ABSTRACT

These notes are based on lectures given by Clément Mouhot at Cambridge in Michaelmas 2013. However, the original material has been significantly supplemented by other resources, most notably Evans’s Partial Differential Equations, due to the scribe’s lack of skill. The notes are still rather rough, and all of the errors should be attributed to the scribe. If you find any, please let me know about them at tonyfeng009@gmail.com
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1.1 FROM ODE TO PDE

The simplest example of a differential equation is the first-order equation

$$\frac{du}{dx} = F(x)$$

where $F : \mathbb{R} \to \mathbb{R}$ is a continuous function. The solution, of course, is given by the theory of integration:

$$u(x) = u(x_0) + \int_{x_0}^{x} F(y) \, dy.$$  

More generally, an $n$-th order ordinary differential equation takes the form

$$F(x, u, u^{(1)}, \ldots, u^{(n)}) = 0.$$  

In the theory of partial differential equations, we generalize this notion to functions of several real variables. We then study solutions to an equation of the form

$$F \left( x_0, x_1, \ldots, x_n, \frac{\partial u}{\partial x_0}, \ldots, \frac{\partial^2 u}{\partial x_i \partial x_j}, \ldots, \frac{\partial^3 u}{\partial x_i \partial x_j \partial x_k} \right) = 0.$$  

There are two (essentially equivalent) viewpoints on how PDE generalize ODE:

1. a PDE is simply an ordinary differential equation in several variables, involving partial derivatives.

2. when one variable can be identified as “time,” the PDE can be interpreted as an ODE trajectory

$$\frac{du}{dt} = G(u) := G \left( x, u, \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j}, \ldots, \frac{\partial^3 u}{\partial x_i \partial x_j \partial x_k} \right)$$

in infinite-dimensional functional spaces.

The first viewpoint is more geometrical, while the second brings in infinite-dimensional functional analysis. Either way, it is clear that complex and deep new phenomena appear, as one is working in a much more subtle geometric space, or in infinite-dimensional vector spaces where, for instance, there can be many different norms, and operators are generally unbounded.

Unlike the theory of ODEs, there are not very many general results for PDEs. So we rely instead on guiding principles; in this course, we focus on the fundamental equations from physics (possibly with careful simplications), emphasizing the concepts and methods useful to PDEs in general.
1.2 THE CAUCHY PROBLEM FOR ODES

1.2.1 The three main theorems of ODE

Suppose \( t \in I \subset \mathbb{R} \), and we are considering the ODE

\[
  u'(t) = F(t, u(t))
\]

where \( u : I \to \mathbb{R}^m \). Written out explicitly, this corresponds to the system of equations

\[
  u'_1(t) = F_1(t, u_1, \ldots, u_m) \\
  \vdots \\
  u'_m(t) = F_m(t, u_1, \ldots, u_m)
\]

plus some initial data. Here \( F \) is called the vector field and a solution is called a flow.

There are three main theorems regarding the existence and uniqueness of solutions to such a system.

**Theorem 1.2.1** (Cauchy-Kovalevskaya). If \( F \) is analytic in \( A \subset I \times \mathbb{R}^m \) then there is a unique local analytic solution to (1) about any point of \( A \).

This is the only result that generalizes in a meaningful way to PDE, and we will study it in the next chapter.

**Theorem 1.2.2** (Picard-Lindelöf/Cauchy-Lipschitz). Suppose \( F \) is continuous (in both variables) and locally Lipschitz in the second argument on \( A \subset I \times \mathbb{R}^m \), i.e. for any \( (t_0, x_0) \in A \) there is an open neighborhood \( U \) and a constant \( C \) such that for all \( (t, x) \) and \( (t, y) \in U \), we have

\[
  |F(t, x) - F(t, y)| \leq C|x - y|.
\]

Then there exists a unique local \( C^1 \)-solution to (1).

The idea of the proof is as follows. We want to write the solution as an integral

\[
  u(t) = u(t_0) + \int_{t_0}^{t} F(s, u(s)) \, ds.
\]

Of course, this doesn’t make sense, so we turn it into an iterated procedure

\[
  u^{(n+1)}(t) = u(t_0) + \int_{t_0}^{t} F(s, u^{(n)}(s)) \, ds.
\]

Phrased differently, we can define a function

\[
  \Phi[u] = \left( t \mapsto u(t_0) + \int_{t_0}^{t} F(s, u(s)) \, ds \right)
\]

and apply fixed-point theorems to this function in order to show that it converges to a solution.

**Theorem 1.2.3** (Cauchy-Peano). In the region \( C \subset I \times \mathbb{R}^m \) where \( F \) is continuous (in both variables), there exist local \( C^1 \) solutions.
Remark 1.2.4. Note that the theorem does not say anything about uniqueness. It is possible that the solutions bifurcate.

How do you prove this? By using an iterative scheme again, together with the Arzela-Ascoli compactness theorem.

Example 1.2.5. The differential equation

\[
\begin{align*}
    u'(t) &= \sqrt{u(t)} \\
    u(0) &= 0
\end{align*}
\]

has infinitely many solutions. Indeed, if we solve by the usual method

\[
\frac{du}{\sqrt{u}} = dt
\]

then we can integrate to recover a solution provided that \(u\) is non-zero. So a family of solutions is given by \(u = \frac{1}{4}(t - t_0)^2\).

Exercise 1.2.6. Consider the ODEs

\[
\begin{align*}
    u'(t) &= 3u(t)^{2/3} \\
    u(0) &= 0
\end{align*}
\]

and

\[
\begin{align*}
    u'(t) &= \frac{4u(t)^3}{u(t)^2 + 4s} \\
    u(0) &= 0.
\end{align*}
\]

For each, see how many solutions you can find.

1.2.2 Continuation of Solutions: Local vs Global

So far we have discussed local existence and uniqueness of solutions. What about global solutions? If you have global existence and local uniqueness, then you can patch together local solutions.

What else could go wrong? The solution could “blow up” in finite time. This is a phenomenon that you should keep in mind for the PDE case, where it can be much more complex.

Example 1.2.7. Consider the ODE

\[
\begin{align*}
    u'(t) &= u^2 \\
    u(0) &= u_0 > 0.
\end{align*}
\]

The solution is \(u(t) = \frac{1}{u_0^{1-t}}\), so we have blowup at time \(T_* = u_0^{-1}\).

Now consider the ODE

\[
\begin{align*}
    u'(t) &= -u^2 \\
    u(0) &= u_0 > 0.
\end{align*}
\]

This does not blow up; intuitively, the speed decreases the larger \(u(t)\) is. Indeed, a solution is of the form \(u(t) = \frac{1}{u_0^{-1} + t}\).
A blow-up requires super-linearity: a function that grows faster than linear at $\infty$. Similarly, a lack of uniqueness is usually related to sublinearity (the vector field grows too slowly). However, as we saw above, the sign of the non-linearity plays a crucial role, and this will hold true for PDEs as well.

One way to avoid blowup in finite time is to impose a uniform Lipchitz condition

$$|F(t,x)| \leq c(|t| |x|).$$

To prove that this leads to finite solutions, one can establish a bound of the form

$$|u(t)|^2 \leq e^{Ct}|u(0)|^2.$$

One then argues that for any finite time $t$, $u(t)$ has limit and hence so does $u'(t)$ (if the vector field is well-behaved). Then one can continue the solution locally beyond the point $t$.

**Exercise 1.2.8.** (a) Prove that the solution to the pendulum equation

$$\begin{cases}
  u''(t) + \sin u = 0 \\
  u(0) = u_0 \in \mathbb{R}
\end{cases}$$

is global. Do the same for the equation

$$\begin{cases}
  u''(t) + \sin(u(t)^2) = 0 \\
  u(0) = u_0 \in \mathbb{R}
\end{cases}$$

**Example 1.2.9.** The linear ODE

$$u'(t) = Au(t)$$

boils down to a study of the matrix $A$. If $A$ is diagonalizable, then this is easy to solve component-by-component. Otherwise, use the Jordan form and solve within each Jordan block; this is more difficult, but doable.

**Example 1.2.10.** The contrast of the equations $u'(t) = \sqrt{u(t)}$ and $u'(t) = u(t)^2$ highlight the principle that sublinearity leads to non-uniqueness and superlinearity leads to finite time blow-up. We will encounter this again when studying PDE. The proofs of uniqueness for PDE are much harder due to the presence of unbounded operators and many different norms.

### 1.3 THE CAUCHY PROBLEM FOR PDES

A prototypical Cauchy problem in PDE takes the form

$$\begin{aligned}
\partial_t u &= F(t, x_1, \ldots, x_\ell, \ldots \partial_i u, \ldots \partial_j u, \ldots) \\
\quad u(0,x) &= u_0(x_1, \ldots, x_\ell)
\end{aligned}$$

where $u = u(t, x_1, \ldots, x_\ell)$.

**Definition 1.3.1.** A Cauchy problem of the form (2) is well-posed if

1. there exists a solution,
2. the solution is unique,

3. the solution depends continuously on the boundary data.

The concept of well-posedness was formulated by Hadamard. The important point is that this question depends only the underlying spaces we are considering. In which space to we ask existence, in which space do we ask uniqueness, and for which topology do we ask continuity? There are many possibilities: strong solutions, weak solutions, entropy solutions... Often these are suggested by the problem itself. On the other hand, we would always like to wind up in a space where the problem is well-posed, which corresponds to “physically meaningful” solution.

What happens to the three main results of ODE when we try to extend them to PDE?

1. The Cauchy-Kovalevskaya theorem does extend, in a limited way. It requires analyticity of $F$, and only gives solutions that are local in time. Furthermore, it has some restrictions: $F$ involving only first-order derivatives (corresponding to “quasilinear equations”), and must satisfy a “non-characteristic condition.”

**Example 1.3.2 (Heat equation).** Consider the heat equation

$$\begin{cases} 
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \\
u(0, x) = \frac{1}{1+x^2}. 
\end{cases}$$

Show that the unique entire series solving this problem has radius of convergence in $x$ equal to 0 for any $t > 0$.

2. The Cauchy-Lipschitz Theorem *seems* okay, in the sense that the proof generalizes quite readily. One needs only fixed-point theorems, which should hold broadly in normed vector spaces. However, it turns out that the Lipschitz condition almost never holds: one is usually consider differential operators in infinite-dimensional spaces, which are virtually always unbounded.

3. The Cauchy-Peano Theorem fails to generalize for the same reason that differential operators are generally unbounded and hence discontinuous. However, the basic technique of using discrete approximations and compactness arguments does work in PDE settings (with much more work).
The Cauchy-Kovalevskaya Theorem is the only one of the three “main theorems of ODE” that generalizes to PDE, we shall see that it is extremely limited in considering only analyticity, and obscures many of the most interesting features of PDE.

2.1 Analytic Functions

We begin with some recollections on analytic functions, starting in the one-variable case.

Definition 2.1.1. Let $\Omega \subset \mathbb{R}$ be an open subset. A function $f : \Omega \to \mathbb{R}$ is analytic at $x_0$ if there exists some open neighborhood $U \subset \Omega$ containing $x_0$ such that for all $x \in \Omega$, the Taylor series for $f(x)$ about $x_0$ converges and is equal to $f$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n.$$ 

We say that $f$ is analytic on $\Omega$ if it is analytic at all $x_0 \in \Omega$.

Proposition 2.1.2. Let $\Omega \subset \mathbb{R}$. A function $f : \Omega \to \mathbb{R}$ is analytic in $\Omega$ if and only if for every compact subset $K \subset \Omega$, there exist constants $c(K)$ and $r > 0$ such that for all $x \in K$ and $n \geq 0$, we have

$$|f^{(n)}(x)| \leq c(K) \frac{n!}{r^n}.$$ 

Proof. Suppose that $f$ is analytic on $\Omega$ and let $x_0 \in \Omega$. Then in fact $f$ defines a complex analytic function on some open neighborhood of $\Omega$ in $\mathbb{C}$. If $K \subset \Omega$ is a compact subset, then $f$ is holomorphic on an open neighborhood $U \subset \mathbb{C}$ containing $K$. Let $M = \sup_{\partial U} f$. Since $K$ is compact, is some $\delta > 0$ separating any point of $K$ from $\partial U$. Then by Cauchy’s formula, we have

$$|f^{(n)}(x)| = \left| \frac{(-1)^n n!}{2\pi i} \int_{\partial U} \frac{f(z)}{(z - x)^{n+1}} dz \right| \leq \frac{n!}{2\pi} \frac{M}{\delta^{n+1}}.$$ 

This is a bound of the desired shape.

Conversely, let $K$ be a compact neighborhood of $x_0$ such that $|f^{(n)}(x)| \leq c(K)n!/r^n$ for all $x \in K$. By shrinking $r$ if necessary, we may assume that $K$ contains $B(x_0, r/2)$. We will show that $f$ agrees with its power series on this open ball. By the theorem on Taylor series with remainder, we have for $x \in B(x_0, r/2)$

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + f^{(n+1)}(x^*) \frac{(x - x_0)^{n+1}}{(n + 1)!}$$

for some $x^* \in B(x_0, r/2)$. By the hypothesis, the remainder term here is

$$\left| f^{(n+1)}(x^*) \frac{(x - x_0)^{n+1}}{(n + 1)!} \right| \leq \frac{c(K)}{2^{n+1}}.$$
which converge to 0 as \( n \to \infty \). Furthermore, hypothesis furnishes a geometric bound on all the summands, so the power series converges in this ball. \( \square \)

**Example 2.1.3.** Note the distinction between a smooth function, which has a Taylor series, and an analytic function, which has a Taylor series and is equal to it.

- Polynomials, the exponential function, and trigonometric functions are real analytic. Since analyticity can be detected by the growth of the power series coefficients, we see that (finite) sums, products, and compositions of analytic functions are analytic.
- The function \( z \mapsto z \) is smooth but not complex-analytic.
- On \( \mathbb{R} \), there are smooth functions that are not analytic, such as the bump function
  \[
  f(x) = \begin{cases} 
  e^{-1/x^2} & x \geq 0, \\
  0 & x \leq 0.
  \end{cases}
  \]

We now consider multivariate analytic functions \( f : \Omega \to \mathbb{R} \) where \( \Omega \subset \mathbb{R}^n \). Let \( (x_1, \ldots, x_n) \) be the coordinates on \( \Omega \). For a multi-index \( \alpha = (\alpha_1, \ldots, \alpha_n) \) on \( \{1, \ldots, n\} \) we define

\[
x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \quad \text{and} \quad D^\alpha f = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}.
\]

We also set \( \alpha! = \alpha_1! \cdots \alpha_n! \). Then the Taylor series for \( f \) about \( x_0 \) is

\[
f(x) = \sum_{|\alpha|=0}^{\infty} \frac{D^\alpha f(x_0)}{\alpha!} (x - x_0)^\alpha.
\]

**Definition 2.1.4.** Let \( \Omega \subset \mathbb{R}^n \) be an open subset. A function \( f : \Omega \to \mathbb{R} \) is **analytic at** \( x_0 \) if there exists some open neighborhood \( U \subset \Omega \) containing \( x_0 \) such that for all \( x \in \Omega \), the Taylor series for \( f(x) \) about \( x_0 \) converges and is equal to \( f \):

\[
f(x) = \sum_{|\alpha|=0}^{\infty} \frac{D^\alpha f(x_0)}{\alpha!} (x - x_0)^\alpha.
\]

We say that \( f \) is **analytic on** \( \Omega \) if it is analytic at all \( x_0 \in \Omega \).

As before, analyticity is equivalent to growth control on the derivatives.

**Proposition 2.1.5.** Let \( \Omega \subset \mathbb{R}^n \). A function \( f : \Omega \to \mathbb{R} \) is analytic in \( \Omega \) if and only if for every compact subset \( K \subset \Omega \), there exist constants \( c(K) \) and \( r > 0 \) such that for all \( x \in K \) and \( n \geq 0 \), we have

\[
|f^{(\alpha)}(x)| \leq c(K) \frac{\alpha!}{r^{|\alpha|}}.
\]

**Proof.** The proof that \( f \) is analytic if the bound holds is exactly the same as before, using the multivariable Taylor theorem with remainder.

In the other direction, one can use the multivariable Cauchy theorem after extending \( f \) to an analytic function of several complex variables:

\[
f^{(\alpha)}(x) = \frac{(-1)^{|\alpha|}\alpha!}{2\pi i} \int_{\partial U} \frac{f(z)}{(z - x)^{\alpha+1}} \, dz
\]

which can be proved by induction, expanding \( f \) as an analytic function in \( x_n \) with coefficients being functions in \( x_1, \ldots, x_{n-1} \). \( \square \)
2.2 THE CAUCHY-KOVALEVSKAYA THEOREM FOR ODES

2.2.1 The scalar case

We will first focus on the scalar case, where we give four proofs. The last one, which is the original method of Cauchy, is the most important, as it generalizes readily to multivariable ODE and PDE.

**Theorem 2.2.1** (Cauchy-Kovalevskaya). Let \( a > 0 \) and \( F: (-b, b) \to \mathbb{R} \) be a real analytic function. Suppose \( u: (-a, a) \to (-b, b) \) is a solution to

\[
\begin{align*}
    u'(t) &= F(u(t)), \\
    u(0) &= 0.
\end{align*}
\]

Then \( u \) is real analytic on \((-a, a)\).

The existence and uniqueness is also guaranteed by the Picard-Lindelöf theorem (which tells us that it is \( C^1 \)). Therefore, we assume that the solution \( u \) exists and is unique, and prove that it is analytic in a neighborhood of 0.

**Proof 1/Exercise.** Imitate the proof of the Picard-Lindelöf theorem by the fixed-point argument, using the iterated scheme

\[
    u_{n+1}(z) = u_{n+1}(0) = \int_0^z F(u_n(z')) \, dz'.
\]

Here we have replaced the real integral by a complex path integral. The goal is to show that the iteration is a contraction in the space of holomorphic functions with the \( L^\infty \) norm. \( \square \)

**Proof 2.** If \( F(0) = 0 \), then \( u = 0 \) solves the equation. Otherwise, we may assume that \( F \) is non-vanishing in some open set \((-b', b') \subset (-b, b)\). For \( y \in (-b', b') \) we set

\[
    G(y) = \int_0^y dx \frac{1}{F(x)}.
\]

Then you can check that \( G \) is analytic in some open neighborhood of 0. Moreover, we compute

\[
    \frac{d}{dt} G(u(t)) = F(u(t))u'(t) = 1.
\]

Since \( G(0) = 0 \), we get \( G(u(t)) = t \). Now we may deduce that \( u(t) = G^{-1}(t) \) on some possibly smaller domain since \( G'(0) = \frac{1}{F(0)} \neq 0 \) and is therefore locally invertible. \( \square \)

**Proof 3.** For \( z \in \mathbb{C} \), let \( u_z(t) \) be a solution to the ODE

\[
    \begin{align*}
        u_z'(t) &= zF(u_z(t)), \\
        u_z(0) &= 0.
    \end{align*}
\]

Picard-Lindelöf implies that \( u_z(t) \) exists for, say, all \( |z| \leq 2 \) and \( |t| \leq \epsilon \) (note that this is uniform in \( z \). If you go back to the proof of Picard-Lindelöf, you see that the existence of solution depends only on the Lipschitz bound, so a uniform Lipschitz bound for \( |z| \leq 2 \) implies a uniform time interval of existence).
We can think of \( u_z(t) \) as solutions to a family ODE, interpolating between the trivial equation for \( z = 1 \) and the problem of interest at \( z = 1 \). Define the differential operators

\[
\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).
\]

Then you can calculate that

\[
\frac{\partial}{\partial t} \left( \frac{\partial u_z(t)}{\partial \bar{z}} \right) = z F'(u_z(t)) \frac{\partial u_z(t)}{\partial \bar{z}}.
\]

So we have the explicit solution

\[
\frac{\partial u_z}{\partial \bar{z}}(t) = \exp \left( \int_0^t z F'(u_z(s)) \, ds \right) \frac{\partial u_z(0)}{\partial \bar{z}}.
\]

Combining this with \( u_z(0) = 0 \) for all \( |z| \leq 2 \), we deduce that \( \frac{\partial u_z}{\partial \bar{z}}(t) = 0 \) for \( t \in (-\varepsilon, \varepsilon) \), so \( u_z(t) \) is complex analytic in \( z \), and we have

\[
u_1(t) = \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} \frac{1^n |\partial^n u_z(t)|_{z=0}}{n!}.
\]

Now observe that \( u(zt) \) solves the same ODE as \( u_z(t) \), so by uniqueness \( u_z(t) = u(zt) \) for \( |t| < \varepsilon, |z| \leq 2 \), and thus

\[
\frac{\partial^n u_z(t)}{\partial z^n} \big|_{z=0} = \frac{\partial^n u(zt)}{\partial z^n} \big|_{z=0} = t^n u^{(n)}(0)
\]

which implies the result. One thing that we didn’t check was the smoothness of \( u \); you do this by induction and bootstrapping from the equation.

We now give Cauchy’s proof of the theorem, using the method of majorants.

Proof. The first insight is to note that the equation tells us all the derivatives of the function \( u \).

\[
u^{(1)}(t) = F'(0)(u^{(0)}(t))
\]

\[
u^{(2)}(t) = F'(1)(u^{(0)}(t))u^{(1)}(t) = F'(1)(u^{(0)}(t))F'(0)(u^{(0)}(t))
\]

\[
u^{(3)}(t) = F'(2)(u(t))F'(0)(u(t))^2 + F'(1)(u(t))^2 F'(0)(u(t))
\]

\[\vdots\]

More generally, \( u^{(n)}(t) = p_n(F'(0)(u(t)), \ldots, F'(n-1)(u(t)) \). The key point is that the polynomial \( p_n \) has non-negative integer coefficients. This is easily established by induction. In fact, there is an explicit formula for this called the Faà di Bruno formula, which says that \( \frac{d^n}{dt^n} F(u(t)) \) is equal to

\[
\sum_{m_1 + \ldots + m_n = 0}^{m_1 + m_2 + \ldots + m_n = n} \frac{n!}{m_1!(1)^{m_1} m_2!(2)^{m_2} \ldots m_n!(n)^{m_n}} F^{(m_1 + m_2 + \ldots + m_n)}(u(t)) \prod_{j=1}^{n} (u^{(j)}(t))^{m_j}.
\]

Anyway, since \( p_n \) has non-negative coefficients we see that

\[
|u^{(n)}(0)| \leq |p_n(F'(0)(u(0)), \ldots, F'(n)(u(0)))| \leq p_n(|F'(0)(0)|, \ldots, |F'(n)(0)|).
\]
Now suppose $G$ is a function such that for all $n \geq 0$, we have
\[ G^{(n)}(0) \geq |F^{(n)}(0)| \geq 0. \]
Such a function is called a majorant function. Then $p_n(|F^{(n)}(0)|, \ldots, |F^{(n)}(0)|) \leq p_n(G^{(0)}(0), \ldots, G^{(n)}(0))$ by the increasing property of the polynomial $p_n$ in each of its variables. Since Proposition 2.1.2 tells us that analyticity can be measured by the growth of the Taylor coefficients, this means that if we have a majorant function $G$ and a solution to the ODE
\[
\begin{align*}
v'(t) &= G(v(t)) \\
v(0) &= 0
\end{align*}
\]
such that $v$ is analytic, then the Taylor coefficients of $v$ will majorize those of $u$, so $u$ will be analytic too!

For the construction of this majorant function, we restrict our attention to some compact subset $K$. We can perform this argument about any point, so we will deduce compactness in all compact subset of the domain, which is what we want. Then Proposition 2.1.2 tells us that for all $n \geq 0$ there exist constants $C, r > 0$ such that
\[ |F^{(n)}(0)| \leq C \frac{n!}{r^n}. \]
We take
\[ G(x) = C \sum_{n=0}^{\infty} \left( \frac{x}{r} \right)^n = \frac{Cr}{r-x} \]
which converges for $|x| < r$ and satisfies $G^{(n)}(0) = C \frac{n!}{r^n} \geq |F^{(n)}(0)|$ by construction.

The auxiliary problem
\[
\begin{align*}
v'(t) &= G(v(t)) = \frac{Cr}{1-v} \\
v(0) &= 0
\end{align*}
\]
can be easily solved using the standard methods:
\[(r-v)dv = Crdt \implies -d(r-v)^2 = 2Crdt \implies v(t) = r - r \sqrt{1 - \frac{2Ct}{r}}.\]
using the initial condition $v(0) = 0$, and this is indeed analytic for $|t| \leq \frac{r}{2C}$, so we are done.

2.2.2 The multivariate case

We now outline an extension of the Cauchy-Kovalevskaya theorem to systems of differential equations.

**Theorem 2.2.2** (Multivariable Cauchy-Kovalevskaya). Let $F : (-b, b)^m \to \mathbb{R}^m$ be real analytic, and $u(t)$ a solution to the system
\[
\begin{align*}
u'(t) &= F(u(t)) \\
u(0) &= 0
\end{align*}
\]
on $(-a, a)$, and $u((-a, a)) \subset (-b, b)^m$. Then $u$ is real analytic on $(-a, a)$. 

Proof. We apply the method of majorants again. Write \( F = (F_1, \ldots, F_m) \). Restricting to a compact subset, we may assume by Proposition 2.15 that

\[
|F_i^{(\alpha)}(x)| \leq C \frac{\alpha!}{r^{\alpha}}.
\]

Then \( F_i \) is majorized by the function

\[
G_i = \frac{Cr}{r - z_1 - \ldots - z_m}
\]

which is real analytic near zero. The solution to the system

\[
\begin{align*}
v'(t) &= G(u(t)) \\
u(0) &= 0
\end{align*}
\]

is given by \( v_1(t) = v_2(t) = \ldots = v_m(t) = w(t) \), where

\[
w(t) = \frac{r}{m} \left[ 1 - \sqrt{1 - \frac{2Cmt}{r}} \right]
\]

is real analytic near zero. The conclusion is deduced from the majorant argument, as before.

\[\square\]

2.3 THE ANALYTIC CAUCHY PROBLEM FOR PDE.

Let us begin by discussing a general description of partial differential equations. A general \( k \)-th order PDE is of the form

\[
f(\nabla^k u, \nabla^{k-1} u, \ldots, \nabla u, u, x) = 0.
\]

(3)

Here \( \nabla^k \) is the vector of all \( k \)-th order partial derivatives. We say that the equation is \textit{quasi-linear} if the PDE is linear in the highest order derivatives:

\[
\sum_{|\alpha|=k} a_\alpha (\nabla^{k-1} u, \nabla^{k-2} u, \ldots, u, x) \partial^\alpha u + a_0 (\nabla^{k-1} u, \nabla^{k-2} u, \ldots, u, x) = 0.
\]

(4)

We say that it is \textit{semi-linear} if the coefficients of the highest order derivatives depend only on \( x \), not on \( u \).

\[
\sum_{|\alpha|=k} a_\alpha(x) \partial^\alpha u + a_0 (\nabla^{k-1} u, \nabla^{k-2} u, \ldots, u, x) = 0.
\]

(5)

We say that it is \textit{linear} if all coefficients depend only on \( x \), not on \( u \).

\[
\sum_{i=0}^{k} \sum_{|\alpha|=i} a_\alpha(x) \partial^\alpha u = 0.
\]

(6)

Finally, we say that it has \textit{constant coefficients} if the coefficients are constant, independent of \( x \).
We seek to set up a general Cauchy problem for PDE. The intuition to keep in mind is that we write down a PDE in a region, along with some prescribed initial conditions at “time 0.” In general, we study a PDE defined on a region \( \Omega \subset \mathbb{R}^\ell \) and the initial data is given along a hypersurface \( \Gamma \), in the differential geometry sense. That is, locally around any point \( x \in \Gamma \), there is an open neighborhood \( U \) of \( x \) and a smooth diffeomorphism \( \phi: U \rightarrow V \subset \mathbb{R}^\ell \) such that \( \phi(\Gamma) = V \cap \{ y_\ell = 0 \} \). Let \( \psi: V \rightarrow U \) denote the (smooth) inverse map. Then \( \psi \) induces an isomorphism of tangent bundles
\[
\psi^*: \mathbb{T}V \rightarrow \mathbb{T}U.
\]
In particular, \( \psi^* \) sends the tangent space at 0, which is spanned by \( \partial \partial_{y_1}, \ldots, \partial \partial_{y_{\ell-1}} \), isomorphically onto \( T_x \Gamma \).

Let \( n(x) = (n_1(x), \ldots, n_\ell(x)) \) be a unit normal vector to \( \Gamma \).

**Definition 2.3.1.** Given \( j \in \mathbb{N} \), the \( j \)-th order normal derivative of \( u \) at \( x \in \Gamma \) is
\[
\frac{\partial^j u}{\partial n^j} := \sum_{|\alpha|=j} D^\alpha u n^\alpha,
\]
where \( n^\alpha = n_1^{\alpha_1} \cdots n_\ell^{\alpha_\ell} \). This is a function on \( \Gamma \).

**Example 2.3.2.** If \( \Omega = \mathbb{R}^n \) and \( \Gamma = \{ x_\ell = 0 \} \), then \( n = (0, \ldots, 0, 1) \), and hence
\[
\frac{\partial^j u}{\partial n^j} = \frac{\partial^j u}{\partial x_\ell^j}.
\]

**Cauchy Problem.** For a \( k \)th order quasilinear PDE (4), the Cauchy problem asks for solutions on \( \Omega \) with the following initial data on \( \Gamma \):
\[
\begin{align*}
    u &= g_0, \\
    \frac{\partial u}{\partial n} &= g_1 \\
    &\vdots \\
    \frac{\partial^{k-1} u}{\partial n^{k-1}} &= g_{k-1}.
\end{align*}
\]
If we hope to solve the PDE with this initial data, we should at least ask if we can even determine all the derivatives (leaving aside the question of whether or not they fit into a convergent power series).

**2.3.2 The case of a flat hypersurface**

We consider the case \( \Gamma = \{ x_\ell = 0 \} \subset \mathbb{R}^\ell \). Then the normal vector is \( n = (0, \ldots, 0, 1) \) and the Cauchy data is \( u = g_0, \frac{\partial u}{\partial x_\ell} = g_1, \ldots, \frac{\partial^{k-1} u}{\partial x_\ell^{k-1}} = g_{k-1} \). Let’s see if we can compute all the derivatives on the boundary from this data.

