1 Introduction and motivation

1.1 The prototypical example

Let $E$ be an elliptic curve over a field $k$; imagine for the moment that $k$ is of characteristic 0, although we will also be very interested in the characteristic $p$ case. The $p$-divisible group of $E$ is easy enough to describe: it is the “direct limit” of the $p^n$-power torsion subgroups of $E$. (For now, let’s brush aside technical issues such as in what category this direct limit takes place.)

$$E[p] \hookrightarrow E[p^2] \hookrightarrow E[p^3] \hookrightarrow \ldots$$

Since we know that $E[p^n](\overline{k}) \cong \mathbb{Z}/p^n \oplus \mathbb{Z}/p^n$, we see that

$$\lim_{\rightarrow} E[p^n](\overline{k}) \cong \mathbb{Q}_p/\mathbb{Z}_p \oplus \mathbb{Q}_p/\mathbb{Z}_p.$$ 

More generally, we can repeat this construction for any group scheme. The key case of interest is an abelian variety, which is a group scheme that is projective. These are the higher-dimensional generalizations of elliptic curves. (Over the complex numbers, you can imagine them as a quotient of $\mathbb{C}^n$ by a lattice, but only for some lattices will the quotient admit the structure of an algebraic variety.)

1.2 Why $p$-divisible groups?

The formalism of $p$-divisible groups may seem a little bizarre at first, but the ends justify the means. One point of view is that they turn out to be a robust bridge between geometry (e.g. of abelian varieties) and linear algebraic data. While the finite torsion subgroups of an abelian variety capture very little information about the ambient variety - for instance, they are mostly insensitive to the isomorphism class of the abelian variety - the $p$-divisible group “knows” a lot about the abelian variety. On the other hand, $p$-divisible groups are equivalent to linear-algebraic data called Dieudonné modules, which will be discussed in the second talk on $p$-divisible groups. This makes them easier to get a handle on.

One articulation of the above philosophy is an important theorem due to Serre–Tate that “deformations of an abelian variety as the same as deformations of its $p$-divisible group”.

More precisely, let $R_0 = R/I$ where $R$ is a ring such that $p^N R = 0$ for some $N > 0$ and $I$ is a nilpotent ideal of $R$. Let $A(R)$ be the category of Abelian schemes over $R$ and let $\text{Def}(R, R_0)$ be the category of triples $(A_0, G, \epsilon)$ so that $G$ is a $p$-divisible group, $A_0$ is an Abelian scheme over $R$ and $\epsilon$ is an isomorphism of $G$ with the $p$-divisible group associated to $A_0$. Then we have
Theorem 1.1 (Serre-Tate). The natural functor $A(R) \to \text{Def}(R, R_0)$ is an equivalence of categories.

1.3 Relation to formal groups

We can also explain how $p$-divisible groups complement the theory of formal groups attached to abelian varieties (this will be elaborated upon later).

Suppose $E/K$ is an elliptic curve over $K$ a perfect field of characteristic $p$. Taking the group law of $E$ in a neighborhood of the identity, we get a formal group associated to $E$. These formal groups can have two possible heights.

- If $E(K)[p]$ (the geometric $p$-torsion of $E$) contains exactly $p$ points then we say $E$ is ordinary and the formal group has height 1.

- If $E(K)[p] = \{0\}$ then we say $E$ is supersingular and the formal group has height 2.

This discrepancy in heights is a first indication that the formal group of $E$ isn’t the best choice of algebraic object associated to $E$. This can be remedied by passing to the $p$-divisible group instead. Roughly speaking, we will think of the $p$-divisible group corresponding to an elliptic curve as being the product of the formal group and the geometric $[p^n]$-torsion. Once we make this precise, we will see that the $p$-divisible group corresponding to $E$ will have height two in all cases.

2 Basic Properties

2.1 Preliminaries on finite group schemes

Let $R$ be a commutative ring. Then a finite group scheme of order $m$ over $R$ is the spectrum of an $R$-algebra $A$, free of rank $m$, with a cogroup structure. Explicitly, this structure is given by maps $\mu : A \to A \otimes A, i : A \to A, e : A \to R$ corresponding to comultiplication, inverse, and identity maps and satisfying the axioms of a cogroup.

