Topological Methods in Algebraic Geometry

Lectures by Burt Totaro

Notes by Tony Feng

Michaelmas 2013

Preface

These are live-texed lecture notes for a course taught in Cambridge during Michaelmas 2013 by Burt Totaro, on a hodgepodge of topics at the intersection of algebraic topology and algebraic geometry. An unfortunate side effect of live-texing is that it tends to introduce a lot of errors, which are solely the fault of the scribe. If you find any, please let me know at tonyfeng009@gmail.com.

Contents

Chapter 1

Principal Bundles and Classifying Spaces

1.1 Principal Bundles in Topology

Let G be a topological group. That is, G is a topological space equipped with continuous maps $G \times G \to G$ (the group operation), a distinguished point $1 \in G$ (the identity), and a map $G \to G$ (the inverse) satisfying the standard associativity, identity, and inverse axioms.

Definition 1.1. A *principal G-bundle* is a topological space E with a continuous action of G such that

1. If $gx = x$, then $g = 1$ (i.e. the action is free), and
2. If $\pi\colon E \to E/G$ is the quotient map, then every point $x \in E/G$ has an open neighborhood U such that $\pi^{-1}(U) \cong G \times U$, and the action of G on $\pi^{-1}(U)$ is given by $g(h, x) = (gh, x)$.

So a principal G-bundle looks like a fiber bundle with fibers G, with locally trivial action of G. However, it is important to note that the fibers are *not* naturally groups, but rather *torsors* for G. That is, we do *not* have a distinguished section of the bundle, so the fibers become identified with G only after choosing some point to be the identity.

Typically, we denote $B = E/G$, and we call the whole fiber bundle $\pi\colon E \to B$ the principal G-bundle. We also write $G \to E \to B$ for this fiber bundle.

Example 1.1. Complex vector bundles of rank n over a topological space are equivalent to principal $\mathrm{GL}(n, \mathbb{C})$-bundles over B. Indeed, given a vector bundle $V \to B$ of rank n, you can construct a principal $\mathrm{GL}(n, \mathbb{C})$-bundle by fixing the vector space $\mathbb{C}^n$, and defining

$$E = \bigcup_{b \in B} \{\text{isomorphisms } \mathbb{C}^n \cong V_b\}.$$

$\mathrm{GL}(n,\mathbb{C})$ acts on E via its action on $\mathbb{C}^n$ (notice that this is completely canonical).

Conversely, given a principal G-bundle $E \to B$ and any space F with an action of G, we can form the space $(E \times F)/G$. This comes with a natural projection map to $E/G \simeq B$ with fiber F, and is therefore an F-bundle over B. Applying this to the special case $G = \mathrm{GL}(n,\mathbb{C})$ and $F = \mathbb{C}^n$, we obtain a natural $\mathbb{C}^n$-bundle over B from a principal $\mathrm{GL}(n,\mathbb{C})$-bundle over B. Furthermore, since $\mathrm{GL}(n,\mathbb{C})$ fixes $0 \in \mathbb{C}^n$, the resulting fiber bundle has a distinguished zero section, namely $x \mapsto (x, 0)$, which gives it a natural vector bundle structure.

Remark 1.2. Another perspective on this equivalence is through transition functions. Both principal $\mathrm{GL}(n,\mathbb{C})$-bundles and rank n vector bundles are equivalent to the data of transition functions in $\mathrm{GL}(n,\mathbb{C})$ forming a cocycle, and this identification is the identity on transition functions.

If we have a group G acting freely on a topological space E, then we would like to be able to say that $E \to E/G$ is a principal G-bundle. However, we have to be more careful about our definition of freeness for the resulting quotient map to be locally trivial, as the following example shows.

Example 1.2. Let the group $\mathbb{R}$ act on $\mathbb{R}^2/\mathbb{Z}^2 \cong S^1 \times S^1$ by $t(x,y) = (x+t, y+\lambda t)$ for some irrational λ. Here $\mathbb{R}$ acts freely on $S^1 \times S^1$ as a set, but this is not a principal bundle because the quotient space has the trivial topology, and so the map is not locally trivial.

In light of this, we introduce a finer notion of free action that is nicely behaved in the topological category.

Definition 1.3. A topological group G as *freely* on a space X if the map

$$\begin{aligned} G \times X &\to X \times X \\ (g,x) &\mapsto (x, gx) \end{aligned}$$

is a homeomorphism onto its image.

Theorem 1.4 (Serre)**.** *For a Lie group G (including discrete groups), if G acts freely (in the above sense) on a metrizable topological space X, then $X \to X/G$ is a principal G-bundle (i.e. it is locally trivial).*

Remark 1.5.

- The trivial G-bundle over B is $G \times B$ with the obvious G-action.
- A principal G-bundle $f\colon E \to B$ is trivial if and only if it has a *section*, meaning a continuous map $s\colon B \to E$ such that $fs = 1_B$. Indeed, given a section s, we can define an isomorphism of principal G-bundles over B from $G \times B$ to E by $(g,b) \mapsto g \cdot s(b)$.

- Given any principal G-bundle $E \to B$ and a continuous map $f\colon Y \to B$, you can define a "pullback" principal G-bundle over Y as the fiber product in the category of topological spaces:

$$\begin{tikzcd} E_Y \arrow[r] \arrow[d] & E \arrow[d] \\ Y \arrow[r] & B \end{tikzcd}$$

1.2 Classifying Spaces

In the following discussions, all spaces are paracompact when necessary.

Definition 1.6. Let G be a topological group. If EG is a contractible space on which G acts freely, then we define the *classifying space* BG of G to be the quotient space EG/G.

It is a fact that such a space EG exists, and any two classifying spaces for G are homotopy equivalent. We won't prove this rigorously, but we will try to elucidate it a little bit. In particular, the classifying space is technically only defined up to homotopy equivalence.

Example 1.3. Let G be a discrete group. A classifying space BG is a connected space whose universal cover is contractible and whose fundamental group is G. So for instance, $S^1 = B\mathbb{Z}$.

In fact, for any "nice" connected space X, the universal cover $\tilde{X} \to X$ is a principal $\pi_1(X)$-bundle. Whenever the universal covering space is contractible, X is the classifying space $B\pi_1(X)$.

Recall that an Eilenberg-Maclane space $K(G,1)$ is a connected space X with $\pi_1(X) \cong G$ and $\pi_i(X) = 0$ for $i \geq 2$. If G is discrete, then this is a classifying space of G. Why?

Let $\widetilde{X}$ denote the universal cover of X. It is a general fact that $\pi_i(\widetilde{X}) \to \pi_1(X)$ is an isomorphism for $i \geq 2$ (this is clear from the long exact sequence of homotopy groups, since the higher homotopy groups of the fiber are trivial). This shows that the classifying space of a discrete group G is a $K(G,1)$. Conversely, if X is a $K(G,1)$, then all the homotopy groups of $\widetilde{X}$ are zero, which implies that it is contractible for nice X (for example, a CW complex), by Whitehead's theorem.

So at least for discrete groups, the classifying spaces are familiar objects.

Example 1.4. A closed oriented surface X of genus ≥ 1 is a $K(G,1)$ where $G = \pi_1(X)$. In particular, the torus is $B(\mathbb{Z}^2)$ and the closed surface of genus 2 is the classifying space for

$$G = \langle A_1, A_2, B_1, B_2; [A_1, A_2][B_1, B_2] = 1 \rangle.$$

Example 1.5. $B(\mathbb{Z}/2) = S^\infty/\{\pm 1\} = \mathbb{RP}^\infty$. Indeed, S^∞ (meaning the increasing union $\cup_n S^n$) is contractible. This is perhaps somewhat surprising, since the S^n are not contractible, but we can contract each S^n within S^{n+1}.

Example 1.6. S^1 is a nice topological group, so it has a classifying space. What is BS^1? Thinking of $S^{2n-1} \subset \mathbb{C}^n$ and $S^1 \subset \mathbb{C}^*$, we get a free action of S^1 on S^{2n-1} by scaling each coordinate. Therefore, $BS^1 = S^\infty/S^1$. We can recognize this space as $\mathbb{CP}^\infty$.

1.3 The classifying space of $\mathrm{GL}(n, \mathbb{C})$

Let $E = \mathrm{Inj}(\mathbb{C}^n, \mathbb{C}^\infty)$ (injective linear maps). Here $\mathbb{C}^\infty$ is the increasing union $\bigcup_{n\geq 0} \mathbb{C}^n$. The space E is contractible [why?]. So

$$BGL(n, \mathbb{C}) = E/\mathrm{GL}(n, \mathbb{C}) = \mathrm{Gr}(n, \infty)$$

This is the "Grassmannian" of n-dimensional C-linear subspaces in $\mathbb{C}^\infty$. Note that we can also think of it as $\bigcup_{N\geq 0} \mathrm{Gr}(n, N)$. The cohomology ring of this space is generated by the *Chern classes*, so we have deduce that $H^*(BGL(n, \mathbb{C}), \mathbb{Z}) = \mathbb{Z}[c_1, \ldots, c_n]$ where $|c_i| = 2i$.

If $H \to G$ is a homomorphism of topological groups which is a homotopy equivalence, then $BH \to BG$ is a homotopy equivalence.

Example 1.7. $U(n) \to \mathrm{GL}(n, \mathbb{C})$ is a homotopy equivalence. (Here $U(n)$ is the set of linear maps $g \colon \mathbb{C}^n \to \mathbb{C}^n$ such that $||gx|| = ||x||$ for all $x \in \mathbb{C}^n$. Note that $g \in U(n) \iff g \cdot \overline{g^t} = 1$.) Why? Given $A \in \mathrm{GL}(n, \mathbb{C})$, we have n linearly independent vectors $A(e_1), \ldots, A(e_n) \in \mathbb{C}^n$. The Gram-Schmidt process produces (canonically) an orthonormal basis from this given one, which form the columns of an element of $U(n)$. More generally, a connected Lie group deformation retracts onto a maximal compact subgroup.

Example 1.8. Let $SU(n) = \{A \in U(n) \colon \det A = 1\}$. The classifying space $BSU(2)$ can be viewed as $\mathbb{HP}^\infty$ (infinite dimensional quaternionic projective space).