The “tangential derivatives” are no problem. If \( \alpha = (\alpha_1, \ldots, \alpha_{\ell-1}, 0) \) and \( 0 \leq j \leq k-1 \) then we have
\[
D^\alpha \partial^j u(x) = D^\alpha g_j.
\]
So far, we have not been able to determine $D^\alpha u$ when $\alpha = (0, \ldots, 0, k)$. For this, we turn to the PDE relation
\[ \sum_{|\alpha| = k} a_0(\nabla^{k-1} u, \ldots, u, x) D^\alpha u + a_0(\nabla^{k-1} u, \ldots, u, x) = 0. \]

If $A(x) := a_{(0, \ldots, 0, k)}(\nabla^{k-1} u, \ldots, u, x) \neq 0$ on $x \in \Gamma$, then we can divide by this coefficient to solve for the term we’re interested in:
\[ \frac{\partial^k u}{\partial x^k} = -\frac{1}{A(x)} \left( \sum_{|\alpha| = k, a_\ell \leq k-1} a_\alpha(\ldots) D^\alpha u + a_0(\ldots) \right). \]

Using this, we can determine $D^\alpha u$ whenever $a_\ell \leq k$, since we can differentiate the right hand side above with respect to any tangential directions. Therefore, the only difficulty is to obtain the normal derivatives.

How can we find $\partial^{k+1}_\ell u$? We now differentiate the PDE itself with respect to $x_\ell$. There will be many new terms, since the coefficients potentially depend on $x$, but the coefficient of $\partial^{k+1}_\ell u$ will still be $A(x)$.
\[ 0 = \sum_{|\alpha| = k} a_\alpha(\ldots) \partial^{k+1}_\ell u + \tilde{a}_0(\ldots) \]

with
\[ \tilde{a}_0(\nabla^k u, \ldots, u, x) = \partial_\ell (a_0(\ldots)) + \sum_{|\alpha| = k} \partial_\ell a_\alpha(\ldots) D^\alpha u. \]

So if $A(x) \neq 0$, we can again solve for the term of interest:
\[ \partial^{k+1}_\ell u = \frac{1}{A(x)} \left( -\tilde{a}_0(\ldots) - \sum_{|\alpha| = k, a_\ell < k-1} a_\alpha(\ldots) D^\alpha \partial_\ell u \right). \]

You can see that we will obtain all higher-order derivatives in this way, by induction, provided that we have the ‘non-characteristic condition’ $A(x) \neq 0$ on $\Gamma$. This is simple in this case because the unit vector doesn’t depend on the point; in general, it clearly depends on the geometry of the surface.

2.3.3 General hypersurfaces

**Definition 2.3.3.** A smooth hypersurface $\Gamma$ and (smooth) boundary data $g_0, \ldots, g_{k-1}$ is non-characteristic for the PDE if
\[ A(x) := \sum_{|\alpha| = k} a_\alpha(\nabla^{k-1} u, \ldots, u, x) n^\alpha(x) \neq 0 \]
for all $x \in \Gamma$. Note that this depends only on $x$, since the relevant values of $u$ are provided by the Cauchy data.

**Theorem 2.3.4.** Assuming the non-characteristic condition, if $u$ is a smooth solution to the PDE and boundary data, then all derivatives of $u$ on $\Gamma$ are determined by finite induction from Cauchy data.
Proof. It suffices to verify this locally. For any \( x \in \Gamma \), we choose a local chart \( \phi : U \to V \subset \mathbb{R}^\ell \) taking \( \Gamma \) to \( V \cap \{ x_\ell = 0 \} \), which has an inverse \( \psi : V \to U \).

Let \( v(y) = u(\psi(y)) \). Then \( v \) satisfies a PDE for the flat hypersurface of the form
\[
\sum_{|\alpha|=k} b_\alpha(\nabla_{y}^{k-1}v, \ldots, v, y) D_\alpha^v + b_0(\nabla_{y}^{k-1}v, \ldots, v, y) = 0.
\]

Since \( \psi \) is a diffeomorphism, the derivatives of \( u \) will be computed if we can compute all the derivatives of \( v \), so we are reduced to the case of flat boundary. The Cauchy data for \( v \) is determined by \( \psi \) and \( g_0, \ldots, g_{k-1} \), so we need only to check the non-characteristic condition.

Now we check the non-characteristic equation. We have \( u(x) = v(\phi(x)) \). By the chain rule,
\[
D_\alpha u = \frac{\partial}{\partial x_\ell}(D\psi)^a \left( \sum_{\alpha} \right) + \text{terms involving lower order derivatives in } y_\ell \text{ (here } D\psi^a \text{ is the last column vector of } D\psi \).
\]

Therefore,
\[
b_{(0, \ldots, 0, k)}(D\psi^a) = \sum_{|\alpha|=k} a_\alpha(D\psi^a)^a.
\]

Since \( D\psi^a \) is parallel to \( \mathbf{n} \), we have that \( b_{(0, \ldots, 0, k)} \) is a non-zero multiple of \( A \). Therefore, the non-characteristic condition for \( u \) is equivalent to that for \( v \), and we are done.

\[
\square
\]

### 2.4 The Cauchy-Kovalevskaya Theorem for PDE

**Theorem 2.4.1.** Let \( \Omega \subset \mathbb{R}^\ell \) be an open subset and \( \Gamma \subset U \) a real-analytic hypersurface. Consider the PDE (4) on \( \Omega \) with real-analytic Cauchy data \( g_0, \ldots, g_{k-1} \).

Suppose furthermore that the non-characteristic condition (2.3.3) holds on \( \Gamma \). Then for any \( x \in \Gamma \), there is a unique analytic solution \( u \) on an open subset of \( U_x \) of \( x \) satisfying the boundary data on \( \Gamma \cap U_x \).

**Proof.** The big difference between this and the ODE version we proved is that we cannot start out with existence and uniqueness of a smooth solution. We make a series of reductions.

1. By flattening the boundary with local analytic charts, we may reduce to the case where \( \Gamma = \{ x_\ell = 0 \} \cap \Omega \).

2. By dividing by \( a_{(0, \ldots, 0, k)} \neq 0 \) locally around \( x \), we reduce to the case where this coefficient is 1.

3. By subtracting a real analytic function compatible with the Cauchy data (but not necessarily satisfying the PDE), we reduce to the case of identically Cauchy data.

4. We reduce to a first-order system as in the ODE case,
\[
u = (u, \partial_1 u, \partial_2 u, \ldots, \partial_{\ell-1} u).
\]

Then we have an equation of the form
\[
\frac{\partial \mathbf{u}}{\partial x_\ell} = \sum_{j=1}^{\ell-1} \mathbf{a}_j(u, x') \frac{\partial \mathbf{u}}{\partial x_j} + a_0(u, x')
\]
where the $b_j$ are $m \times n$ matrices, and $x' = (x_1, \ldots, x_{\ell-1})$. The boundary data is simply $u = 0$ on $\Gamma$. Note that we may assume that the coefficients depend only on $x'$ (i.e. not on $x_\ell$) by adding $x_\ell$ to our vector $u$ if necessary.

The second step is to compute the derivatives at the point 0. By the same sorts of computations as before, we find that $D^\alpha u(0)$ is a polynomial with non-negative coefficients involving the lower-order derivatives of $u$ and the derivatives of the coefficients. Note that this already gives uninqueness of any analytic solution.

Now, we are not given local existence of a solution, but we do have a candidate. Consider the Taylor series

$$
\tilde{u}(x_1, \ldots, x_\ell) = \sum_{\alpha \geq 0} D^\alpha u(0) \frac{x^\alpha}{\alpha!}.
$$

Assume first that this series has positive radius of convergence in $\mathbb{R}^\ell$, i.e. converges to an analytic function on some open neighborhood. By the computations above, this series is constructed so that it satisfies the PDE. So it only remains to show that the series does have some positive radius of convergence. Here, we use the method of majorants again.

Finally, we must argue that the coefficients are indeed bounded by those of an analytic function.

1. Using the real analyticity of the coefficients $a_j$’s, we may find $C, r > 0$ such that

$$
|a_j^{(a)}| \leq \frac{Ca!}{r|x|} M_1
$$

where $M_1$ is the matrix with all 1’s (here we mean that each entry is majorized). Then if we set

$$
g(z_1, \ldots, z_m, x_1, \ldots, x_{\ell-1}) = \frac{Cr}{r - (x_1 + \ldots + x_{\ell-1}) - (z_1 + \ldots + z_m)}
$$

(since the coefficients $a_j(u, x)$ depends on $\ell - 1 + m$ variables) we will have

$$
|a_j| \leq g M_1 =: a_j^*.
$$

2. We consider the auxiliary problem

$$
\frac{\partial \mathbf{v}}{\partial x_\ell} = \sum_{j=1}^{\ell-1} a_j^*(\mathbf{v}, x) + a_0^*(\mathbf{v}, x)
$$

$v = 0$ on $\Gamma$.

By symmetry again, we have $v_1 = \ldots = v_m = w(x_1 + \ldots + x_{\ell-1}, x_\ell) \zeta$ so that the PDE reduces to

$$
\frac{\partial w}{\partial t} = \frac{Cr}{r - \zeta - mw} ((l - 1) \zeta \partial_\zeta w + 1).
$$

with $w(\zeta, 0) = 0$. For $\ell \geq 3$, the solution is

$$
w(\zeta, t) = \frac{1}{\ell m} \left( (r - \zeta) - \sqrt{(r - \zeta)^2 - 2\ell m C r t} \right).
$$
2.5 COUNTEREXAMPLES AND WELL-POSEDNESS

2.5.1 Failures of the Cauchy-Kovalevskaya Theorem

Consider the heat equation on \( \mathbb{R}^2 \).

\[
\partial_t u = \partial^2_x u,
\]
\[
u(0, x) = g(x).
\]

In our previous notation, here \( \Gamma = \{ t = 0 \} \), \( a_{0,2} = -1 \), and \( n = (1,0) \). The non-characteristic equation is \( \sum_{|\alpha|=2} a_{\alpha} n^\alpha = 0 \), so the time surface is characteristic everywhere. That means that any analytic solution has zero radius of convergence! This reflects the fact that the equation cannot be reversed in time.

**Example 2.5.1.** Consider the heat equation with initial data \( g(x) = \frac{1}{1+x} \). Let us search for an analytic solution of the form

\[
u(t, x) = \sum_{m,n \geq 0} \frac{t^m}{m!} \frac{x^n}{n!} a_{m,n}.
\]

From the PDE one fights the relation

\[
\alpha_{m+1,n} = \alpha_{m,n+2} \quad \forall n, m \geq 0
\]

(the right hand side of the PDE shifts by two spatial indices, and the left hand side shifts by one time index) and the initial conditions imply

\[
a_{0,2n+1} = 0 \quad \text{and} \quad a_{0,2n} = (-1)^n (2n)!
\]

Using these relations, one finds that \( a_{m,2n} = (-1)^{m+n} (2m + 2n)! \), and using Stirling’s formula, we see that the coefficients blow up faster than geometrically.

However, we know that the heat equation can be solved. What goes wrong? The point is that the heat equation is well-posed for \( t > 0 \) and ill-posed for \( t < 0 \), so we cannot solve along any open set containing \( t = 0 \).

The point is that we are equating \( \partial_t^k u = \partial_x^2 u \), but the right hand side grows like \( C_{\frac{(2k)!}{k!}} \) and the left hand side like \( C_{\frac{k!}{k!}} \), so we cannot have analyticity. This immediately suggests that if we have more space derivatives than time derivatives, then the solution cannot be analytic, and indeed such a PDE is automatically characteristic all along the hypersurface.

**Exercise 2.5.2.** Consider the wave equation \( \partial_t^2 - \partial_x^2 u = 0 \). You can solve this with the D’Alembert formula.

\[
|u(x,t)| \leq \sup_{[x-t,x+t]} |\phi| + \sup_{[x-t,x+t]} |\psi|
\]

where \( u(x,0) = \phi(x) \) and \( \partial_t u(x,0) = \psi(x) \).

2.5.2 What is wrong with analyticity?

If we restrict ourselves to an analytic theory of solutions, we would miss out on a lot of interesting phenomena in PDE. For instance, analytic solutions are
extremely *rigid*, being determined by their values on any open set. They are not suitable for describing phenomena like wave propagation, in which the function is influenced only by a specific region. As another example, we have the phenomenon of *elliptic regularity*, which describes the tendency of elliptic solutions to be smooth, and even analytic, once they possess some minimal level of regularity. This would obviously be obscured if we only paid attention to analytic solutions.

It is more natural to work with solutions that have only just enough regularity so that the PDE is well-defined (or even less - this leads to the idea of “weak solutions”). If we consider non-analytic solutions, we see a lot of interesting questions arise. For instance, does uniqueness hold if we allow non-analytic solutions?

One might ask: what if we approximate non-analytic data by analytic data (say via Stone-Weierstrass) and study the limit? Things can be very badly behaved, as was demonstrated by Hadamard.

**Example 2.5.3 (Hadamard).** Consider the problem
\[
\begin{align*}
\partial_t^2 u + \partial_x^2 u &= 0 \\
u(x, 0) &= 0 \\
\partial_t u(x, 0) &= a_\omega \cos(\omega x)
\end{align*}
\]

Then a solution is
\[
u(x, t) = \frac{a_\omega}{\omega} \cos(\omega x) \sinh(\omega t).
\]

Choose \(a_\omega = e^{-\sqrt{\omega}}\) for \(\omega > 0\). By making \(\omega\) very large, we can arrange that the initial data be arbitrarily small. However, \(u(x, t)\) is dominated by the exponential term \(e^{\omega t}\) coming from \(\sinh(\omega t)\). In particular, at \(t = 1\) we can make the solution arbitrarily large by our large choice of \(\omega\).

In fact, this example shows that the solution can blow up even if we demand that \(||\partial_t^2 u||\) be small for all derivatives \(k\) up to some fixed constant.

### 2.6 Classification of PDE

#### 2.6.1 Characteristic Hypersurfaces

Let \(P\) be a scalar linear differential operator of the form
\[
P u = \sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha u, \quad x \in \mathbb{R}^\ell.
\]

**Definition 2.6.1.** The *total symbol* of \(P\) is
\[
\sum_{|\alpha| \leq k} a_\alpha(x) (\xi)^\alpha = \sigma(x, \xi), \quad \xi \in \mathbb{R}^\ell
\]

If the coefficients \(a_\alpha\) were constants, this would be the corresponding operator in Fourier space (derivatives go to multiplication by the Fourier variable).

**Definition 2.6.2.** The *principal symbol* of \(P\) is the highest-degree term of the total symbol:
\[
\sigma_P(x, \xi) = \sum_{|\alpha| = k} a_\alpha(x) \xi^\alpha.
\]
The non-characteristic condition at \( x \) becomes, in this terminology,

\[
\sigma_p(x, n(x)) \neq 0.
\]

The total and principal symbols are more properly understood as sections of the vector bundle \( \text{Sym}(T\Omega) \), i.e. polynomial functions on the cotangent space.

**Definition 2.6.3.** We define the characteristic cone for \( P \) at \( x \) to be

\[
\mathcal{C}_x = \{ \zeta \in \mathbb{R}^\ell : \sigma_p(x, \zeta) = 0 \}.
\]

(This is called a cone because it is the zero set of a homogeneous function.)

A reformulation of the characteristic condition is that a hypersurface is characteristic at \( x \) if and only if \( n(x) \in \mathcal{C}_x \).

### 2.6.2 The Main Linear PDEs

Now let’s consider examples for the main linear PDEs.

1. **Laplace equation.** (Elliptic)

   \[
   \Delta u := \sum_{j=1}^{\ell} \frac{\partial^2 u}{\partial x_j^2} = 0.
   \]

   The principal symbol is \( \zeta_1^2 + \ldots + \zeta_\ell^2 \). The characteristic cone at any \( x \) is \( \{ (0, \ldots, 0) \} \). So all hypersurfaces \( \Gamma \) are non-characteristic. More generally, this is the defining feature of *elliptic* equations. A similar example is the *Poisson equation* \( \Delta u = f \), which has the same principal symbol.

2. **Wave equation.** (Hyperbolic)

   \[
   \Box u := (-\partial_t^2 + \sum_{j=1}^{\ell} \partial_j^2) u = 0.
   \]

   Here the principal symbol is \( \sigma_p(x, \zeta) = \zeta_1^2 + \ldots \zeta_\ell^2 - \zeta_\ell^2 \). The characteristic cones are called “light cones.”

3. **Heat equation.** (Parabolic)

   \[
   \partial_t u = (\sum_{j=1}^{\ell} \partial_j^2) u.
   \]

   The principal symbol is \( \zeta_1^2 + \ldots + \zeta_\ell^2 = 0 \), so the characteristic cone at any \( x \) is \( \zeta_1 = \zeta_2 = \ldots = \zeta_\ell = 0 \).

4. **Schrödinger equation.** (Dispersive)

   \[
   i\partial_t u + (\sum_{j=1}^{\ell} \partial_j^2) u = 0.
   \]

   The principal symbol is \( \zeta_1^2 + \ldots + \zeta_\ell^2 = 0 \), and the characteristic equation is again \( \zeta_1 = \zeta_2 = \ldots = \zeta_\ell = 0 \).
5. **Transport equation.** (Hyperbolic)

\[
\sum_{j=1}^{\ell} c_j(x) \partial_j u = 0.
\]

The principal symbol is

\[
\sigma_p(x, \xi) = \sum_{j=1}^{\ell} c_j(x) \xi_j.
\]

We have \( C_x = c(x)^\perp \), where \( c(x) = (c_1(x), \ldots, c_\ell(x)) \). For \( \ell \geq 2 \), there is always a characteristic hypersurface at any \( x \).

### 2.6.3 Basic classification

Roughly speaking, there are four broad classes of PDE, which try to identify equations similar to, respectively, the Laplace, wave (and transport), heat, and Schrödinger equations.

1. **Elliptic.** No characteristic hypersurface, analytic solutions. The Cauchy problem is ill-posed. Corresponds to a stationary problem in physics.

2. **Hyperbolic.** Many characteristic surfaces, which are transversal between time and space.

3. **Parabolic.** There is a distinguished time variable \( t \). The initial time is characteristic, and we have ellipticity with respect to the other variables. It is ill-posed backwards in time.

4. **Dispersive.** Very similar to transport/wave equations, but there is a crucial difference. For the wave equation

\[
\partial_t^2 u = \partial_x^2 u
\]

\[
u(0, x) = \cos(kx)
\]

\[
\partial_t u(0, x) = 0
\]

a solution is \( u(t, x) = \cos(k(t + x)) \); the point is that the speed is independent of \( k \).

For the Schrödinger equation

\[
i \partial_t u = -\partial_x^2 u
\]

\[
u(0, x) = e^{ikx}
\]

we have a solution

\[
u(t, x) = \exp(i(kx - |k|^2 t))
\]

so the speed \( |k| \) depends on the frequency. Physically, this means that there is wave packet dispersion.
Example 2.6.4. For semilinear second-order PDE over \( \mathbb{R} \), which are what we mostly study, we can give a more concrete classification. In this case, the principle will be homogeneous quadratic form

\[
\sum a^{ij}(x)\zeta_i\zeta_j.
\]

By the equality of mixed partials, we can assume that this quadratic form is symmetric, i.e. \( a^{ij} = a^{ji} \). Then the geometry of the characteristic hypersurfaces at any point \( x \) is determined by the signature of the quadratic form. In these terms, the PDE is \textit{elliptic} if the form has signature \((n,0)\); \textit{hyperbolic} if the form has signature \((n-1,1)\), and \textit{parabolic} if the form has signature \((n-1,0)\).

In the two-variable example, this is quite explicit. The quadratic form is

\[
A(x)\xi_1^2 + B(x)\xi_1\xi_2 + C(x)\xi_2^2.
\]

The signature is determined by the quadratic formula:

- \text{elliptic} \iff B^2 - 4AC < 0,
- \text{hyperbolic} \iff B^2 - 4AC > 0,
- \text{parabolic} \iff B^2 = 4AC.
ELLiptic Equations

3.1 Ellipticity

We now turn our attention to elliptic PDE. Recall that a $k$-th order linear differential operator $P$ is elliptic at $x \in \mathbb{R}^n$ if $\sigma_k(x, \xi) \neq 0$ for all non-zero $\xi$, i.e. if all hypersurfaces through $x$ are non-characteristic.

We will focus on semilinear second-order linear operators of the form

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x) \partial_i \partial_j u + \text{lower order terms.}$$

We assume without loss of generality that $a^{ij} = a^{ji}$. The ellipticity condition at $x$ is then that

$$\sum_{i,j=1}^{n} a^{ij}(x) \xi_i \xi_j \neq 0 \text{ for all } \xi \neq 0.$$  

The most characteristic feature of elliptic equations is elliptic regularity, which refers to the phenomenon that their solutions possess an extraordinary amount of regularity. Let us first discuss this heuristically. A general heuristic is that singularities propagate along the characteristics; since an elliptic equation has no characteristics, we expect no propagation of singularities.

More concretely, lack of regularity is linked to oscillation. Indeed, recall that the regularity of a function is related to the rate of decay of its Fourier transform, which has to do with the level of oscillation. To make this slightly more precise, consider the differential operator

$$P = \sum_{|\alpha| = k} a_\alpha \partial^\alpha_x$$

where we assume, for now, that the $a_\alpha$ are constant. Then $\sigma_P(x, \xi) = \sigma_P(\xi) = \sum_{|\alpha| = k} a_\alpha \xi^\alpha$. If this has a nontrivial $\xi_0$, let’s consider how it operates on the plane wave, $\psi = e^{i\lambda x \cdot \xi}$ for $\lambda \gg 1$. It is easily computed that

$$P(e^{i\lambda x \cdot \xi_0}) = \sigma_P(\lambda \xi_0) e^{i\lambda x \cdot \xi_0} = 0.$$  

Therefore, we can get as large oscillations as we want by letting $\lambda \to \infty$. If the coefficients are not constant, $P = \sum_{|\alpha| = k} a_\alpha(x) \partial^\alpha_x$ then $u(x) = e^{i\lambda x \cdot \xi}$ is still an approximate solution around $x_0$ such that $\sigma_P(x_0, \xi) = 0$. So the absence of characteristics prohibits this sort of violent oscillation.

We now state a qualitative formulation of elliptic regularity for the Dirichlet operator, which is the prototype of all elliptic operators.

Theorem 3.1.1 (Elliptic regularity, qualitative version). Let $u : \mathbb{R}^n \to \mathbb{R}$ be a $C^2$ function satisfies $\Delta u = f$ where $f$ is $C^\infty$ on $\mathbb{R}^n$. Then $u \in C^\infty(\mathbb{R}^n)$.

This is familiar in the case $f = 0$, in which case $u$ is called a harmonic function. The proof is organized three steps.
1. First, one introduces the notion of *Sobolev spaces*, which measure regularity by integrals rather than derivatives.

2. Then one establishes *a priori estimates* admitting the existence of solutions.

3. Then one goes back and justifies the a priori estimate by a “regularization argument.” This is the most technical and painful part of the proof, but it is usually more straightforward (and, of course, necessary for rigor).

### 3.2 Sobolev Spaces

As mentioned already, Sobolev spaces provide a means to measure regularity through integrals, and are thus a key technical tool for phrasing and using a priori estimates.

#### 3.2.1 Hölder spaces

Let $\Omega \subset \mathbb{R}^n$ be an open subset. For a function $f : \Omega \to \mathbb{R}$ we write

$$||f||_{C(\overline{\Omega})} = \sup_{x \in \Omega} |f(x)|.$$

Now fix a constant $0 < \gamma \leq 1$. Recall that a function $f : \Omega \to \mathbb{R}$ is said to be *Hölder continuous with exponent* $\gamma$ on $\Omega$ if there exists a constant $C$ such that for all $x, y \in \Omega$ we have

$$\frac{|f(x) - f(y)|}{|x - y|^\gamma} \leq C.$$

The infimum of all such $C$ is called the $\gamma$-Hölder seminorm, and denoted $[u]_{C^{0,\gamma}(\Omega)}$. Note that it is only a seminorm (consider the constant functions).

**Definition 3.2.1.** Fix $0 < \gamma \leq 1$. We define the *Hölder space* $C^{k,\gamma}(\overline{\Omega})$ to be the subspace of functions $f \in C^k(\overline{\Omega})$ for which the norm

$$||f||_{C^{k,\gamma}(\Omega)} = \sum_{|\alpha| = 0}^k ||D^\alpha f||_{C^k(\overline{\Omega})} + \sum_{|\alpha| = k} ||D^\alpha f||_{C^{0,\gamma}}.$$

**Theorem 3.2.2.** $C^{k,\gamma}(\overline{\Omega})$ is a Banach space.

#### 3.2.2 Sobolev spaces

We give several equivalent definitions of Sobolev spaces on $\Omega$. First recall the definition of a weak derivative.

**Definition 3.2.3.** Suppose $f, g \in L^1_{\text{loc}}(\Omega)$ and $\alpha$ is multi-index in $\{1, \ldots, n\}$. We say that $g$ is the $\alpha$-th weak partial derivative of $f$, and denote $g = D^\alpha f$ if for all $\phi \in C^\infty_0(\Omega)$ we have

$$\int_{\Omega} f D^\alpha v = (-1)^{|\alpha|} \int_{\Omega} g \phi, dx.$$

In other words, $v$ is the *distributional derivative* of $u$. 
Definition 3.2.4 (Sobolev spaces). We define \( W^{s,p}(\Omega) \subset L^p(\Omega) \) to be the subspace of functions \( u \) such that for all \( |\alpha| \leq k \), \( D^\alpha u \) exists in the weak sense and is in \( L^p(\Omega) \).

Definition 3.2.5 (Sobolev spaces, II). For \( \varphi \in C^\infty(\Omega) \), we define the norm
\[
\|\varphi\|_{W^{s,p}(\Omega)} = \left( \sum_{|\alpha| \leq s} \|D^\alpha \varphi\|_{L^p(\Omega)}^p \right)^{1/p}
\]
when it exists and is finite. We then define the Sobolev space \( W^{s,p} \) to be the completion of the space of such \( \varphi \) in \( L^p(\Omega) \).

Remark 3.2.6. When \( \Omega = \mathbb{R}^n \), we can define \( W^{s,p} \) to be the completion of \( \varphi \in C^\infty(\Omega) \) under \( \|\cdot\|_{W^{s,p}(\Omega)} \).

Definition 3.2.7. When \( p = 2 \), we denote \( H^s(\Omega) := W^{s,2}(\Omega) \).

Definition 3.2.8 (Sobolev spaces, III). We say that \( f \in L^p(\mathbb{R}^n) \) belongs to \( W^{s,p}(\Omega) \) if and only if there exists a constant \( C > 0 \) so that for all \( \varphi \in C^\infty(\Omega) \cap L^p(\Omega) \) and \( |\alpha| \leq s \), we have
\[
\left| \int_{\Omega} f D^\alpha \varphi(x) \right| \leq C \|\varphi\|_{L^2(\Omega)}.
\]
The infimum of all such \( C \) is \( \|f\|_{W^{s,p}(\Omega)} \).

Definition 3.2.9 (Sobolev spaces, IV). We say that \( f \in L^p(\mathbb{R}^n) \) belongs to \( W^{s,p}(\Omega) \) if and only if there exists a constant \( C > 0 \) such that
\[
\left( \int_{\Omega} |\hat{f}(\xi)|^2 (1 + |\xi|^2)^s d\xi \right)^{1/2} \leq C.
\]
The Sobolev norm is the infimum of all such constants \( C \).

Theorem 3.2.10. These definitions are all equivalent.

Proof. \( \blacksquare \)

Proposition 3.2.11. \( W^{s,p}(\Omega) \) is a Hilbert space.

Proof. \( \blacksquare \)

Definition 3.2.12. We define \( W_0^{s,p}(\Omega) \) to be the closure of \( C^\infty(\Omega) \) in \( W^{s,p}(\Omega) \).

This can be thought of the space of functions in \( W^{s,p}(\Omega) \) that “vanish on the boundary.”

Definition 3.2.13. For \( f \in W^{s,p}(\Omega) \), we define the homogeneous Sobolev seminorm
\[
\|f\|_{W^{s,p}_0(\Omega)} = \left( \sum_{|\alpha| = s} \|f\|_{L^p(\Omega)}^p \right)^{1/p}.
\]
3.2.3 Sobolev inequalities

**Theorem 3.2.14** (General Sobolev Inequalities). Let $\Omega = \mathbb{R}^n$ or a bounded open subset of $\mathbb{R}^n$ with $C^1$ boundary. Let $u \in W^{k,p}(\Omega)$.

1. If $k < \frac{n}{p}$, set $p^* = \frac{np}{n-kp}$. Then we have an embedding

$$W^{k,p}(\Omega) \hookrightarrow L^{p^*}(\Omega).$$

Moreover, there exists a constant $C$ such that for all $f \in W^{k,p}_0(\Omega)$ we have

$$||f||_{L^{p^*}(\Omega)} \leq C||f||_{W^{k,p}(\Omega)}.$$

2. If $k > \frac{n}{p}$, set $\gamma = 1 + \left[\frac{n}{p}\right] - \frac{n}{p}$, or any positive number less than 1 if this is 0. Then we have an embedding

$$W^{k,p}(\Omega) \hookrightarrow C^{k-\left[\frac{n}{p}\right]-1,\gamma}(\Omega).$$

Moreover, there exists a constant $C$ such that for all $f \in W^{k,p}_0(\Omega)$ we have

$$||f||_{C^{k-\left[\frac{n}{p}\right]-1,\gamma}(\Omega)} \leq C||f||_{W^{k,p}(\Omega)}.$$

**Example 3.2.15.** Let’s see what this looks like in the case that we are most interested in, which is $p = 2$. First, we consider the case where $k < \frac{n}{2} = \frac{n}{2}$. Then we have

$$H^k(\Omega) \hookrightarrow L^{2^{*k}}(\Omega).$$

If $k > \frac{n}{2}$ and $n$ is odd, then we have

$$H^k(\Omega) \hookrightarrow C^{k-\frac{n}{2}}(\Omega) := C^{k-\left[\frac{n}{2}\right]-1,1/2}(\Omega).$$

If $k > \frac{n}{2}$ and $n$ is even, then we have

$$H^k(\Omega) \hookrightarrow C^{k-n/2-1,\epsilon}(\Omega)$$

for any $0 < \epsilon < 1$.