Example 2.1. The group scheme $\mu_m$ is associated to the ring $A = R[x]/(x^m - 1)$ with comultiplication $\mu(x) = x \otimes x$. This is the kernel of $[m] : G_m \to G_m$. Notice that this scheme contains $m$ points if $(m, p) = 1$ for $p$ the characteristic of $R$ but only 1 if $m = p^k$.

Example 2.2. Given a finite group $G$, we can form the constant group scheme $G$ corresponding to the group ring $R[G]$ and comultiplication

$$\mu(e_g) = \sum_{g=\sigma\tau\in G} e_\sigma e_\tau.$$ 

If we think of a finite group scheme as a family of finite groups, then this is, as suggested by the name, the constant family.

Example 2.3. Given an Abelian variety $A$ of dimension $g$ over a field $K$, the kernel of $[p^n] : A \to A$ is a finite group scheme of order $p^{2ng}$. 

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2.2 p-divisible groups

Definition 2.4. Let $p$ a prime and $h$ a non-negative integer. A $p$-divisible group of height $h$ is an inductive system $(G_v, i_v), v \geq 0$ where $G_v$ is a finite group scheme over $R$ of order $p^{hv}$ and $i_v : G_v \hookrightarrow G_{v+1}$ identifies $G_v$ with $G_{v+1}[p^v]$, i.e. so that we have exact sequences

$$0 \to G_v \xrightarrow{i_v} G_{v+1} \xrightarrow{[p^v]} G_{v+1}.$$ 

Remark 2.5. Let’s explore some of the additional structure that is implicit in our definition of $p$-divisible groups. Composing $i_v$ maps and keeping in mind that they are maps of groups and so commute with $[p]$, we get sequences

$$0 \to G_\mu \xrightarrow{i_{\mu,v}} G_{\mu+v} \xrightarrow{[p^v]} G_{\mu+v}.$$ 

Since $G_{\mu+v}[p^v] \to G_{\mu+v}$ factors through the $p^\nu$-torsion, we get a unique map $j_{\mu,v}$ satisfying $[p]^\mu = i_{\mu,v} \circ j_{\mu,v}$.

This gives short exact sequences

$$0 \to G_v \xrightarrow{i_{\mu,v}} G_{v,\mu} \xrightarrow{j_{\mu,v}} G_{\mu} \to 0.$$ 

In particular, $G_\mu$ is “divisible by $p^\nu$” by some group higher up in the inductive limit.

**Fundamental construction.** If $G$ is a group scheme such that $[p] : G \to G$ is surjective, then we can form the $p$-divisible group with $G_v = G[p^v]$ and $i_v : G[p^v] \hookrightarrow G[p^{v+1}]$ the natural inclusion. Here are some examples of this construction:

**Example 2.6.** The system $(\mathbb{Z}/p^n\mathbb{Z}, i_v)$ given by $\mathbb{Z}/p^n\mathbb{Z} \xrightarrow{i_v} \mathbb{Z}/p^{n+1}\mathbb{Z}$ corresponding to the standard inclusion of finite groups gives a $p$-divisible group of height 1 which we denote $\mathbb{Q}_p/\mathbb{Z}_p$.

**Example 2.7.** The system $(\mu_{p^n}, i_v)$ given by the standard inclusion is a $p$-divisible group of height 1 which we denote $\mu_{p\infty}$.

**Example 2.8.** $(\ker([p^n] : A \to A), i_v)$ for $A$ an Abelian variety of dimension $g$ is a $p$-divisible group of height $2g$.

2.3 Cartier duality

Let $G = \text{Spec} A$ a finite group scheme over $R$ with comultiplication $\mu : A \to A \otimes A$ and multiplication $m : A \otimes A \to A$. If we write $A' = \text{Hom}_{R-mod}(A, R)$ then $\mu, m$ induce maps $\mu' : A' \otimes A' \to A', m' : A' \to A' \otimes A'$ which give $G^* := \text{Spec} A'$ a group scheme structure.