Proof. Note that $SU(2)$ is the group of unit quaternions. (In particular, $SU(2)$ is diffeomorphic to the 3-sphere.) The group $SU(2)$ acts freely on $S^\infty \subset \mathbb{H}^\infty$, so $BSU(2) \simeq S^\infty/SU(2) = \mathbb{HP}^\infty$. $\square$

Note that

$$\begin{aligned}\mathbb{RP}^\infty &= \text{pt.} \cup (1-\text{cell}) \cup (2-\text{cell}) \cup \dots \\ \mathbb{CP}^\infty &= \text{pt.} \cup (2-\text{cell}) \cup (4-\text{cell}) \cup \dots \\ \mathbb{HP}^\infty &= \text{pt.} \cup (4-\text{cell}) \cup (8-\text{cell}) \cup \dots\end{aligned}$$

It turns out that $H^*(BS^1, \mathbb{Z}) \cong \mathbb{Z}[u], |u| = 2$ and $H^*(BSU(2), \mathbb{Z}) \cong \mathbb{Z}[v], |v| = 4$. That this is true additively is clear from the cell decompositions above; the ring structure is trickier.

Example 1.9. How do you construct BG for any compact Lie group G? Such a group has a faithful complex representation (by the Peter-Weyl theorem). That is, we can find an embedding $G \hookrightarrow \mathrm{GL}(n, \mathbb{C})$ for some n. We can further: G embeds as a subgroup of $U(n)$ for some n (start with the standard inner product on $\mathbb{C}^n$ and average it with respect to Haar measure on G to get a G-invariant inner product on $\mathbb{C}^n$).

Then G acts freely on $EU(n) = \mathrm{Inj}(\mathbb{C}^n, \mathbb{C}^\infty)$ which is contractible, so

$$BG = EU(n)/G \to EU(n)/U(n).$$

In this way, we get a fiber bundle

$$U(n)/G \to BG \to BU(n).$$

1.4 Fiber bundles

Theorem 1.7 (cf. Husemöller, Fiber bundles.)**.** *Let $E \to B \times [0,1]$ be a principal G-bundle, with B a paracompact space. Let $\pi\colon B \times [0,1] \to B$ be the projection map. Then $E \cong \pi^*(E|_{B\times 0})$.*

Corollary 1.8. *If $f_0, f_1\colon X \to Y$ are homotopic maps, and E is a principal G-bundle over Y, then $f_0^*E \cong f_1^*E$ as principal G-bundles.*

Proof. We have a homotopy $F\colon X \times [0,1] \to Y$. By the theorem, F^*E on $X \times [0,1]$ is pulled back from a G-bundle over X. In particular, the restrictions of this bundle to $X \times \{0\}$ and $X \times \{1\}$ are isomorphic. □

Corollary 1.9. *If $f\colon X \to Y$ is a homotopy equivalence, then f^* is a bijection from isomorphism classes of G-bundles on Y to those on X.*

Proof. By definition, we have a continuous map $g\colon Y \to X$ such that $fg \sim 1_Y$ and $gf \sim 1_X$. □

1.4.1 The long exact sequence of a fiber bundle.

Let $F \to E \to B$ be a fiber bundle. That is, $\pi\colon E \to B$ is a continuous map, and B is covered by open sets U such that there is a homeomorphism $\pi^{-1}U \cong U \times F$ over U.

$$\begin{tikzcd} \pi^{-1}U \arrow[r, "\sim"] \arrow[d] & U \times F \arrow[d] \\ U \arrow[r, "="] & U \end{tikzcd}$$

Then there is a long exact sequence of homotopy groups (once we fix a base point on the spaces involved):

$$\dots \pi_i F \to \pi_i E \to \pi_i B \to \pi_{i-1} F \to \dots$$

One can view the boundary map as the homomorphism associated to a certain continuous map $\Omega B \to F$. Note that $\pi_i(\Omega X) \cong \pi_{i+1}(X)$, since the suspension Σ and the loop space Ω are adjoint functors on the category of pointed spaces.

Example 1.10 (The path fibration)**.** For any connected space X, there is a fibration of the form

$$\Omega X \to PX \to X$$

where $PX = \{f\colon [0,1] \to X \text{ with } f(0) = x_0\}$, i.e. the space of all paths in X starting at x_0. The map sends $f \mapsto f(1) \in X$. This is well-behaved enough that we get a long exact sequence of homotopy groups:

$$\dots \pi_i \Omega X \to \pi_i PX \to \pi_i X \to \pi_{i-1} \Omega X \to \dots$$

but the path space PX is contractible, so the long exact sequence tells us that $\pi_i X \cong \pi_{i-1} \Omega X$.

1.5 Universal Property of Classifying Spaces

Theorem 1.10. *Let X be a paracompact space that is homotopy equivalent to a CW complex (e.g. any CW complex or manifold). Let G be a topological group. Then there is a bijection*

$$[X, BG] \leftrightarrow \{ \textit{ isomorphism classes of principal } G\textit{-bundles over } X \ \}.$$

Remark 1.11. We can extend this the pointed case: fix a base point $x_0 \in X$ and consider based homotopy classes. Likewise, consider principal G-bundles over X that are trivialized over x_0, that is, with a choice of a point in the fiber over x_0.

Proof Sketch. A continuous map $f\colon X \to BG$ gives a principal G-bundle on X by pulling back the *universal* principal G-bundle on BG, $EG \to BG$. We already know that homotopic maps give isomorphic G-bundles.

Conversely, let $E \to X$ be a principal G-bundle. Consider the principal G-bundle $(E \times EG) \to (E \times EG)/G$ (the G-action is diagonal on the product).

Remark 1.12. This is a free action even if only *one* of the actions is free. Indeed, if the map $G \times E \to E \times E$ is an isomorphism, then an inverse to the map

$$G \times (E \times EG) \to (E \times EG) \times (E \times EG) \cong (E \times E) \times EG \times EG$$

is obtained by combining this isomorphism with the projection map $EG \times EG \to EG$ to the first coordinate.

There are obvious continuous maps

$$\begin{tikzcd} & (E \times EG)/G \arrow[dl, "f"'] \arrow[dr, "g"] & \\ E/G = X & & EG/G = BG. \end{tikzcd}$$

Notice that the map f is a homotopy equivalence, because EG is contractible. Indeed, f is a fiber bundle with fiber EG. One can use the long exact sequence of homotopy groups to see that the two spaces have the same homotopy groups, and the result then follows by Whitehead's theorem. Therefore, we get a homotopy inverse map $X \to (E \times EG)/G \to BG$.

Let's now try to see the surjectivity of this assocation. Given a principal G-bundle on X, we want to construct a map $X \to BG$ that pull backs EG to our given bundle.

The key point is that the map $E \times EG \to EG$ is G-equivariant, so if you pull back the universal G-bundle from BG to $(E \times EG)/G$ then you get the universal G-bundle $E \times EG \to (E \times EG)/G$. Similarly, since the map $E \times EG \to E$ is G-equivariant, the pullback of $E \times EG$ to E/G is just E.

What about injectivity: if two maps $f_1, f_2\colon X \to BG$ satisfy $f_1^*(EG) = f_2^*(EG)$, why is $f_1 \sim f_2$? Now we use the cell complex hypothesis and build up the result cell-by-cell. Since every principal G-bundle over a disc is trivial, we have two maps $e_1, e_2\colon D^n \times G \to EG$ and we know that $e_1 = e_2$ on $\partial D^n \times G$ (by induction), and we want a homotopy between these two maps.

Observe that a G-equivariant map $D^n \times G \to EG$ is the same as a continuous map $D^n \to EG$. So we can view e_1, e_2 as two continuous maps $D^n \to EG$, agreeing on their boundaries, which together give a map $S^n \to EG$. Since EG is contractible, they must be homotopic.

□

Example 1.11. Let G be a discrete group, so that $BG = K(G,1)$. For a connected space X, this theorem says that

$$[X, K(G,1)] = \left\{ \substack{\text{isomorphism classes of} \\ \text{principal } G\text{-bundles over } X} \right\} = \mathrm{Hom}(\pi_1(X), G).$$

To see this last identification, observe that a G-bundle over X is a covering space with G as its group of deck transformations.

For example, if $G = \mathbb{Z}$, then we get that $[X, S^1] = \mathrm{Hom}(\pi_1 X, \mathbb{Z})$ for connected X.

Definition 1.13. A space X is a $K(G,n)$ (*Eilenberg-Maclane space*) if

$$\pi_i X \cong \begin{cases} G & i = nm \\ 0 & i \neq n \end{cases}$$

where G is discrete, and abelian if $n \geq 2$.

Theorem 1.14. *For any (reasonable) space X and any Eilenberg-Maclane space $K(G,n)$,*

$$[X, K(G,n)] \cong H^n(X, G).$$

This agrees with the previous result because, for X connected,

$$[X, S^1] = [X, K(\mathbb{Z}, 1)] = H^1(X, \mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(H_1(X, \mathbb{Z}), \mathbb{Z}) = \mathrm{Hom}(\pi_1 X, \mathbb{Z}).$$

Remark 1.15. For $n = 1$, it's reasonable to define, for any topological group G,

$$H^1(X, G) := [X, BG].$$

One can prove theorems such as the above by "obstruction theory." We saw a hint of this above: decomposing X as a cell complex, one slowly builds up maps cell-by-cell. Any discrepancies that arise on the boundaries can be analyzed using homotopy theory, which is simplified by the vanishing of most of the homotopy groups of $K(G,n)$.

Example 1.12. For a space X,

$$[X, \mathbb{CP}^\infty] = [X, BS^1] = \{\text{principal } S^1\text{-bundles over } X/\cong\}.$$

We can also think of $\mathbb{CP}^\infty$ as $BGL(1, \mathbb{C}) = B\mathbb{C}^\times$, so that the above is in natural bijection with the set of principal $\mathrm{GL}(1, \mathbb{C})$ bundles over X. But we explained how principal $GL(1, \mathbb{C})$-bundles over X correspond to complex line bundles, so this is the same as isomorphism classes of *complex line bundles* on X.