Here’s a way to remember this result. Suppose we seek a bound of the form

$$||f||_{L^q} \leq C||Df||_{L^p} \text{ for all } f \in C_c^{\infty}(\Omega).$$

Consider replacing $f$ by $f_\lambda(x) = f(\lambda x)$. Then the left hand side scales as $\lambda^{-n/q}$ and the right hand side scales as $\lambda^{1-n/p}$, so we must have $-n/q = 1 - n/p$, i.e.

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{n}.$$

**Definition 3.2.16.** If $1 \leq p < n$, the Sobolev conjugate $p^*$ of $p$ is

$$p^* = \frac{np}{n-p}.$$
3.3 Elliptic Regularity: Dirichlet Operator

**Theorem 3.2.17** (Gagliardo-Nirenberg-Sobolev inequality). There exists a constant $C$ such that for $u \in C^1_c(\mathbb{R}^n)$, we have

$$||u||_{L^{p^*}} \leq C||Du||_{L^p}.$$  

From this we can use a density argument to deduce the following embedding theorem.

**Theorem 3.2.18.** Suppose $1 \leq p < n$ and $p \leq q \leq p^*$. Then $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ and

$$||u||_{L^q} \leq C||u||_{W^{1,p}}$$

for all $u \in W^{1,p}(\mathbb{R}^n)$ for some constant $C = C(n, p, q)$.

One can iterate this to get more derivatives. The cost of obtaining the first derivative can be measured by $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$, so for $k$ derivatives we have

$$||u||_{L^{(p^*-1, kn^{-1})^{-1}}} \leq C||u||_{W^{k,p}}.$$  

Here $(p^{-1} - kn^{-1})^{-1} = \frac{np}{n-kp}$, so that gives the first part of the Sobolev embedding theorem.

**Remark 3.2.19.** If $\Omega$ is bounded, we have $L^q(\Omega) \hookrightarrow L^{q'}(\Omega)$ for $1 \leq q' \leq q$, so we also get embeddings of $W^{k,p}$ into $L^{q'}(\Omega)$.

If $p > n$, then we can obtain uniform bounds on the derivatives of a function in $W^{1,p}$.

**Theorem 3.2.20** (Morrey’s inequality). Assume $n < p \leq \infty$. Then there exists a constant $C$ depending only on $p$ and $n$ such that

$$||u||_{C^{0, \gamma}(\mathbb{R}^n)} \leq C||u||_{W^{1,p}(\mathbb{R}^n)}$$

for all $u \in C^1(\mathbb{R}^n)$ where $\gamma = 1 - n/p$.

The theorem basically says that we can get pointwise regularity from weak regularity by sacrificing $\frac{n}{p}$. It is useful to write $C^{1-\gamma/p}$ for $C^{1, \gamma}$. Then in general, we get

$$||u||_{C^{1-\gamma/p}(\mathbb{R}^n)} \leq C||u||_{W^{k,p}(\mathbb{R}^n)}$$

for all $u \in C^1(\mathbb{R}^n)$.

By $C^{k-\frac{\gamma}{p}}(\mathbb{U})$ we mean the Hölder space $C^{k-\frac{|u/p|-1}{p}, \gamma}$ where $\gamma$ is the fractional part, with the caveat that if $\frac{u}{p}$ is an integer then have to take $\gamma$ to be any positive constant less than 1.

3.3 Elliptic Regularity: Dirichlet Operator

We are now ready to outline a proof of elliptic regularity for the Poisson problem. This is just to give an idea of how the arguments work; the result here will be subsumed by a more general and powerful theory later.
3.3 Elliptic regularity: dirichlet operator

3.3.1 The a priori estimate

We assume for now that we have a solution \( u \in C^\infty_c(\mathbb{R}^n) \).

**Proposition 3.3.1.** We have for all \( s \in \mathbb{N} \),

\[
|||u|||_{H^{s+2}(\mathbb{R}^n)} = |||f|||_{H^s(\mathbb{R}^n)}.
\]

**Proof.** Squaring and integrating the equation \( \Delta u = f \), we have

\[
\int_{\mathbb{R}^n} f^2 = \int_{\mathbb{R}^n} \sum_{i,j} (\partial_i^2 u)(\partial_j^2 u)
\]

\[
= \sum_{i,j} \int_{\mathbb{R}^n} (\partial_i \partial_j u)^2
\]

\[
= |||u|||_{H^2(\mathbb{R}^n)}^2.
\]

That establishes the result in the case \( s = 2 \). More generally, we can differentiate the Poisson equation to see that \( \Delta D^\alpha u = D^\alpha \Delta u = D^\alpha f \), and the same calculation shows that

\[
|||D^\alpha|||_{L^2(\mathbb{R}^n)} = |||D^\alpha u|||_{H^2(\mathbb{R}^n)}.
\]

Summing over \( |\alpha| = s \), we deduce the result. \( \square \)

3.3.2 Justification by regularization

Now we must go back and justify the a priori estimate, which we made under differentiability estimates. In fact, the a priori estimate is usually the key part of the proof, and the justification is just some (long and painful) technical maneuvering. The general principle is that if a quantity can be controlled a priori, then it exists - “existence follows from the estimate.”

**Proposition 3.3.2.** Suppose that \( \Delta u = f \) on \( \mathbb{R}^n \) where \( u \) is \( C^2 \) and \( f \) is \( C^\infty_c(\mathbb{R}^n) \). Then \( u \in C^\infty_c(\mathbb{R}^n) \).

**Proof.** We claim that \( u \in H^s(\mathbb{R}^n) \) for all \( s \), and then we will be done by the Sobolev inequalities. So we may assume that \( u \in H^{s+1}(\mathbb{R}^n) \) for some fixed \( s \), and show that \( u \in H^{s+2}(\mathbb{R}^n) \).

We construct a mollifier or approximation to the identity. Let \( \chi \) be a function, meaning that \( \chi \) is a non-negative function in \( C^\infty_c(B_1(0)) \) (where \( B \) is the unit ball in \( \mathbb{R}^n \)) such that \( \int_{\mathbb{R}^n} \chi = 1 \). (A typical example is constructed using the smooth bump function \( e^{-1/x^2} \).) Then we define

\[
\chi^\varepsilon(x) = \frac{1}{\varepsilon^n} \chi(x/\varepsilon).
\]

This is a bump function supported on \( B_\varepsilon(0) \), tending towards a “delta function” at 0. We call it an “approximation to the identity” because it furnishes smooth approximations. To be more precise, suppose \( g: \Omega \rightarrow \mathbb{R} \) is locally integrable. For \( \varepsilon \) small enough, we can define

\[
g^\varepsilon(x) = \chi^\varepsilon \ast g(x) := \int_{\mathbb{R}^n} g(x-y) \chi^\varepsilon(y) dy = \int_{\mathbb{R}^n} \chi^\varepsilon(x-y) g(y) dy.
\]
Lemma 3.3.3. If \( g \in L^2_{\text{loc}}(\mathbb{R}^n) \), then \( g^\epsilon \in C_c^\infty(\mathbb{R}^n) \), and \( g^\epsilon \to g \) in \( L^2 \).

Proof. That \( g^\epsilon \in C_c^\infty(\mathbb{R}^n) \) follows from the second representation above, and differentiation under the integral sign. To verify pointwise convergence, observe that

\[
g^\epsilon(x) - g(x) = \int_{\mathbb{R}^n} g(x-y) \chi^\epsilon(y) \, dy - g(x) = \int_{\mathbb{R}^n} [g(x-y) - g(x)] \chi^\epsilon(y) \, dy
\]

Then taking the \( L^2 \) norm, we have

\[
||g^\epsilon - g||_{L^2} \leq \sup_{y \leq \epsilon} ||g(\cdot - y) - g||_{L^2}
\]

That this last expression tends to 0 as \( \epsilon \to 0 \) is the fundamental Lebesgue differentiation theorem.

Corollary 3.3.4. If \( g \in H^s(\mathbb{R}^n) \), then

\[
\lim_{\epsilon \to 0} ||g^\epsilon - g||_{H^s(\mathbb{R}^n)} = 0.
\]

Now we are prepared to tackle the proof. First we define \( \chi_R(x) := \chi(Rx) \), where we think of \( R \) as being quite large. Given \( u \) and \( f \) as above, we set

\[
\tilde{u} = u\chi_R \quad \tilde{f} = \Delta \tilde{u}.
\]

Observe that

\[
\tilde{f} = (\Delta u)\chi_R + (\nabla u) \cdot (\nabla \chi_R) + u(\Delta \chi_R)
\]

so \( \tilde{f} \in H^s(\mathbb{R}^n) \) by the assumption that \( u \in H^{s+1}(\mathbb{R}^n) \). Since \( \chi_R \) is supported on \( B_R(0) \), \( \tilde{u} \) is compactly supported. This is the “localization” step. Now our functions are compactly supported but not necessarily smooth, but the functions \( \tilde{u}^\epsilon \) and \( \tilde{f}^\epsilon \) are smooth and compactly supported, and moreover satisfy \( \Delta \tilde{u}^\epsilon = \tilde{f}^\epsilon \) (at least away from the boundary of the support). Applying the earlier estimates, we have

\[
||\tilde{u}^\epsilon - \tilde{u}||_{H^{s+2}(\mathbb{R}^n)} = ||\tilde{f}^\epsilon - \tilde{f}||_{H^s(\mathbb{R}^n)}.
\]

By the preceding Corollary, \( \tilde{f}^\epsilon \to \tilde{f} \) as \( \epsilon \to 0 \), so the right hand side above is a Cauchy sequence. Therefore, the left hand side is as well, and we conclude that \( \tilde{u}^\epsilon \to u' \in H^{s+2}(\mathbb{R}^n) \). On the other hand, we know that \( \tilde{u}^\epsilon \to \tilde{u} \in L^2(\mathbb{R}^n) \) again by Lemma 3.3.3, and \( H^s(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n) \), so we must have \( \tilde{u} = u' \), i.e. \( \tilde{u} \in H^{s+2}(\mathbb{R}^n) \). By letting \( R \to \infty \), we have \( \tilde{u} \to u \), and we are done. \( \square \)
3.3.3 Well-posedness of the Cauchy problem

Consider the Cauchy problem for the Poisson equation

\[ \Delta u = f \]
\[ u(0, x) = u_0(x) \]
\[ \partial_1 u(0, x) = u_1(x). \]

The Cauchy-Kovalevskaya Theorem tells us that this problem is well-posed (locally) for analytic initial data. However, if we leave the setting of analytic functions, then elliptic regularity immediately tells us that it is ill-posed. Indeed, suppose that \( u_0 \in C^k \) and \( u_1 \in C^{k-1} \) but not \( C^k \). Then there cannot be a \( C^2 \) solution, since it would necessarily be \( C^\infty \).

It is not only the existence aspect that fails. The example of Hadamard discussed earlier in Example 2.5.3 shows that solutions to the Laplace equation \( \Delta u = 0 \) need not depend continuously on their initial data.

**Example 3.3.5.** We can easily extend this example to the Poisson problem \( \Delta u = f \) for any analytic \( f \). By Cauchy-Kovalevskaya, we have local existence of solutions, and then we can add arbitrarily bad solutions to the homogeneous problem \( \Delta u = 0 \) such as constructed by Hadamard.

3.4 HILBERT SPACES

3.4.1 Operators on Hilbert spaces

**Definition 3.4.1.** Let \( \mathcal{H}, \mathcal{H}' \) be Hilbert spaces. The norm of a linear map \( T : \mathcal{H} \to \mathcal{H}' \) is

\[ ||T|| = \sup_{h \in \mathcal{H}} \frac{||Th||_{\mathcal{H}'}}{||h||_{\mathcal{H}}} \]

We say \( T \) is bounded if \( ||T|| < \infty \).

**Remark 3.4.2.** There are other (equivalent) formulations of the norm for bounded operators in the literature. Another that we use is: \( T \) is bounded if there exists some \( C \) such that

\[ ||Th|| \leq C||h|| \text{ for all } h \in \mathcal{H} \]

and \( ||T|| \) is the infimum of all such \( C \).

A bounded map of Hilbert spaces is evidently continuous, and in fact the two notions are equivalent. Indeed, if \( T : \mathcal{H} \to \mathcal{H}' \) is continuous, then the pre-image of the unit ball in \( \mathcal{H}' \) contains an open ball of radius \( r > 0 \) in \( \mathcal{H} \), implying that \( ||T|| \leq \frac{1}{r} \).

We are often interested in the case when \( \mathcal{H}' = \mathcal{H} \), in which case a map \( T : \mathcal{H} \to \mathcal{H} \) is called an operator on \( \mathcal{H} \).

**Proposition 3.4.3 (Projection onto closed convex subset).** Let \( \mathcal{H} \) be a Hilbert space and \( C \subset \mathcal{H} \) a closed, non-empty, convex subset of \( \mathcal{H} \). For each \( x \in \mathcal{H} \), there exists a unique \( P_C(x) \in C \) such that

\[ ||P_C(x) - x|| = \min_{y \in C} ||y - x||. \]

Furthermore, the map \( P_C : \mathcal{H} \to C \) is 1-Lipschitz (hence bounded).
Proof. After a translation, we may assume that $x$ is the origin. Set $\ell = \min_{y \in C} \|y\|$ and let $y_1, \ldots, y_n$ be in $C$ such that

$$\|y_n\| \to \ell.$$  

We claim that the sequence is Cauchy. Indeed,

$$\|y_n - y_m\|^2 = 2\|y_n\|^2 + 2\|y_m\|^2 - \|y_n + y_m\|^2.$$  

Since $C$ is convex, $\frac{y_n + y_m}{2} \in C$ and hence $\|y_n + y_m\| \to 4\ell^2$. Substituting this above, we have

$$\|y_n - y_m\|^2 \leq 2\|y_n\|^2 + 2\|y_m\|^2 - 4\ell^2$$

which is small for all sufficiently large $m, n$. Therefore, $y_n \to y \in C$ since $C$ is closed.

For uniqueness, let $y$ and $y'$ be two such minimizers. Then also $ty + (1 - t)y' \in C$ for any $t \in (0, 1)$ by convexity, and

$$\|ty + (1 - t)y'\|^2 = t^2\|y\|^2 + (1 - t)^2\|y'\|^2 + 2t(1 - t)\langle y, y' \rangle \geq \ell.$$  

We can re-arrange this as

$$(2t^2 - 2t)\ell + 2t(1 - t)\langle y, y' \rangle \geq 0$$

which simplifies, after some algebra, to $\langle y, y' \rangle \geq \ell$. By Cauchy-Schwarz, this is only possible if $y = y'$.

For the Lipschitz property, let $x, y \in H$ and $\tilde{x}, \tilde{y}$ denote their projections. Then

$$\|\tilde{x} - x\|^2 \leq \|t\tilde{x} + (1 - t)\tilde{y} - x\|^2$$

$$= \|\tilde{x} - x + (1 - t)(\tilde{y} - \tilde{x})\|^2$$

$$= \|\tilde{x} - x\|^2 + (1 - t)^2\|\tilde{y} - \tilde{x}\|^2 + 2(1 - t)\langle \tilde{x} - x, \tilde{y} - \tilde{x} \rangle.$$  

Simplifying, we obtain

$$0 \leq (1 - t)\|\tilde{y} - \tilde{x}\|^2 + 2(\tilde{x} - x, \tilde{y} - \tilde{x}).$$

By a symmetric argument, we also have

$$0 \leq (1 - t)\|\tilde{x} - \tilde{y}\|^2 + 2(\tilde{y} - y, \tilde{x} - \tilde{y}).$$

Summing, we find that

$$2(1 - t)\|\tilde{y} - \tilde{x}\|^2 \geq 2\|\tilde{y} - \tilde{x}\|^2 + 2\langle y - x, \tilde{y} - \tilde{x} \rangle.$$  

Rearranging and applying Cauchy-Schwarz, we deduce that

$$t\|\tilde{y} - \tilde{x}\|^2 \leq \langle y - x, \tilde{y} - \tilde{x} \rangle \leq (\|y - x\|)(\|\tilde{y} - \tilde{x}\|)$$

which implies the Lipschitz property. 

\[ \square \]

**Corollary 3.4.4** (Projection to closed subspace). Let $S \subset H$ be a closed subspace, then

$$H = S \oplus S^\perp.$$
Proof. For \( h \in \mathcal{H} \), we let \( P_S(h) \) denote the projection onto \( S \) as given by Proposition 3.4.3. We claim that \( h - P_S(h) \in S^\perp \). For any \( s \in S \), we have

\[
||h - P_S(h)|| \leq ||h - P_S(h) + es|| = ||h - P_S(h)||^2 + 2e \langle h - P_S(h), s \rangle + e||s||^2
\]

so we must have \( \langle h - P_S(h), s \rangle = 0 \).

Therefore, any \( h \in H \) may be written as

\[
h = h_1 + h_2 \quad h_1 \in S, h_2 \in S^\perp.
\]

If \( h = h_1' + h_2' \) is another such decomposition, then \( h_1 - h_1' = h_2 - h_2' \in S \cap S^\perp = 0 \), so \( h_1 = h_1' \) and \( h_2 = h_2' \).

\( \square \)

3.4.2 The Riesz representation theorem

Theorem 3.4.5 (Riesz Representation Theorem). Let \( \mathcal{H} \) be a Hilbert space with norm \( \langle u, v \rangle \). If \( f : \mathcal{H} \to \mathbb{R} \) is a bounded linear functional, then

\[
f(v) = \langle u, v \rangle
\]

for some unique \( u \in \mathcal{H} \). Moreover, \( ||f|| = ||u|| \).

Remark 3.4.6. We denote by \( H^* \) the space of bounded linear operators \( \mathcal{H} \to \mathbb{R} \). Then the theorem can be viewed as saying that \( u \mapsto \langle u, - \rangle \) is an isometry of \( \mathcal{H} \) onto \( H^* \).

Proof. If \( f \) is identically zero, then the conclusion is obvious, so we assume otherwise. For motivation, think about the finite-dimensional case. We can think of the linear functional defining a coordinate on \( \mathcal{H} \), say \( x_1 \). How do you recognize the unit vector corresponding to this coordinate? It is the smallest vector that maps to 1!

Motivated by this discussion, let \( C_f \) be the set

\[
C_f = \{ y \in \mathcal{H} : f(y) = 1 \}.
\]

This is a convex set since \( f \) is linear. By Proposition 3.4.3, there exists some \( y_0 \in C_f \) of minimal norm. Then we claim that \( y_0 \in (\ker f)^\perp \). If \( x \in (\ker f)^\perp \), then \( y_0 + tx \in C_f \) for any \( t \in \mathbb{R} \). We have

\[
||y_0||^2 \leq ||y_0 + tx||^2 = \langle y_0 + tx, y_0 + tx \rangle = ||y_0||^2 + t \langle x, y_0 \rangle + t^2 ||x||^2.
\]

Since this holds for all \( t \), we must have \( \langle x, y_0 \rangle = 0 \) (otherwise by taking \( t \) close enough to zero, we will violate the inequality).

Observe that for any \( v \in \mathcal{H} \), we have \( v - f(v)y_0 \in (\ker f)^\perp \), so

\[
\langle v, y_0 \rangle = \langle f(v)y_0, y_0 \rangle = f(v)||y_0||^2.
\]

Then \( u := \frac{y_0}{||y_0||^2} \) satisfies

\[
\langle v, u \rangle = f(v).
\]

\( \square \)
Remark 3.4.7. One can give a slicker proof (which is essentially the same) using Corollary 3.4.4. We can assume that \( f \) is non-zero. Since \( f \) is continuous, \( S := \ker f \) is closed and non-empty and we may write
\[
\mathcal{H} = S \oplus S^\perp.
\]
If we choose any \( y_0 \in S^\perp \) such that \( f(y_0) = 1 \), then we claim that \( u = \frac{f(y_0)}{||y_0||^2} \) is the desired element. Indeed, any \( v \in \mathcal{H} \) may be written as
\[
v = f(v)y_0 + (v - f(v)y_0)
\]
where the second term is in \( S \). Then
\[
\langle v, u \rangle = \langle f(v)y_0, u \rangle = f(v).
\]

3.5 Laplace’s Equation: Weak Solutions

3.5.1 The Poincaré Inequality for \( H^1_0 \)

We would like to be able to give a uniform bound on functions \( u \) in some space in terms of \( Du \). Such inequalities are called Poincaré inequalities. We need to impose some restrictions in order to do this. First, we must impose some boundary vanishing, since the constant functions have vanishing derivatives but could be arbitrarily large. Second, we will obviously have difficulty if the domains are unbounded, since over large distances functions can grow slowly and still become quite large.

Therefore, we focus our attention on a Sobolev space \( H^1_0(\Omega) \) where \( \Omega \subset \mathbb{R}^n \) a domain that is bounded in some direction. Note that this implies that \( u \in H^1_0(\Omega) \) vanishes on \( \partial \Omega \) in the trace sense.

**Theorem 3.5.1** (Poincaré inequality). Let \( \Omega \) be a region that is bounded in one direction. Then
\[
||u||_{L^2(\Omega)} \leq C||Du||_{L^2(\Omega)} \text{ for all } u \in H^1_0(\Omega).
\]
for some constant \( C \) depending only on \( \Omega \).

**Proof.** Without loss of generality, we may assume that \( \Omega \) is bounded in the \( x_n \) direction, say between \( x_n = 0 \) and \( x_n = a \). Then for any \( (x', y) \in \Omega \), we have
\[
|u(x', y)| = \left| \int_0^y \frac{\partial u}{\partial x_n}(x', x_n) \, dx_n \right|
\]
\[
\leq a \left| \int_0^a \frac{\partial u}{\partial x_n}(x', x_n) \, dx_n \right|
\]
\[
\leq a^{1/2} \left( \int_0^a \left| Du(x', x_n) \right|^2 \, dx_n \right)^{1/2}.
\]
Therefore,
\[
|u(x', y)|^2 \leq a \left( \int_0^a \left| Du(x', x_n) \right|^2 \, dx_n \right).
\]
Integrating over $\Omega$, we see that
\[
||u||^2_{L^2(\Omega)} = \int_{\Omega} |u(x',y)|^2 \, dy \, dx' \\
\leq \int a \left( \int_0^a \int |Du(x',x_n)|^2 \, dx_n \right) \, dy \, dx' \\
\leq a^2 \int \Omega |Du(x',x_n)|^2.
\]

The point of this inequality is that it allows us to compare the standard norm on $H_0^1(\Omega)$ with the “homogeneous norm” involving only the derivatives.

**Corollary 3.5.2.** If $\Omega \subset \mathbb{R}^n$ is bounded in some direction, then $H_0^1(\Omega)$ equipped with the inner product
\[
(u,v)_0 = \int_{\Omega} uv
\]
is a Hilbert space isomorphic to $H_0^1(\Omega)$ with the standard inner product.

**Proof.** Denote the standard inner product by
\[
(u,v) = \int_{\Omega} uv + Du \cdot Dv.
\]
We obviously have $(u,u)_0 \leq (u,u)$. On the other hand, Poincare’s inequality shows that
\[
(u,u) \leq \int |u|^2 + |Du|^2 \leq (1 + C^2) \int |Du|^2,
\]
so we find that there exists a constant $C$ (depending only on $\Omega$) for which
\[
(u,u)_0 \leq (u,u) \leq C(u,u)_0 \text{ for all } u \in H_0^1(\Omega).
\]

### 3.5.2 Existence of weak solutions

We are now ready to prove existence of weak solutions to Laplace’s equation.

**Theorem 3.5.3.** Let $\Omega \subset \mathbb{R}^n$ be an open set that is bounded in some direction and $f \in H^{-1}(\Omega)$. Then there is a unique weak solution $u \in H_0^1(\Omega)$ of the equation $-\Delta u = f$.

**Proof.** As before, we define the inner product on $H_0^1(\Omega)$
\[
(w,v)_0 = \int_{\Omega} Dw \cdot Dv.
\]
By definition, a weak solution to $-\Delta u = f$ is some $u$ satisfying
\[
(u,\phi)_0 = \langle f, \phi \rangle \text{ for all } \phi \in H_0^1(\Omega)
\]
By assumption, the function $\phi \mapsto \langle f, \phi \rangle$ is a bounded linear functional on the Hilbert space $(H_0^1(\Omega), (,)_0)$ so we may apply the Riesz Representation Theorem 3.4.5 to deduce that there exists a unique $u \in H_0^1(\Omega)$ such that
\[
(u,\phi)_0 = \langle f, \phi \rangle.
\]
It is clear that the same argument will work for other symmetric linear elliptic PDE.

**Example 3.5.4.** Consider the Dirichlet problem

\[-\Delta + u = f \quad \text{in } \Omega,\]
\[u = 0 \quad \text{on } \partial \Omega.\]

Then \(u \in H^1_0(\Omega)\) is a weak solution if

\[
\int_{\Omega} (Du \cdot D\phi + u\phi) dx = \langle f, \phi \rangle \quad \text{for all } \phi \in H^1_0(\Omega).
\]

This is equivalent to \((u, \phi) = \langle f, \phi \rangle\) where we are the *standard* norm on \(H^1_0(\Omega)\), so the Riesz representation theorem implies the existence of a unique weak solution. Note that we do not even need Poincaré’s inequality since we are working with the standard norm! In particular, the result is valid on potentially unbounded domains. In \(\mathbb{R}^n\), for instance, we have \(H^1_0(\mathbb{R}^n) = H^1(\mathbb{R}^n)\).

Moreover, it is clear that if \(u\) is a weak solution then \(||u||_{H^1} = ||f||_{H^{-1}},\) so \(-\Delta + I\) is an isometry of \(H^1(\mathbb{R}^n)\) onto \(H^{-1}(\mathbb{R}^n)\).

3.6 **The Lax-Milgram Theorem and General Elliptic PDE**

3.6.1 **The weak formulation**

We now consider a general second-order semi-linear elliptic PDE

\[
Lu := -\sum_{ij} \partial_j(a^{ij}(x)\partial_i u) + \sum_i b^i(x)\partial_i u + c(x)u = f
\]

(7)

on a domain \(U\), with the *uniform ellipticity* assumption

\[
\sum_{ij} a^{ij}(x)\xi_i\xi_j \geq \theta ||\xi||^2 \quad \text{for all } \xi \in U.
\]

(8)

(Ellipticity corresponds to the condition \(\sum_{ij} a^{ij}(x)\xi_i\xi_j > 0\) for \(\xi \neq 0\).) We all suppose that all the coefficients \(a^{ij}(x), b^i(x), c(x) \in L^\infty(U)\).

We consider a weak formulation for the problem. Let

\[
B(u, v) = \int \sum_{ij} a^{ij}(\partial_i u)(\partial_j v) + \int \sum_i b^i(\partial_i u)v + \int cuv.
\]

(9)

**Definition 3.6.1.** We say that \(u \in H^1(\Omega)\) is a weak solution to the PDE (7) if

\[
B(u, \phi) = \int f \phi
\]

for all \(\phi \in H^1(\Omega)\).

The proof of existence of weak solutions in general is similar to what we did for the Laplace, but this time \(B(u, v)\) does not define an inner product since it is not symmetric. The Lax-Milgram Theorem is a way of generalizing the Riesz Representation Theorem to non-symmetric bilinear forms.
3.6.2 The Lax-Milgram Theorem

We let \((H, \langle \cdot, \cdot \rangle)\) be a (real) Hilbert space and \(\langle \cdot, \cdot \rangle : H \times H^* \to \mathbb{R}\) the pairing between \(H\) and its dual space.

**Theorem 3.6.2 (Lax-Milgram).** Let \(B : H \times H \to \mathbb{R}\) a bilinear form. Suppose further that there exist constants \(C_1, C_2\) such that for all \(u, v \in H\) we have

\[
C_1 \|u\|^2 \leq B(u, u)
\]

and

\[
|B(u, v)| \leq C_2 \|u\| \|v\|
\]

Then for every bounded linear functional \(f : H \to \mathbb{R}\), there exists a unique \(u \in H\) such that

\[
B(u, v) = \langle f, v \rangle \text{ for all } v \in H.
\]

**Remark 3.6.3.** In the case where \(B(u, v)\) is symmetric, the two conditions say that \(B\) induces a norm equivalent to the given one on \(H\).

**Proof.** The idea of the proof is quite simple: show that \(v \mapsto B(u, v)\) and \(v \mapsto \langle f, v \rangle\) are both bounded operators, and use the Riesz Representation Theorem to write them both as an inner product.

1. Let’s first argue carry this out for the linear functional \(v \mapsto B(u, v)\). For any fixed \(u\), this is bounded by the hypothesis, so there exists some \(Au \in H\) such that

\[
B(u, v) = \langle Au, v \rangle.
\]

We claim that the map \(u \mapsto Au\) is itself a bounded linear operator. The linearity follows from the fact that \(B\) is a bilinear form. On the other hand, we have by hypothesis that

\[
C_1 \|u\|^2 \leq B(u, Au) \leq C_2 \|u\| \|Au\|.
\]

This implies that \(\|Au\| \leq C \|u\|\) for some constant \(C\), establishing boundedness.

2. Next we claim that \(A\) is actually a bijection. Note that

\[
C_1 \|u\|^2 \leq B(u, u) = \langle Au, u \rangle \leq C_2 \|Au\| \|u\|
\]

so \(C \|u\| \leq \|Au\|\) for some \(C > 0\). That already forces \(A\) to be injective.

The inequality (10) also implies that the image of \(A\) is closed. Indeed, if we have a Cauchy sequence \(Au_n\) tending to a limit \(v \in H\), then (10) implies that

\[
\|u_m - u_n\| \leq C \|Au_m - Au_n\| \to 0
\]

so \(u_n \to u \in H\), whose image is necessarily \(v\) since \(A\) is continuous.