It is a small exercise in algebra to show that the functor of points is

$$G^*(T) = \text{Hom}_T(G_T, \mathbb{G}_{m,T})$$

is the “character group” of $G$. We call $G^*$ the Cartier dual of $G$. The functor $G \mapsto G^*$ is exact and $(G^*)^*$ and $G$ are canonically isomorphic.
Given a $p$-divisible group $(G_v, i_v)$ and exact sequences
\[ G_{v+1} \xrightarrow{[p]^{v+1}} G_v \xrightarrow{[p]} G_v \xrightarrow{} 0 \]
we get a dual sequence
\[ 0 \to G^*_v \xrightarrow{[p]^*} G^*_v \xrightarrow{[p']^*} G^*_v \xrightarrow{} G^*_v. \]

We’d like these sequences to define the structure of a $p$-divisible group, and we observe that they do so since $[p]_{G^*} = [p]_{G^*}$ on account of this map being an iteration of the multiplication and comultiplication maps.

**Definition 2.9.** Given a $p$-divisible group $(G_v, i_v)$ we define the dual $p$-divisible group $(G^*_v, j_{v,v+1})$ according to the discussion above.

**Example 2.10.**
1. The dual of $\mathbb{Q}_p/\mathbb{Z}_p$ is $\mu_{p^n}$.
2. If $G$ is the $p$-divisible group associated to the Abelian variety $A$, then $G^*$ is the $p$-divisible group associated to the Abelian variety $A^*$. This follows from the fact that $A[p^n]^* = A^*[p^n]$.
3. Since elliptic curves are self-dual, so are their associated $p$-divisible groups.

## 3 Some background on finite flat group schemes

Since $p$-divisible groups are built out of finite flat group schemes, we take a small detour to discuss the basics of this theory.

### 3.1 Étale schemes

We say that a finite flat group scheme $G = \text{Spec } A$ over Spec $R$ is étale if $A$ is an étale $R$-algebra. There are many ways of defining what this means, but we’ll take as our working definition will be that $A$ is a finite flat $R$-module, and the module of relative ("vertical") differentials $\Omega^1_{A/R}$ is 0.

Intuitively, this says that $\text{Spec } R \to \text{Spec } A$ is a "finite covering map". This is probably what you first imagine when you envision a finite group scheme.

**Example 3.1.** If $G$ is a finite group, there is a “constant group scheme $G_R$” over Spec $R$ which one should think of as, well, just that. Its ring of functions is the ring of maps $G \to R$.

For $S$ connected and $\overline{\pi} \to S$ any geometric point there is an equivalence of categories between finite étale $S$-schemes and $\pi_1(S, \overline{\pi})$-sets; this restricts to an equivalence of categories
\[ \{\text{finite étale group schemes }/S\} \leftrightarrow \{\text{groups with } \pi_1(S, \overline{\pi})\text{-action}\}. \]

**Theorem 3.2.** Over a field of characteristic 0, all finite flat group schemes are étale.

**Proof sketch.** The idea is to show that any non-trivial “tangent” directions propagate to an infinite group scheme. Let $I \subset A$ be the augmentation ideal of the Hopf algebra, and pick generators $x_1, \ldots, x_n$ for $I/I^2$. Then we have a surjective map
\[ k[X_1, \ldots, X_n] \to \text{Gr}_I(A) := A/I \oplus I/I^2 \oplus I^2/I^3 \oplus \ldots \]

sending $X_i \mapsto x_i$. 

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An important fact about affine group schemes is that the map $A \to (I/I^2) \otimes_R A$, where the first coordinate is projection onto $I$ followed by quotienting by $I^2$, is the universal derivation. This tells us that there are enough derivations. In particular we can find $D_i: A \to A$ such that $D_i(x_j) = \delta_{ij}$. The above map intertwines $\frac{d}{dx_i}$ with $D_i$, so its kernel is stable under the derivations $\frac{d}{dx_i}$. But if there were anything non-zero in the kernel, we could differentiate it to eventually obtain a scalar, which would generate the unit ideal. (This last step is what fails in positive characteristic!)

### 3.2 Connected schemes

At the “opposite” end of the spectrum one has connected group schemes. An étale group scheme has the property (and in fact is characterized by it) that the identity section is open; in particular, an étale group scheme is “as disconnected as possible”. By contrast, a connected group scheme is one that is, well, connected. This is slightly unintuitive because it’s only a characteristic $p$ phenomenon (in characteristic 0, all finite flat group schemes are étale), so let’s take a look at an example.