Also, note that $\mathbb{CP}^\infty$ is a $K(\mathbb{Z}, 2)$ space. (More generally, consider the fiber bundle for a topological group G:

$$G \to EG \to BG.$$

The long exact sequence shows that $\pi_i BG \cong \pi_{i-1} G$. Therefore, if G is $K(G', n)$ then BG is a $K(G', n+1)$.) We have deduced the following theorem.

Theorem 1.16. *There is a bijection*

$$\{\textit{isomorphism classes of complex line bundles over } X\} \cong H^2(X, \mathbb{Z}).$$

The cohomology class associated to a line bundle L is called the "first Chern class" of that line bundle, and denoted $c_1(L)$. One can think of $c_1(L)$ as the "Euler class" of the associated oriented real bundle of rank 2 over X. Therefore, if X is a smooth manifold and s is a section transverse to the zero section, then $c_1(L)$ is the cohomology class of the zero set of a section s of L (or rather, Poincaré dual to it) in $H^2(X, \mathbb{Z})$.

Remark 1.17. You can define the Chern classes of any $\mathbb{C}$-bundle in a similar way. If $E \to X$ is a complex vector bundle of rank n, we have shown that isomorphism classes of rank n vector bundles are the same as principal $\mathrm{GL}(n, \mathbb{C})$ bundles over X, which is the same as maps $[X, BGL(n, \mathbb{C})]$. This furnishes a natural way to attach invariants to these vector bundles, by pulling back cohomology classes of

$$H^*(BGL(n, \mathbb{C}), \mathbb{Z}) \cong \mathbb{Z}[c_1, \ldots, c_n] \qquad |c_i| = 2i$$

so any $\mathbb{C}$-vector bundle E of rank n determines cohomology classes $c_1(E)$, $c_2(E), \ldots, c_n(E) \in H^*(X, \mathbb{Z})$ by pulling back the c_i.

1.6 Fibrations

First, we say a few additional things about fibrations.

Definition 1.18. A map $E \to B$ has the *homotopy lifting property* with respect to a space Y if for any maps $f\colon Y \times [0,1] \to B$ and $g\colon Y \to E$ such that $\pi \circ g = f|_{Y\times 0}$, f lifts to a map $F\colon Y \times [0,1] \to E$ such that $\pi \circ F = f$.

$$\begin{tikzcd} Y \times [0,1] \arrow[r, dotted, "F"] \arrow[dr, "f"'] & E \arrow[d] \\ & B. \end{tikzcd}$$

Definition 1.19. A *(Serre) fibration* is a continuous map $\pi\colon E \to B$ satisfying the homotopy lifting property for $Y = [0,1]^n$, for any $n \geq 0$.

Example 1.13. Any fiber bundle is a fibration.

Example 1.14. For any space X, the path fibration $\Omega X \to PX \to X$ is a fibration, but not necessarily a fiber bundle.

Some properties of fibrations.

- For B path-connected, all fibers of a fibration $\pi\colon E \to B$ are homotopy equivalent. This is a relaxation of the condition for fiber bundles.

- A fibration $\pi\colon E \to B$ gives a long exact sequence of homotopy groups

$$\ldots \to \pi_i F \to \pi_i E \to \pi_i B \to \pi_{i-1} F \to \ldots$$

  where F is the pre-image of any point (the homotopy groups are well-defined by the previous statement.

Remark 1.20. Any continuous map is "equivalent" to a fibration, where by equivalent we mean that we can replace spaces by homotopy-equivalent spaces. Explicitly, suppose $E \to B$ is a continuous map and let E_2 be the space of pairs (e, α) where $e \in E$ and α is a path in B starting at $f(e)$. The space E_2 is evidently homotopy equivalent to E, because you can contract each path to the constant path at $f(e)$. However, there is a map $E_2 \to B$ sending α to $\alpha(1)$, and one can check that this is a fibration.

Definition 1.21. The *homotopy fiber* of a map $f\colon E \to B$ is a fiber of the fibration $E_2 \to B$. This is well-defined up to homotopy equivalence.

Example 1.15. The homotopy fiber of the inclusion $* \to X$ is ΩX, since the previous construction gives the fibration $\Omega X \to PX \to X$.

Example 1.16. For any fibration $F \to E \to B$, the homotopy fiber of $E \to B$ is homotopy equivalent to F.

Corollary 1.22. *For any topological group G, $\Omega(BG)$ is homotopy equivalent to G.*

Proof. We have a fiber bundle $G \to EG \to BG$ where EG is contractible, so the homotopy fiber of $* \to BG$ is homotopy equivalent to $\Omega(BG)$. On the other hand, the homotopy fiber of $EG \to BG$ is homotopy equivalent to G by the preceding example. $\square$

The rough picture is that there is a remarkable equivalence in homotopy theory between two problems. One is to classify connected topological spaces up to homotopy equivalence. The other is to classify topological groups up to "homotopy equivalence as topological groups". As a technical matter, it may be more convenient to work with topological monoids M such that $\pi_0(M)$ is a group. The two problems are related by: given a connected space X, one can form the loop space ΩX, which is a topological monoid whose π_0 is a group (precisely $\pi_1 X$). One can go backwards by taking the classifying space, so this is an equivalence of categories.

Example 1.17. $\Omega(\mathbb{CP}^\infty) \simeq S^1$ because $\mathbb{CP}^\infty = BS^1$.

Chapter 2

Spectral Sequences

In this chapter, we will introduce spectral sequences, with an emphasis on examples and applications. The slogan is that spectral sequences are a generalization of long exact sequences.

We will focus on the Serre spectral sequence of a fibration. Given a fibration $F \to E \to B$, how are H^*F, H^*E, and H^*B related? We might hope that H^*E can be recovered from H^*F and H^*B. For instance, if $E = F \times B$ and B and F are CW-complexes of finite type, and the cohomology coefficients are a field k, then the Kunneth formula says that

$$H^i(E, k) \cong \bigoplus_{j=0}^{n} H^j(F, k) \otimes_k H^{i-j}(B, k).$$

In general, the relationship between these three cohomology rings is highly nontrivial. If we were considering homotopy groups instead of (co)homology, then we would get a long exact sequence. In the cohomology setting, the best we can hope for is a *spectral sequence.*

Definition 2.1. A (first-quadrant) *spectral sequence* is a collection of abelian groups $E_r^{p,q}$ for integers $r \geq 2, p \geq 0, q \geq 0$ and homomorphisms $d_r^{p,q} \colon E_r^{p,q} \to E_r^{p+r,q-(r-1)}$ such that:

1. $(d_r)^2 = 0$,
2. $E_{r+1}^{p,q} = \frac{\ker(d_r \text{ on } E_r^{p,q})}{\operatorname{im}(d_r \text{ to } E_r^{p,q})}$.

$$\begin{tikzcd}
E_2^{0,2} \arrow[drr] & E_2^{1,2} \arrow[drr] & E_2^{2,2} & E_2^{3,2} & \cdots \\
E_2^{0,1} \arrow[drr] & E_2^{1,1} \arrow[drr] & E_2^{2,1} & E_2^{3,1} & \cdots \\
E_2^{0,0} & E_2^{1,0} & E_2^{2,0} & E_2^{3,0} & \cdots
\end{tikzcd}$$

We think of the spectral sequence as being a collection of "pages" indexed by r. Each page consists of a two-dimensional array of abelian groups, and has its own differential d_r that goes r to the right and $r-1$ down.

The groups on the pages get successively "smaller", since the group $E_{r+1}^{p,q}$ is a quotient of a subgroup of $E_r^{p,q}$. However, it is possible that they will eventually stop changing.

Theorem 2.2 (Leray, Serre). *For a fibration $F \to E \to B$, there is a spectral sequence such that*

- $E_2^{p,q} \cong H^p(B, H^q(F))$,
- *for each p, q, there exists r such that $E_r^{p,q} \cong E_{r+1}^{p,q} \cong \ldots =: E_\infty^{p,q}$, and*
- *there is a filtration of $H^j(E, R)$ with quotient groups being the groups $E_\infty^{p,j-p}$ for $p = 0, 1, \ldots, j$.*

There is a complication here. If $\pi_1 B$ does not act trivially on H^*F, then the cohomology on the E_2 page has to be considered as the cohomology of B with coefficients in a locally constant sheaf on B. So we will try to stick to the case where this actually is trivial, especially when $\pi_1 B$ is itself trivial.

Example 2.1. Consider an oriented sphere bundle $S^n \to E \to B$. Then the E_2 page of the spectral sequence has zeros everywhere except in row 0 and row n, and the only possibly nonzero differential is $d_{n+1}\colon H^iB \to H^{i+n+1}B$. In particular, $d_{n+1}(1 \in H^0B) = \chi(E)$ in $H^{n+1}(B, \mathbb{Z})$ is called the *Euler class* of this sphere bundle.

$$\begin{tikzcd}
H^0B \arrow[ddddrrrrr, "d_{n+1}"] & H^1B & H^2B & H^3B & & \\
0 & 0 & 0 & 0 & \ldots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & 0 & 0 & 0 & \ldots & \\
H^0B & H^1B & H^2B & H^3B & \ldots & H^{n+1}B
\end{tikzcd}$$

For any $x \in H^iB$, $d_{n+1}(x) = x\chi(E) \in H^{i+n+1}(B)$ (we will explain this later). So this one cohomology class determines how the cohomology of E is related to that of B. This gives a long exact sequence

$$\ldots H^{i-(n+1)} \xrightarrow{\chi(E)} H^i(B) \to H^iE \to H^{i-n}B \xrightarrow{\chi(E)} H^{i+1}B \to \ldots$$

2.1 Spectral sequence of a filtered complex

One way that spectral sequences come up in nature is when you have a *filtered* chain complex. To elaborate, suppose that

$$\ldots \to A^{-1} \to A^0 \to A^1 \to \ldots$$

is a (co)-chain complex of abelian groups (meaning that the homomorphisms d satisfy $d^2 = 0$). Suppose each A^i has a decreasing filtration, meaning subgroups $F^j(A^i)$ for all integers j such that $F^{j+1}(A^i) \subset F^j(A^i)$. Suppose further that the differentials are compatible with the filtration, in the sense that $d(F^j(A^i)) \subset F^j(A^{i+1})$. Consider the associated graded of the filtration,

$$\mathrm{gr}^j(A^i) = F^j(A^i)/F^{j+1}(A^i)$$

giving the associated chain complex (by compatibility)

$$C^j : \ldots \to \mathrm{gr}^j(A^i) \to \mathrm{gr}^j(A^{i+1}) \to \ldots$$

Then there is a spectral sequence from the cohomology of the complexes C^j, for $j \in \mathbb{Z}$, converging to the cohomology of $A^\bullet$.