Suppose \(A(H) \neq H\). Since \(A(H)\) is closed, we have a splitting \(H \cong A(H) \oplus \ker A\). But if \(u \in A(H)\), then

\[
0 = \langle Au, u \rangle = B(u, u) \geq C_1 \|u\|^2 \implies u = 0.
\]
3. Finally, the fact that $f$ is bounded implies that there exists some $w$ such that

$$\langle f, v \rangle = \langle w, v \rangle \text{ for all } v \in H.$$ 

Since $A$ is a bijection, we have $w = Au$ for some $u$, hence

$$B(u, v) = (Au, v) = (w, v) = \langle f, v \rangle \text{ for all } v \in H.$$ 

\[\square\]

### 3.6.3 Existence of weak solutions

**Proposition 3.6.4.** Let $B$ be the bilinear form on $H_0^1(U)$ defined by (9), where the coefficients satisfy (8). Then there exist constants $C_1, C_2 > 0$ and $\gamma \in \mathbb{R}$ such that for all $u, v \in H_0^1(U)$ we have

$$C_1||u||_{H_0^1}^2 \leq B(u, u) + \gamma||u||_{L^2}^2$$

and

$$|B(u, v)| \leq C_2||u||_{H_0^1}||v||_{H_0^1}.$$ 

**Proof.** We establish the second estimate first.

$$|B(u, v)| = \left| \int \sum a^{ij} \partial_i u \partial_j v + \int \sum b^j v \partial_j u + \int cuv \right|$$

$$\leq \sum ||a^{ij}||_{L^\infty} \int |Du||Dv| + \sum ||b^j||_{L^\infty} \int |v||Du| + \int |c||u||v|$$

$$\ll ||Du||_{L^2}||Dv||_{L^2} + ||Du||_{L^2}||v||_{L^2} + ||u||_{L^2}||v||_{L^2}$$

$$\leq C_2||u||_{H_0^1}||v||_{H_0^1}.$$

For the first estimate, we note by the uniform ellipticity assumption that

$$\theta||Du||_{L^2(U)}^2 \leq B(u, u) - \int \sum b^j u \partial_j u - \int cu^2$$

$$\leq B(u, u) + \sum ||b^j||_{L^\infty} \int u|Du| + ||c||_{L^\infty} \int u^2.$$ 

Using the weighted AM-GM inequality in the form $u|Du| \leq \theta|Du|^2 + \frac{C}{4\epsilon}|u|^2$ for $\epsilon$ small enough, we obtain a bound of the form

$$C||Du||_{L^2(U)}^2 \leq B(u, u) + \gamma||u||_{L^2}^2.$$ 

By Poincaré’s inequality (Theorem 3.5.1), we have $C_2||u||_{L^2(U)}^2 \leq C||Du||_{L^2(U)}^2$ for some constant $C_2$, which gives the estimate we want. 

We would like to apply the Lax-Milgram Theorem 3.6.2 to our bilinear form

$$B(u, v) = \int \sum a^{ij} \partial_i u \partial_j v + \int \sum b^j (\partial_i u)v + \int cuv.$$ 

Unfortunately, the hypotheses are not quite satisfied, since the lower bound

$$C_1||u||_{H_0^1}^2 \leq B(u, u) + \gamma||u||_{L^2}^2$$
is not good enough for Lax-Milgram if \( \gamma \) is positive. By tracing through the proof, one can check that we can take \( \gamma = 0 \) if \( b, c = 0 \). In general we cannot necessarily conclude the existence of solutions. We can, however, deduce the following.

**Theorem 3.6.5 (Existence and uniqueness of weak solutions).** Under the above assumptions, there exists some \( \gamma > \mathbb{R} \) such that for all \( \mu \geq \gamma \), and all \( f \in H_0^1(U) \), the PDE

\[
Lu + \mu u = f
\]

has a unique solution \( u \in H_0^1(U) \).

**Proof.** We introduce the bilinear form

\[
B_\mu(u, v) = B(u, v) + \mu uv
\]

where \( B \) is as in Proposition 3.6.4. Then \( u \) solves

\[
Lu + \mu uv = f
\]

if and only if \( B_\mu(u, v) = f \), and now by Proposition 3.6.4 the form \( B' \) satisfies the conditions to apply the Lax-Milgram Theorem 3.6.2. Doing so, we obtain the desired conclusion.

\[\square\]

### 3.7 THE FREDHOLM ALTERNATIVE

#### 3.7.1 Recasting the problem

We just barely failed to prove existence of weak solutions to (uniformly) elliptic PDE. To say that the PDE \( Lu = f \) has a unique weak solution \( u \in H_0^1(U) \) for all \( f \in H_0^1(U) \) would amount to saying that the map \( L \) is invertible. While we cannot prove that \( L \) is invertible (indeed, it may not be), we have seen (Theorem 3.6.5) that if \( \mu \) is sufficiently large, then \( L + \mu I \) is invertible. We might then attempt to factorize

\[
L = (L + \mu I - \mu I)
\]

so that our PDE becomes

\[
(L + \mu I)u - \mu u + f.
\]

Applying \( (L + \mu I)^{-1} \), we can rewrite this as

\[
(I - \mu (L + \mu I)^{-1})u = (L + \mu I)^{-1} f.
\]

Setting \( K := \mu(L + \mu I)^{-1} \) and \( h := (L + \mu I)^{-1} f \), we are reduced to studying the question

\[
(I - K)u = h. \tag{11}
\]

This may not seem like a simplification, but it is significant but it turns out that the operator \( K \) is compact, meaning that it is small in some sense. Therefore, \( I - K \) can be viewed as a perturbation of the identity.
3.7.2 Compact operators

**Definition 3.7.1.** Let $\mathcal{H}, \mathcal{H}'$ be Hilbert spaces. We say that a map $T: \mathcal{H} \to \mathcal{H}'$ is compact if it is bounded and it sends the unit ball in $\mathcal{H}$ to a relatively compact set in $\mathcal{H}'$. If $\mathcal{H}' = \mathcal{H}$, we call $T$ a compact operator.

**Example 3.7.2.** If $\mathcal{H}$ is finite-dimensional, then the unit ball is already compact, so any linear map has this property. More generally, any bounded linear operator with finite-dimensional range is compact.

On the other hand, if $\mathcal{H}$ is infinite-dimensional then the unit ball is not relatively compact, so for instance the identity operator is not compact. Thus, in infinite-dimensional Hilbert spaces a compact operator is “small.”

**Example 3.7.3.** The map $K$ on $\ell^2$ given by

$$K: (x_1, x_2, x_3, \ldots) \mapsto (x_1, \frac{1}{2} x_2, \frac{1}{3} x_3, \ldots)$$

is compact.

**Definition 3.7.4.** A sequence $(u_n)_n \subset \mathcal{H}$ converges weakly to $u \in \mathcal{H}$ if

$$\langle u_n, f \rangle \to \langle u, f \rangle$$

for all bounded $f \in \mathcal{H}^*$. 

**Remark 3.7.5.** Note that by the Riesz Representation Theorem 3.4.5, this is equivalent to

$$\langle u_n, v \rangle \to \langle u, v \rangle$$

for all $v \in \mathcal{H}$.

It is easy to check that any weakly convergence sequence is bounded.

**Lemma 3.7.6.** If $u_n \rightharpoonup u$ weakly in $\mathcal{H}$ and $K$ is a compact operator on $\mathcal{H}$, then $Ku_n \to Ku$ strongly in $\mathcal{H}$.

**Proof.** If not, then we could extract a subsequence $(u_{n_j})_j$ such that no subsequence of $(Ku_{n_j})_j$ converges to $Ku$. But since $(u_{n_j})_j$ is bounded, $(Ku_{n_j})_j$ has a subsequence converging strongly to some limit, and since $u_{n_j} \rightharpoonup u$, this limit must be $Ku$, a contradiction.

**Definition 3.7.7.** A sequence $(u_n)_n \subset \mathcal{H}$ is weakly precompact if there exists a subsequence $(u_{n_j})_j$ and $u \in H$ such that

$$\langle u_{n_j}, v \rangle \to \langle u, v \rangle$$

for all $v \in H^*$.

**Theorem 3.7.8.** Let $X$ be a reflexive Banach space (i.e. $X'' = X$). If $(u_n)_n \subset X$ is a bounded sequence, then it is weakly precompact.

**Proof.** In the Hilbert space case, this is quite easy: choose an orthonormal basis $(e_1, e_2, \ldots)$. We can extract a subsequence of $(u_n)$ that converging in the $e_1$ coordinate, then a further subsequence converging in the $e_2$ coordinate, etc. and use a diagonal argument.

In general, one argues by general topology. Taking $I = [-1, 1]$, we form the set $D = \prod_{x \in X} I$. Since $I$ is compact, Tychonoff’s theorem implies that $D$ is compact.

If $B$ denotes the unit ball of $X^*$, then we have a map $B \to D$ taking $f \mapsto (f(x))_{x \in X}$. This is clearly injective and continuous, with continuous inverse. So it suffices to show that the image is closed. But indeed, if $(f_\alpha)_{x \in X}$ is a net in the image converging to $(\lambda_x)_{x \in X}$, then the functional $x \mapsto \lambda_x$ is in $B$ and has image $(\lambda_x)_{x \in X}$. 

Recall that the adjoint of an operator $T$ is the linear operator $T^*$ satisfying $(Tx, y) = (x, T^*y)$ for all $x, y \in \mathcal{H}$.

**Lemma 3.7.9.** If $K$ is a compact operator on $\mathcal{H}$, then its adjoint $K^*$ is also compact.

**Proof.** It suffices to show that $K^*$ takes any bounded sequence to a precompact sequence. Let $(u_n)_n \in \mathcal{H}$ be a bounded sequence. By Theorem 3.7.8, after passing to a subsequence we may assume that $u_n \rightharpoonup u$ weakly for some $u \in \mathcal{H}$. We will extract a subsequence $u_{n_i}$ such that $K^* u_{n_i} \to K^* u$.

Observe that

$$||K^* u - K^* u_n|| = \langle K^* u - K^* u_n, K^* u - K^* u_n \rangle = \langle KK^* u - KK^* u_n, u - u_n \rangle.$$  

Now, we have $K^* u_n \rightharpoonup K^* u$ weakly, so by Lemma 3.7.6 we have $KK^* u_n \to KK^* u$ strongly. Inserting that above, and applying Cauchy-Schwarz, we deduce

$$\langle KK^* u - KK^* u_n, u - u_n \rangle \to 0.$$  

\[\Box\]

**Proposition 3.7.10.** Let $K$ be a compact operator on a Hilbert space $\mathcal{H}$. Then $I + K$ has closed range, finite-dimensional kernel, and finite-dimensional cokernel (the cokernel is the orthogonal complement of the range, by definition).

**Proof.** To show that $\ker(I + K)$ is finite-dimensional, it suffices to show that its unit ball is compact. Let $(x_n)_n$ be a sequence in $\ker(I + K)$ satisfying $|x_n| \leq 1$ for all $n$. Then we have

$$x_n = -Kx_n$$

but since the $x_n$ are bounded and $K$ is compact, Theorem 3.7.8 and Lemma 3.7.6 imply that there is a subsequence $(x_{n_i})_i$ such that $Kx_{n_i}$ converges strongly, so $x_{n_i}$ converges strongly by the equation. Since kernels are always closed, we deduce that any sequence of the unit ball in $\ker(I + K)$ has a convergent subsequence, hence it is compact.

Next, let us show that $I + K$ is coercive on $\ker(I + K)^\perp$, i.e. there exists $\lambda > 0$ such that

$$||x + Kx|| \geq \lambda ||x||$$

for all non-zero $x \in \ker(I + K)^\perp$.

If not, then there exists a sequence $(x_n)_n$ in $\ker(I + K)^\perp$ such that $||x_n|| = 1$ for all $n$, and

$$||x_n + Kx_n|| \to 0.$$  

Since the sequence $(x_n)_n$ is bounded, Theorem (3.7.8) implies that it has a weakly converging subsequence. Passing to this subsequence, we have $x_n \rightharpoonup x$ weakly, so obviously $x \in \ker(I + K)^\perp$. On the other hand, Lemma 3.7.6 implies that $Kx_n \to Kx$ strongly, so from the preceding equation we deduce that $x \in \ker(I + K)$ as well. Clearly this is only possible if $x = 0$, which is a contradiction because $||x_n|| = 1$ for all $n$.  

\[\Box\]
3.7.3 Fredholm theory

**Definition 3.7.11.** Let $\mathcal{H}, \mathcal{H}'$ be Hilbert space. A linear map $T: \mathcal{H} \to \mathcal{H}'$ is said to be **Fredholm** if $\ker T$ is finite-dimensional and $\coker T$ is finite-dimensional (so $T$ has closed range). If furthermore $\mathcal{H} = \mathcal{H}'$, we say that $T$ is a **Fredholm operator**.

**Example 3.7.12.** By Proposition 3.7.10, if $K$ is a compact operator then $I + K$ is a Fredholm operator.

**Remark 3.7.13.** An equivalent definition of Fredholm operator is “invertible modulo compact operators,” i.e. $T$ is Fredholm if there exists an operator $S$ such that $I - TS$ and $I - ST$ are both compact.

**Definition 3.7.14.** If $T$ is a Fredholm operator, then we define the **index** of $T$ to be

$$\text{ind } T = \dim \ker T - \dim \coker T.$$  

**Example 3.7.15.** Any operator on a finite-dimensional Hilbert space is Fredholm, and has index 0. The identity operator on any Hilbert space is Fredholm with index 0.

**Theorem 3.7.16.** The index is locally constant on Fredholm operators in the operator norm.

**Proof.** The proof is via a perturbation argument. Write $$\mathcal{H} \cong C \oplus \ker T \cong \text{range}(T) \oplus \coker T.$$

With respect to this decomposition, we have

$$T = \begin{pmatrix} T_{11} & 0 \\ 0 & 0 \end{pmatrix}.$$  

Suppose we have a small perturbation operator

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$  

Then

$$T + B = \begin{pmatrix} T_{11} + B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$  

We “diagonalize” this using invertible operators. For $B$ very small, $B_{11}$ will be small so since $T$ is invertible, $T_{11} + B_{11}$ will be invertible.

$$\begin{pmatrix} 1 & 0 \\ -B_{21}(T + B_{11})^{-1} & 1 \end{pmatrix} \begin{pmatrix} T_{11} + B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} T_{11} + B_{11} & B_{12} \\ 0 & -B_{21}(T_{11} + B_{11})^{-1}B_{12} + B_{22} \end{pmatrix}.$$  

Next, we right-multiply

$$\begin{pmatrix} T_{11} + B_{11} & B_{12} \\ 0 & -B_{21}(T_{11} + B_{11})^{-1}B_{12} + B_{22} \end{pmatrix} \begin{pmatrix} I & -(T_{11} + B_{11})^{-1}B_{12} \\ 0 & I \end{pmatrix} = \begin{pmatrix} I + B_{11} & 0 \\ 0 & A \end{pmatrix}.$$  

Since multiplying by invertible operators doesn’t change the index and $\text{ind } (I) = 0$, we have $\text{ind } (T + B) = \text{ind } A$ where now $A$ is map between finite-dimensional spaces $\ker T$ and $\coker T$, hence has index 0. \qed
Corollary 3.7.17. Let $K$ be a compact operator on a Hilbert space $\mathcal{H}$. Then
\[
\dim \ker (I + K) = \dim \coker (I + K) < \infty.
\]

Proof. $I + tK$ is Fredholm for all $t \in [0, 1]$, so Theorem 3.7.16 tells us that $\text{ind} (I + tK)$ is a locally constant function from $[0, 1]$ to $\mathbb{Z}$. Obviously, the only such function is constant, and at $t = 0$ it is obviously 0. $\square$

Example 3.7.18. The projection map on $\ell^2$
\[
P: (x_1, x_2, x_3, \ldots) \mapsto (0, x_2, x_3, \ldots)
\]
is Fredholm with $\dim \ker P = \dim \coker P = 1$, hence $\text{ind} P = 0$.

Theorem 3.7.19 (Fredholm Alternative). Let $K: H \to H$ be a compact linear operator. Then
\begin{enumerate}
\item $\ker(I + K)$ is finite-dimensional,
\item $\text{range}(I + K)$ is closed,
\item $\text{range}(I + K) = \ker(I + K^*)$,
\item $\ker(I + K) = 0 \iff \ker(I + K^*) = 0$,
\item $\dim \ker(I + K) = \dim \ker(I + K^*)$.
\end{enumerate}

Proof. Only (3) and (5) remain to be proved. In fact, (3) is an immediate consequence of the general fact that for any operator on $\mathcal{H}$, we have $\overline{\text{range}(A)} = \ker(A^*)^\perp$.

More generally, this is true with $I + K$ replaced by a Fredholm operator.

Corollary 3.7.20. If $T$ is a Fredholm operator, then $T^*$ is Fredholm with
\[
\dim \ker T = \dim \coker T^*, \quad \dim \ker T^* = \dim \coker T, \quad \text{ind} T = - \text{ind} T^*.
\]

Example 3.7.21. We define the left and right shift maps on $\ell^2$ by
\[
L: (x_1, x_2, x_3, \ldots) \mapsto (x_2, x_3, \ldots)
\]
and
\[
R: (x_1, x_2, x_3, \ldots) \mapsto (0, x_1, x_2, \ldots).
\]
Note that $(Lx, y) = (x, Ry)$, so $L$ and $R$ are adjoint.

Observe $\dim \ker L = 1$ and $\dim \coker L = 0$, so $L$ has index 1. On the other hand, $\dim \ker R = 0$ and $\dim \coker R = 1$, so $R$ has index $-1$. So indeed, Corollary 3.7.20 holds in this case.

By considering $L^n$ and $R^n$, we see that a Fredholm operator can have any integer index.
3.7.4 The Fredholm alternative

Now we return to (11):

\[ u - Ku = h. \]

We check that the operator \( K \) is compact; it clearly suffices to show that \((L + \mu I)^{-1}\) is compact. Define the bilinear form \( B_\mu \) as before. By the Riesz Representation Theorem 3.4.5, for all \( g \in H_0^1(U) \) there exists \( u \in H_0^1(U) \) such that

\[ B_\mu(u, v) = (g, v) \quad \text{for all } v \in H_0^1(U). \]

In these terms, \( u = (L + \mu I)^{-1}g \). Recall from Proposition 3.6.4 that there exists some uniform constant \( C \) such that

\[ ||u||_{H^1_0(U)} \leq CB_\mu(u, u) = C(g, u) \leq ||g|| \cdot ||u||_{H^1_0(U)} \]

which shows that

\[ ||(L + \mu I)^{-1}g||_{H^1_0(U)} \leq ||g||. \]

That implies that \( L + \mu I \) is a bounded operator from \( H_0^1(U) \) into \( H_0^1(U) \), and then the result is immediate from the following theorem.

**Theorem 3.7.22** (Rellich-Kondrachov, \( p = 2 \)). Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set and suppose that \( \partial \Omega \) is \( C^1 \). Then the inclusion

\[ H^1(\Omega) \hookrightarrow L^2(\Omega) \]

is compact.

This is the special case \( p = 2 \) of the following more general formulation.

**Theorem 3.7.23** (Rellich-Kondrachov Compactness Theorem). Let \( U \subset \mathbb{R}^n \) be a bounded open set and suppose that \( \partial U \) is \( C^1 \). Then the inclusion

\[ W^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \]

is compact for each \( 1 \leq q \leq p^* \).

**Proof.** The inclusion \( W^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \) is due to the Sobolev embeddings (since \( \Omega \) is bounded). Let \( (u_k)_k \) be a bounded sequence in \( W^{1,p}(\Omega) \); we wish to show that it has a subsequence converging in \( L^q(\Omega) \).

Let \( \chi_\epsilon \) be the family of standard mollifiers. Setting \( u^\epsilon_k = \chi_\epsilon * u_k \), we claim that the family \( \{u^\epsilon_k\} \) is uniformly bounded and equicontinuous for each fixed \( \epsilon \). Indeed,

\[ |u^\epsilon_k(x)| \leq \int \chi_\epsilon(x-y)u_k(y) \, dy \leq \epsilon^{-n}||\chi||_{L^\infty} ||u_k||_{L^1}. \]

Since \( ||\chi||_{L^\infty} \) is finite and \( ||u_k||_{L^1} \leq ||u_k||_{L^p} \leq ||u_k||_{W^{1,p}} \) (again using the boundedness of the domain), this is uniformly bounded. Similarly,

\[ |Du^\epsilon_k(x)| \leq \int D\chi_\epsilon(x-y)u_k(y) \, dy \leq \epsilon^{-n-1}||\chi||_{L^\infty} ||u_k||_{L^1} \]

is bounded, so the family is equicontinuous.

By the Arzela-Ascoli theorem, for a fixed \( \epsilon \) we can extract a subsequence \( (u^\epsilon_{k_i})_i \) converging uniformly on compact subsets, i.e.

\[ \lim_{\sup(i,j) \to \infty} ||u^\epsilon_{k_i} - u^\epsilon_{k_j}||_{L^\infty} = 0 \] (12)
Lemma 3.7.24. We have \( \lim_{\epsilon \to 0} u_k^\epsilon \to u_k \) uniformly in \( L^q(\Omega) \).

Proof. By approximation, it suffices to establish the inequality under the assumption that \( u_k \) is smooth. First observe that as above, \( \| u_k^\epsilon - u_k \|_{L^1(\Omega)} \) is uniformly bounded. Now, we have

\[
|u_k^\epsilon(x) - u_k(x)| \leq \int_{\Omega} \chi_\epsilon(y) |u_k(x-y) - u_k(x)| \, dy \\
\leq \epsilon \int \chi(y) |u_k(x-\epsilon y) - u_k(x)| \, dy \\
= \int \chi(y) \int_0^1 \left| \frac{d}{dt} u_k(x - \epsilon ty) \right| \, dt \, dy.
\]

Integrating both sides over \( V \), we find that

\[
\| u_k^\epsilon - u_k \|_{L^1} \leq \epsilon \| D u_k \|_{L^1}.
\]

Hence

\[
\| u_k^\epsilon - u_k \|_{L^1} \leq \epsilon \| D u_k \|_{L^1} \leq C \| D u_k \|_{W^{1,p}}.
\]

Therefore, we have \( \| u_k^\epsilon - u_k \|_{L^1} \to 0 \) uniformly in \( L^1 \). By using the interpolation inequality \( \| u \|_{L^q} \leq \| u \|_{L^p} \| u \|_{L^{p_\theta}}^{1-\theta} \) together with the Sobolev inequality to bound the second factor, we deduce the convergence in \( L^q \) as well.

By the Lemma, by choosing \( \epsilon \) very small we may guarantee that \( \| u_n^\epsilon - u_n \|_{L^q} \) is uniformly small, and combining that with (12) we see that for any fixed \( \delta \), we can choose a subsequence \( (n_i)_i \) such that

\[
\lim_{\sup (i,j) \to \infty} \| u_{n_i} - u_{n_j} \|_{L^q} < \delta.
\]

Applying the usual diagonal argument, we obtain a further subsequence converging uniformly in \( L^q \).

We may now apply Theorem 3.7.19 to deduce:

Theorem 3.7.25. Consider a second-order elliptic equation

\[ Lu = f \]

where \( L \) satisfies the same conditions we assumed above. Then exactly one of the following holds:

1. For each \( f \in H^1_0(U) \), there exists a unique weak solution \( u \in H^1_0(U) \) of the problem \( Lu = f \), or else
2. there exists a non-zero weak solution \( u \in H^1_0(U) \) of the homogeneous problem \( Lu = 0 \).

Furthermore, in the second case the dimension of the space \( N \subset H^1_0(U) \) of weak solutions to \( Lu = f \) is finite and equals the dimension of the space \( N^* \subset H^1_0(U) \) of solutions to \( L^* v = f \), where

\[
L^* v = -\sum_{i,j} \partial_i (a^{ij} \partial_j v) - \sum_i b^i \partial_i v + \left( c - \sum_i \partial_i b^i \right) v.
\]

In this case, there exists a solution to \( Lu = f \) if and only if

\[
(f, v) = 0 \text{ for all } v \in N^*.
\]
3.8 Interior Elliptic Regularity

Elliptic PDE have the remarkable property that their solutions tend to automatically be smoother than is necessary to formulate the PDE. You have probably already seen this phenomenon in the example of holomorphic functions, or more generally harmonic functions, which are smooth as soon as they are $C^2$.

We consider an elliptic operator

$$Lu = -\sum_{i,j} \partial_j (a^{ij}(x) \partial_i u) + \sum_i b^i(x) \partial_i u + c(x) u.$$ 

with the uniform ellipticity assumption

$$\sum_{i,j} a^{ij}(x) \xi_i \xi_j \geq \theta ||\xi||^2$$ for all $\xi \in \mathbb{R}^n$.

We suppose that we have a weak solution to the PDE $Lu = f$ on the bounded domain $U \subset \mathbb{R}^n$, which means that for all $v$ in the relevant function space we have

$$\sum_{i,j} \int_U a^{ij}(\partial_i u)(\partial_j v) + \sum_i \int_U b^i(\partial_i u)v + \int_U cuv = \int_U f v$$

(13)

### 3.8.1 The bootstrap argument

The key result is the following a priori estimate.

**Theorem 3.8.1.** Let $L$ be as above, with $a^{ij} \in C^1(U)$ and $b^i, c \in L^\infty(U)$ for all $i, j = 1, \ldots, n$. If $f \in L^2(U)$ and $u \in H^1(U)$ weakly solves $Lu = f$, then $u \in H^2_{loc}(U)$ and for all $V \subset U$ we have the estimate

$$||u||_{H^2(V)} \leq C(||f||_{L^2(U)} + ||u||_{L^2(U)}).$$

where the implicit constant depends only on $V, U$, and $L$.

**Remark 3.8.2.** Note that Theorem 3.8.3 tells us that $u$ actually solves the PDE almost everywhere. Since $u$ has (weak) second derivatives, this actually makes sense, and we can integrate by parts in the definition of weak solution to see that $\langle Lu - f, v \rangle = 0$ for all $v \in C^\infty_c(U)$, hence $Lu = f$ almost everywhere.

This theorem is the main technical ingredient in establishing (interior) elliptic regularity. We will give the proof in the next section. Recall that for the Poisson equation, assuming that $u$ is $C^2$ we deduced $||D^2 u||_{L^2(U)} = ||f||_{L^2(U)}$. Of course, this argument wasn’t really valid, since we assumed second-order differentiability that is not available, but it illustrates the point.

Theorem 3.8.1 says that we get higher order derivatives on $u$ for free, if the initial data is smooth enough. Once we have it, we can use a simple bootstrapping argument to deduce higher order regularity.

**Theorem 3.8.3.** Let $L$ be as above, with $a^{ij} \in C^{m+1}(U)$ and $b^i, c \in C^m(U)$ for all $i, j = 1, \ldots, n$. If $f \in H^m(U)$ and $u \in H^1(U)$ weakly solves $Lu = f$, then $u \in H^{m+2}_{loc}(U)$ and for all $V \subset U$ we have the estimate

$$||u||_{H^{m+2}(V)} \leq C(||f||_{H^m(U)} + ||u||_{L^2(U)}).$$

where the implicit constant depends only $V, U,$ and $L$. 
We now embark on a proof of Theorem 3.8.1, so we may assume that we already have \( u \in H^{m+1}_{\text{loc}}(U) \), so we only need to control the derivatives of \( u \) of order \( m+2 \).

Let \( \alpha \) be any multi-index for \( \{1, \ldots, n\} \) of degree \( m \). Set \( \tilde{u} = \partial_\alpha u \). Given our assumption, controlling \( |u|_{H^{m+2}(U)} \) is the same as controlling \( |\tilde{u}|_{H^{m}(U)} \) for all possible \( \alpha \).

Let \( v \in C_c^\infty(U) \) and \( \tilde{v} = (-1)^m \partial_\alpha v \). Then since \( u \) is weakly solves \( Lu = f \), we have

\[
\sum_{i,j} \int_U a^{ij}(\partial_i u)(\partial_j \tilde{v}) + \sum_i \int_U b^i(\partial_i u)\tilde{v} + \int_U c u \tilde{v} = \int_U f \tilde{v}.
\]

We integrate by parts in the first integral to transfer all derivatives to \( u \). The product rule, there will be many extra terms involving lower order derivates of \( u \), so the result is

\[
\sum_{i,j} \int_U a^{ij}(\partial_i \partial_\alpha u)\partial_j v = \int_U \tilde{f} v
\]

where

\[
\tilde{f} = \partial_\alpha f - \sum_i \partial_\alpha (b^i u) - \partial_\alpha (cu) - \sum_{i,j} \sum_{\beta < \alpha} \left( \frac{\alpha}{\beta} \right) \partial_j [(\partial_\beta \partial_i u)(\partial_{\alpha - \beta} a^{ij})].
\]

The point is that all terms in \( \tilde{f} \) are bounded by our assumptions. Specifically, we only take derivatives of \( a^{ij} \) of order at most \( m+1 \), and derivatives of \( b^i, c \) of order at most \( m \), so these can all be bounded absolutely by some constant. Also, the expression involves derivatives of \( f \) and \( u \) of order at most \( m+1 \), so we have

\[
||\tilde{f}||_{L^2(V)} \ll ||\partial_\alpha f||_{L^2(U)} + ||u||_{H^{m+1}(U)} \ll ||f||_{H^m(U)} + ||u||_{L^2(U)}.
\]

Now \( \partial_\alpha u \) solves an elliptic PDE satisfying the conditions of Theorem 3.8.1, so \( \partial_\alpha u \in H^2_{\text{loc}}(U) \) and for any \( V \subset U \subset W \subset U \) we have

\[
||\partial_\alpha u||_{H^2(V)} \ll ||\tilde{f}||_{L^2(W)} + ||u||_{L^2(U)} \ll ||f||_{H^m(U)} + ||u||_{L^2(U)}.
\]

This establishes that \( u \) has locally bounded weak derivatives of order \( m+2 \) and thus lies in \( H^{m+2}_{\text{loc}}(U) \). Combining the bound above for the order \( m+2 \) derivatives plus the induction hypothesis, we obtain the bound in the theorem.

\[ \square \]

**Theorem 3.8.4.** Let \( L \) be as above, with \( a^{ij}, b^i, c \in C^\infty(U) \) for all \( i, j = 1, \ldots, n \). If \( f \in C^\infty(U) \) and \( u \in H^1(U) \) weakly solves \( Lu = f \), then \( u \in C^\infty(U) \).