**Example 3.3.** Recall that $\mu_p = \text{Spec } k[t]/t^p - 1$. Over $\mathbb{F}_p$, we have

$$k[t]/t^p - 1 = k[t]/(t - 1)^p,$$

which is a “fat point”; in particular, it is supported at a single point and so must be connected.

**Example 3.4.** Over $\mathbb{Z}_p$ (which has “characteristic” 0!), $\mu_p$ is still connected, since it is connected over the special fiber.

A connected group scheme cannot map non-trivially to an étale one, since it would factor through the connected component. Conversely, an étale group scheme cannot map non-trivially to a connected one; in fact any map from a reduced variety to a non-reduced one factors through the reduction (at the level of rings, this is saying that a map from a ring to a reduced one kills the nilradical).

**Remark 3.5.** One can push the argument above further to show that any finite flat group scheme which is connected has order a power of $p$.

### 3.3 The connected-étale sequence

For the results of this section, we assume that our base is a henselian local ring. (For instance, a field or a discrete valuation ring.)

**Proposition 3.6.** If $G$ is any group over a Henselian noetherian local ring then there is a short exact sequence

$$0 \to G^0 \to G \to G^{\text{ét}} \to 0$$

with $G^0$ connected and $G^{\text{ét}}$ étale.

One chooses $G^0$ to be the connected component of the identity of $G$, and takes $G^{\text{ét}}$ to be the quotient $G/G^0$. The latter is étale because its identity section is open. What is not obvious is that $G^0$ really is a subgroup; this follows from understanding the connected components in terms of orbits of the étale fundamental group on the special fibers.

**Remark 3.7.** At the level of algebras, if $G = \text{Spec } A$ is finite flat over $R$ then $G^{\text{ét}} = \text{Spec } A^{\text{ét}}$ where $A^{\text{ét}}$ is the maximal subalgebra of $A$ étale over $R$. Of course, it requires proof that such a subalgebra exists.
3.4 The $F$ and $V$ maps

I find most discussions of the Frobenius maps to be very confusing, so I’m going to go through it in painstaking detail. Suppose $X = \text{Spec } A$ is a scheme over a ring $R$ in characteristic $p$ (you can safely assume that $R = F_q$ if you like). As you know, in this situation the map $F_A: a \mapsto a^p$ is actually a ring homomorphism $A \to A$. However, it does not define a map of schemes over $\text{Spec } R$ in general, since it does not act as the identity on $R$. Equivalently, this is not an $R$-algebra homomorphism.

Now for something which may seem confusing at first, but you’ll eventually see is basically a tautology. Consider the $R$-algebra $A^{(p)} = A \otimes_{R,F_R} R$, where this means we take the tensor product of $A$ and $R$ where the latter is viewed as an $R$-algebra not by the identity map, but by the map $F_R$. Explicitly, in $A^{(p)}$ if $a \in A$ and $\lambda \in R$, then $a \otimes \lambda^p = a^p \otimes 1$.

Geometrically, you can think of $X^{(p)} := \text{Spec } A^{(p)}$ as the fibered product

$$
\begin{array}{ccc}
X^{(p)} & \rightarrow & X \\
\downarrow & & \downarrow \\
\text{Spec } R & \rightarrow & \text{Spec } R
\end{array}
$$

Now here’s the point: we have an $R$-algebra homomorphism $A^{(p)} \to A$ sending $a \otimes \lambda \mapsto a^p \lambda$. This defines a relative Frobenius $F_{X/R}: X \to X^{(p)}$.

Example 3.8. Suppose $k = F_q$ and $A = k[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$, i.e. imagine that $X$ is the variety cut out by $f_1 = \ldots = f_m = 0$. Intuitively, $X$ consists of tuples $(t_1, \ldots, t_n) \in k^n$ satisfying the equations $f_1, \ldots, f_m$. Then the relative Frobenius map acts by raising the “coordinates” $t_1, \ldots, t_n$ to the $p$th power. Of course, after doing this the points no longer need satisfy the equations $f_1, \ldots, f_m$ anymore. However, since

$$
f_i(t_1, \ldots, t_n)^p = f_i^{(p)}(t_1^p, \ldots, t_n^p)
$$

where $f_i^{(p)}$ is the polynomial obtained by raising the coefficients of $f_i$ to the $p$th power, the points will satisfy $f_1^{(p)} = \ldots = f_m^{(p)} = 0$. The variety $X^{(p)}$ is precisely that cut out by these equations.