The Serre spectral sequence can be constructed in this way. Given a fibration $F \to E \to B$ with B a CW-complex, we want to know the cohomology of

$$C^\bullet E : \ldots \to C^j E \to C^{j+1} E \to \ldots$$

You can filter this chain complex by the inverse images of the j-skeletons of B, for $j \in \mathbb{Z}$. This gives a spectral sequence with

$$E_1^{p,q} = C^p(B, H^q F).$$

So on E_2, the groups are $H^p(B, H^q F)$.

How is $H^* E$ related to the E_∞ page of the Serre spectral sequence of a fibration $F \to E \to B$? Recall that we said that the terms along the diagonals $p+q = n$ are the quotients of a filtration on $H^n E$. More precisely, $H^p E$ maps onto $E_\infty^{0,p}$, and the kernel of that maps onto $E_\infty^{1,p-1}$, ... the kernel of the last homomorphism is $E_\infty^{p,0}$ (so the latter is a subspace).

Here's a trick to remember which direction does the filtration go. Note that $E_2^{p,0} = H^p(B, \mathbb{Z})$ if F is connected. It only receives maps in the spectral sequence, so $E_\infty^{p,0}$ is a quotient of $H^p(B, \mathbb{Z})$. The homomorphism from this group to $H^p E$ is induced from the pullback map $H^p B \to H^p E$. Likewise, $E_2^{0,q} = H^0(B, H^q F) = H^q F$. So $E_\infty^{0,q}$ is a subgroup of $H^q F$. The homomorphism from $H^q E$ to this group is the pullback homomorphism $H^q E \to H^q F$.

The Serre spectral sequence fits well with the cup product on cohomology. Each page E_r of the spectral sequence is a graded commutative ring

($xy = (-1)^{|x||y|}yx$ where $x \in E_r^{a,b}$ has $|x| = a + b$). Moreover,

$$d_r(xy) = d_r(x) \cdot y + (-1)^{|x|} x d_r(y).$$

This is the natural sign to expect, if you think of d_r has a "symbol of degree one" (it raises the total degree by one).

On the E_2 page, the product is the "obvious" one on $H^*B \otimes_k H^*F$ (if we consider cohomology with coefficients in a field). Moreover, the product on each page E_r determines the product on the page E_{r+1}.

Finally, the ring structure on $E_\infty^{*,*}$ is the associated graded ring to the product on H^*E, which makes sense because

$$F^j(H^*E)F^\ell(H^*E) \subset F^{j+\ell}(H^*E).$$

Example 2.2. Let $S^n \to E \to B$ be an oriented sphere bundle. Write x for the element element of $E^2_{0,n}$ that corresponds to 1 in H^0B.

$$\begin{tikzcd}
x \in H^0B \arrow[rrrrrdddd, "d_{n+1}"] & H^1B & H^2B & H^3B & & \\
0 & 0 & 0 & 0 & \cdots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & 0 & 0 & 0 & \cdots & \\
H^0B & H^1B & H^2B & H^3B & \cdots & H^{n+1}B
\end{tikzcd}$$

Clearly $d_{n+1} = 0$ on row 0, and every element on row n is equal to xy for some y in row 0. So $d_{n+1}(xy) = d_{n+1}(x)y$, where $d_{n+1}(x)$ is the Euler class of the sphere bundle.

This situation commonly arises from taking the sphere bundle of an oriented real vector bundle E of rank $n+1$. Thus, we have associated a cohomology class in $H^{n+1}(B, \mathbb{Z})$ to an oriented real vector bundle of rank $n+1$; this is called the Euler class of that vector bundle.

Note that the element x^2 in the spectral sequence is zero (it lives in a group which is zero). So $0 = d_{n+1}(x^2) = (1 + (-1)^n)d_{n+1}(x)x$. If n is odd, this says nothing. If n is even, then $2d_{n+1}(x)x = 0$, so $2d_{n+1}(x) = 0$. That is, the Euler class of an oriented real vector bundle of odd rank is killed by 2. As a result, the Euler class is more useful for real bundles of even rank.

Example 2.3. Suppose that the sphere S^n has the structure of a topological group. What can you say about $H^*(BS^n, \mathbb{Z})$? (In fact, it is a famous theorem that this only happens when $n = 0, 1$, or 3.)

We have a fibration $S^n \to ES^n \to BS^n$. Suppose $n \geq 1$. Since we know that

$$\pi_i S^n = \begin{cases} 0 & i < n, \\ \mathbb{Z} & i = n. \end{cases}$$

it follows that

$$\pi_i BS^n = \begin{cases} 0 & i < n+1 \\ \mathbb{Z} & i = n+1 \end{cases}.$$

In particular, BS^n is simply-connected, so the Serre spectral sequence has a simple E_2 page. We get that $H^*(BS^n, H^*(S^n, \mathbb{Z}))$ converges to $H^*(\mathrm{pt}, \mathbb{Z})$, whose only non-zero group is $\mathbb{Z}$ in degree 0. That implies that all of the maps d_{n+1} must be isomorphisms. But d_{n+1} is multiplication by the Euler class. Therefore, $H^* BS^n$ is a polynomial ring generated by the Euler class.

This shows that if S^n is a topological group and $n \geq 1$, then n must be odd, since if n were even then we would have $|\chi| = n + 1$ so $\chi^2 = -\chi^2$ by graded-commutativity.

Example 2.4. What is $H^*(\Omega S^n, \mathbb{Z})$? Note that $\pi_i(\Omega S^1) = \pi_{i+1} S^1$, which we know, so $\Omega S^1 \simeq \mathbb{Z}$ as a discrete space.

Now suppose $n \geq 2$, so that S^n is simply connected. We have a fibration $\Omega S^n \to PS^n \to S^n$ so we get a spectral sequence

$$E_2^{**} = H^*(S^n, H^*(\Omega S^n, \mathbb{Z})) \implies H^*(\mathrm{pt}, \mathbb{Z}).$$

The only non-vanishing differential is d_n, and the only cohomology group that can survive is H^0. Therefore, $H^i(\Omega S^n, \mathbb{Z}) \cong H^{i-(n-1)}(\Omega S^n, \mathbb{Z})$ is an isomorphism for $i \geq n$. Furthermore, $H^0(\Omega S^n, \mathbb{Z}) \cong \mathbb{Z}$ and $H^i(\Omega S^n, \mathbb{Z}) = 0$ for $0 < i < n - 1$. This shows that $H^i(\Omega S^n, \mathbb{Z})$ is $\mathbb{Z}$ in multiples of $n - 1$ and 0 otherwise.

Let's find the ring structure. Suppose $n = 2m + 1$, and let y be the element of $H^{n-1}(\Omega S^n)$ with differential $d_n(y) = x$ generating $H^0(\Omega S^n)$. Then $d_n(y^k) = ky^{k-1} d_n(y)$, so $d_n^k(y^k) = k!x$. This shows that $H^{k(n-1)}(\Omega S^n) \cong \mathbb{Z}[\frac{y^k}{k!}]$. Thus we see that the cohomology ring $H^*(\Omega S^{2m+1}, \mathbb{Z})$ is the "free divided-power algebra on one generator." With $\mathbb{Q}$ coefficients, this is simply a polynomial ring! So $H^*(\Omega S^{2m+1}, \mathbb{Q}) \cong \mathbb{Q}[y]$, $|y| = 2m$.

Exercise 2.1. Compute $H^*(\Omega S^{2m}, \mathbb{Z})$ as a $\mathbb{Z}$-algebra.

2.2 The rational cohomology of Eilenberg-Maclane spaces

There are two viewpoints in homotopy theory. The first is that any space is built by gluing spheres (definition of a CW-complex), from which the basic objects are cofibrations $X \to Y \to Y/X$. Every space is approximately a "wedge of spheres," but with twisting from the gluing maps. The ways an a-cell can be attached to a b-sphere are described by the homotopy group $\pi_{a-1}(S^b)$. So a lot of homotopy theory would be determined if we knew the homotopy groups of spheres (whereas their homology is simple).

The second is that any space is built from Eilenberg-MacLane spaces, and the foundational objects are fibrations. In this philosophy, all spaces are approximately products of Eilenberg-Maclane spaces, but possibly with twisting from fibrations. The homotopy groups of Eilenberg-MacLane spaces are simple, while their cohomology is complicated (but known!).

We have $\Omega K(A, n) \simeq K(A, n-1)$ for an abelian group A and $n \geq 1$. By the path fibration on $K(A, n)$ we see that $K(A, n) = BK(A, n-1)$. The path fibration is homotopy equivalent to $K(A, n-1) \to * \to K(A, n)$.

Theorem 2.3. *$H^*(K(\mathbb{Z}, 2m+1), \mathbb{Z}) = \mathbb{Q}\langle x_{2m+1}\rangle$ and $H^*(K(\mathbb{Z}, 2m), \mathbb{Q}) \cong \mathbb{Q}[y_{2m}]$.*

Notation: $\mathbb{Q}\langle x_a, x_b, \ldots\rangle$ denotes the free graded-commutative $\mathbb{Q}$-algebra with generators in degrees $a, b, \ldots$. For a odd, $\mathbb{Q}\langle x_a\rangle$ is just the exterior algebra $\mathbb{Q} \oplus \mathbb{Q}x_a$.

Remark 2.4. We are already familiar with this theorem in the case $K(\mathbb{Z}, 1) \simeq S^1$ and $K(\mathbb{Z}, 2) \simeq \mathbb{CP}^\infty$.