**Proof.** By Theorem 3.8.3, we find that \( u \in H^m_{\text{loc}}(U) \) for all \( m \). By the Sobolev embedding theorem ??, we have \( u \in C^m(U) \) for all \( m \).

\[ \square \]

### 3.8.2 The key a priori estimate

We now embark on a proof of Theorem 3.8.1. What we would like to do is take the weak formulation (13) and set \( v = -\partial_k^2 u \), so that we would obtain an estimate of the form

\[
\sum_{i,j} \int_U a^{ij}(\partial_i \partial_k u)(\partial_j \partial_k u) + \sum_{i} \int_U (\partial_i a^{i j})(\partial_j u)(\partial_k \partial_k u) = -\int_U f \partial_k^2 u.
\]
By the uniform ellipticity assumption plus Cauchy-Schwarz, this will give a bound on $\|u\|_{H^2}$ in terms of $\|u\|_{H^{1,2}}$ and $\|f\|_{L^2}$.

There are a couple of important technical points. First, we assume only $u \in H^1(U)$, so we have no control on $u$ near the boundary of $U$. Therefore, we must use some cutoff function to restrict to a region we can control. Second, we do not know that $u$ is twice differentiable, so we must replace a derivative with a "discrete derivative."

**Definition 3.8.5.** We define the difference quotient

$$D_h^k u(x) = \frac{u(x + he_k) - u(x)}{h}.$$ 

We also define the vector

$$D^h u(x) = (D_1^h u(x), \ldots, D_n^h u(x)).$$

This makes sense when $x + he_k$ lies in the domain of definition for $u$. The following basic properties of the difference quotient are easy to check from the definition.

**Lemma 3.8.6.** The difference quotient has the following properties.

1. (Commutativity with derivatives)

$$\partial_i D_h^k u(x) = D_h^k \partial_i u(x).$$

2. (Integration by parts) If $u \in L^p(\mathbb{R})$ and $v \in L^q(\mathbb{R})$ where $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\int_U (D_h^k u)v = -\int_U u D_h^k v.$$  

3. (Product rule)

$$D_h^k (uv) = (D_h^k u)v + u_h^k (D_h^k v)$$

where $u_h^k = u(x + he_k)$.

The difference quotient has a couple of more subtle properties that will be important. Since we are using the difference quotient as a "surrogate derivative," we want to be able to relate it to the actual derivative. First, if the $\partial_k u$ actually exists and then we expect to be able to control the difference quotient in terms of its derivative, essentially by the Mean Value Theorem.

**Proposition 3.8.7.** Suppose $1 \leq p < \infty$ and $u \in W^{1,p}(U)$. Then for each $V \subset U$ we have

$$\|D^h u\|_{L^p(V)} \leq C \|Du\|_{L^p(U)}$$

for all $0 < |h| < \frac{1}{2} \text{dist}(V, \partial U)$.

**Proof.** By the density of smooth functions, it suffices to establish the inequality under the assumption that $u$ is smooth. In that case, we may use the identity

$$\frac{u(x + he_k) - u(x)}{h} = \int_0^h \partial_t u(x + te_k) dt.$$
we have
\[
||D^h_k u||_{L^p(V)} = \int_V \left| \int_0^h \partial_t u(x + te_k) \, dt \right|^p \\
\leq C \int_V \int_0^h |\partial_t u(x + te_k)|^p \, dt \\
\leq C \int_0^h \int_U |Du|^p \\
= C||Du||_{L^p(U)}.
\]

Next, we want to be able to establish conditions under which differentiability can be deduced if the difference quotient is nice enough. Think of the classical definition
\[
\partial_k u(x) = \lim_{h \to 0} D^h_k u(x).
\]
This limit might fail to exist because the difference quotient blows up as \( h \to 0 \).
If the difference quotients are uniformly bounded (in \( h \)), however, we know this can’t happen, and in fact it turns out that the (weak) derivative exists.

**Proposition 3.8.8.** Suppose \( u \in L^p(U) \) for some \( 1 < p < \infty \). If for \( V \subset \subset W \) there exists a constant \( C \) such that
\[
||D^h_k u||_{L^p(V)} \leq C
\]
for all \( 0 < |h| < \frac{1}{2} \text{dist}(V, \partial U) \) then \( u \in W^{1,p}(V) \) and
\[
||Du||_{L^p(V)} \leq C.
\]

**Proof.** Since \( ||D^h_k u|| \leq C \), and bounded balls are compact in \( L^p(U) \), there is a subsequence \( h_i \to 0 \) such that \( D^h_{k_i} u \) converges weakly to some \( v \). We claim that \( \partial_i u \) exists and is equal to \( v \). Indeed, by the definition of weak convergence for any \( \phi \in C_0^\infty(U) \) we have
\[
\int_U (D^h_{k_i} u) \phi \to 0.
\]
Using Lemma 3.8.6,
\[
\int_U v \phi = \int_U \lim_{i \to \infty} (D^h_{k_i} u) \phi \\
= \int_U u(\lim_{i \to \infty} D^{-h_i} \phi) \\
= -\int_U u \partial_i \phi
\]
which is the defining property of the weak derivative.

**Proof of Theorem 3.8.1.** We now have the tools necessary to prove the theorem. The argument is quite technically involved, but simple enough in spirit.

1. **Basic setup.** To obtain the a priori estimate, we wanted to use the test function \( v = -\partial^2_k u \) in the definition for weak solution. However, we first need to localize the function away from \( \partial U \), and to replace the derivative with the difference quotient, since we are not given differentiability of \( u \).
So choose an intermediate set \( W \) such that \( V \subset W \subset U \), and a smooth cutoff function \( \eta \) supported on \( W \) such that \( \eta \equiv 1 \) on \( V \) and \( 0 \leq \eta \leq 1 \) everywhere in \( U \). Then set \( v = -D_k^{-h}(\eta^2 D_k^h u) \), where \( h < \frac{1}{4} \text{dist}(V, \partial U) \). Using this as our test function in (13) (since it lies in \( H^1(U) \) by hypothesis), we obtain

\[
\sum_{ij} \int_U a^{ij}(\partial_i u) \partial_j(-D_k^{-h}(\eta^2 D_k^h u)) = \int_U \tilde{f} D_k^{-h}(\eta^2 D_k^h u)
\]

where

\[
\tilde{f} = f - \sum_i b^i \partial_i u - cu.
\]

We call the left hand side \( A \) and the right hand side \( B \), and estimate these separately.

2. Estimating \( A \). Applying the integration by parts for the difference quotient, we have

\[
A = \sum_{ij} \int_U D_k^h(a^{ij}\partial_i u) \partial_j(\eta^2 D_k^h u).
\]

If we distribute the (discrete) derivatives through, we will obtain many terms from the product rule. We are really interested in isolating a term that looks like \( \int_U |Du|^2 \), so we write \( A = A_1 + A_2 \) where

\[
A_1 = \sum_{ij} \int_U \eta^2(D_k^h \partial_i u)(\partial_j D_k^h u)
\]

and

\[
A_2 = \sum_{ij} \int_U D_k^h(a^{ij})(\partial_i u) \partial_j(\eta^2 D_k^h u) + \sum_{ij} \int_U (2\eta \partial_i \eta)(a^{ij})(D_k^h \partial_i u)(D_k^h u).
\]

(\( A_1 \) is one of the four terms that comes out from applying the product rule twice, and \( A_2 \) consists of the other three.) By the uniform ellipticity assumption,

\[
A_1 \geq \theta \sum_{ij} \int_U \eta^2|D_k^h Du|^2.
\]

This is the main term we want, so we seek to bound above the contribution from \( A_2 \).

\[
A_2 = \sum_{ij} \int_U \eta^2 D_k^h(a^{ij})(\partial_i u)(D_k^h \partial_j u)
\]

\[
+ \sum_{ij} \int_U (2\eta \partial_i \eta)D_k^h(a^{ij})(\partial_i u)(D_k^h u)
\]

\[
+ \sum_{ij} \int_U (2\eta \partial_i \eta)(a^{ij})_k(D_k^h \partial_i u)(D_k^h u).
\]

The condition that \( a^{ij} \) is \( C^1 \) implies that \( D_k^h(a^{ij}) \) is bounded by an absolute constant, which we can take to be \( \max_{x \in U} |\partial_k a^{ij}(x)| \). So we can
absorb $\sum_{ij} a_{ij}$ and its different quotients into some large constant $C$. By increasing $C$ further to absorb $2 \partial_j \eta$, we obtain an equality of the form

$$A_2 \leq C \int_U \eta \left[(\partial_i u) (D_k^h \partial_j u) + (\partial_j u) (D_k^h u) + (D_k^h \partial_i u) (D_k^h u)\right].$$

The summands that appear here are all products of two terms from among $\partial_i u$, $D_k^h \partial_j u$, and $D_k^h u$. We use the weighted AM-GM inequality

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}$$

to convert this into an equality in terms of $L^2$ norms:

$$A_2 \leq \epsilon \int_U \eta^2 |D_k^h \partial_j u|^2 + C \left( \int_U |D_k^h u|^2 + \int_U |\partial_i u|^2 \right).$$

If we take $\epsilon$ to be small, say $\epsilon = \frac{\theta}{2}$, then combining this with our bound on $A_1$ and using Proposition 3.8.7 to bound $|||D_k^h u|||_{L^2}$ in terms of $|||Du|||_{L^2}$

$$A \geq \frac{\theta}{2} \int_U \eta^2 |D_k^h \partial_j u|^2 - C \int_U |Du|^2. \tag{14}$$

3. Estimating $B$. By definition,

$$B = \int_U (f - \sum_i b^i \partial_i u - cu)v$$

so we have (by using weighted AM-GM again):

$$|B| \ll \int_U (|f| + |Du| + |u|)|v| \leq \epsilon \int_U |v|^2 + C \left( \int_U |f|^2 + |Du|^2 + |u|^2 \right). \tag{15}$$

To control this, we need to estimate the $L^2$ norm of $v = D^{-h}_k(\eta^2 D_k^h u)$. Applying Proposition 3.8.7 repeatedly, we obtain

$$\int_U |v|^2 = \int_U |D_k^{-h}(\eta^2 D_k^h u)|^2 \leq C \int_U |D(\eta^2 D_k^h u)|^2$$

$$\ll C \int_U |D_k^h u|^2 + \eta^2 |D_k^h Du|^2 \leq C \int_U |Du|^2 + \eta^2 |D_k^h Du|^2.$$

Now substituting this into (15) and fiddling the constants, we find that

$$|B| \leq \epsilon \int_U \eta^2 |D_k^h Du|^2 + C \int_U |f|^2 + |Du|^2 + |u|^2.$$

By taking $\epsilon$ to be sufficiently small, say $\frac{\theta}{4}$, we can arrange that

$$|B| \leq \frac{\theta}{4} \int_U \eta^2 |D_k^h Du|^2 + C \int_U |f|^2 + |Du|^2 + |u|^2. \tag{16}$$
4. Combining the estimates. Putting together (14) and (16), we find that
\[
\int_V |D_k^h Du|^2 \leq \frac{\theta}{4} \int_U \eta^2 |D_k^h Du|^2 \leq C \int_U |f|^2 + |Du|^2 + |u|^2.
\]
By Proposition 3.8.8, we deduce that \(Du \in H^1_{loc}(U)\) hence \(u \in H^2_{loc}(U)\), with explicit bound
\[
||u||_{H^p(V)} \leq C(||f||_{L^2(U)} + ||u||_{H^1(U)}).
\]

5. Refining the bound. The \(H^2\) bound that we just obtained is slightly weaker than that asserted in the theorem, since it involves \(||u||_{H^1(U)}\) instead of just \(||u||_{L^2(U)}\). Observe that the same argument we have given above shows that if \(V \subset W \subset U\), then
\[
||u||_{H^2(V)} \leq C(||f||_{L^2(W)} + ||u||_{H^1(W)}).
\]
To finish off the proof, it suffices to obtain an upper bound for \(||u||_{H^1(W)}\) in terms of \(||f||_{L^2(U)}\) and \(||u||_{L^2(U)}\). To do this, one substitutes \(v = \eta^2 u\) for an appropriate cutoff function \(\eta\) into (13) and applies completely analogous arguments to those that we have given above, but everything is easier since no difference quotients are required. The result is that there exists a constant \(C\) such that
\[
\int_U \eta^2 |D u|^2 \leq C \int_U f^2 + u^2.
\]
\[\square\]

3.9 BOUNDARY ELLIPTIC REGULARITY

So far, we have established local regularity for weak solutions \(u \in H^1(U)\). We now investigate what happens near the boundary.

3.9.1 Trace operators

By our definition, a function \(u \in H^1_0(U)\) is represented by an \(L^2\) function, and is thus defined only up to sets of measure 0. Since \(\partial U\) does have measure zero, it is nontrivial to digest the meaning of “\(u|_{\partial U}\)” It turns out that there is a “trace” operator that does allow us to make sense of the boundary values.

**Theorem 3.9.1 (Trace Theorem).** Assume \(U\) is bounded and \(\partial U\) is \(C^1\). Then there exists a bounded linear operator
\[
T: W^{1,p}(U) \to L^p(\partial U)
\]
such that
1. \(Tu = u|_{\partial U}\) if \(u \in W^{1,p}(U) \cap C(\overline{U})\), and
2. \(||Tu||_{L^p(\partial U)} \leq C||u||_{W^{1,p}(U)}\).
Proof Sketch. The first condition tells us how to describe $Tu$ for $u \in C^1(U)$, and one checks that the inequality

$$||Tu||_{L^p(\partial U)} \leq C ||u||_{W^{1,p}(U)}.$$ 

holds for all such $u$. If $u \in W^{1,p}(U)$ then there exist $u_m \in C^\infty(\overline{U})$ converging to $u$ in $W^{1,p}(U)$. The inequality tells us that the sequence $Tu_m$ converges in $L^p(\partial U)$, and we define $Tu$ to be the limit. $\square$

Theorem 3.9.2 (Trace zero functions). Assume $U$ is bounded and $\partial U$ is $C^1$. If $u \in W^{1,p}(U)$ then

$$Tu = 0 \iff u \in W^{1,p}_0(U).$$

3.9.2 Flattening the boundary

To obtain nice results, we need to impose some smoothness assumptions on the boundary. We assume that $\partial U$ is (at least) $C^2$, meaning that there are local $C^2$ charts about each point of $\partial U$ taking $U$ to the upper-half unit of the unit ball in $\mathbb{R}^n$, $B(0,1) \cap \mathbb{R}^n_+$, where $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \mathbb{R}^+$. Since we have assumed that $U$ is bounded, $\partial U$ is compact so in fact we reduce to the case where there are only finitely many charts involved. Furthermore, by our investigation of interior regularity it will suffice to establish estimates in a neighborhood of the boundary.

We would like to be able to use atlas to reduce to the case where $U$ is itself $B(0,1) \cap \mathbb{R}^n_+$, but we need to guarantee that the hypotheses on the initial data for the PDE are preserved. Explicitly, suppose that we begin with a PDE

$$Lu = -\sum_{ij} \frac{\partial}{\partial x_j} (a^{ij} \frac{\partial u}{\partial x_i}) + \sum_i b^i \frac{\partial u}{\partial x_i} + cu = f \text{ in } U$$

where $a^{ij} \in C^{m+1}(\overline{U})$, $b^i, c^i \in C^m(\overline{U})$.

Given a chart $\psi : B(0,1) \cap \mathbb{R}^n_+ \to \overline{U}$ around a given basepoint $p \in \partial U$ sending 0 to $p$, we set $\tilde{u}(y) = u(\psi(y))$ and $\tilde{f}(y) = f(\psi(y))$. Then $\tilde{u}$ will satisfy a PDE of the form

$$\tilde{L}\tilde{u} := -\sum_{ij} \frac{\partial}{\partial y_k} (\tilde{a}^{ij} \frac{\partial \tilde{u}}{\partial y_j}) + \sum_k \tilde{b}^k \frac{\partial \tilde{u}}{\partial y_k} + \tilde{c}\tilde{u}.$$ 

We want to verify that the coefficients $\tilde{a}^{ij}, \tilde{b}^i, \tilde{c}^i$ have the same level of regularity. It is easier to work with the non-divergence form of the PDE, which corresponds to the change of coefficients $b^i \mapsto b^i - \sum_j \frac{\partial a^{ij}}{\partial x_j}$. In particular, if we reset the coefficients of the non-divergence PDE to

$$Lu = \sum_{ij} a^{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \sum_i b^i \frac{\partial u}{\partial x_i} + cu$$

then we still have $a^{ij} \in C^{m+1}(\overline{U})$, $b^i, c^i \in C^m(\overline{U})$.

Let $\phi$ denote the inverse of $\psi$. Then

$$\frac{\partial u}{\partial x_i} = \sum_k \frac{\partial \tilde{u}}{\partial y_k} \frac{\partial y_k}{\partial x_i} = \sum_k \frac{\partial \tilde{u}}{\partial y_k} (D\phi)^k_i.$$
Similarly, \( \frac{\partial u}{\partial x_j} = \sum \frac{\partial u}{\partial y_i} (D\Phi)^{ij} \). Therefore, the PDE becomes

\[
\tilde{L} \tilde{u} = \sum_{i,j,k,l} a^{ij}(\psi(y))(D\Phi)^{ij}_{k}(D\Phi)^{k}_{\ell} \frac{\partial \tilde{u}}{\partial y_k} \frac{\partial \tilde{u}}{\partial y_{\ell}} + \sum_{i} b^{i}(\psi(y))(D\Phi)^{i}_{i} \frac{\partial \tilde{u}}{\partial y_i} + c(\psi(y)) \tilde{u}.
\]

Therefore, we see that the new coefficients will have the same regularity properties as long as \( \psi \) is \( C^{m+2} \). If this is the case, we say that \( \partial U \) is \( C^{m+2} \). This discussion establishes the following result.

Finally, we need to check that this change of variables preserves uniform ellipticity. Note that

\[
\sum_{k,l=1}^{n} a^{kj} \xi_k \xi_l = \sum_{k,l} a^{ij}(\psi(y))(D\Phi)^{ij}_{k}((D\Phi)^{k}_{\ell})(D\Phi)^{i}_{\ell} \xi_k \xi_l \\
= \sum_{i,j} a^{ij}(\psi(y))((D\Phi)^{i}_{i})(D\Phi)^{j}_{j} \xi_i \xi_j \\
\geq \theta |(D\Phi)\xi|^2
\]

where the last line follows from the uniform ellipticity assumption on \( L \). Since \( D\Phi \) is non-singular we have \( |D\Phi\xi|^2 \geq \theta' |\xi|^2 \) for some \( \theta' > 0 \) (here the compactness of the domain is important), which gives us a uniform ellipticity constant for \( \tilde{L} \).

**Proposition 3.9.3.** Let \( L \) be the operator defined by

\[
Lu = -\sum_{i,j} \frac{\partial}{\partial x_j} (a^{ij} \frac{\partial u}{\partial x_i}) + \sum_{i} b^{i} \frac{\partial u}{\partial x_i} + cu
\]

where \( a^{ij} \in C^{m+1}(\overline{U}) \), \( b^{i}, c^{i} \in C^{m}(\overline{U}) \). Suppose \( \partial U \) is \( C^{k+2} \) and \( \psi : B(0,1) \cap \mathbb{R}^n \rightarrow U \) is a local chart. If \( \tilde{u}(y) := u(\psi(y)) \), then

\[
Lu(\psi(y)) = \tilde{L} \tilde{u}(y) = -\sum_{k,l=1}^{n} \frac{\partial}{\partial y_l} (\tilde{a}^{kl} \frac{\partial \tilde{u}}{\partial y_k}) + \sum_{k} \tilde{b}^{k} \frac{\partial \tilde{u}}{\partial y_k} + \tilde{c} \tilde{u}
\]

where \( \tilde{a}^{ij} \in C^{m+1}(\overline{U}) \), \( \tilde{b}^{i}, \tilde{c}^{i} \in C^{m}(\overline{U}) \).

Furthermore, if \( L \) is uniformly elliptic then so is \( \tilde{L} \).

### 3.9.3 Boundary regularity

**Theorem 3.9.4.** Let \( L \) be as above, with \( a^{ij} \in C^1(\overline{U}) \) and \( b^{i}, c^{i} \in L^{\infty}(U) \) for all \( i,j = 1, \ldots, n \). Assume \( \partial U \) is \( C^{2} \). If \( f \in L^{2}(U) \) and \( u \in H^{1}_{0}(U) \) is a weak solution for the PDE \( Lu = f \), then \( u \in H^2(U) \) and we have

\[
\|u\|_{H^2(U)} \leq C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}).
\]

where the implicit constant depends only on \( U \) and \( L \).

The difference between this and Theorem 3.8.1 is that we have imposed some niceness conditions on the boundary, namely that \( \partial U \) is \( C^{2} \) and \( u \in H^{1}_{0}(U) \) (so its trace vanishes on \( \partial U \)) and \( a^{ij} \in C^1(\overline{U}) \), in return for an absolute bound on \( \|u\|_{H^2(U)} \).
Proof. The argument is very similar to the proof of Theorem 3.8.3. By Proposition 3.9.3, we reduce to the case where \( U = B(0, 1) \cap \mathbb{R}^n_+ \). As already discussed, by compactness of \( \partial U \) and interior regularity in \( U \), it suffices to prove that if \( V \) is an open neighborhood of 0 then we have \( u \in H^2(V) \), and
\[
|u|_{H^2(V)} \leq C(\|f\|_{L^2(U)} + |u|_{L^2(U)}).
\]
So we choose \( V = B(0,1/2) \) and a smooth cutoff function \( \eta \) such that \( \eta \equiv 1 \) on \( V \) and \( \eta \equiv 0 \) on \( \mathbb{R}^n - B(0,3/4) \) and \( 0 \leq \eta \leq 1 \). This time the cutoff function vanishes away from the boundary, and is used to ensure that the difference quotient \( D_k^h u \) is defined for sufficiently small \( h \).

As before, the definition of weak solution says that for all \( v \in H_0^1(U) \), we have
\[
\sum_{i,j=1}^n \int_U a^{ij}(\partial_i u)(\partial_j v) = \int_U \tilde{f} v
\]
where
\[
\tilde{f} := f - \sum_{i=1}^n b^i \partial_i u - cu.
\]
If \( k < n \) then we choose \( v = -D_k^{-h}(\eta^2 D_k^h u) \) as before. We take \( k < n \) is so that there will be no problems with the difference quotients leaving the region \( U \). Also, since \( u = 0 \) along \( \{x_n = 0\} \) in the trace sense, and \( \eta \) vanishes near \( \partial U \), we have \( v \in H_0^1(U) \), so this is a valid choice. Substituting into (17), and writing it as \( A = B \) as before, we follow the same argument as in the proof of Theorem 3.8.3 to show that
\[
A \geq \frac{\theta}{2} \int_U \eta^2 |D_k^h u|^2 - C \int_U |Du|^2.
\]
and
\[
B \leq \frac{\theta}{4} \int_U \eta^2 |D_k^h u|^2 + C \int_U f^2 + u^2 + |Du|^2
\]
which furnishes a bound
\[
\int_U |D_k^h u|^2 \leq C \int_U f^2 + u^2 + |Du|^2.
\]
By Proposition 3.8.8 (technically, we do not have \( V \subset U \), but the same proof works in this case) we have again \( \partial_k u \in H^1(V) \) for \( k = 1, \ldots, n-1 \) and we have an estimate of the form
\[
\|\partial_k \partial_l u\|_{L^2(V)} \leq C(\|f\|_{L^2(U)} + |u|_{H^1(U)}).
\]
as long as \( k \neq n \).

To control \( \partial_n^2 u \), we use the PDE \( Lu = f \), which we know holds almost everywhere by the interior regularity results. We may rearrange \( Lu \) as \( a_{nn} \partial_n^2 u \) plus terms that involve fewer than two \( \partial_n \) derivatives of \( u \), and are therefore bounded by our previous estimates. Furthermore, \( a_{nn} \geq \theta \) by uniform ellipticity, so we again obtain a bound of the form
\[
|\partial_n^2 u| \leq C \left( \int |f|^2 + |u|^2 + |Du|^2 \right).
\]
Combining this with the above bounds on the other second-order derivatives, and using the same method as before to convert the \( H^1 \) upper bound into an \( L^2 \) upper bound, we are done. \( \square \)
As before, we can use bootstrap off this to deduce higher order regularity.

**Theorem 3.9.5.** Let $L$ be as above, with $a^{ij} \in C^{m+1}(\overline{U})$ and $b^i, c \in C^m(\overline{U})$ for all $i, j = 1, \ldots, n$. Assume that $\partial U$ is $C^{m+2}$. If $f \in H^m(U)$ and $u \in H^1(U)$ is a weak solution for the PDE $Lu = f$, then $u \in H^{m+2}(U)$ and for all $V \subset \subset U$ we have the estimate

$$||u||_{H^{m+2}(U)} \leq C(||f||_{H^m(U)} + ||u||_{L^2(U)}),$$

where the implicit constant depends only on $U$ and $L$.

**Proof.** By Proposition 3.9.3, we reduce to the case where $U = B(0,1) \cap \mathbb{R}^n$. We proceed by induction on $m$. The case $m = 0$ follows from Theorem 3.9.4, so we may assume that we already have $u \in H^{m+1}(U)$, so we only need to control the derivatives of $u$ of order $m + 2$.

Let $a$ be any multi-index for $\{1, \ldots, n\}$ of degree $m$ but not involving $n$. As in the proof of Theorem 3.8.3, we set $\tilde{v} = (-1)^m \partial_a v$, and we integrate by parts to find an elliptic equation solved by $\partial_a u \in H^1_0(U)$ (that the derivative has trace zero is the point of requiring $a$ not to involve $n$). Moreover, this elliptic equation satisfies the hypotheses of Theorem 3.9.4 with respect to a function $\tilde{f}$ defined analogously as in the proof of Theorem 3.8.1. Therefore, we obtain a bound of the form

$$||\partial_a u||_{H^1(U)} \leq C(||f||_{H^m(U)} + ||u||_{L^2(U)}).$$

Now we have to control the derivatives with respect to $x_n$. Again, we do this by using the PDE $Lu = f$ and induction, much as in the proof of the Cauchy-Kovalevskaya theorem. Suppose by induction that

$$||\partial_\beta u||_{L^2(U)} \leq C(||f||_{H^m(U)} + ||u||_{L^2(U)})$$

where $\beta$ has degree $m + 2$ and involves at most $j$ derivatives with respect to $x_n$. (We know the result already for $j \leq 2$ by Theorem 3.9.4.) Write $\beta = \gamma + \delta$ where $\delta = (0, \ldots, 0, 2)$. Since $u \in H^{m+2}_{\text{loc}}(U)$ and $Lu = f$, we have $\partial_\gamma Lu = \partial_\gamma f$, and $\partial_\delta Lu$ is $a^m \partial_\beta u$ plus terms involving fewer derivatives of $u$ with respect to $x_n$ and derivatives of order $m + 2$; in other words, the other terms are bounded in the desired way by the induction hypothesis. By the uniform ellipticity assumption, $a^m \geq \theta > 0$ so by the induction hypotheses

$$||\partial_\beta u||_{L^2(U)} \leq C(||f||_{H^{m+1}(U)} + ||u||_{L^2(U)}).$$

□

**Theorem 3.9.6.** Let $L$ be as above, with $a^{ij}, b^i, c \in C^\infty(\overline{U})$ for all $i, j = 1, \ldots, n$. If $f \in C^\infty(\overline{U})$ and $u \in H^1(U)$ solves the PDE $Lu = f$, then $u \in C^\infty(\overline{U})$.

**Proof.** By Theorem 3.9.5, we find that $u \in H^m(U)$ for all $m$. By the Sobolev embedding theorem ??, we have $u \in C^m(\overline{U})$ for all $m$.

□

### 3.10 Maximum Principles

Recall that harmonic functions satisfy a whole slew of nice properties in addition to analyticity, including the mean value property and the maximum principle. In this section, we study maximum principles for solutions to general elliptic PDE.
3.10.1 The weak maximum principle

Here we prove a “weak maximum principle” for a solution to the elliptic PDE
\[- \sum_{i,j} a^{ij} \partial_i \partial_j u + \sum_i b^i u + cu = 0.\]

Note that we have changed the equation into “non-divergence form.”

The intuition is quite simple. At an interior extremum for \(u\), we have \(Du = 0\). Furthermore, since \((a^{ij})\) is positive-definite, the first term will be positive. Therefore, we obtain a contradiction if we assume that \(c \leq 0\).

Let’s now formulate the principle more precisely. We assume that \(u \in C^2\), in order to make sense of \(Du\) and \(D^2u\). By the regularity theory just proved, this follows automatically for any weak solution as long as the coefficients are sufficiently regular. We continue to assume that \(U \subset \mathbb{R}^n\) is an open, bounded subset. We can make sense of \(u|_{\partial U}\) via trace operators, as in Theorem 3.9.1.

**Theorem 3.10.1.** Assume \(u \in C^2(U) \cap C(\overline{U})\) and \(c \equiv 0\) on \(U\).

1. If \(Lu \leq 0\) on \(U\), then
   \[\max_{\overline{U}} u = \max_{\partial U} u.\]

2. If \(Lu \geq 0\) on \(U\), then
   \[\min_{\overline{U}} u = \min_{\partial U} u.\]

**Remark 3.10.2.** Notice that we have not required \(u\) to be a solution of the PDE. If \(Lu \leq 0\) then we say that \(u\) is a subsolution, and if \(Lu \geq 0\) then we say that \(u\) is a supersolution.

**Proof.** The second case follows from the first by replacing \(u\) with \(-u\), so we may just prove the first assertion.