In particular, if $G$ is a finite flat group scheme over $R$ then we obtain a finite flat scheme $G^{(p)}$ over $R$. You can check that $G^{(p)}$ again has the structure of a finite flat group scheme; the morphisms and commutative diagrams involved are just base-changes of those defining the group structure on $G$:  

$$
\begin{array}{ccc}
G^{(p)} \times G^{(p)} & \rightarrow & G \times G \\
\downarrow & & \downarrow \\
G^{(p)} & \rightarrow & G \\
\downarrow & & \downarrow \\
R & \rightarrow & R
\end{array}
$$
We define the (relative) Frobenius \( F : G \rightarrow G^{(p)} \).

On Cartier duals, we also have \( F^* : G^* \rightarrow G^*^{(p)} = G^{(p)*} \).

We can define the Verschiebung map

\[
V : G^{(p)} \rightarrow G
\]

to be the dual of \( F^* \).

**Remark 3.9.** Some people might consider this to be cheating. In the literature, the map \( V \) is defined in some other way and then shown to coincide with the dual of \( F^* \).

**Some facts.**

- The map \( F : G \rightarrow G^{(p)} \) is a monomorphism (and equivalently, by order considerations, an isomorphism) if and only if \( G \) is étale.
  
  *(Proof sketch: Reduce to the case of \( G = \text{Spec } A \) over an algebraically closed field \( k \).
  
  The idea is that Frobenius acts as \( 0 \) on the tangent space (basically because \( d(a^p) = 0 \) in characteristic \( p \)). But the tangent spaces are \( \mathfrak{m}/\mathfrak{m}^2 \), so they are killed under a monomorphisms if and only if \( \mathfrak{m}/\mathfrak{m}^2 = 0 \). This can only happen if \( A \) is a product of \( k \).)

- The compositions \( V \circ F : G^{(p)} \rightarrow G^{(p)} \) and \( F \circ V : G^{(p)} \rightarrow G^{(p)} \) are multiplication by \( p \) in the respective group laws.

### 4 More on \( p \)-divisible groups

#### 4.1 The connected-étale sequence

We assume that we are working over a complete noetherian local ring \( R \). Let \( G = (G_v, i_v) \) be a \( p \)-divisible group. For each finite group \( G_v \), we have the connected-étale sequence

\[
0 \rightarrow G_v^0 \rightarrow G_v \rightarrow G_v^\text{et} \rightarrow 0.
\]

These fit together into a connected-étale sequence for the entire \( p \)-divisible group: that is, there is a short exact sequence of \( p \)-divisible groups

\[
0 \rightarrow G^0 = (G^0_v) \rightarrow G = (G_v) \rightarrow G^\text{et} = (G_v^\text{et}) \rightarrow 0.
\]

To check this, we have to verify that \( i_v : G_v^0 \rightarrow G_v^{0+1} \) may be identified with the inclusion of the \( p^v \)-torsion, and \( i_v : G_v^{\text{et}} \rightarrow G_v^{\text{et}+1} \) may be identified with the inclusion of the \( p^v \)-torsion. These are straightforward using the basic properties. For example, the restriction \( i_v|_{G_v^0} \) maps into \( G_v^{0+1} \) because the composition to \( G_v^{0+1} \) is a map from a connected group scheme to an étale one, which is necessarily trivial. We leave the details to the reader.

Philosophically, the connected-étale sequence decomposes a \( p \)-divisible group into two parts which can both be understood in a simpler way. For instance, recall the equivalence of categories

\[
\{\text{finite étale group schemes }/S\} \leftrightarrow \{\text{groups with } \pi_1(S, \overline{s})\text{-action}\}.
\]
Therefore, an étale $p$-divisible group is simply a group which is $p$-divisible in the usual sense with some finiteness properties, which is furthermore equipped with $\pi_1(S, x)$-action.