Proof. As in the analysis of H^*BS^n if S^n is a topological group, suppose we know that $H^*(K(\mathbb{Z}, 2m-1), \mathbb{Q}) \cong \mathbb{Q}\langle x_{2m-1}\rangle$. Then consider the fiber bundle $K(\mathbb{Z}, 2m-1) \to * \to K(\mathbb{Z}, 2m)$. The E_2 page of the corresponding spectral sequence is zero except in rows 0 and $2m-1$, and the cohomology of the total space is 0 except in degree 0, so the only nonzero differential is d_{2m} and it is an isomorphism in each entry.

$$
\begin{tikzcd}
0 & 0 & 0 & 0 & 0 \\
\mathbb{Q}x \arrow[ddrr] & \cdots & \mathbb{Q}xy \arrow[ddrr] & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & 0 \\
\mathbb{Q} & \cdots & \mathbb{Q}y & \cdots & \mathbb{Q}y^2
\end{tikzcd}
$$

Therefore we conclude as before that $H^*(K(\mathbb{Z},2m),\mathbb{Q}) \cong \mathbb{Q}[y_{2m}]$ (the point is that the spectral sequence tells us multiplication by y_{2m} is an isomorphism). Next, suppose we know that

$$H^*(K(\mathbb{Z},2m),\mathbb{Q}) \cong \mathbb{Q}[y_{2m}].$$

We have a fibration

$$K(\mathbb{Z},2m) \to * \to K(\mathbb{Z},2m+1)$$

so we have a spectral sequence whose E_2 page is nonzero only in rows a multiple of $2m$. Since the cohomology of the total space vanishes except in degree 0, we see that $H^*(K(\mathbb{Z},2m+1),\mathbb{Q}) = 0$ in degrees $1,\ldots,2m$ and $\mathbb{Q}x$ in degree $2m+1$, where $x = d_{2m+1}(y)$.

$$\begin{tikzcd}
\mathbb{Q}y^2 \arrow[ddrrr] & 0 & \cdots & \mathbb{Q}xy^2 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbb{Q}y \arrow[ddrrr] & 0 & \cdots & \mathbb{Q}xy & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbb{Q} & 0 & \cdots & \mathbb{Q}x & \cdots
\end{tikzcd}$$

By the compatibility between differentials and products, we have $d_{2m+1}(y^r) = rxy^{r-1}$ for all $r \geq 1$. Since we are considering rational coefficients, it follows that d_{2m+1} is an isomorphism from column 0 to column $2m+1$, except at at the $0,0$ entry.

As a result, we can show that the rational cohomology of $K(\mathbb{Z},2m+1)$ is zero in all degrees greater than $2m+1$. If it had nonzero cohomology in degree j, where j is the smallest number greater than $2m+1$ such that this happens, then no differentials could hit the group $E_2^{j,0}$, contradicting the fact that $E_\infty^{j,0}$ must be zero. This completes the inductive proof that $H^*(K(\mathbb{Z},2m+1),\mathbb{Q}) \cong \mathbb{Q}\langle x_{2m+1}\rangle$. $\square$

Corollary 2.5. *Let X be any simply connected space with $H^*(X,\mathbb{Q})$ a free graded commutative $\mathbb{Q}$-algebra. Then X is $\mathbb{Q}$-homotopy equivalent to a product of Eilenberg-Maclane spaces.*

What this means is that there is a map $X \to \prod K(A_n,n)$ that induces an isomorphism on $\mathbb{Q}$-cohomology.

Proof. Suppose that $H^*(X, \mathbb{Q}) \cong \mathbb{Q}\langle x_{a_1}, x_{a_2}, \ldots \rangle$. Replacing each x_{a_i} by a multiple if necessary, we may assume that each x_{a_i} comes from $H^*(X, \mathbb{Z})$. Since $[X, K(A, n)] \cong H^n(X, A)$, each x_{a_i} gives a map $X \to K(\mathbb{Z}, a_i)$. That gives a map from X to the product, such that the generator of $H^{a_i}(K(\mathbb{Z}, a_i), \mathbb{Z}) \cong \mathbb{Z}$ pulls back to $x_{a_i} \in H^*(X, \mathbb{Z})$. By the Künneth formula, the pullback map is an isomorphism on rational cohomology. □

Theorem 2.6 (Serre)**.** *The groups $\pi_i S^n$ are finitely generated for all n. More generally, this holds for any simply-connected CW-complex of finite type.*

Proof. The key ingredient here is the Hurewicz theorem, which says that if $\pi_i X = 0$ for $0 \le i \le n-1$ with $n \ge 2$, then the map $\pi_i(X) \to H_i(X)$ is an isomorphism for $1 \le i \le n$. Technically, the argument should be stated using of the Leray-Serre spectral sequence for homology rather than cohomology. Since we haven't stated that, we will informally switch between homology and cohomology.

Let X be a simply connected CW complex of finite type. Then all homology groups $H_i(X, \mathbb{Z})$ are finitely generated. Suppose that $A = \pi_n(X)$ is the first nonzero homotopy group. By Hurewicz, A is finitely generated. Consider the fibration $F \to X \to K(A, n)$. We know that $H^i(X, \mathbb{Z})$ are finitely generated abelian groups, and we want to show that $H^i(F, \mathbb{Z})$ are also finitely generated abelian groups. Why is this useful? The long exact sequence for a fibration says that

$$0 \to \pi_{n+1} X \cong \pi_n F \to 0 \to \pi_n X \cong \pi_n(K(A, n)) \to 0.$$

So the homotopy groups of the fiber are those of X kicked up one dimension. If the cohomology of F is finitely generated, then its lowest nontrivial homotopy group is isomorphic to its lowest nontrivial cohomology group, which is the *second*-lowest nontrivial homotopy group for X.

Towards this end, we must first establish that $H^i(K(A, n), \mathbb{Z})$ is finitely generated for A finitely generated. This is a straightforward spectral sequence argument, like others we have seen. Granting this, one then looks at the spectral sequence

$$H^*(K(A, n), H^*(F, \mathbb{Z})) \implies H^*(X, \mathbb{Z}).$$

Could it happen that $H^i F$ is not finitely generated? Consider the least i for which this is the case. It has to end up finitely generated on the E_∞ page, but by the minimality assumptions it only has maps to and receives maps from finitely generated groups, so this is impossible.

2.3 References

- R. Bott and L. Tu, *Differential forms in algebraic topology*
- J. McCleary, *A user's guide to spectral sequences*
- P. Griffiths and J. Harris, *Principles of algebraic geometry*

□

Chapter 3

Rational Homotopy Theory

3.1 Rational Homotopy Equivalences

Definition 3.1. A map $f\colon X \to Y$ of simply connected spaces is a *rational homotopy equivalence* if any of the following equivalent conditions hold:

1. $f_*\colon \pi_i(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to \pi_i(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$ is an isomorphism for all i.
2. $f_*\colon H_*(X, \mathbb{Q}) \to H_*(Y, \mathbb{Q})$ is an isomorphism for all i.
3. $f^*\colon H^*(Y, \mathbb{Q}) \to H^*(X, \mathbb{Q})$ is an isomorphism for all i.

(2) $\iff$ (3) is clear from the universal coefficient theorem. (1) $\iff$ (2) is proved by a spectral sequence argument, e.g. using induction on Postnikov towers:

$$F_2 \to X \to K(\pi_a(X), a).$$

Remark 3.2. Part of the reason this works so neatly is that $\mathbb{Q}$ is a flat $\mathbb{Z}$-module (flat is equivalent to torsion-free over $\mathbb{Z}$). So, for example, for a fibration $F \to E \to B$ we have a long exact sequence

$$\ldots \to \pi_i F \otimes_{\mathbb{Z}} \mathbb{Q} \to \pi_i E \otimes \mathbb{Q} \to \pi_i B \otimes \mathbb{Q} \to \ldots \to$$

Another example of a flat $\mathbb{Z}$-module is the integers localized at a prime number p, $\mathbb{Z}_{(p)}$. We can define a *p-local homotopy equivalence* by equivalent conditions as above.

Example 3.1. A simply-connected space X has $\pi_*(X) \otimes \mathbb{Q} = 0 \iff \tilde{H}_*(X, \mathbb{Q}) = 0$.

Note: if there is a $\mathbb{Q}$-homotopy equivalence $X \to Y$, it does not follow that there is a $\mathbb{Q}$-homotopy equivalence $Y \to X$.

Definition 3.3. A 1-connected space X is *rational* if the following equivalent conditions hold:

1. $\pi_* X$ is a $\mathbb{Q}$-vector space.

2. $\tilde{H}_*(X, \mathbb{Z})$ is a $\mathbb{Q}$-vector space.

The proof that these are actually equivalent uses the fact that $\tilde{H}_i(K(\mathbb{Q}, n), \mathbb{F}_p) = 0$. By the universal coefficient theorem, $H_i(K(\mathbb{Q}, n), \mathbb{Z})$ are $\mathbb{Q}$-vector spaces. This shows that (1) $\implies$ (2) in the simplest case of Eilenberg-Maclane spaces, and then one uses the usual spectral sequence / fibration arguments. The other direction is similar.

Example 3.2.
- $K(\mathbb{Q}, n)$ is a rational space.
- For any $n \geq 1$, the rational n-sphere is

$$S^n_{\mathbb{Q}} = \left(\bigvee_{k \geq 1} S^n_k \right) \cup \left(\coprod_{k \geq 2} D^{n+1}_k \right)$$

  where the attaching map for the kth cell is the map $S^n \to S^n_{k-1} \vee S^n_k$ given by $f_{k-1} - k f_k$, where $f_{k-1} \colon S^n \to S^n_{k-1}$ and $f_k \colon S^n \to S^n_k$ are the identity.

  Here $H_i(S^n_{\mathbb{Q}}, \mathbb{Z}) = 0$ for $i \neq 0, n$ and

$$H_n(S^n, \mathbb{Z}) = \varinjlim (\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{4} \dots) \cong \mathbb{Q}.$$

Theorem 3.4. *Let X be a simply connected CW-complex. Then there is a 1-connected rational CW-complex $X_{\mathbb{Q}}$ with a map $X \to X_{\mathbb{Q}}$ which is a $\mathbb{Q}$-homotopy equivalence (so $\pi_i(X_{\mathbb{Q}}) \cong \pi_i X \otimes \mathbb{Q}$) and $H_i(X_{\mathbb{Q}}, \mathbb{Z}) \cong H_i(X, \mathbb{Q})$ for $i > 0$). Moreover, this map is universal with respect to maps from X into rational spaces: for any rational space Y and map $X \to Y$, there is a unique (up to homotopy) map $X_{\mathbb{Q}} \to Y$ making the diagram commute:*

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & X_{\mathbb{Q}} \arrow[d] \\ & Y \end{tikzcd}$$

Proof. Since X is 1-connected, it is isomorphic to a CW-complex with a 0-cell and no 1-cells. Construct $X_{\mathbb{Q}}$ step by step, replacing each attaching map $\alpha \colon S^n \to X_n$ by a map $S^n_{\mathbb{Q}} \to (X_n)_{\mathbb{Q}}$.