First suppose that we have the strict inequality \(Lu < 0\) and the maximum is achieved at some interior point \(x_0\). Then \(Du = 0\), so we have
\[- \sum_{i,j} a^{ij}(x_0) \partial_i \partial_j u(x_0) < 0.\]

Since \(x_0\) is a local maximum, it must be the case that \(D^2 u(x_0)\) is nonpositive-definite. Since \(A = (a^{ij}(x_0))\) is symmetric and positive definite, we can diagonalize it via a change of variables: \(A = O^T D O\) where \(O\) is an orthogonal matrix and \(D\) is diagonal. By performing the change of variables \(y = O(x - x_0)\), we may assume that \(A = D\) is diagonal with positive entries. Then
\[- \sum_{i,j} a^{ij}(x_0) \partial_i \partial_j u(x_0) = - \sum_i a^{ii} \partial_i^2 u(x_0) > 0,\]
which contradicts the assumption \(Lu < 0\).

Now, suppose that \(Lu \leq 0\). Let \(u_\varepsilon(x) = u(x) + \varepsilon e^{\lambda x_1}\). Then
\[Lu_\varepsilon = Lu - L(\varepsilon e^{\lambda x_1}) = Lu - \varepsilon e^{\lambda x_1}(-\lambda^2 a^{11} + \lambda b^1) < 0\]
for \(\lambda\) large enough, so we can apply the preceding argument to \(u_\varepsilon\) and then take \(\varepsilon \to 0\) to deduce the general result. \(\Box\)
We can relax the assumptions of the theorem slightly.

**Theorem 3.10.3.** Assume $u \in C^2(U) \cap C(\overline{U})$ and $c \geq 0$ on $U$. Let $u^+(x) = \max\{0, u(x)\}$ be the positive part of $u$.

1. If $Lu \leq 0$ on $U$, then
   $$\max_{\partial U} u = \max_{\partial U} u^+.$$

2. If $Lu \geq 0$ on $U$, then
   $$\min_{\partial U} u = \min_{\partial U} u^+.$$

**Proof.** Let $V \subset U$ be the subset where $u(x) > 0$. If $V$ is empty then the result is trivial, so we assume that it is nonempty. On $V$ we have
   $$Lu - cu \leq -cu \leq 0$$
and $L$ has no zeroth order term, so the preceding theorem guarantees that
   $$\max_{\partial V} u = \max_{\partial V} u.$$
Notice that $u$ vanishes at any point of $\partial V$ contained in $U$, so
   $$\max_{\partial V} u = \max_{\partial U} u$$
and we are done.

The second part follows from the first by considering $-u$.

3.11 EXAMPLES

**Problem 1**

**Theorem 3.11.1** (Poincaré-Wirtinger). Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Show that there exists a constant $C(\Omega)$ depending on the domain such that for all $u \in H^1(\Omega)$, we have
   $$\int_{\Omega} |u - \overline{u}| dx \leq C(\Omega) \int_{\Omega} |\nabla u|^2 dx$$
where $\overline{u} = \frac{1}{|\Omega|} \int_{\Omega} u(x) dx$.

**Proof.** We gave an argument for the inequality
   $$\int_{\Omega} |u|^2 dx \leq C(\Omega) \int_{\Omega} |\nabla u|^2 dx$$
under the assumption that $u = 0$ on $\partial \Omega$. We want to reduce our problem to this.

**Lemma 3.11.2.** Let $\Omega \subset \mathbb{R}^d$. For $V \subset H^1(\Omega)$ a closed linear subspace, whose only constant function is 0, then for all $u \in v$, we have
   $$\int_{\Omega} |u|^2 dx \leq C(\Omega) \int_{\Omega} |\nabla u|^2 dx.$$
Proof. If not, there is a sequence $v_n \in V$ such that $|v_n|_{L^2} = 1$ but $|Dv_n|_{L^2} \to 0$. By the Rellich-Kondrachov compactness theorem, we can pass to a subsequence converging to a limit $v \in L^2(\Omega)$. We claim that $Dv = 0$, hence $v$ is constant. To see this, observe that for any $\phi \in C_c^\infty(\Omega)$, we have

$$\int v \phi_{x_i} = \lim_{k \to \infty} \int v_k \phi_{x_i} = - \lim_{k \to \infty} \int (v_k)_{x_i} \phi = 0.$$ 

To complete the argument, define

$$V = \{ \phi \in H^1(\Omega) : \int \Omega u = 0 \}.$$ 

This is closed, and $v := u - \pi \in V$. Applying the Lemma finishes off the proof.

Problem 2

**Theorem 3.11.3** (Cacciopoli). Suppose $\Omega \subset \mathbb{R}^n$ is open. Let $x_0 \in \Omega$ and $0 < \rho \bar{\rho}$ such that $B(x_0, \bar{\rho}) \subset \Omega$. Suppose that $u \in H^1(\Omega)$ satisfies

$$-\Delta u + b \cdot \nabla u + au = 0 \text{ in } \Omega,$$

where $a, b_i \in \mathbb{R}$. Show that there exists a constant $C$ such that

$$\int_{B(x_0, \rho)} |\nabla u|^2 dx \leq \frac{C}{\rho^2 - \bar{\rho}^2} \int_{B(x_0, \bar{\rho})} |u|^2 dx.$$

Proof. This is basically a special case of some calculations we performed in proving elliptic regularity. Let $\eta$ be a smooth cutoff function which is $\equiv 1$ in $B(x_0, \rho)$ and supported in $B(x_0, \bar{\rho})$. Since $\eta$ decays from 1 to 0 in a distance of $\bar{\rho} - \rho$, we can arrange that $|\eta'(r)| \leq \frac{1}{\bar{\rho} - \rho}$. Taking the test function $\varphi = \eta^2 u$ in the weak formulation, we have

$$\int \sum_i u_{x_i}(\eta^2 u)_{x_i} + \int \eta \sum_i b^i u_{x_i} u + \int \eta cu^2 = 0.$$ 

Rearranging and replacing constants, we find that

$$\int \eta^2 |\nabla u|^2 \leq C \int \eta'(r) |u| \sum_i |u_{x_i}| + |u|^2 \leq \frac{C}{\bar{\rho} - \rho} \int |u| \sum_i |u_{x_i}| + |u|^2.$$ 

Applying the AM-GM with $\varepsilon$ trick $|u||u_{x_i}| \leq \varepsilon |u_{x_i}|^2 + \frac{1}{\varepsilon} |u|^2$ and re-arranging as before, we arrive at the desired form of inequality.

Problem 6

Consider the following Neumann problem:

$$-\nabla \cdot (A(x) \nabla u) + b(x) \cdot \nabla (u) = f \quad \Omega$$

$$-A(x) \nabla u \cdot n = g \quad \partial \Omega$$
where \( f \in L^2(\Omega), g \in H^1(\Omega) \), \( A \) is a uniformly elliptic matrix satisfying \( \alpha_0 |\xi|^2 \leq A_{ij}(x)\xi_i\xi_j \) and \( b(x) \in L^\infty(\Omega) \) satisfies \( \nabla \cdot b = 0 \) in \( \Omega \) and \( b \cdot n = 0 \) on \( \partial \Omega \). Prove that there is a unique solution \( u \in H^1(\Omega) \) if and only if

\[
\int_\Omega f(x) \, dx + \int_{\partial \Omega} g(x) \, d\sigma(x) = 0.
\]

First assume that \( g = 0 \), so we can apply the Fredholm alternative: the PDE admits a solution if and only if \( f \) is orthogonal to all solutions to the homogeneous adjoint problem, which in this case is

\[
- \nabla \cdot (A(x)\nabla u) + b(x) \cdot \nabla u = f \quad \Omega
\]
\[
- A(x)\nabla u \cdot n = 0 \quad \partial \Omega
\]

Integrating, we find that

\[
\int A(x)\nabla u \cdot \nabla u = 0
\]

which forces \( \nabla u = 0 \) by the ellipticity assumption, so the only solutions are the constants. The Fredholm Alternative says that the only solutions are a solution to this inhomogeneous problem plus solutions to the homogeneous problem, i.e. there is a unique solution up to constants.

For the general case, let \( v \) be a function such that

\[
-A\nabla u \cdot n = g.
\]

Then the PDE is equivalent to the zero-boundary problem:

\[
- \nabla \cdot (A(x)\nabla u) + b(x) \cdot \nabla u = \tilde{f} \quad \Omega
\]
\[
- A(x)\nabla u \cdot n = 0 \quad \partial \Omega
\]

where

\[
\tilde{f} = f - b(x) \cdot \nabla v - \nabla \cdot (A(x)\nabla v).
\]

Applying the preceding discussion, a solution exists if and only if

\[
\int_\Omega f + b(x) \cdot \nabla v - \nabla \cdot (A(x)\nabla v) = 0.
\]

Since \( \nabla \cdot b = 0 \) in \( \Omega \), the extra terms are

\[
b(x) \cdot \nabla v - \nabla \cdot (A(x)\nabla v) = \int_{\partial \Omega} v(x)b(x) \cdot n - \int_{\partial \Omega} A(x)\nabla v \cdot n.
\]

Since \( b(x) \cdot n = 0 \) and \( -A(x)\nabla v \cdot n = g \), this is exactly the asserted condition.
4.1 INTRODUCTION TO TRANSPORT EQUATIONS

In this chapter we study transport equations, especially scalar transport equations and wave equations. These are equations that model the transport of matter (air, water, etc.) or information (waves).

Let \( T \in \mathbb{R}_+ \cup \{+\infty\} \) and \( A_i(t, x, u), 1 \leq i \leq d \) be \( N \times N \) matrices smooth in \( (t, x, u) \in [0, T] \times U \times \mathbb{R}^n \). We study solutions to the system of equations.

\[
\frac{\partial u}{\partial t} + \sum_{i=1}^{d} A_i(t, x, u) \frac{\partial u}{\partial x_i} = 0. \tag{18}
\]

When \( d = 1 \), we call this equation monodimensional, and when \( N = 1 \) we call it scalar. We will mainly focus on monodimensional scalar equations, although much of the theory generalizes in a natural way.

We return to the formalism of Cauchy problems. The fundamental questions are of existence, uniqueness, and regularity.

**Problem** (Existence). Given initial data \( u_0 \) on \( \mathbb{R}^d \), does there exist a solution \( u(x, t) \) on \( \mathbb{R}^d \times [0, T) \) to (18) with initial data \( u(x, 0) = u_0(x) \)?

Of course, we have to specify what functional space we are searching in for solutions. When \( T = \infty \), we say that the solution is global. When the solution is regular enough so that the derivatives in the equation exist in the classical sense, we call it classical (or strong).

**Problem** (Uniqueness). Given initial data \( u_0 \) on \( \mathbb{R}^d \) and solutions \( u(x, t) \), \( v(x, t) \) on \( \mathbb{R}^d \times [0, T) \) with initial data \( u_0 \), do we have \( u_1 = u_2 \)?

This is a subtle question. Not only do we need to specify the functional spaces in which uniqueness is asked, but we can have nontrivial relations among these. For instance, it is conceivable that we have unique smooth solutions but non-unique \( L^\infty \) solutions, but if a smooth solution exists then all \( L^\infty \) solutions are unique and equal to it (we will indeed see that something like this occurs for linear transport equations). This is called a “weak-strong uniqueness principle.”

**Problem** (Regularity). Given \( u_0 \) on \( \mathbb{R}^d \) and a solution \( u(x, t) \) on \( \mathbb{R}^d \times [0, T) \) with initial data \( u_0 \), if the initial data has a certain regularity (\( C^k \), Hölder, etc.) does the solution then enjoy the same regularity?

We saw that this does occur for elliptic equations; this is the phenomenon of elliptic regularity. It is not true for transport equations in general, which is captured by the mathematical theory of shocks.
4.1.1 Examples

We now give some examples of transport equations.

1. Linear transport equation. For \( c \in \mathbb{R} \), we can consider the monodimensional linear transport equation

\[
\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}_+,
\]

\[
u(x,0) = u_0(x) \quad \text{on } \mathbb{R} \times 0.
\]

Here \( u \) models the density of particles flowing along a line with velocity \( c \). More generally, we can consider linear transport equations of the form

\[
\frac{\partial u}{\partial t} + F(x,t) \cdot \nabla_x u = 0 \quad \mathbb{R}^n \times \mathbb{R}_+,
\]

\[
u(x,0) = u_0(x) \quad \mathbb{R}^n \times 0.
\]

2. Burgers equation. The Burgers’ equation models traffic flow:

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0 \quad \mathbb{R} \times \mathbb{R}_+,
\]

\[
u(x,0) = u_0(x) \quad \mathbb{R} \times 0.
\]

More generally, we will study transport of the form

\[
\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 \quad \mathbb{R} \times \mathbb{R}_+,
\]

\[
u(x,0) = u_0(x) \quad \mathbb{R} \times 0.
\]

4.2 Classical transport equations

In this section, we study transport equations of the form

\[
\partial_t u + A(t,x,u) \cdot \nabla_x u = 0 \quad t > 0, x \in \mathbb{R},
\]

\[
u(0,x) = u_0(x) \quad t = 0, x \in \mathbb{R}.
\]

where \( \nabla_x u = (\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}) \). For now, we restrict our attention to classical solutions \( u \in C^1(\mathbb{R} \times \mathbb{R}_+) \), where the derivatives are all interpreted in the classical sense. When \( A(t,x,u) = A(t,x) \) defines a linear PDE, this problem admits a very elegant and nice solution using the method of characteristics.

4.2.1 Linear transport equations

As a simple example, let us consider the scalar equation

\[
\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}_+,
\]

\[
u(x,0) = u_0(x) \quad \text{on } \mathbb{R} \times 0.
\]

The key to solving this equation is to observe that \( \partial_t u + c \partial_x u \) represents a directional derivative of \( u \) in \((t,x)\) space: it is \( Du \cdot (c,1) \). Therefore, we can
transform the partial differential equation into an *ordinary* differential equation by restricting ourselves to the lines in this distinguished direction.

Consider the curve \((t(s), x(s)) = (x_0 + cs, s)\) parametrizing the line of slope \((c, 1)\) emanating from \((x_0, 0)\). By the preceding discussion, the PDE may be interpreted as saying that the directional derivative of \(u\) along this curve vanishes, i.e. \(u\) is constant along this curve. To formulate this idea precisely, we define

\[
\tilde{u}(s) := u(x(s), t(s)) = u(x_0 + cs, s)
\]

Then observe that

\[
\tilde{u}'(s) = \frac{\partial u}{\partial t}(x(s), t(s)) + c \frac{\partial u}{\partial x}(x(s), t(s)) = 0.
\]

Therefore, \(\tilde{u}'(s)\) is constant, and its value may be found by taking \(s = 0\), from which we find \(\tilde{u}(s) = u_0(x_0)\). Reversing our steps, we find that the solution to the PDE is

\[
u(x, t) = u_0(x - ct)\]

**Theorem 4.2.1.** Suppose \(u_0 \in C^1(\mathbb{R})\). Then the PDE

\[
\partial_t u + c \partial_x u = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}_+,
\]

\[
u(0, x) = u_0(x) \quad \text{on } \mathbb{R} \times 0
\]

admits a unique solution \(u \in C^1(\mathbb{R} \times \mathbb{R}_+),\) which is given by the explicit formula

\[
u(x, t) = u_0(x - ct).
\]

**Proof.** Existence may be checked by differentiating the explicit formula. Uniqueness follows from the observation that any solution must be constant along the lines of slope \((1, c)\), as we used in constructing this explicit representation.

Now let us tackle the *inhomogeneous* formulation:

\[
\partial_t u + c \partial_x u = h(x, t) \quad \text{in } \mathbb{R} \times \mathbb{R}_+,
\]

\[
u(0, x) = g(x) \quad \text{on } \mathbb{R} \times 0.
\]

Now the PDE can be interpreted as saying that the directional derivative of \(u\) along certain curves is \(h(x, t)\), so \(u\) should be obtained by integrating \(h(x, t)\) along these curves. This is an instance of the *Duhamel principle*, which says that an inhomogeneous PDE can be interpreted as a series of homogeneous PDE over time, shifted by the constant \(h(x, t)\). Therefore, the solution is the same as the solution of the homogeneous PDE plus the contribution from these accumulated shifting constants.

**Theorem 4.2.2.** Suppose \(u_0 \in C^1(\mathbb{R})\) and \(h(x, t) \in C^0(\mathbb{R} \times \mathbb{R}_+)\). Then the PDE

\[
\partial_t u + c \partial_x u = h(x, t) \quad \text{in } \mathbb{R} \times \mathbb{R},
\]

\[
u(0, x) = u_0(x) \quad \text{on } \mathbb{R} \times 0
\]

admits a unique solution in \(C^1(\mathbb{R} \times \mathbb{R}_+),\) given explicitly by the formula

\[
u(x, t) = u_0(x - ct) + \int_0^t h(x - ct + cs, s) \, ds.
\]
Proof. Again, we define
\[ \tilde{u}(s) = u(x + cs, t + s) \]
so that
\[ \tilde{u}'(s) = \partial_t u(x + cs, t + s) + c \partial_x u(x + cs, t + s) = h(x + cs, t + s). \]
Integrating from \( s = -t \) to \( s = 0 \), we find that
\[ u(x, s) - u(x - ct, 0) = \int_{-t}^0 h(x + cs, t + s) \, ds \]
which is the asserted formula after a change of variables. \( \square \)

From the explicit representation of the solution, we see that \( u \) automatically acquires the same regularity as \( u_0 \) and \( h \).

**Corollary 4.2.3.** Suppose \( u_0 \in C^k(\mathbb{R}) \) and \( h(x, t) \in C^{k-1}(\mathbb{R} \times \mathbb{R}_+) \) and \( u(x, t) \in C^1(\mathbb{R} \times \mathbb{R}_+) \) solves the PDE
\[
\partial_t u + c \partial_x u = h(x, t) \quad \text{in } \mathbb{R} \times \mathbb{R}, \\
u(0, x) = u_0(x) \quad \text{on } \mathbb{R} \times 0.
\]
Then \( u(x, t) \in C^k(\mathbb{R} \times \mathbb{R}_+) \).

### 4.2.2 The method of characteristics

The key observation we used to solve the scalar transport equation was that we could convert the PDE into a family of ODE, each of which was easy to solve. In general, this technique of converting a partial differential equation into a family of ordinary differential equations is called the *method of characteristics*.

To illustrate, let us consider a simple example:
\[
a(x, y, u) \frac{\partial u}{\partial x} + b(x, y, u) \frac{\partial u}{\partial y} + c(x, y, u) = 0. \tag{19}
\]

We can think of the locus \( (x, y, u(x, y)) \) as cutting out a surface in \( \mathbb{R}^3 \), with normal vector \( \left( \frac{\partial u}{\partial x}(x, y), \frac{\partial u}{\partial y}(x, y), -1 \right) \). The equation (19) tells us that the normal vector is orthogonal to the vector field \( (a(x, y, u), b(x, y, u), c(x, y, u)) \), so the vector field is tangent to the solution surface. In other words, the solution surface \( (x, y, u(x, y)) \) is a union of *integral curves* for the vector field \( (a(x, y, u), b(x, y, u), c(x, y, u)) \). These integral curves, which we call the characteristic curves of the PDE, are determined by solving the ordinary differential equation
\[
\gamma'(s) = (a(\gamma(s)), b(\gamma(s)), c(\gamma(s)))
\]
where \( \gamma(s) = (x(s), y(s), u(s)) \).

In the constant-coefficient, scalar transport equation
\[
\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0
\]
we found that the integral curves for the constant vector field \( (c, 1) \) were of the form \( (x + cs, t + s) \).
For a general linear transport equation
\[ \frac{\partial u}{\partial t} + F(x, t) \cdot \nabla_x u = 0 \]
the characteristic curves are determined by solving
\[ \gamma'(s) = F(\gamma(s), t) \]
(which is just the defining equation for the integral curves of \( F \)).

**Proposition 4.2.4** (Existence of characteristic curves). Suppose that \( F(x, t) \in C^1(\mathbb{R}^n \times \mathbb{R}_+) \) and there exists a constant \( L \) such that
\[ |\nabla_x F(x, t)| \leq L \text{ for all } t \geq 0, x \in \mathbb{R}^n. \]
Then there exists a map
\[ Z(s, t, x) : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n \]
satisfying
\[ \frac{\partial}{\partial t} Z(s, t, x) = F(Z(s, t, x), x), \]
\[ Z(s, s, x) = x, \]
and for fixed \( s \) and \( t \), the map
\[ Z_{s,t} : x \mapsto Z(s, t, x) \]
is a \( C^1 \) diffeomorphism.

Furthermore, we have for any \( t_1, t_2, t_3 \geq 0 \), we have
\[ Z_{t_2,t_3} \circ Z_{t_1,t_2} = Z_{t_1,t_3}. \]  \hspace{1cm} (20)

**Proof.** The uniqueness and local existence of solutions is guaranteed by the Picard-Lindelöf Theorem. It is important to know that the local existence depends uniformly on the Lipschitz parameter, so by our uniform Lipschitz parameter and uniqueness, the local solutions patch to a global solution.

The Picard-Lindelöf Theorem also implies that \( Z_{s,t} \) is \( C^1 \). The fact that its inverse exists is also \( C^1 \) follows from the identity (20) since it implies that
\[ Z_{s,t} \circ Z_{t,s} = \text{Id}. \]

So it only remains to establish (20). But observe that \( Z(t_2, t_3, Z(t_1, t_2, x)) \) is also an integral curve for the vector field \( F(t, x) \) with the same initial conditions, hence it agree with \( Z(t_1, t_3, x) \) by uniqueness of solutions. \( \square \)

From the existence of characteristic curves, we can generalize our earlier results on existence, uniqueness, and regularity of solutions to any linear PDE by extending \( u \) along characteristic curves from the initial conditions.

**Remark 4.2.5.** In the general theory for linear PDE, one can use the characteristic method to obtain existence and uniqueness of solutions provided that the characteristics do not vanish, i.e. if the characteristic curves are not stationary (if the characteristic curves are stationary, then obviously one cannot extend the solution globally along them). In our case, we have a distinguished variable \( t \) so our characteristic curves are actually integral curves for the vector field \( (F(x, t), 1) \), which is obviously non-vanishing.
Theorem 4.2.6. Let $u_0 \in C^1(\mathbb{R}^d)$ and suppose that $F$ satisfies the hypotheses of Proposition 4.2.4. Then the PDE
\[
\frac{\partial u}{\partial t} + F(x, t) \cdot \nabla_x u = 0 \quad \mathbb{R}^n \times \mathbb{R}_+,
\]
\[ u(x, 0) = u_0(x) \quad \mathbb{R}^n \times 0,
\]
admits a unique global classical solution $u \in C^1(\mathbb{R}^d \times \mathbb{R}_+)$, which is given explicitly by
\[ u(x, t) = u_0(Z_{t,0}(x)) \]
where $Z_{s,t}$ is as defined in Proposition 4.2.4.

Proof. We can show existence by checking that our explicit formula solves the PDE. The explicit formula is clearly equivalent to the implicit formula
\[ u(Z_{0,t}(x), t) = u_0(x) \]
since $Z_{t,0}$ and $Z_{0,t}$ are inverses. The initial condition is satisfied by the fact that $Z_{s,t}(x) = x$ for any $s$, and
\[
\frac{d}{dt} u(Z_{0,t}(x), t) = \frac{\partial u}{\partial t}(Z_{0,t}(x), t) + \frac{\partial Z_{0,t}(x)}{\partial t}(Z_{0,t}(x), t) \cdot \frac{\partial}{\partial x} u(Z_{0,t}(x), t)
\]
\[ = \frac{\partial u}{\partial t}(Z_{0,t}(x), t) + F(x, t) \cdot \nabla_x u(Z_{0,t}(x), t).\]

Since we set $u(Z_{0,t}(x), t) = u_0(x)$, there is in fact no $t$-dependence and the derivative vanishes, verifying the PDE. This calculation just expresses the fact that $u$ is constant along the characteristic curves, which we arranged by construction.

For uniqueness, we simply have to observe that the preceding calculation shows that any solution satisfies
\[
\frac{d}{dt} u(Z_{0,t}(x), t) = 0.
\]
Setting $t = 0$, we find that $u(Z_{0,t}(x), t) = u_0(x)$. $\square$

It is also straightforward to generalize this argument for PDE with source term.

Theorem 4.2.7. Let $u_0 \in C^1(\mathbb{R}^d), h \in C^1(\mathbb{R}^n \times \mathbb{R}_+)$ and suppose that $F$ satisfies the hypotheses of Proposition 4.2.4. Then the PDE
\[
\frac{\partial u}{\partial t} + F(x, t) \cdot \nabla_x u = h(x, t) \quad \mathbb{R}^n \times \mathbb{R}_+,
\]
\[ u(x, 0) = u_0(x) \quad \mathbb{R}^n \times 0,
\]
admits a unique global classical solution $u \in C^1(\mathbb{R}^d \times \mathbb{R}_+)$, which is given explicitly by
\[ u(x, t) = u_0(Z_{t,0}(x)) + \int_0^t h(Z_{t,s}(x), s) \, ds,
\]
where $Z_{s,t}$ is as defined in Proposition 4.2.4.
Now let’s try to extend the characteristic method to quasilinear PDE of the form
\[ \frac{\partial u}{\partial t}(x,t) + \mathbf{F}(x,t,u) \cdot \nabla_x (u,t) = 0. \]
At first, this seems problematic because we can no longer interpret \( \mathbf{F} \) as a vector field on \( \mathbb{R}^n \times \mathbb{R}_+ \) since it depends on \( u \) itself. But let’s plough on anyway: we are seeking characteristic curves satisfying
\[ \frac{\partial}{\partial t} Z(0,t,x) = \mathbf{F}(Z(0,t,x), t, u(Z(0,t,x), t)). \]
Now we can use the fact that we know \( u \) is constant along characteristic curves, so if such a curve exists then it would have the property that \( u(Z(0,t,x), t) = u_0(x) \). Therefore, we may substitute this above to find that
\[ \frac{\partial}{\partial t} Z(0,t,x) = \mathbf{F}(Z(0,t,x), t, u_0(x)). \] (21)
This is an equation that we looks reasonable to solve (under suitable hypotheses). However, the dependence on the initial point \( x \) in the PDE is problematic, and it means that the characteristic curves may not be invertible as we had before; geometrically, we can have crossing characteristics.
To see this, consider the simple example where \( \mathbf{F}(x,t,u) = \mathbf{F}(u) \). Then (21) reduces to
\[ \frac{\partial}{\partial t} Z(0,t,x) = \mathbf{F}(u_0(x)). \] (22)
so we see that the characteristics are lines of the form \( (x + s\mathbf{F}(u_0(x)), s) \). In particular, if \( x_l < x_r \) but \( \mathbf{F}(u_0(x_l)) > \mathbf{F}(u_0(x_r)) \), then the characteristic curves emanating from \( (x_l,0) \) and \( (x_r,0) \) will cross in finite time.

**Example 4.2.8.** The Burgers equation is
\[ u_t + \left( \frac{1}{2} u^2 \right)_x = 0. \]
We can rewrite this as \( u_t + u(u_x) = 0 \). Applying the preceding discussion, we find that the characteristics are the lines \( (x + u_0(x)s, s) \). If \( u_0 \) fails to be monotone nondecreasing, then the characteristic curves will cross.

### 4.2.3 Quasilinear transport equations

We have just seen that the characteristic method can fail for quasilinear PDE. This suggests that global solutions fail to exist, which we now prove rigorously by studying the PDE in a neighborhood of crossing characteristics. We restrict our attention to scalar PDE of the form
\[ \frac{\partial u}{\partial t}(x,t) + \frac{\partial f(u)}{\partial x}(x,t) = 0 \quad \mathbb{R} \times [0,T] \] (23)
\[ u(x,0) = u_0(x) \quad \mathbb{R} \times 0. \]
where \( f \in C^1(\mathbb{R}) \) (we allow \( T = \infty \), but we anticipate the possibility that classical solutions may only exist for finite \( T \)). To put this equation in the form of our earlier analysis, we rewrite it as
\[ \frac{\partial u}{\partial t}(x,t) + f'(u) \frac{\partial u}{\partial x}(x,t) = 0. \]
Then (22) shows that the characteristics are of the form \((x + f'(u_0(x)))t, t)\). In our earlier notation, \(Z_{0,t}(x) = x + f'(u_0(x))t\). We would like to use the characteristics in order to define an explicit solution, as before, but this will not be well-defined if the characteristics cross. Specifically, if \(x_l < x_r\) and \(Z_{0,t}(x_l) = Z_{0,t}(x_r)\) then the explicit formula attempts to define

\[ u_0(x_l) = u(Z_{0,t}(x_l), t) = u(Z_{0,t}(x_r), t) = u_0(x_r) \]

which is obviously problematic if \(u_0(x_l) \neq u_0(x_r)\).

To understand the region on which the classical solution is defined, we ask ourselves when is the first time that characteristics will cross. This is determined by

\[ T_* = \inf_{x_l \neq x_r} Z_{0,t}(x_l) = Z_{0,t}(x_r). \] (24)

Studying the equality for specific \(x_l, x_r\), we see that

\[ x_l + f'(u(x_l))t = x_r + f'(u(x_r))t \iff t = -\frac{x_l - x_r}{f'(u(x_l)) - f'(u(x_r))}. \]

By the mean value theorem,

\[ \frac{x_l - x_r}{f'(u(x_l)) - f'(u(x_r))} = \frac{1}{\frac{d}{dx}(f' \circ u(x^*))} \]

for some \(x^* \in [x_l, x_r]\). Therefore, (24) can be reformulated as

\[ T_* = \inf_x \frac{1}{\frac{d}{dx}(f' \circ u(x^*))} = -\left( \inf_x \frac{d}{dx}(f' \circ u_0(x)) \right)^{-1}. \]

**Theorem 4.2.9.** Suppose \(f \in C^2(\mathbb{R})\) and \(f' \in L^\infty(\mathbb{R}), u_0 \in C^1(\mathbb{R})\) and \(u_0, u'_0 \in L^\infty(\mathbb{R})\). Define \(T_* \in \mathbb{R}_+ \cup \infty\) by

\[ T_* = \begin{cases} \infty & f' \circ u_0 \text{ non-decreasing}, \\ -\left( \inf_x \frac{d}{dx}(f' \circ u_0(x)) \right)^{-1} & \text{otherwise}. \end{cases} \]

Then there exists a unique classical solution \(u\) to (23) on \([0, T_*]\), which is given implicitly by

\[ u(Z_{0,t}(x), t) = u_0(x) \]

where \(Z_{0,t}(x) = x + f'(u_0(x))t\) as above.