When $R$ is a complete noetherian local ring with residue field $k$ of characteristic $p$, the connected part can also be understood as arising from formal groups by the fundamental construction of taking $p$-power torsion. More precisely, let $\Gamma$ be a formal group (law) over $R$, i.e. a power series

$$F(X_1, \ldots, X_n, Y_1, \ldots, Y_n)$$

satisfying properties that make it look like the a commutative group law on a Lie group.

Then we can form an associated $p$-divisible group

$$\Gamma(p) = (\Gamma_p, i_v)$$

where $\Gamma_p$ is the $p^v$-torsion of $\Gamma$. (Concretely, if $\Gamma = \text{Spf}\ A$ then $\Gamma_p = \text{Spec}\ A/[p^v]^*(I)A$, where $I$ is the augmentation ideal.)

**Theorem 4.1** (Tate). The association $\Gamma \mapsto \Gamma(p)$ defines an equivalence of categories between the category of divisible commutative formal Lie groups over $R$ and the category of connected $p$-divisible groups over $R$.

The proof is rather involved, but the idea is simple enough. The point is to show that a connected $p$-divisible group comes from a smooth formal group. If $G_v = \text{Spec}\ A_v$, then one tries to define $\Gamma = \text{Spf}\ \varprojlim A_v$. The task is then to show that $\varprojlim A_v$ is a power series ring, which involves some gymnastics in commutative algebra. The flatness assumption allows us to reduce to the residue field.

**Definition 4.2.** If $G = (G_v)$ is a $p$-divisible group, then we define the dimension of $G$ to be the dimension of the formal group associated to $G^0$.

**Proposition 4.3.** Let $G$ be a $p$-divisible group of dimension $d$ and height $h$. If $d' = \dim G^*$, then we have

$$h = d + d'.$$

**Proof.** Thanks to the factorization

$$[p]: G \xrightarrow{F} G^{(p)} \xrightarrow{V} G$$

we have a short exact sequence

$$0 \to \ker F \to \ker[p] \xrightarrow{F} \ker V \to 0.$$

By definition, $\deg\ker[p] = p^h$.

We have to calculate $\deg\ker F$. Since $F$ is injective on $G^0$, we have $\ker F \subseteq G^0$. Therefore, $\ker F$ coincides with the analogous map on a smooth formal group. Since $F$ corresponds to

$$k[[T_1, \ldots, T_d]] \to k[[T_1, \ldots, T_d]]$$

sending $T_i \mapsto T_i^p$, we can see by inspection that $\ker F$ has degree $p^d$. Dually, $\ker V$ has degree $p^{d'}$. So the short exact sequence implies

$$p^d p^{d'} = p^h$$

hence the result.

**Example 4.4.** The $p$-divisible group $\mathbb{G}_m(p)$ is connected, with $d = h = 1$. Its dual is the étale $p$-divisible group $\mathbb{Q}_p/\mathbb{Z}_p$, which has $d = 0$ and $h = 1$. 

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4.2 The $p$-divisible group of an abelian variety

Let $G = (G_v, i_v)$ be the $p$-divisible group of an abelian variety $X$. We have the connected-étale sequence

$$0 \to G^0 \to G \to G^{\text{ét}} \to 0.$$ 

What can we say about $G^0$ and $G^{\text{ét}}$? If $X$ is an abelian variety and $G$ its attached $p$-divisible group. Then the formal completion $\hat{X}$ can be identified with the $p$-divisible group attached to $G^0$. We will elaborate on what it means below, but the intuition is that $G^0$ has a formal group law which looks like the group law on $X$ near the identity. The height of $G^0$ can be any integer between $n$ and $2n$; for an elliptic curve these are the “ordinary” and “supersingular” cases. Therefore, $\dim G^0 = \dim X = d$. Since $G$ has height $h = 2d$, we also have $d' = d$.

**Example 4.5.** The Tate curve is an elliptic curve over a $p$-adic field, let’s just say $\mathbb{Q}_p$ for simplicity - which has a uniformization $\mathbb{G}_m/q^\mathbb{Z}$. This is an exponential version of the classical uniformization

$$\mathbb{C}/\mathbb{Z} \oplus \mathbb{Z} \exp \to \mathbb{C}^*/e^{2\pi i \tau}.$$ 

This uniformization is in the sense of “rigid analytic varieties”, which we don’t want to fully explain. However, a particular consequence of it is that

$$C(\mathbb{Q}_p) = \mathbb{Q}_p^*/q^\mathbb{Z}$$

so that the $p^n$ torsion is

$$0 \to \mu_{p^n} \to C[p^n] \to (q^{1/p^n}) \to 0$$

an extension of $\mathbb{Z}/p^n$ by $\mu_{p^n}$.