By induction, we have already constructed the rational space $(X_n)_{\mathbb{Q}}$ with a map $X_n \to (X_n)_{\mathbb{Q}}$. We know that $\pi_n(X_n) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \pi_n((X_n)_{\mathbb{Q}})$. Map $\bigvee_{k \geq 1} S^n_k \to (X_n)_{\mathbb{Q}}$ by mapping the kth sphere via $\frac{1}{k!}\alpha$. By this choice, the map can be extended over all the $n+1$ cells of $S^n_{\mathbb{Q}}$. So we have constructed a map $S^n_{\mathbb{Q}} \to (X_n)_{\mathbb{Q}}$ that extends α. I'll skip the proof of the universal propert of $X_{\mathbb{Q}}$. $\square$

3.2 Rational Homotopy Groups of Spheres

Example 3.3. Note that $K(\mathbb{Z},n)_{\mathbb{Q}} = K(\mathbb{Q},n)$ (by considering the homotopy groups of $K(\mathbb{Z},n)$). There is a map $S^n \to K(\mathbb{Z},n)$ that induces an isomorphism on $H_n(\cdot,\mathbb{Z})$. The map $S^{2a+1} \to K(\mathbb{Z},2a+1)$ is a rational homotopy equivalence (since it is on $\mathbb{Q}$-homology - we computed this in previous lectures). Rationalizing, we get a map

$$(S^{2a+1})_{\mathbb{Q}} \to K(\mathbb{Z},2a+1)_{\mathbb{Q}}$$

inducing an isomorphism on homotopy groups, hence a homotopy equivalence. One sees here the power of rationalization: it takes rational homotopy equivalences into genuine homotopy equivalences.

Although only S^0, S^1, S^3 are loop spaces, we've shown that all odd-dimensional spheres are "rationally" loop spaces.

Corollary 3.5.

$$\pi_i(S^{2a+1}) \cong \begin{cases} 0 & i < 2a+1 \\ \mathbb{Z} & i = 2a+1 \\ \textit{finite abelian group} & i > 2a+1 \end{cases}.$$

Remark 3.6. A 1-connected space X of finite type has $H^*(X,\mathbb{Q})$ a free graded-commutative algebra if and only if $X_{\mathbb{Q}}$ is homotopy equivalent to a product of Eilenberg-Maclane spaces.

Consider the homotopy fiber of

$$F \to S^{2a} \to K(\mathbb{Z},2a).$$

One can compute $H^*(F,\mathbb{Q})$ by the spectral sequence of this fibration (with rational coefficients).

$$\begin{tikzcd}
\mathbb{Q} \arrow[ddddrrrrr, "\approx", "d_{4a}"'] & 0 & 0 & \cdots & & \\
\vdots & \vdots & \vdots & \vdots & & \\
? \arrow[ddrrr, "0", "d_{2a}"'] & 0 & 0 & \cdots & & \\
\vdots & \vdots & \vdots & \vdots & & \\
\mathbb{Q} & 0 & \cdots & \mathbb{Q} & \cdots & \mathbb{Q}
\end{tikzcd}$$

By the usual observations, $H^i(F, \mathbb{Q}) = 0$ for $0 < i < 2a - 1$. In fact, it must *also* be zero for $i = 2a - 1$, since we know that the map $S^{2a} \to K(Z, 2a)$ is an isomorphism at $i = 2a$. The next potentially nonzero group is at $i = 4a - 1$, where we see that d_{4a} must be an isomorphism since the higher cohomology of S^{2a} is trivial. This kills off the rest of the cohomology of $K(\mathbb{Z}, 2a)$, and so F cannot have any higher cohomology. We see that $H^*(F, \mathbb{Q}) \cong H^*(S^{4a-1}, \mathbb{Q})$ so $F_{\mathbb{Q}} \simeq K(\mathbb{Q}, 4a - 1)$.

Theorem 3.7.

$$\pi_i(S^{2a}) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & i = 2a \\ \mathbb{Q} & i = 4a - 1 \\ 0 & \textit{otherwise} \end{cases}.$$

In the course of the proof, we constructed a fibration

$$K(\mathbb{Q}, 4n - 1) \to S^{2n}_{\mathbb{Q}} \to K(\mathbb{Q}, 2n).$$

In low dimensions, we can see the two nonzero rational homotopy groups of an even-dimensional sphere more concretely using the Hopf fibrations:

$$S^1 \to S^3 \to S^2,$$

$$S^3 \to S^7 \to S^4,$$

$$S^7 \to S^{15} \to S^8$$

These come from $\mathbb{C}, \mathbb{H}$, and the octonions Ca. Indeed, think of $S^2 = \mathbb{CP}^1, S^4 = \mathbb{HP}^1, S^8 = \mathrm{Ca}\mathbf{P}^1$. The map is from the sphere in $\mathbb{C}^2$, $\mathbb{H}^2$, or Ca^2 to the projective line. The octonions are not associative, which has the effect that one can define $\mathrm{Ca}\mathbf{P}^1$ and $\mathrm{Ca}\mathbf{P}^2$ but not higher-dimensional projective spaces over the octonions.

One can get from the fibration $K(\mathbb{Q}, 4n - 1) \to S^{2n}_{\mathbb{Q}} \to K(\mathbb{Q}, 2n)$ to a rationalized version of the Hopf fibrations by *looping*. Given any fibration $F \to E \to B$, the homotopy fiber of the inclusion $F \to E$ is homotopy equivalent to ΩB, so have have another fibration:

$$\Omega B \to F \to E.$$

This can be constructed by hand using the description of the homotopy fiber we gave earlier. Explicitly, define the space

$$F_2 = \{(f, p) \colon f \in F \text{ and } p \text{ is a path in } E \text{ with } p(0) = f\}.$$

Then $F_2 \simeq F$, but the map $F_2 \to E$ given by $(f, p) \mapsto p(1)$ is a fibration. So the homotopy fiber Y of $F \to E$ is

$$Y = \{(f, p) \colon f \in F, p \colon [0, 1] \to E, p(0) = f, p(1) = f_0\}$$

for $f_0 \in F$ fixed. Then there is a natural map $Y \to \Omega B$ given by projecting the path p down to B. To check that this is a homotopy equivalence, we compare the homotopy groups by comparing the LES of the fibration $F \to E \to B$ and $Y \to F \to E$. We see that Y and ΩB have isomorphic homotopy groups, hence are homotopy equivalent.

Thus, starting from any continuous map $f \colon X \to Y$, we can let F be the homotopy fiber of f, and then we have a whole sequence of homotopy fiber sequences going off to the left:

$$\cdots \to \Omega^2 Y \to \Omega F \to \Omega X \to \Omega Y \to F \to X \to Y.$$

We showed earlier using a spectral sequence that for $n \geq 1$, the even-dimensional sphere S^{2n} is not a topological group. Since the argument works using rational cohomology, the same argument shows that $S^{2n}_{\mathbb{Q}}$ is not a loop space. (In particular, $S^{2n}_{\mathbb{Q}}$ is not the product of $K(\mathbb{Q}, 4n - 1)$ and $K(\mathbb{Q}, 2n)$, as we can also see directly from its cohomology.)

A different point of view on why $S^{2n}_{\mathbb{Q}}$ is not a loop space uses the Whitehead product. For any space X, there is a product

$$[,] \colon \pi_a X \times \pi_b X \to \pi_{a+b-1} X.$$

To define this, think of $S^a \times S^b$ as a cell complex with one S^a, one S^b, and one $(a+b)$-cell with attaching map $S^{a+b-1} \to S^a \vee S^b$. When $a = b = 1$, this is the familiar cell decomposition of the torus $(S^1)^2$. So given any (pointed) maps $S^a \to X, S^b \to X$, one gets a map $S^{a+b-1} \to X$ by composition

$$S^{a+b-1} \to S^a \vee S^b \to X.$$

This is the Whitehead product.

Example 3.4.

- Take $a = b = 1$, so the Whitehead product is $\pi_1 X \times \pi_1 X \to \pi_1 X$. To see what this is, one just has to understand the attaching map for the torus, which is $(a, b) \mapsto aba^{-1}b^{-1}$. So the Whitehead product gives the commutator on $\pi_1 X$.
- For $a = 1$ and $b > 1$, the Whitehead product is

  $$\pi_1 X \times \pi_b X \to \pi_b X.$$

  Explicitly, the group $\pi_1 X$ acts on the abelian group $\pi_b X$ by "conjugation". given $\alpha \colon S^1 \to X$ and $\beta \colon S^b \to X$, and the Whitehead product is $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$. Thus the Whitehead product is zero if and only if $\pi_1 X$ acts trivially on $\pi_b X$.

Theorem 3.8. *For X simply connected, the Whitehead product makes $\pi_* X$ into a graded Lie algebra: it is equipped with a product*

$$[\cdot, \cdot]\colon \pi_a X \times \pi_b X \to \pi_{a+b-1} X$$

satisfying:

$$[x, y] = (-1)^{|x||y|}[y, x],$$
$$0 = (-1)^{ca}[[x, y], z] + (-1)^{ab}[[y, z], x] + (-1)^{bc}[[z, x], y]$$

for $x \in \pi_a X, y \in \pi_b X, z \in \pi_c X$.

One interpretation of the Whitehead product is that it measures "how commutative the group operation is on ΩX." (As usual, we refer to this as a group operation even though it is more properly considered as a monoid operation.) Indeed, the Whitehead product can be viewed as a product $\pi_a \Omega X \times \pi_b \Omega X \to \pi_{a+b} \Omega X$. I claim that the Whitehead product measures the nontriviality of the commutator map $\Omega X \wedge \Omega X \to \Omega X$ sending $(\alpha, \beta) \mapsto \alpha\beta\alpha^{-1}\beta^{-1}$. Here $X \wedge Y$ denotes the *smash product* $(X \times Y)/(X \vee Y)$ of pointed spaces X and Y. Precisely, given $S^a \to \Omega X$ and $S^b \to \Omega X$, the Whitehead product corresponds to the map $S^{a+b} \cong S^a \wedge S^b \to \Omega X \wedge \Omega X \to \Omega X$.