**Proof.** We claim that \(Z_{0,t}(x)\) is again a \(C^1\) diffeomorphism on the domain, satisfying

\[ \frac{d}{dt} Z_{0,t}(x) = f'(u(Z_{0,t}(x), t)) = f'(u_0(x)). \]

Granting this, the expression \(u(Z_{0,t}(x), t) = u_0(x)\) gives a well-defined solution on the entire domain: since \(Z_{0,t}(\pm \infty) = \pm \infty\), continuity ensures that every point \((y, t)\) is \((Z_{0,t}(x), t)\) for some \(x\), and it is well-defined since \(Z_{t,0} = Z_{0,t}^{-1}\) exists. Then existence and uniqueness are checked exactly as in the proof of Theorem 4.2.6, so it now suffices to verify the claim. But we have explicitly \(Z_{0,t}(x) = x + tf'(u_0(x))\), so

\[ Z'_{0,t}(x) = 1 + t \frac{d}{dx} f'(u_0(x)) \geq 1 + \min_x \frac{d}{dx} f'(u_0(x)) = 1 - \frac{t}{T_*} > 0. \]
Example 4.2.10. Consider again the Burgers equation
\[ u_t + \left( \frac{1}{2} u^2 \right)_x = 0. \]
Then \( f'(u_0(x)) = u_0(x) \), so \( T_* = \infty \) if \( u_0'(x) \geq 0 \) for all \( x \), or else
\[
T_* = \begin{cases} 
\infty & u_0'(x) \geq 0 \text{ for all } x, \\
- \frac{1}{\inf_x u_0'(x)} & \text{otherwise}.
\end{cases}
\]

4.2.4 Finite time blowup

We have just constructed classical solutions to (23) on the domain \( \mathbb{R} \times [0, T_*) \). We now show that the classical solution cannot be extended beyond time \( T_* \).

Theorem 4.2.11. Suppose \( f \in C^2(\mathbb{R}) \) and \( f' \in L^\infty(\mathbb{R}), u_0 \in C^1(\mathbb{R}) \) and \( u_0, u_0' \in L^\infty(\mathbb{R}) \). Define
\[
T_* = \begin{cases} 
\infty & f' \circ u_0 \text{ non-decreasing}, \\
- \left( \min_x \frac{d}{dx} f' \circ u_0(x) \right)^{-1} & \text{otherwise}.
\end{cases}
\]
If \( T_* < \infty \), then there does not exist a classical solution \( u \in C^1(\mathbb{R} \times [0, T]) \) to (23) for any \( T \geq T_* \).

We will give two arguments. 

First proof. Let \( (x_n)_n \) be a sequence of points such that
\[
\frac{d}{dx} (f'(u_0(x))) \to -1/T_*.
\]
(25)
The first proof is based on examining the behavior of the solution near \( (x_n, T_*) \). By Theorem 4.2.9, we now that any solution must satisfy
\[ u(Z_{0,t}(x), t) = u_0(x) \text{ for } t < T_. \]
Differentiating with respect to \( x \), we find that
\[ \frac{\partial u}{\partial x}(Z_{0,t}(x), t)(Z_{0,t}'(x)) = u_0'(x). \]
Now, \( Z_{0,t}'(x) = 1 + t(f'(u_0(x))) \) so for \( x = x_n \) we have
\[
\lim_{t \to T_*} Z_{0,t}'(x_n) \to 1 - \lim_{t \to T_*} \frac{t}{T_*} = 0.
\]
Therefore, we may conclude that \( \frac{\partial u}{\partial x}(Z_{0,t}(x_n), t) \) blows up if we can establish that \( u_0'(x_n) \) is bounded below. Returning to (25) we see that
\[ f''(u_0(x_n))u_0'(x_n) \to -1/T_* .\]
By the assumption \( f'' \in C^0(\mathbb{R}) \), we have \( |f''(y)| \leq M \) for some absolute constant \( M \), which shows that
\[ u_0'(x_n) \to - \frac{1}{f''(u_0(x_n))T_*} < \frac{1}{T_* M}. \]
Second proof. Our second solution gives some insight into the blowup behavior. Differentiating the PDE in $x$, we obtain

$$\frac{\partial u_x}{\partial t} + f'(u) \frac{\partial u_x}{\partial t} + f''(u) u_x^2 = 0.$$  

If we re-arrange this as

$$\frac{\partial u_x}{\partial t} + f'(u) \frac{\partial u_x}{\partial x} = -f''(u) u_x^2$$  

then we see that this has the form of a quasilinear transport equation in $u$ with source term! As found in Theorem 4.2.9, $u$ is constant along the characteristic curves, which are $C^1$ diffeomorphisms for $t < T_\ast$. So let us fix some $x_0$ and define

$$\gamma(s) = (x_0 + s(f'(u_0(x_0)), s).$$

Let $w(s) = u_x(\gamma(s))$. Noting that $u(\gamma(s)) = u_0(x_0)$ for $s < T_\ast$, we have by (26) we have the ordinary differential equation

$$w'(s) = \frac{\partial u_x}{\partial t}(\gamma(s)) + f'(u_0(x_0)) \frac{\partial u_x}{\partial x}(\gamma(s)) = -f''(u_0(x_0)) w(s)^2$$

$$w(0) = \frac{\partial u_0}{\partial x}(x_0).$$

This is an instance of the family of ODE

$$w'(s) = -bw(s)^2$$

whose (unique) solution is

$$w(s) = \frac{a}{1 + abs}.$$ 

Using this above, we find that

$$w(s) = \frac{\partial_x u_0(x_0)}{1 + f''(u_0(x_0)) \partial_x u_0(x_0) s} = \frac{\partial_x u_0(x_0)}{1 + s \frac{\partial}{\partial x}(f' \circ u)(x_0)}.$$ 

Arguing as before, we can choose $x_0$ along a sequence of points such that the denominator tends to 0 as $s \to T_\ast$. 

Remark 4.2.12. These two solutions are essentially the same, but the second gives a more explicit description of the derivative. In the lecture, the second solution seemed slicker because we “cheated” a little by hiding the $\frac{\partial u_0}{\partial x}(x_0)$ factor away in the change of variables, which is obviously only valid if we can establish its non-vanishing.

4.3 weak solutions

We have seen that even quasilinear transport equations may not admit global classical solutions. Therefore the Cauchy problem is ill posed for classical solutions, and we seek to expand our space under consideration to one where it will be well-posed.

The natural first attempt is to look for distributional solution, as we did for elliptic equations. To discover what this notion is, we follow the standard trick of introducing a smooth test function and transferring all derivatives to it.
4.3.1 Linear transport equations

Let us start with the linear transport equations, for which we already obtained a satisfactory classical theory. Just to warm up, let us begin with our simplest example

\[
\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0 \quad t > 0, x \in \mathbb{R},
\]

\[
u(0, x) = u_0(x) \quad t = 0, x \in \mathbb{R}.
\]

Therefore, we define:

\[
\psi(\nu) = \int_{\mathbb{R}} \psi(u) \, dx.
\]

Therefore, we need only to check the initial condition. But now let us look at the preceding observation, we have solutions. By reversing the steps in deriving the weak formulation, we conclude that for any \( \phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}_+) \), we have

\[
- \left( \int_{\mathbb{R}} \int_{\mathbb{R}_+} u(\phi_t + c \phi_x) + \int_{\mathbb{R}} u_0(x) \phi \right) = \int_{\mathbb{R}} \int_{\mathbb{R}_+} (u_t + cu_x) \phi + \int_{\mathbb{R}} (u(0, x) - u_0(x)) \phi.
\]

Therefore, we define:

**Definition 4.3.1.** We say that \( u \in L^\infty(\mathbb{R} \times \mathbb{R}_+) \) is a weak solution to (27) if for all \( \phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}_+) \), we have

\[
\int_{\mathbb{R}} \int_{\mathbb{R}_+} u(\phi_t + c \phi_x) \, dx \, dt + \int_{\mathbb{R}} u_0(x) \phi(x, 0) \, dx = 0.
\]

When one defines a weak notion of solution, one should always check that it does indeed extend the classical notion, i.e. if classical solutions are weak solutions, and if weak solutions are regular then they are classical.

**Theorem 4.3.2.** Let \( u_0 \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \). If \( u \) is a classical solution to (27), then \( u \) is also a weak solution. Conversely, if \( u \) is a weak solution to (27) and \( u \in C^1(\mathbb{R} \times \mathbb{R}_+) \), then \( u \) is a classical solution.

**Proof.** By reversing the steps in deriving the weak formulation, we conclude that for any \( \phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}_+) \), and \( u \in C^1(\mathbb{R} \times \mathbb{R}_+) \), we have

\[
- \left( \int_{\mathbb{R}} \int_{\mathbb{R}_+} u(\phi_t + c \phi_x) + \int_{\mathbb{R}} u_0(x) \phi \right) = \int_{\mathbb{R}} \int_{\mathbb{R}_+} (u_t + cu_x) \phi + \int_{\mathbb{R}} (u(0, x) - u_0(x)) \phi.
\]

If \( u \) is a classical solution, then clearly the right hand side is 0, so \( u \) is a weak solution. Suppose \( u \in C^1(\mathbb{R} \times \mathbb{R}_+) \) is a weak solution. By choosing \( \phi \) to vanish along \( t = 0 \), we see that \( u_t + cu_x = 0 \) for all \( t > 0 \), hence on all of \( \mathbb{R} \times \mathbb{R}_+ \). Therefore, we need only to check the initial condition. But now let \( \psi \in C_c^\infty(\mathbb{R}) \), and extend \( \psi \) to some function \( \phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}_+) \) satisfying \( \phi(x, 0) = \psi \). By the preceding observation, we have

\[
\int_{\mathbb{R}} (u(0, x) - u_0(x)) \psi = 0
\]

and since this holds for all such \( \psi \), we may conclude that \( u(0, x) = u_0(x) \). \( \square \)

We now show that the Cauchy problem for (27) is also well-posed for weak solutions.
Theorem 4.3.3. Suppose \( u_0 \in L^\infty(\mathbb{R}) \). Then the Cauchy problem (27) admits a unique weak solution \( u \in L^\infty(\mathbb{R} \times \mathbb{R}_+) \), which is given explicitly (almost everywhere) by

\[
    u(x, t) = u_0(x - ct).
\]

Proof. First let us establish existence. The idea is quite simple: we know that \( u \) should be constant along the characteristic curves, so we decompose the region into a union of characteristic curves. Along each, the integrand becomes a total derivative. Making the change of variables \( y = x - ct \), we have for all \( \varphi \in C_c^\infty(\mathbb{R} \times \mathbb{R}_+) \)

\[
    \int_{\mathbb{R}} \int_{\mathbb{R}_+} u(x, t)(\varphi_x + c\varphi_t) \, dt \, dx = \int_{\mathbb{R}} \int_{\mathbb{R}_+} u_0(x - ct)(\varphi_x(y + ct, t) + c\varphi_t(y + ct, t)) \, dt \, dx
    = \int_{\mathbb{R}} \int_{\mathbb{R}_+} u_0(y) \varphi(y + ct, t) \, dy \, dx
    = -\int_{\mathbb{R}} u_0(y) \varphi(y, 0) \, dy
\]

which is the defining relation for weak solutions.

Now we establish uniqueness. If \( u \) and \( v \) are two solutions to (27), then their difference is a solution to (27) with \( u_0(x) := 0 \), so it suffices to check that the only such solutions are 0 almost everywhere. Suppose that \( u \in L^\infty(\mathbb{R} \times \mathbb{R}_+) \) is a solution to (27) with \( u_0 \equiv 0 \). By definition, for all \( \varphi \in C_c^\infty(\mathbb{R} \times \mathbb{R}_+) \) we have

\[
    \int_{\mathbb{R}} \int_{\mathbb{R}_+} u(x, t)(\varphi_x + c\varphi_t) \, dt \, dx = 0.
\]

It suffices to show that for all \( \psi \in C_c^\infty(\mathbb{R} \times \mathbb{R}_+) \), there exists \( \varphi \in C_c^\infty(\mathbb{R} \times \mathbb{R}_+) \) such that \( \varphi_x + c\varphi_t = \psi \). Now that we are working with smooth functions, we can apply the theory developed for classical solutions. According to Theorem 4.2.2, there is a smooth solution (not necessarily compactly supported) for any smooth initial data, given explicitly by

\[
    \varphi(x, t) = \psi_0(x - ct) + \int_0^t \psi(x - c(s - t), s) \, ds.
\]

The trick is to choose \( \psi_0 \) so that \( \varphi(x, t) \) will be compactly supported. By hypothesis, \( \psi(x, t) \) is supported in some band \( 0 \leq t < T \). We choose

\[
    \psi_0(x - ct) = -\int_0^T \psi(x - c(s - t), s) \, ds
\]

so that

\[
    \varphi(x, t) = \int_0^t \psi(x - c(s - t), s) \, ds.
\]

If \( t \geq T \), then the integrand vanishes for all \( s \) hence so does \( \varphi(x, t) \). If \( t \leq T \), then \( |x - c(s - t)| \geq |x| - cT \), so the integrand vanishes for all sufficiently large \( x \) (uniformly in \( t \)), hence \( \varphi \) has compact support.

Now that we have warmed up on this example, it is easy to extend the theory to general linear transport equations using the method of characteristics. We consider the Cauchy problem

\[
    \frac{\partial u}{\partial t} + F(x, t) \cdot \nabla_x u = 0 \quad \mathbb{R}^n \times \mathbb{R}_+,
    u(x, 0) = u_0(x) \quad \mathbb{R}^n \times 0,
\]

where we impose the conditions
1. \( F \in C^1(\mathbb{R}^n \times \mathbb{R}_+) \),
2. \(|F(x,t)| \leq L\) for all \((x,t) \in \mathbb{R}^n \times \mathbb{R}_+\),
3. \(\nabla_x \cdot F(t,x) = 0\) for all \((x,t) \in \mathbb{R}^n \times \mathbb{R}_+\).

(The third condition is new, and is introduced so that we don’t get additional terms when integrating by parts.)

**Definition 4.3.4.** We say that \( u \in L^\infty(\mathbb{R}^n \times \mathbb{R}_+) \) is a weak solution to (28) if for all \( \varphi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}_+) \), we have

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}_+} u(x,t) (\varphi_t + F(x,t) \cdot \nabla_x \varphi) \, dt \, dx + \int_{\mathbb{R}^n} u_0(x,0) \varphi(x,0) \, dx = 0.
\]

**Theorem 4.3.5.** Let \( u_0 \in C^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \). If \( u \) is a classical solution to (28) then \( u \) is a weak solution. If \( u \in C^1(\mathbb{R}^n \times \mathbb{R}_+) \) is a weak solution to (28) then \( u \) is a classical solution.

**Proof.** This is a straightforward generalization of Theorem 4.3.2. Suppose \( u \in C^1(\mathbb{R}^n \times \mathbb{R}_+) \). Then by the divergence theorem,

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}_+} (u_t + F(x,t) \cdot \nabla_x u) \varphi = -\int_{\mathbb{R}_+} u(x,0) \varphi(x,0) - \int_{\mathbb{R}^n} u(\varphi_t + F(x,t) \cdot \nabla_x \varphi).
\]

From this we see that if \( u \) is a classical solution, then \( u \) is a weak solution. Conversely, if \( u \in C^1(\mathbb{R}^n \times \mathbb{R}_+) \) is a weak solution, then

\[
-\left( \int_{\mathbb{R}^n} u_0(x) \varphi(x,0) + \int_{\mathbb{R}^n} u(\varphi_t + F(x,t) \cdot \varphi) \right) = \int_{\mathbb{R}^n} \int_{\mathbb{R}_+} (u_t + F(x,t) \cdot \nabla_x u) \varphi + \int_{\mathbb{R}^n} (u(0,x) - u_0(x)) \varphi.
\]

By taking \( \varphi \) to vanish along \( t = 0 \), we deduce that \( u_t + F(x,t) \cdot \nabla_x u = 0 \) on \( \mathbb{R}^n \times \mathbb{R}_+ \). To deduce the boundary condition, we take \( \psi \in C_c^\infty(\mathbb{R}^n) \) and extend it to \( \varphi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}_+) \) so that \( \varphi(x) = \varphi(x,0) \). This shows that

\[
\int_{\mathbb{R}^n} (u(0,x) - u_0(x)) \varphi = 0
\]

for all such \( \varphi \), implying \( u(0,x) = u_0(x) \). \( \square \)

We now establish well-posedness for the general equation. Let \( Z_{s,t} : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be the characteristic curves as before, satisfying

\[
\frac{\partial}{\partial t} Z(s,t,x) = F(x,t),
\]

\[
Z(s,s,x) = x.
\]

**Theorem 4.3.6 (Well-posedness of weak solutions).** Suppose \( u_0 \in L^\infty(\mathbb{R}^n) \). Then the Cauchy problem (28) admits a unique weak solution \( u \in L^\infty(\mathbb{R}^n) \), which is given explicitly (almost everywhere) by

\[
u(x,t) = u_0(Z_{t,0}(x)).\]
Proof. This is a straightforward generalization of Theorem 4.3.3. For existence, we just need to check the explicit formula; to do this, we again change variables to integrate along the characteristic curves, where $u$ is constant. Set $y = Z_{t,0}(x)$. Then

$$
\phi_t(x,t) + F(x,t) \cdot \nabla_x \phi(x,t) = \phi_t(Z_{0,t}(y),t) + F \cdot \nabla_x \phi(Z_{0,t}(y),t) = \partial_t \phi(Z_{0,t}(y),t).
$$

Therefore, if $u(x,t) = u_0(y)$ we have

$$
\int_{\mathbb{R}^n} \int_{\mathbb{R}^+} u(\phi_t + F \cdot \nabla_x \phi) \, dt \, dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^+} u(Z_{0,t}(y),t) \partial_t \phi(Z_{0,t}(y),t) \, dt \, dx
$$

$$
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^+} u_0(y) \partial_t \phi(Z_{0,t}(y),t) \, dt \, dy
$$

$$
= - \int_{\mathbb{R}^n} u_0(y) \phi(y,0) \, dy
$$

which is the defining condition for weak solutions.

Now let us consider uniqueness. Again, this reduces to showing that any solution to (28) with $u_0 \equiv 0$ vanishes almost everywhere, given that for all $\phi \in C^\infty_c(\mathbb{R}^n)$ we have

$$
\int_{\mathbb{R}^n} \int_{\mathbb{R}^+} u(\phi_t + F \cdot \nabla_x \phi) = 0.
$$

So again, it suffices to show that for all $\psi \in C^\infty_c(\mathbb{R}^n)$, there exists $\phi \in C^\infty_c(\mathbb{R}^n)$ with $\phi_t + F \cdot \nabla_x \phi = \psi$ with $\phi(x,0) = \psi_0$. Since we are now in the classical realm, we may apply Theorem 4.2.7 to deduce that there is a solution for any smooth initial data, given explicitly by

$$
\phi(x,t) = \phi_0(Z_{t,0}(x),0) + \int_0^t \psi(Z_{t,s}(x),s) \, ds.
$$

If $\psi$ is supported in the band $0 \leq t < T$, then we take

$$
\phi_0(Z_{t,0}(x),0) = - \int_0^T \psi(Z_{t,s}(x),s) \, ds.
$$

With this choice, $\phi(x,t)$ will be compactly supported by the same argument as before, and we are done.

4.3.2 The Rankine-Hugoniot condition

Unfortunately, it turns out that weak solutions are too large a space to be looking in. When considering the Cauchy problem for classical solutions, we found that we do not always have existence; when considering the Cauchy problem for weak solutions, we will find that we do not always have uniqueness. To see this, we will study a specific kind of Cauchy problem.

We consider the quasilinear PDE

$$
\frac{\partial u}{\partial t}(x,t) + \frac{\partial f(u)}{\partial x}(x,t) = 0 \quad \mathbb{R} \times [0,T],
$$

$$
u(x,0) = u_0(x) \quad \mathbb{R} \times 0.
$$
We introduce the notation \( \Omega \). We suppose that \( n \) is positive. The weak solution is simply that the integrand vanishes:

\[
\int_{\Omega} u \varphi_t + F(u) \varphi_x \, dt \, dx + \int_{\Omega} u_0(x) \varphi(x,0) \, dx = 0.
\]

This discussion in fact applies to an open subset \( \Omega \subset \mathbb{R} \times \mathbb{R}_+ \) with the obvious generalizations. We attempt to construct a discontinuous weak solution that is smooth on either side of its curve of discontinuity. More precisely, let \( \Gamma \) be a \( C^1 \) curve dividing \( \Omega \) into two regions, say \( \Omega_l \) and \( \Omega_r \), which intersects the line \( \{ t = 0 \} \) transversely at finitely many points.

![Diagram](image.png)

We suppose that \( u \in L^\infty(\mathbb{R} \times \mathbb{R}_+) \) is actually smooth on \( \Omega_l \) and \( \Omega_r \) separately, and that it satisfies the PDE on points in those regions. What condition is needed along \( \Omega \) for \( u \) to patch to a weak solution on all of \( \Omega \)？

Although \( u \) is not continuous along \( \Gamma \), the smoothness implies that it has left and right limits, so denote \( u_l(x) = \lim_{y \to x^-} u(x) \) and \( u_r(x) = \lim_{y \to x^+} u(y) \). Now, for any \( \varphi \in C^\infty_c(\Omega) \) we have (applying the Divergence Theorem)

\[
\int_\Omega u \varphi_t + F(u) \varphi_x \, dt \, dx = \int_{\Omega_l} u \varphi_t + F(u) \varphi_x \, dt \, dx + \int_{\Omega_r} u \varphi_t + F(u) \varphi_x \, dt \, dx
\]

\[
= -\int_{\Omega_l} (u_l + F(u)_x) \varphi \, dt \, dx + \int_{\Gamma} (u_l, u_l) \cdot n_l \, d\sigma
\]

\[
-\int_{\Omega_r} (u_r + F(u)_x) \varphi \, dt \, dx + \int_{\Gamma} (u_r, u_r) \cdot n_r \, d\sigma
\]

\[
-\int_{\Omega \cap \{ t = 0 \}} u(x,0) \varphi(x,0) \, dx,
\]

where \( n_l \) and \( n_r \) are the outward unit normal vectors along \( \Gamma \). In particular, \( n_r = -n_l \). Since we have \( u_l + F(u)_x = 0 \) in the separate regions, the above equation simplifies to

\[
\int_\Omega u \varphi_t + F(u) \varphi_x \, dt \, dx + \int_{\Omega \cap \{ t = 0 \}} u_0(x) \varphi(x,0) \, dx = \int_{\Gamma} \varphi(F(u_l) - F(u_r), u_l - u_r) \cdot n_l \, d\sigma.
\]

Write \( n_l = (n_x, n_t) \). Since this holds for all \( \varphi \in C^\infty_c(\Omega) \), the condition for a weak solution is simply that the integrand vanishes:

\[
F(u_l) - F(u_r))n_x + (u_l - u_r)n_t = 0.
\]

We introduce the notation \( [u] = u_l - u_r \) and \( [F] = F(u_l) - F(u_r) \), so that the equation may be rewritten as

\[
[F]n_x + [u]n_t = 0 \quad \text{for all} \quad (x,t) \in \Omega.
\]
This is called the Rankine-Hugoniot condition.

**Theorem 4.3.8 (Rankine-Hugoniot condition).** With the notation and assumptions above, \( u \in L^1(\Omega) \) defines a weak solution to (4.2.9) on \( \Omega \) if and only if \( u \) satisfies (30).

Now suppose that \( \Gamma \) is parametrized as \( x = \eta(t) \). Then a tangent to the \( \Gamma \) is \((\dot{\eta}(t), 1)\) so a unit normal is \( \frac{1}{\sqrt{1+\dot{\eta}(t)^2}}(-1, \dot{\eta}(t)) \). Letting \( \sigma = \dot{\eta}(t) \) and substituting this into (30), we obtain the following formulation of the Rankine-Hugoniot condition:

\[
[F] = \sigma [u].
\]

Although we have suppressed it in the notation, recall that this is an equation of functions on the curve, so equality must hold at each point of \( \Gamma \).

**Example 4.3.9.** Let us revisit the Burgers equation

\[
 u_t + \left( \frac{1}{2} u^2 \right)_x = 0.
\]

If \( u_0 \equiv 0 \), we have the obvious classical solution \( u \equiv 0 \).

However, using the Rankine-Hugoniot condition we can construct an infinite of solutions:

\[
u(x, t) =
\begin{cases}
0 & x < -pt, \\
-2p & -pt < x < 0, \\
2p & 0 < x < pt, \\
0 & pt < x.
\end{cases}
\]

Let's examine the leftmost curve of discontinuity, \( x = -pt \), to verify the Rankine-Hugoniot condition. In the notation above, \( \sigma = -p \) and \( [u] = 2p \), \( [F] = 0 - \frac{1}{2}(2p)^2 = -2p^2 \), so the condition \( [F] = \sigma [u] \) is indeed satisfied. A similar analysis at the curve \( x = pt \) shows that \( u \) is a weak solution.

### 4.4 Entropy Solutions

In terms of well-posedness, we have now seen that classical solutions are too strong (no existence) and weak solutions are too weak (no uniqueness). We search for an intermediate space for which the Cauchy problem will be well posed. This is the concept of *entropy solution.*
4.4.1 First examples

The idea of the entropy solution is to rule out certain “nonphysical” weak solutions. In order to understand this, we should try to understand some of the physical intuition behind the singularities caused by crossing characteristics.

Example 4.4.1 (Shocks). We consider the Burgers equation

\[ u_t + \left( \frac{u^2}{2} \right)_x = 0 \]

with initial data

\[
    u_0(x) = \begin{cases} 
    1 & x \leq 0 \\
    1 - x & 0 \leq x \leq 1 \\
    0 & x \geq 1 
    \end{cases}
\]

The characteristics are \((x + su_0(x), s)\). They first cross at time \(t = 1\), so before then we have the classical solution for \(t < 1\):

\[
    u(x, t) = \begin{cases} 
    1 & x \leq t \\
    \frac{1-x}{1-t} & t \leq x \leq 1 \text{ for } t < 1. \\
    0 & x \geq 1 
    \end{cases}
\]

What happens afterwards?

By symmetry, we expect a curve of discontinuity with slope 2, interpolating between the two regions where \(u = 0\) and \(u = 1\). Indeed, if the curve is parametrized by \(x = c(t)\), then the Rankine-Hugoniot condition is precisely

\[
[F] = c'(t)[u] \implies \frac{1}{2} = c(t).
\]

That suggests a singular curve \((\frac{1}{2} + \frac{t}{2}, t)\).

Physically, the picture suggests a shockwave as the two families of characteristic curves collide.
Example 4.4.2 (Rarefaction waves). Now we consider the Burgers equation with initial condition

\[ u_0(x) = \begin{cases} 
0 & x < 0, \\
1 & x \geq 0. 
\end{cases} \]

Here the characteristic method fails because the characteristics are “underdetermined.”

One natural way to “fill in” the rest of the characteristics is to vary the slope \( x/t \) continuously from 0 to 1.

The solution we have just sketched geometrically has the explicit description

\[ u_1(x, t) = \begin{cases} 
0 & x < 0, \\
1 & x > t, \\
\frac{x}{t} & 0 < x < t.
\end{cases} \]
Obviously $u_1$ satisfies the Burger equation in the regions $x < 0$ and $x > t$. In the region $0 < x < t$, we can check:

$$\frac{\partial u_1}{\partial t} + \frac{1}{2} \left( \frac{\partial u_2}{\partial x} \right)^2 = -\frac{x}{t^2} + \frac{1}{2} \left( \frac{x}{t} \right)^2 = 0.$$  

We then only need to check the Rankine-Hugoniot condition on the curves $x = 0$ and $x = t$, but in fact the solution is continuous everywhere, so it is clearly satisfied.

One can also produce a discontinuous Rankine-Hugoniot solution, like what we did for the shockwave. If the curve of discontinuity is $x(t) = c(t)$, then the Rankine-Hugoniot condition is

$$\frac{1}{2} = \dot{c}(t)$$

so we have a shockwave along a line of slope 2, as before.

The explicit formula for this solution is

$$u(x,t) = \begin{cases} 
0 & x < t, \\
1 & x > t.
\end{cases}$$

This solution appears “unphysical” in the sense that there was a shock without any physical cause.

4.4.2 Entropy solutions

The idea of entropy solutions is to capture the intuition that the entropy of a physical system only increases. Suppose $\Phi(u)$ is a smooth convex function, which we think of as a measure of entropy for the solution $u$. Actually, the usual physical entropy functions are concave, so we think of $\Phi(u)$ as being the negative of the entropy.
**Definition 4.4.3.** Let $\Psi$ be a smooth function such that

$$\Phi'(u)f'(u) = \Psi'(u).$$

Then we call $(\Phi, \Psi)$ an entropy/flux pair.

The idea is that for a conservative system

$$u_t + f(u)_x = 0$$

we also have a “conservation of entropy”

$$\Phi(u)_t + \Psi(u)_x = \Phi'(u)u_t + \Psi'(u)u_x$$

$$= -f'(u)\Phi'(u)u_x + \Psi'(u)u_x$$

$$= (\Psi'(u) - f'(u)\Phi'(u))u_x$$

$$= 0.$$

In general, when our system can undergo a shock, we don’t necessarily require that the entropy be conserved, but we expect $\Phi(u)$ to decrease (recall that this is the negative of the physical entropy, which increases). So that corresponds to the condition

$$\Phi(u)_t + \Psi(u)_x \leq 0.$$

Now, this can be measured in a weak sense.