The $p$-divisible group is then an extension

$$0 \to \mathbb{G}_m(p) \to G \to \mathbb{Z}_p \to 0$$

which is its connected-étale decomposition.

4.3 Formal completion

Here we justify our claim that the connected part of the $p$-divisible group attached to an abelian variety $X$ is its “formal completion”. This will be almost immediate once we understand what that means. Unfortunately, we can’t really do justice to it; that would require discussing the framework of formal schemes more than we want to do.

Let $X$ be an abelian variety. We will describe a process of extracting from $X$ a formal group (law). Intuitively, this is some kind of localization process, which contains information about $A$ in a “small neighborhood of the identity”.

So how should we do this? Unfortunately, the technical framework of formal groups is a bit delicate. The “right” approach will probably take us too far afield, so we’ll adopt a clumsier, but more intuitive, point of view.

Recall that a scheme over a field $k$ can be interpreted a functor from $k$-algebras to sets. A formal scheme will be a functor on a more limited category of $k$-algebras which represent “infinitesimal” schemes. More precisely, we consider the category of Artin local rings $R$ over $k$. (Think $R = k[e]/e^n$.) A formal scheme will be a functor from this category to sets, satisfying conditions analogous to those cutting out schemes among functors from general rings to sets.
An Artin local ring has a unique maximal ideal $m$; we can think of it as a point with some amount of infinitesimal “direction” specified. In particular, the spectrum of an Artin local ring has a single closed point.

Now, we define the formal completion along the identity of an abelian variety $X$ to be the functor which takes $R$ to the points of $X(R)$ which send the maximal ideal to the identity. This is the sense in which it captures the “infinitesimal neighborhood” of $X$.

**Proposition 4.6.** This functor is “represented” by a power-series ring $k[[X_1,\ldots,X_n]]$, in the sense that

$$\hat{X}(R) = \text{Hom}_{\text{cont}}(k[[X_1,\ldots,X_n]], R).$$

**Proof.** Any map $\text{Spec } R \to X$ factors through an (in fact, any) affine neighborhood of $e$. (As discussed, $\text{Spec } R$ is a single point!) Therefore, it factors through $\mathcal{O}_{X,e}$. By the Artinian property, any such map $\text{Spec } R \to X$ factors through $\text{Spec } \mathcal{O}_{X,e}/m^n_{X,e}$ for some $n$. So all such maps come from

$$\text{Hom}(\lim_{\leftarrow} \mathcal{O}_{X,e}/m^n_{X,e}, R).$$

Now, since $X$ is smooth, the completion of its local rings are isomorphic to power series rings.

Choosing such an isomorphism

$$\lim_{\leftarrow} \mathcal{O}_{X,e}/m^n_{X,e} \cong k[[T_1,\ldots,T_d]]$$

The group operations on

$$X(\mathcal{O}_{X,e}/m^n_{X,e} \otimes \mathcal{O}_{X,e}/m^n_{X,e})$$

fit together into a group law $X \times X \to X$ which defines a formal group law $F$ on $k[[T_1,\ldots,T_d]].$

With this understanding in place, our earlier claim is nearly a tautology. The formal completion $\hat{X}$ is the functor taking an Artin local ring $R$ to the maps $\text{Spec } R \to \lim X$ which send the (closed) point to $e$. Its $p$-divisible group is the functor taking $R$ to maps $\text{Spec } R \to \lim X[p^n]$, sending the (closed) point to $e$.

This is the subfunctor of the $p$-divisible group of $X$, which takes $R$ to $\text{Hom}(\text{Spec } R, X[p^n])$ without the condition that the closed point is sent to $e$. But since $\text{Spec } R$ is connected, this condition forces any such map to factor through the connected component of $\lim X[p^n]$. So the latter condition is the same as the condition of factoring through the connected component.