Corollary 3.9. *If $X = \Omega Y$, then $\Omega X = \Omega^2 Y$ is homotopy commutative.*

The is basically the same argument as the proof that the group $\pi_2 Y$ is abelian.

So if X is a loop space, the Whitehead product on its homotopy groups is zero. We can use this to see that $S^{2n}_{\mathbb{Q}}$ is not a loop space. Namely, let X be a generator of $\pi_{2a} S^{2n}$. Then I claim that $[x, x] \in \pi_{4n-1} S^{2n}$ is nonzero. Why? The Whitehead product describes the attaching map for the top dimensional cell in the CW complex

$$X = S^{2n} \times S^{2n}/(x, x_0) \sim (x_0, x)$$

where $x_0 \in S^{2n}$ is a base point. We want to show that the attaching map $S^{4n-1} \to S^{2n}$ is not 0 in $\pi_*(S^{2n})$. We use the "Hopf invariant." The idea is to measure the nontriviality by the cup product in some cohomology. If the $\mathbb{Q}$-cohomology ring of X is not just that of $S^{2n} \vee S^{4n}$, then $[x, x] \neq 0$ in $\pi_{4n-1}(S^{2n} \otimes \mathbb{Q})$. This is left as an exercise, using the obvious map $S^{2n} \times S^{2n} \to X$ to show that the square of the generator in H^{2n} is not zero.

Finally, we briefly mention the *Eckmann-Hilton duality.* This is a surprising analogy, starting with the analogy between fibrations and cofibrations, not a precise mathematical statement. The idea is clarified by

Quillen's notion of a *model category*, roughly speaking a category in which one can do homotopy theory, with fibrations, cofibrations, and weak equivalences. The point is that Quillen's axioms are self-dual, so that the opposite category to a model category is again a model category, with fibrations and cofibrations switched.

Homotopy groups	Cohomology groups
Fibration	Cofibration
Whitehead product	Cup product
Graded Lie algebra	Graded-commutative algebra
Loop space	Suspension
Eilenberg-Maclane space	Sphere

3.3 References

- R. Bott and L. Tu, *Differential forms in algebraic Topology.*
- Y. Félix, J. Oprea, and D. Tanré, *Algebraic models in geometry.*
- Y. Félix, S. Halperin, and J.-C. Thomas, *Rational homotopy theory.*

Chapter 4

Topology and geometry of Lie groups

4.1 Some examples

In this chapter, we will study the homotopy type of compact Lie groups. One general phenomenon that will observe in examples is captured by the following theorem.

Theorem 4.1 (Hopf). *For any compact Lie group G,*

$$H^*(G, \mathbb{Q}) \cong H^*(S^{2a_1-1} \times \ldots \times S^{2a_n-1}, \mathbb{Q})$$

i.e. $H^*(G, \mathbb{Q})$ is a free graded-commutative algebra on odd-dimensional generators.

This follows from the algebraic fact that any finite dimensional graded-commutative Hopf algebra in characteristic zero is an exterior algebra on odd-dimensional generators as a ring (forgetting the rest of the Hopf algebra structure). That is, $H^*(G)$ has the usual ring structure from the cup product, but also a comultiplication coming from the group structure $G \times G \to G$.

Let's just check this in some important examples. The idea is to use the extra structure of a fibration to inductively compute the cohomology groups. This is a generalization of the idea that you learn in a first course on algebraic topology to compute the cohomology of projective space or Grassmannians using their canonical vector bundles.

What is $H^*(U(n), \mathbb{Z})$? (Recall that the unitary group $U(n)$ is homotopy equivalent to $\mathrm{GL}(n, \mathbb{C})$.) We have a transitive $U(n)$ action on S^{2n-1}, and the stabilizer is the subgroup of $U(n)$ fixing the orthogonal complement of a unit vector in $\mathbb{C}^n$, which is $U(n-1)$. Therefore, we have a diffeomorphism $S^{2n-1} \cong U(n)/U(n-1)$, and we get a fibration:

$$U(n-1) \to U(n) \to S^{2n-1}.$$

It is an easy spectral sequence argument to read off the cohomology groups of $U(n)$.

Theorem 4.2. $H^*(U(n), \mathbb{Z}) \cong \mathbb{Z}\langle x_1, x_3, \ldots, x_{2n-1}\rangle$.

Proof. Use the spectral sequence for the fibration

$$U(n-1) \to U(n) \to S^{2n-1}.$$

In $\mathbb{Z}$-chomology,

$$E_2 = H^*(S^{2n-1}, H^*(U(n-1), \mathbb{Z})) \implies H^*(U(n), \mathbb{Z}).$$

We induct. There are only two nonzero columns, in degree 0 and $2n-1$.

$$\begin{array}{cccc}
H^{2n}U(n-1) & 0 & \ldots & H^{2n}U(n-1) \\
\vdots & \vdots & \vdots & \vdots \\
H^1U(n-1) & 0 & \ldots & H^1U(n-1) \\
\mathbb{Z} & 0 & \ldots & \mathbb{Z}
\end{array}$$

We see from the spectral sequence that $H^*U(n) \to H^*U(n-1)$ is onto in degrees $\leq 2n-1$. Since the ring $H^*(U(n-1), \mathbb{Z})$ is generated by elements of degree at most $2n-3$, the ring homomorphism $H^*U(n) \to H^*U(n-1)$ must be surjective in all degrees. So all differentials in the spectral sequence are zero! Let $x_1, \ldots, x_{2n-3}$ be elements of $H^*U(n)$ that restrict to the generators of $H^*U(n-1)$, and let $x_{2n-1} \in H^{2n-1}(U(n))$ be the pullback of a generator of $H^{2n-1}S^{2n-1}$. Then

$$H^*U(n) = \mathbb{Z}\langle x_1, x_3, \ldots, x_{2n-1}\rangle.$$

There is a slightly subtle point: we know that $2x_\ell^2 = 0$, because the cohomology ring of $U(n)$ must be graded-commutative, but a priori we don't know that $x_\ell^2 = 0$. However, it is clear from the spectral sequence that $H^*(U(n), \mathbb{Z})$ is torsion free, so we do in fact have $x_\ell^2 = 0$. $\square$

Corollary 4.3. $U(n)_\mathbb{Q} \simeq S^1_\mathbb{Q} \times \ldots \times S^{2n-1}_\mathbb{Q}$. *In particular,*

$$\pi_i U(n) \otimes_\mathbb{Z} \mathbb{Q} \cong \begin{cases} \mathbb{Q} & i \in \{1, 3, 5, \ldots, 2n-1\}, \\ 0 & \textit{otherwise} \end{cases}$$

Theorem 4.4. $H^*(BU(n), \mathbb{Z}) \cong \mathbb{Z}[c_1, \ldots, c_n], \quad |c_i| = 2i.$

A polynomial ring typically has many different choices of generators, but the proof will in fact give an explicit set of generators. These are the usual Chern classes.

Proof. For any closed subgroup H in a topological group G, there is a fibration

$$G/H \to BH \to BG.$$

Indeed, thinking of BG as EG/G, we can take EG/H as a model for BH, in which case the result is obvious. (Remark: looping this fibration gives some other interesting fibrations, e.g. looping once gives $G \to G/H \to BH$, and looping again gives $H \to G \to G/H$).

So we have a fibration

$$S^{2n-1} \to BU(n-1) \to BU(n).$$

This is the sphere bundle in the natural complex vector bundle of rank n over $BU(n)$. Now we proceed by induction, so assume that $H^*BU(n-1) \cong \mathbb{Z}[c_1, \ldots, c_{n-1}]$ where $|c_i| = 2i$. The Serre spectral sequence says that $H^*(BU(n), H^*S^{2n-1}) \implies H^*BU(n-1)$ (note that $BU(n-1)$ is simply-connected):

$$\begin{tikzcd}
H^0BU(n) \arrow[ddrrr] & H^1BU(n) & \cdots & H^{2n}BU(n) \\
\vdots & \vdots & \vdots & \vdots \\
H^0BU(n) & H^1BU(n) & \cdots & H^{2n}BU(n)
\end{tikzcd}$$

Let $c_n \in H^{2n}(BU(n), \mathbb{Z})$ be the Euler class of the universal $\mathbb{C}^n$-bundle over $BU(n)$. That is, if $x \in E_2^{0,2n-1} \cong H^0BU(n)$ corresponds to 1, $d_{2n}(x) = c_n$. So every differential is multiplication by c_n.

The spectral sequence shows that $H^*BU(n) \to H^*BU(n-1)$ is an isomorphism in degrees up to $2n-2$. Abusing notation, let $c_1, \ldots, c_{n-1}$ be the unique elements of $H^*BU(n)$ that restrict to the generators $c_1, \ldots, c_{n-1}$ of $H^*BU(n-1)$. Since the ring $H^*BU(n-1)$ is generated by $c_1, \ldots, c_{n-1}$, it follows that $H^*BU(n) \to H^*BU(n-1)$ is surjective in all degrees. Therefore, the differential d_{2n} from row $2n$ to row 0 must be *injective*, since the only contribution to the E^∞ page must come from $H^{2n}BU(n)$. Finally,

$$H^*(BU(n-1)) = \mathbb{Z}[c_1, \ldots, c_{n-1}] = H^*BU(n)/(c_n).$$

The fact that multiplication by c_n is injective implies that $H^*BU(n) = \mathbb{Z}[c_1, \ldots, c_{n-1}, c_n]$ (any relation would descend to the zero relation in $H^*BU(n-1)$, hence must be divisible by c_n).

□

4.2 Lie groups and algebraic groups

We want similar results on $H^*(G,\mathbb{Q})$ and $H^*(BG,\mathbb{Q})$ for any compact Lie group G. We will do this by bringing in some algebraic geometry.