We are ready to give the formal definition of entropy solutions. We consider a PDE of the form

$$u_t + f(u)_x = 0 \quad \text{R} \times [0, T) \quad (32)$$

$$u(x) = u_0(x) \quad \text{R} \times 0.$$

**Definition 4.4.4.** Suppose $f \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. We say that $u \in L^\infty(\Omega)$ is an entropy solution to (32) if for all entropy/flux pairs, we have for all positive functions $\varphi \in C^\infty_c(\mathbb{R} \times [0, T))$ we have the inequality

$$\int_\mathbb{R} \int_{[0, T)} \Phi(u)\varphi_t + \Psi(u)\varphi_x \, dt \, dx + \int_\mathbb{R} \Phi(u_0(x))\varphi(x, 0) \, dx \geq 0. \quad (33)$$

**Remark 4.4.5.**

1. Note that the test function must be positive!

2. An alternate formulation is to demand that $u(\cdot, t) \to g$ in $L^1(\mathbb{R})$ as $t \to 0$, and for all positive test functions $\varphi \in C^\infty_c(\mathbb{R} \times [0, T))$ we have the inequality

$$\int_\mathbb{R} \int_{[0, T)} \Phi(u)\varphi_t + \Psi(u)\varphi_x \, dt \, dx \geq 0. \quad (34)$$

This is the definition that Evans uses.

3. It is easy to construct lots of entropy/flux pairs. For any $\Phi$, we can take

$$\Psi(z) = \int_{z_0}^z \Phi'(w)f'(w) \, dw.$$

For good measure, let’s verify that this solution concept is indeed compatible with the ones we already have.

**Proposition 4.4.6.** A classical solution to (32) is an entropy solution.
Proof. The argument we gave to motivate the definition already shows that if $u$ is a classical solution, then we can integrate by parts to deduce that for all $\phi \in C_0^\infty(\mathbb{R} \times [0, T))$,
\[
\int_{\mathbb{R}} \int_{[0, T)} \Phi(u) \partial_t \phi + \Phi(u) \partial_x \phi + \int_{\mathbb{R}} u_0(x) \phi(x, 0) = 0
\]
\[\square\]

Proposition 4.4.7. An entropy solution to (32) is a weak solution.

Proof. We take the entropy/flux pairs $\Phi(u) = \pm u$ and $\Psi(u) = \pm F(u)$ (or smooth approximations thereof, if necessary). The definition of entropy solution shows that for all positive $\phi \in C_0^\infty(\mathbb{R} \times [0, T))$, we have
\[
\int_{\mathbb{R}} \int_{[0, T)} (\pm u) \partial_t \phi + (\pm F(u)) \partial_x \phi \, dt \, dx + \int_{\mathbb{R}} \pm u(x, 0) \phi(x, 0) \, dx \geq 0.
\]
Obviously, this is only possible if both inequalities are in fact equalities. \[\square\]

4.4.3 Viscosity solutions

As we have seen, it may be difficult to solve a PDE of the form in (32). One very clever approach is the method of vanishing viscosity, which studies the equation perturbed by a “viscosity term”:
\[
u^\epsilon_t + F(u^\epsilon)_x - \epsilon u^\epsilon_{xx} = 0 \quad R \times [0, T)
\]
\[u^\epsilon(x) = u_0(x) \quad \mathbb{R} \times 0.
\]
By introducing this second-order term, the equations (35) admit classical (even smooth) solutions. Intuitively, we expect a “physically correct” solution to our original problem (32) to be the limit of the smooth solutions $u^\epsilon$.

To illustrate this idea, we assume that
1. $\{u^\epsilon\}$ is uniformly bounded for $0 \leq \epsilon \leq 1$, and
2. $u^\epsilon \rightarrow u$ almost everywhere as $\epsilon \rightarrow 0$.

In practice, this second condition is highly nontrivial to verify, but let’s see where it leads us. Of course, even if the $u^\epsilon$ are all smooth, we cannot necessarily deduce that $u$ will be classical. However, it is easy to show that the limit will still be a weak solution.

Lemma 4.4.8. Under the assumptions (4.4.3), $u$ is a weak solution to (32).

Proof. Let $\phi \in C_0^\infty(\mathbb{R} \times [0, T))$ be a test function. Integrating by parts, we find that
\[
0 = \int_{\mathbb{R}} \int_{[0, T)} (u^\epsilon_t + F(u^\epsilon)_x - \epsilon u^\epsilon_{xx}) \phi \, dt \, dx
\]
\[
= - \int_{\mathbb{R}} \int_{[0, T)} u^\epsilon \phi_t + f(u^\epsilon) \phi_x \, dt \, dx - \int_{\mathbb{R}} u_0(x) \phi(x, 0) \, dx
\]
\[
\overset{\epsilon \rightarrow 0}{\longrightarrow} \int_{\mathbb{R}} \int_{[0, T)} u \phi_t + f(u) \phi_x \, dt \, dx - \int_{\mathbb{R}} u_0(x) \phi(x, 0) \, dx
\]
where the limit is justified by dominated convergence. \[\square\]
The more interesting result is that the limit will be an entropy solution.

**Theorem 4.4.9.** Under the assumptions (4.4.3), \( u \) is an entropy solution to (32).

**Proof.** Let \( \Phi, \Psi \) be an entropy/flux pair. Following the calculation we performed when introducing the definition of entropy solutions, we find:

\[
\Phi(u^e)_t + \Psi(u^e)_x = \Phi'(u^e)u^e_t + \Psi'(u^e)u^e_x
\]

\[
= \Phi'(u^e)(\epsilon u^e_x - f'(u^e)u^e_x) + \Psi'(u^e)u^e_x
\]

\[
= \epsilon \Phi'(u^e)u^e_{xx}.
\]

This last term looks like it came from the second derivative of \( \Phi(u^e) \). We compute

\[
\Phi(u^e)_{xx} = (\Phi'(u^e)u^e_x)_x = \Phi'(u^e)u^e_{xx} + \Phi''(u^e)(u^e_x)^2.
\]

In particular, since \( \Phi \) is convex we have \( \Phi(u^e)_{xx} \geq \Phi'(u^e)u^e_{xx} \). Now substituting this above and integrating, we find that for all positive functions \( \varphi \in C^2_c(\mathbb{R} \times [0,T)) \), we have

\[
\int_{\mathbb{R}} \int_{[0,T]} \Phi(u^e) \varphi_t + \Psi(u^e) \varphi_x + \int_{\mathbb{R}} \Phi(u_0(x)) \varphi(x,0) = -\int_{\mathbb{R}} \int_{[0,T]} (\Phi(u^e)_t + \Psi(u^e)_x) \varphi
\]

\[
\geq -\epsilon \int_{\mathbb{R}} \int_{[0,T]} \Phi(u^e)_{xx} \varphi
\]

\[
= -\epsilon \int_{\mathbb{R}} \int_{[0,T]} \Phi(u^e) \varphi_{xx}.
\]

Now applying the dominated convergence theorem and the assumptions (4.4.3), we deduce that

\[
\int_{\mathbb{R}} \int_{[0,T]} \Phi(u^e) \varphi_t + \Psi(u^e) \varphi_x \, dt \, dx + \int_{\mathbb{R}} \Phi(u_0(x)) \varphi(x,0) \, dx \geq 0.
\]

\[ \square \]

### 4.4.4 Kruzkov’s Theorem

It turns out that the Cauchy problem (32) is well-posed for entropic solutions. This and much more follows from an extremely powerful result called **Kruzkov’s Theorem.** This is hard even to state completely, let alone prove, so we describe only some of the main features.

**Theorem 4.4.10** (Kruzkov). Suppose \( u_0 \in L^\infty(\mathbb{R}) \). Then there exists a unique \( u \in L^\infty(\mathbb{R} \times [0,T]) \cap C([0,T), L^1(\mathbb{R})) \) solving the PDE (32), and this solution satisfies

\[ \|u\|_{L^\infty(\mathbb{R} \times [0,T])} = \|u_0\|_{L^\infty(\mathbb{R})}. \]

More precisely, suppose \( u_0, v_0 \in L^\infty(\mathbb{R}) \) and \( u, v \) are the solutions with these respective initial data. Let \( M = \sup \{|f'(x)| : x \in [\inf(u_0, v_0), \sup(u_0, v_0)]\} \). Then for all \( t > 0 \) and every interval \([a, b]\) we have

\[
\int_a^b |v(x,t) - u(x,t)| \, dx \leq \int_{a-Mt}^{b+Mt} |v_0(x) - u_0(x,t)| \, dx
\]

and also

\[
\int_{\mathbb{R}} v(x,t) - u(x,t) \, dx = \int_{\mathbb{R}} v_0(x) - u_0(x) \, dx
\]

(36)
We will sketch a proof of the inequality
\[
\int_a^b |v(x, t) - u(x, t)| \, dx \leq \int_{a-Mt}^{b+Mt} |v_0(x, t) - u_0(x, t)| \, dx.
\]
(37)
and also (36). By taking the limit over \(a, b\) we obtain
\[
||v(\cdot, t) - u(\cdot, t)||_{L^1(\mathbb{R})} \leq ||v_0 - u_0||_{L^1(\mathbb{R})}
\]
which immediately implies uniqueness.

**Remark 4.4.11.** The equation (37) exhibits a fundamental property of the transport equation (and hyperbolic equations more generally), which is *finite speed of propagation*. One manifestation of this is the following.

**Corollary 4.4.12.** In the notation above, if \(u_0\) has support in \([-B, B]\) then \(u(t, \cdot)\) has support in \([-B - Mt, B + Mt]\).

**Proof.** If \([a', b']\) is an interval disjoint from \([-B - Mt, B + Mt]\), then we may assume without loss of generality that \(b' \leq -B - Mt\). Then \(b' + Mt \leq -B\), so (37) with \(v \equiv 0\) and \(v_0 \equiv 0\) implies that
\[
\int_{a'}^{b'} |u(x, t)| \, dx \leq \int_{a-Mt}^{b'+Mt} |u_0(x)| \, dx = 0.
\]
\(\square\)

Now for the proof of (37). To begin, we choose some special entropy/flux pairs. Imagine that we could take the entropy \(\Phi(z) = |z - \alpha|\) (it is convex and continuous, but not differentiable in the classical sense). Then \(\Phi'(z) = \text{sgn}(z - \alpha)\), and
\[
\Psi(z) = \int_a^z \Phi'(w)f'(w) \, dw = \text{sgn}(z - \alpha)(F(z) - F(\alpha)).
\]
This isn’t strictly justified, but by approximating \(|z - \alpha|\) with a smooth function we deduce the following.

**Lemma 4.4.13.** If \(u\) is an entropic solution, then for any \(k \in \mathbb{R}\) and positive \(\varphi \in C_0^\infty(\mathbb{R} \times [0, T])\) we have
\[
\int_{\mathbb{R}} \int_{[0,T]} \varphi_t |u - k| + \varphi_x (f(u) - f(k)) \text{sgn}(u - k) \, dt \, dx + \int_{\mathbb{R}} |u_0(x) - k| \varphi(x, 0) \, dx \geq 0.
\]

In fact, the implication goes both ways: a solution is entropic if it satisfies the above inequality for all \(k\) and \(\varphi\). This is essentially because we can approximate any convex function with piecewise linear functions.

The key technical step is to enhance this up by replacing \(k\) with \(v\). Let \(\Omega = \mathbb{R} \times [0, T]\).

**Proposition 4.4.14.** Let \(u\) and \(v\) be entropy solutions of (32) with initial data \(u_0\) and \(v_0\). For all positive \(\varphi \in C_0^\infty(\mathbb{R} \times [0, T])\), we have
\[
\int_{\Omega} |u - v| \varphi_t + (f(u) - f(v)) \varphi_x \text{sgn}(u - v) \, dt \, dx + \int_{\mathbb{R}} |u_0(x) - v_0(x)| \varphi(x, 0) \, dx \geq 0.
\]
Proof. Choose a positive function $\eta \in C^\infty_c(\Omega \times \Omega)$. We apply Lemma 4.4.13 with respect to $\alpha = v(y,s)$ and test function $\eta(\cdot,\cdot, y, s)$. We then apply Lemma 4.4.13 again reversing the roles of $u$ and $v$, and with test function $\eta(x, t, \cdot, \cdot)$. Summing the results, we have

$$0 \leq \int \int \int |u(x,t) - v(y,s)|((\eta_t + \eta_s)(x, t, y, s)$$

$$+ \int \int \int \text{sgn}(u(x,t) - v(y,s))(f(u(x,t)) - f(v(y,s)))(\eta_x + \eta_y)(x, t, y, s)$$

$$+ \int \int \int |u_0(x) - v(y,s)|\eta(x,0,y,s) + \int \int \int |u(x,t) - v_0(y)|\eta(x, t, y, 0)$$

Now we choose $\eta$ to approximate $\varphi(x,t)\delta(x-y,t-s)$, so that the inequality tends to

$$0 \leq \int \int |u - v|\varphi_t + \text{sgn}(u - v)(f(u) - f(v))\varphi_x + \int_R |u_0(x) - v_0(x)|\varphi(x, 0)$$

which is the claimed result. \qed

Now we are about ready to complete the proof of the estimate (37). By translating, we may reduce to the case where $a = -b$. Let $T$ be the trapezoid

$$T = \{(x,t) \in \Omega: -b - M(s-t) < x < b + M(s-t)\}.$$

Let $\chi(t)$ be a smooth, positive cutoff function which is 1 for $t < s$ and supported in $\{t < s + b/M\}$. Let $\theta(x)$ be a smooth, positive cutoff function which
is 1 for $|x| < b$. Note that $\theta(|x| + M(s - t))$ is a smooth, positive cutoff function for the trapezoid $T$.

We apply Proposition 4.4.14 with the choice $\varphi(x, s) = \chi(s)\theta(|x| + M(s - t))$.

Let us analyze the integrand:

$$|u - v|\varphi_t + F(u - v)\varphi_x = |u - v|\chi'(t)\theta(|x| + M(s - t)) - |u - v|\chi(t)\theta'(|x| + M(s - t))$$

$$+ F(u - v)\chi(t)\theta'(|x| + M(s - t)) \text{ sgn } x$$

$$= \chi'(t)\theta(|x| + Mt)|u - v|$$

$$+ \chi(t)\theta'(|x| + M(s - t))(F(u - v) - M|u - v|)$$

Since $|F(u - v)| \leq M|u - v|$, the second term is negative and we conclude that $|u - v|\varphi_t + F(u - v)\varphi_x \leq |u - v|\chi'(t)\theta(|x| + M(s - t))$.

Then by Proposition 4.4.14 we have

$$\int \int |u - v|\chi'(t)\theta(|x| + M(s - t)) + \int u_0(x) - v_0(x)\theta(|x| + Ms) \geq 0. \quad (38)$$

Let $T_{t'} = T \cap \{t = t'\}$ denote the slice of the trapezoid at time $t'$. Now, as we let $\theta(x)$ approach the characteristic function of $[-b, b], \theta(|x| + M(s - t))$ becomes arbitrarily close to the characteristic function of $T$. Applying this in (38) gives

$$\int_{t=0}^{T} \chi'(t)||u(\cdot, t) - v(\cdot, t)||_{L^1(T_t)} \, dt + ||u_0 - v_0||_{L^1(T_0)} \geq 0.$$}

If we then choose $\chi$ arbitrarily close to a downwards step function at time $s$, then its derivative is arbitrarily close to the negative delta function at time $s$, and we deduce that

$$||u_0 - v_0||_{L^1(T_0)} \geq ||u(\cdot, t) - v(\cdot, t)||_{L^1(T_t)}.$$}

Finally, we establish (36). Choose test functions of the form

$$\varphi^\epsilon(x, t) = \chi(t)\theta(\epsilon x)$$

where $\theta$ is supported in a neighborhood of the origin, and $\chi$ is as before. Then by the definition of weak solution,

$$0 = \int_{\Omega} (u - v)\varphi_t^\epsilon + (f(u) - f(v))\varphi_x^\epsilon \, dt \, dx + \int_{\mathbb{R}} (u_0(x) - v_0(x))\varphi^\epsilon(x, 0) \, dx$$

$$= \int_{\Omega} (u - v)\chi'(t)\theta(\epsilon x) + \epsilon \int_{\Omega} (f(u) - f(v))\chi(t)\theta'(\epsilon x) \, dt \, dx$$

$$+ \int_{\mathbb{R}} (u_0(x) - v_0(x))\varphi^\epsilon(x, 0) \, dx$$
Since $|f(u) - f(v)| \leq M|u - v|$, each term above is uniformly integrable, so we can take limit $\epsilon \to 0$ inside the integrals to obtain
\[
\int_{\Omega} (u - v)\chi'(t) \, dx \, dt + \chi(0) \int_{\mathbb{R}} (u_0 - v_0) \, dx = 0.
\]
Choosing $\chi$ to approximate a step function at time $s$, so that $\chi'$ approximates $-\delta(t-s)$, we deduce (36).

4.4.5 Rankine-Hugoniot type conditions

Let us consider writing down a Rankine-Hugoniot type condition for entropic solutions. Adopting the notation from §4.3.2, we have for all positive test functions $\varphi \in C_0^\infty(\Omega)$,
\[
0 \leq \int_{\Omega} \varphi_t \Phi(u) + \varphi_x \Upsilon(u)
= \left( \int_{\Omega_1} + \int_{\Omega_2} \right) \varphi_t \Phi(u) + \varphi_x \Upsilon(u)
= - \left( \int_{\Omega_1} + \int_{\Omega_2} \right) (\Phi(u)_1 + \Upsilon(u)_x) \varphi + \int_{\Gamma} (\Upsilon(u), \Phi(u)) \cdot n_1 \varphi
\leq \int_{\Gamma} [\Upsilon(u)]_t (\Phi(u)) \cdot n_1 \varphi.
\]
Since this holds for all positive test functions, we may conclude that
\[
[\Upsilon(u)]_t n_x + [\Phi(u)]_t n_t \geq 0.
\]
If our curve $\Gamma$ is parametrized by $x(t) = \eta(t)$, then we may take $n_1 \propto (1, -\eta(t))$.
Setting $\sigma = \eta$ as before, we can reformulate the condition as
\[
\sigma |\Phi(u)| \leq [\Upsilon(u)].
\]
By taking $\Phi(u)$ to approximate $|u - \alpha|$ for any constant $\alpha$, we have
\[
[(f(u) - f(\alpha)) \text{sgn}(u - \alpha)] \leq \sigma |u - \alpha|.
\]
(The same trick was used in Lemma 4.4.13.) Recall that it turns out that this family of inequalities is also sufficient.
Consider this inequality at a point of $\Gamma$ where $u_l \neq u_r$. By choosing $\alpha$ very small, $\text{sgn}(u_l - \alpha) = \text{sgn}(u_r - \alpha)$ is constant at this point and we deduce that
\[
\sigma = \frac{[f(u)]}{|u|}.
\]
By choosing $\alpha$ in the interval $[u_l, u_r]$, say $\alpha = \tau u_l + (1 - \tau) u_r$ for some $\tau \in (0,1)$, we (eventually) deduce that
\[
((1 - \tau)f(u_l) + \tau f(u_r) - f((1 - \tau)u_l + \tau u_r)) \text{sgn}(u_l - u_r) \geq 0.
\]
Therefore, we have two cases.

1. If $u_l < u_r$, then $f$ restricted to $[u_l, u_r]$ must lie above its chord.
2. If $u_l > u_r$, then $f$ restricted to $[u_r, u_l]$ must lie below its chord.
This rules out the “nonphysical” weak solution we found earlier for the Burgers equation in Example 4.4.2.

A further consequence of the inequality is that

\[ f'(u_l) \geq \sigma \geq f'(u_r) \]

which is called Lax’s entropy condition. Physically, this means that the characteristics intersecting along \( \Gamma \) cannot emerge from \( \Gamma \), but must emanate from points “in the past” (again, contrast this with the solution \( u_2 \) from Example 4.4.2). If the inequalities are strict, as is the case when \( f \) is strictly convex or concave, then it means that the entropy solution can be calculated by the method of characteristics.

### 4.4.6 Riemann Problems

The Riemann problem is a special Cauchy problem with initial data of the form

\[
\begin{align*}
    u_0(x) &= \begin{cases} 
        u_l & x < 0, \\
        u_r & x > 0.
    \end{cases}
\end{align*}
\]

The point is that the solutions are invariant under homotheties \( (x,t) \mapsto (ax,at) \). More generally, if \( u(x,t) \) is the entropy solution for \( u_0(x) \), then \( u(ax,at) \) is the entropy solution for \( u_0(ax) \). Therefore, the solutions are of the form \( u(x,t) = v(x/t) \). When does this satisfy (32)? We calculate

\[
\begin{align*}
    \partial_t v(x/t) + \partial_x F(v(x/t)) &= v'(x/t) \left( -\frac{x}{t^2} \right) + F'(v(x/t))v'(x/t)\frac{1}{t} \\
    &= v'(x/t) \left( F'(v(x/t)) - \frac{x}{t} \right).
\end{align*}
\]

Therefore, if \( v'(x/t) \neq 0 \) we have \( F'(v(x/t)) = x/t \), hence \( v = (F')^{-1} \).
Theorem 4.4.15. Suppose $f$ is strictly convex and $C^2$. Then the unique entropy solution of the Riemann problem (32) with initial data (39) is given as follows.

1. If $u_l > u_r$, then

\[
  u(x, t) = \begin{cases} 
  u_l & \frac{x}{t} < \sigma, \\
  u_r & \frac{x}{t} > \sigma 
  \end{cases}
\]

where $\sigma = \frac{f(u_l) - f(u_r)}{u_l - u_r}$.

2. If $u_l < u_r$, then

\[
  u(x, t) = \begin{cases} 
  u_l & \frac{x}{t} < f'(u_l), \\
  v(x/t) & f'(u_l) < \frac{x}{t} < f'(u_r), \\
  u_r & \frac{x}{t} > f'(u_r) 
  \end{cases}
\]

where $v(x) = (f')^{-1}(x)$.

Example 4.4.16. Consider the Burgers equation again, with initial data

\[
  u_0(x) = \begin{cases} 
  0 & x < 0 \\
  1 & 0 \leq x \leq 1, \\
  0 & x > 1 
  \end{cases}
\]

For $t \leq 2$, the entropic solution can be found by juxtaposing the entropic solutions we found in Example 4.4.1 and Example 4.4.2.

What happens at $t = 2$? From the diagram we see that the rarefaction curve from the origin meets the shock curve from $(1, 0)$, forming another shock curve $(x(t), t)$. What is this curve? We have

\[
  \dot{x}(t) = \sigma = \frac{F(u)}{|u|} = \frac{1}{2} \left( \frac{x}{t} \right)^2 = \frac{x}{2t} = \frac{x(t)}{t}.
\]

Solving, we find that $x(t) = (2t)^{1/2}$. Therefore, the solution for $t \geq 2$ is

\[
  u(x, t) = \begin{cases} 
  0 & x < 0, \\
  \frac{x}{t} & 0 < x < (2t)^{1/2}, \\
  0 & x > (2t)^{1/2} 
  \end{cases}
\]
5.1 Introduction to the Wave Equation

The wave equation in \( n + 1 \) dimensions is

\[
\partial_t^2 u - \partial_1^2 u - \ldots - \partial_n^2 u = 0. \tag{40}
\]

where \( u = u(t, x) \) and \( x = (x_1, \ldots, x_n) \). We write \( \Box = \partial_t^2 - \partial_1^2 - \ldots - \partial_n^2 \), which is called the d’Alembertian operator. The wave equation has a distinguished time direction, so we write \( \mathbb{R}^t = \mathbb{R}^{1+n} \) to emphasize this.

We can think of the wave equation as a system of linear transport equations. Indeed, consider for the moment the case \( n = 1 \). Observe that we can factorize the “difference of squares”

\[
\partial_t^2 u - \partial_1^2 u = 0. \tag{41}
\]

If we write \( v = (u, \partial_t u - \partial_x u) =: (v_1, v_2) \) then the PDE (41) is equivalent to the system

\[
\begin{align*}
\partial_t v_1 + \partial_x v_1 & = v_2 \\
\partial_t v_2 - \partial_x v_2 & = 0.
\end{align*}
\]

In higher dimensions, one writes down an operator \( D_x = \sqrt{-\Delta_x} \) and converts (40) into the system

\[
\begin{align*}
\partial_t v_1 + iD_x v_1 & = v_2 \\
\partial_t v_2 + iD_x v_2 & = 0.
\end{align*}
\]

For this reason, the wave equation behaves like a linear hyperbolic equation, even though it has degree 2. The construction of the operator \( D_x \) is nontrivial; one of Dirac’s great insights was to construct this using spinors in \( \mathbb{R}^{1+3} \), leading to the Dirac equation.

5.2 Second-Order Hyperbolic Equations

In contrast to elliptic equations, hyperbolic equations have “as many characteristic hypersurfaces as possible.” We will focus on a specific class of hyperbolic equations of the form

\[
\begin{align*}
-\partial_t^2 + Pu & = f & (0, T) \times \Omega, \\
u & = 0 & \partial[0, T] \times \Omega, \\
u & = u_0 & 0 \times \Omega, \\
\partial_t u & = u_1 & 0 \times \Omega.
\end{align*}
\]

where

\[
P u = - \sum_{i,j=1}^n \partial_j (a^{ij}(t,x) \partial_i u) + \sum_{i=1}^n b^i(t,x) \partial_i u + c(t,x) u.
\]
Here $\Omega \subset \mathbb{R}^n$ is an open subset. Although this appears to be a Cauchy problem, it has Dirichlet conditions on the spatial boundary.

**Definition 5.2.1.** We say that the PDE (42) is hyperbolic if $P$ is elliptic, i.e. for all $(t, x) \in [0, T] \times \Omega$ and $\xi \in \mathbb{R}^n$ we have

$$\sum_{i,j=1}^{n} a^{ij}(t, x)\xi_i \xi_j > 0.$$  

We say that it is uniformly elliptic if there exists some uniform constant $\theta$ such that

$$\sum_{i,j=1}^{n} a^{ij}(t, x)\xi_i \xi_j > \theta |\xi|^2.$$  

Said differently, the principal symbol for the equation is a quadratic form of signature $(n, 1)$.

The set $0 \times U$ is non-characteristic for the PDE (42) (and these are the only non-characteristic points) so we might expect well-posedness for this Cauchy problem. The Cauchy-Kovalevskaya theorem guarantees local existence of analytic solutions given analytic initial data.

### 5.3 Energy Estimates

#### 5.3.1 Homogeneous Equation

We now return to the wave equation $\Box u = f$, and perform some a priori estimates. These are useful in establishing existence and uniqueness; for now we just assume sufficient regularity and decay to justify the calculations. For simplicity, we work on $\Omega = \mathbb{R}^n$.

Multiply the equation (40) by $\partial_t u$ and integrate:

$$0 = \int_{\mathbb{R}^n} \int_0^T \partial_t u (\partial_t^2 u - \Delta_x u) \, dt \, dx = \int_{\mathbb{R}^n} \int_0^T \partial_t^2 u \left( \frac{1}{2} u_t^2 \right) - \partial_t u (\nabla_x \cdot \nabla_x u) \, dt \, dx$$

$$= \int_{\mathbb{R}^n} \int_0^T \partial_t \left( \frac{1}{2} u_t^2 \right) + [\partial_t (\nabla_x u) \cdot \nabla_x u] \, dt \, dx + \int_{\mathbb{R}^n} \partial_t u (\nabla_x u \cdot \mathbf{n}) \, dx$$

$$= \int_{\mathbb{R}^n} \left[ \frac{1}{2} u_t^2 + \frac{1}{2} |\nabla_x u|^2 \right]_0^T \, dx + \int_{\mathbb{R}^n} \partial_t u (\nabla_x u \cdot \mathbf{n}) \, dx$$

Assuming that $\partial_t u = u_1$ vanishes along $t = 0$, we see that the function

$$E(t) := \int_{\mathbb{R}^n} \frac{1}{2} u_t(x, t)^2 + \frac{1}{2} |\nabla_x u(x, t)|^2 \, dx$$

is constant in time. This is already enough to imply the uniqueness of solutions to the wave equation.
5.3.2 Local Energy Estimates

We now develop some local estimates that reveal a cone of dependence, like what we found in Kruzkov’s Theorem. We define the cone

$$C = \bigcup_{t \leq T} \{t\} \times B(x_0, R + T - t).$$

Let $C_t = \{t\} \times B(x_0, R + T - t)$ denote the slice at time $t$. We define the local energy

$$E(t) := \int_{C_t} \frac{1}{2} |u_t(t, x)|^2 + \frac{1}{2} |\nabla_x u(t, x)|^2 \, dx.$$

We assume enough regularity and decay so that this is well-defined. Then we have

$$\frac{dE(t)}{dt} = \int_{C_t} \frac{1}{2} |u_t| + \frac{1}{2} |\nabla_x u| \, dx$$

Integrating the second term by parts, we find that

$$\int_{C_t} \partial_t \nabla_x u \cdot \nabla_x u \, dx = \int_{\partial C_t} \partial_t u (\nabla_x u \cdot n) \, d\sigma - \int_{C_t} \partial_t u \Delta_x u \, dx.$$

Substituting this above, we find that

$$\frac{dE(t)}{dt} = \int_{C_t} u_t (\nabla_x u \cdot n) \, d\sigma - \int_{\partial C_t} \partial_t u (\nabla_x u \cdot n) \, d\sigma - \int_{C_t} \partial_t u \Delta_x u \, dx.$$

By Cauchy-Schwarz,

$$|\partial_t u (\nabla_x u \cdot n)| \leq |\partial_t u| |\nabla_x u| \leq \frac{1}{2} |u_t(t, x)|^2 + \frac{1}{2} |\nabla_x u(t, x)|^2,$$

so we see that

$$\frac{dE(t)}{dt} \leq 0.$$

Therefore, for any time $0 < t < T$ we have

$$E(T) \leq E(0).$$

Note that this implies “finite speed of propagation” as we saw in the transport equation. If the initial data is supported in some set, then at all times the solution will be supported in a “light cone” emanating from that set.