Finally, we must show that if D is compatible with the holomorphic structure on E , then D^* is compatible with the holomorphic structure on D^* . By Proposition 11.6, it suffices to check that the coefficients of D^* are $(1, 0)$ forms in any local holomorphic frame. If (s_1, \dots, s_j) is a local holomorphic frame for E , then Proposition 11.6 tells us that the coefficients Θ_{ij} in this frame are all $(1, 0)$ -forms. By the previous computation, for the dual holomorphic frame (s_1^*, \dots, s_j^*) the coefficients $\Theta_{ij}^* = \Theta_{ji}$ are also $(1, 0)$ -forms.

11. *If E is a holomorphic vector bundle on a complex manifold X and $F \subset E$ is a holomorphic sub-bundle, then a hermitian metric on E induces one on F and we have a direct sum decomposition of complex smooth bundles $E = F \oplus F^\perp$. If D_E is the associated connection on E , show that the composite (in obvious notation) $\pi_F \circ D_E$ is the associated connection on F .*

We must check that $D_F := \pi_F \circ D_E$ is compatible with the holomorphic and hermitian structures on F . For the first, we want to show that $D_F'' = \bar{\partial}_F$. But it is clear that $D_F'' = \pi_F \circ D_E''$. You can easily check that $\bar{\partial}_F = \pi_F \circ \bar{\partial}_E$ either by checking the the right hand side satisfies the defining conditions (Leibniz rule and agreement with $\bar{\partial}$ on \mathcal{A}^0) or by examining our explicit construction. The result can also be interpreted via our local description (Proposition 11.6): in those terms, it is equivalent to the assertion that a symmetric submatrix of a matrix of $(1, 0)$ forms is still pure of type $(1, 0)$, which is obvious.

For the compatibility with the the hermitian structure, observe that for $\alpha, \beta \in F$, we have $(\alpha, \beta)_F = (\alpha, \beta)_E$, so

$$\begin{aligned} d(\alpha, \beta)_F &= d(\alpha, \beta)_E \\ &= (D_E\alpha, \beta) + (\alpha, D_E\beta) \\ &= (\pi_F D_E\alpha, \beta) + (\alpha, \pi_F D_E\beta) \\ &= (D_F\alpha, \beta) + (\alpha, D_F\beta). \end{aligned}$$

The result can also be interpreted via our local description (Proposition 11.8): in those terms, it is equivalent to the assertion that a submatrix of a skew-Hermitian matrix is skew-Hermitian, which is obvious.