Theorem 4.5. *There is a complexification functor from compact Lie groups K to complex algebraic groups, $G \simeq K_{\mathbb{C}}$ (on Lie algebras, $\mathrm{Lie}(G) = \mathrm{Lie}(K) \otimes_{\mathbb{R}} \mathbb{C}$). This gives a bijection from isomorphism classes of compact connected Lie groups to isomorphism classes of complex reductive groups.*

Example 4.1.

- $(S^1)_{\mathbb{C}} \cong \mathbb{G}_m$, the multiplicative group. Here $\mathbb{G}_m(\mathbb{C}) \cong \mathbb{C}^\times$.
- $U(n)_{\mathbb{C}} \cong \mathrm{GL}(n,\mathbb{C})$. Note that $\mathrm{Lie}\, U(n) = \{A \in \mathfrak{gl}_n \colon A^* = -A\}$. Any matrix can be written as as a Hermitian plus skew-Hermitian matrix, which we can write as $A = B + iC$ where both B and C are skew-Hermitian.
- $SU(n)_{\mathbb{C}} \cong \mathrm{SL}(n,\mathbb{C})$.
- $O(n)_{\mathbb{C}} \cong O(n,\mathbb{C})$.
- $\mathrm{Sp}(n)_{\mathbb{C}} \cong \mathrm{Sp}(2n,\mathbb{C})$.

Theorem 4.6. *For a compact Lie group K, the inclusion $K \to K_{\mathbb{C}}$ is a homotopy equivalence.*

Proof. G/K is a symmetric space of noncompact type, so it is a simply connected complete Riemannian manifold with nonpositive curvature. That shows that G/K is diffeomorphic to $\mathbb{R}^n$, and we have a fiber bundle $K \to G \to G/K$. □

Aside: if G is a real Lie group, then G has a unique maximal compact subgroup K up to conjugation, and $K \hookrightarrow G$ is a homotopy equivalence. In fact, the quotient is diffeomorphic to $\mathbb{R}^n$. For example, $\mathrm{SL}(2,\mathbb{R}) \simeq SO(2) \cong S^1$.

Definition 4.7. Among compact Lie groups, a <u>torus</u> T is a group isomorphic to $(S^1)^n$ for some $n \geq 0$. Among complex reductive groups, a <u>torus</u> T is a group isomorphic to $(\mathbb{G}_m)^n$ for some $n \geq 0$.

For any compact Lie group K, K acts on its flag manifold K/T, where T is a maximal torus in K. (Any two maximal tori are conjugate.) In fact, K/T can be viewed as a smooth complex projective variety, by the following theorem.

Definition 4.8. A *Borel subgroup* B in a complex reductive group G is a maximal connected solvable complex algebraic subgroup of G.

Theorem 4.9. *There is an isomorphism*

$$K/T \cong G/B.$$

Again, a Borel subgroup is unique up to conjugation.

Example 4.2. A Borel subgroup in $\mathrm{GL}(n, \mathbb{C})$ is the subgroup of upper-triangular matrices.

$$B = \left\{ \begin{pmatrix} * & \dots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & * \end{pmatrix} \right\} \subset \mathrm{GL}(n, \mathbb{C}).$$

If we take $K = U(n)$, then

$$T = \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & * \end{pmatrix} \right\} \subset B.$$

and $K_{\mathbb{C}} = \mathrm{GL}(n, \mathbb{C})$. Note that $T_{\mathbb{C}} \cong \mathbb{G}_m^n$. Then

$$\mathrm{GL}(n, \mathbb{C})/B = \{0 \subset V_1 \subset V_2 \subset \ldots \subset V_n = \mathbb{C}^n \colon \dim V_j = j\}.$$

This is called a flag variety. In particular, we see that

- $\mathrm{GL}(2)/B \cong \mathbb{P}^1_C$.
- $\mathrm{GL}(3)/B$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$. Indeed, there is a $\mathbb{P}^2$ of planes in $\mathbb{C}^3$, and a $\mathbb{P}^1$ of lines in each plane.

Remark 4.10. The theorem gives another proof that $K \simeq K_{\mathbb{C}} = G$. Indeed, if we can show that $T \simeq B$ is a homotopy equivalence, then we can show that $K \simeq G$: comparing the fibrations

$$\begin{tikzcd} T \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & K/T \arrow[d] \\ B_{\mathbb{C}} \arrow[r] & G_{\mathbb{C}} \arrow[r] & G_{\mathbb{C}}/B_{\mathbb{C}} \end{tikzcd}$$

and using the long exact sequence on π_*, it suffices to show that $T \to B$ is a homotopy equivalence. The point is that (as we see in the case of $\mathrm{GL}(n, \mathbb{C})$) a connected solvable linear algebraic group B has a normal subgroup U (the "unipotent radical") which is an extension of copies of $\mathbb{G}_a$, and is therefore contractible. Moreover, B is the semidirect product of the torus $T_{\mathbb{C}}$ and U, so $B/U \cong T_{\mathbb{C}} \cong (\mathbb{C}^*)^n \simeq (S^1)^n$.

Example 4.3. Let G be a connected real Lie group, $K \subset G$ a maximal compact subgroup. Then G/K is diffeomorphic to $\mathbb{R}^N$, and has a natural Riemannian metric with nonpositive curvature.

- $\mathrm{SL}(2,\mathbb{R})/SO(2) \cong$ the hyperbolic plane.
- $\mathrm{SL}(2,\mathbb{C})/SU(2) \cong$ hyperbolic 3-space.

4.3 The Weyl Group

Definition 4.11. Let K be a compact connected Lie group. Let T be a maximal torus in K. Then the *Weyl group* of K is $N_K(T)/T$.

Fact: W is a finite group, and the action of W on $\mathrm{Lie}(T) \cong \mathbb{R}^n$ is generated by reflections.

Definition 4.12. The *rank* of K is $\dim_{\mathbb{R}} T_{\mathbb{R}}$.

Example. For $G = \mathrm{SL}(2)$ (or $SU(2)$), we have $\operatorname{rank} G = 1$ and $W = \mathbb{Z}/2$.

Example. For the rank two semisimple groups, we have:

- $G = \mathrm{SL}(3,\mathbb{C})$, $W = S_3$.
- $G = \mathrm{Sp}(4,\mathbb{C})$, $W = D_8$ (dihedral group of order 8).
- $G = G_2$, $W = D_{12}$.

Lemma 4.13. *Let G be a compact Lie group with maximal torus T. Then the image of $H^*(BG,\mathbb{Z})$ in $H^*(BT,\mathbb{Z})$ is contained in $H^*(BT,\mathbb{Z})^W$.*

Proof. By the inclusions of groups $T \subset N(T) \subset G$, we have homomorphisms $H^*BG \to H^*BN(T) \to H^*BT$. We claim that the image of $H^*BN(T) \to H^*BT$ is already contained in the W-invariants. The reason is that W acts trivially on $H^*BN(T)$. Indeed, the map $BT \to BN(T)$ is a finite covering with fibers W (recall the fibration $G/H \to BH \to BG$), so the pullback of a cohomology class from the base is necessarily W-invariant. □

Example 4.4. For $G = U(n)$, $H^*(BU(n),\mathbb{Z}) = \mathbb{Z}[c_1,\dots,c_n]$ where $|c_i| = 2i$, and

$$H^*(BT,\mathbb{Z}) = \mathbb{Z}[t_1,\dots,t_n], |t_i| = 2.$$

Here $W = S_n$. We claim that there is a canonical identification $H^2(BT,\mathbb{R}) \cong \mathrm{Lie}(T)^*$. Indeed, any $\mathbb{R}$-linear map $\mathrm{Lie}(T) \to \mathbb{R}$ may be recognized as $d\alpha$, for some representation $\alpha\colon T \to S^1$. This in turns gives a map $BT \to BS^1$, hence a line bundle on BT, which we associate with its Chern class in $H^2(BT,\mathbb{R})$.

The upshot is that W acts on $H^2(BT, \mathbb{R})$ as a finite group generated by reflections (dual to its action on Lie(T)). In this case, $H^*(BU(n), \mathbb{Z}) \cong H^*(BT, \mathbb{Z})^W \cong \mathbb{Z}[e_1, \ldots, e_n]$, the symmetric polynomials.

To see this in action, we ask: for $\mathbb{C}$-line bundles on a space X, what is $c(L_1 \oplus \ldots \oplus L_n)$? Writing $t_i = c(L_i) \in H^2(X)$, then the Whitney sum formula says that

$$\begin{aligned} c(L_1 \oplus \ldots \oplus L_n) &= c(L_1)c(L_2)\ldots c(L_n) \\ &= (1+t_1) + \ldots (1+t_n) \\ &= \sum_{j=0}^{n} \sum_{1 \le i_1 \le \ldots \le i_j \le n} t_{i_1} \ldots t_{i_j} \end{aligned}$$

so the Chern classes map to the elementary symmetric functions. This is as expected, since we proved that the image should be in the W-invariants. We see that in this example, the map in fact goes *isomorphically* to the W-invariants. We want to explain how this generalizes to all compact Lie groups, at least for rational cohomology.

Theorem 4.14 (Bruhat decomposition)**.** *Let G be a reductive group over a field k. Let $B \subset G$ be a Borel subgroup (which exists if k is algebraically closed, for example). Then the orbits of B on the flag manifold G/B are in 1-1 correspondence with W, via*

$$W = T \backslash N(T)/T \to B \backslash G/B$$

Moreover, the orbits of B on G/B are all cells, meaning they are affine spaces $\mathbb{A}^j$ for some j.

Example 4.5. Take $G = \mathrm{SL}(2, \mathbb{C})$. Then $G/B \cong \mathbb{P}^1$ (we can take B to be the group of upper-triangular matrices), and the B-orbits on G/B are the point ∞ and the affine line $\mathbb{A}^1$.

Corollary 4.15. *$H^*(G/B, \mathbb{Z})$ is a free abelian group, concentrated in even degrees. Also, $\chi(G/B) = |W|$.*

The Weyl group W acts on K/T by $kT \mapsto kwT$ for any $w \in N(T)$, inducing a representation on the cohomology groups. Moreover, this representation has dimension $|W|$.

Remark 4.16. However, this does not preserve the complex structure on K/T. Example: $SU(2)/S^1 \cong S^2$, and W acts on S^2 by $x \mapsto -x$, which is not complex-analytic. (One way to see this is that it reverses orientation; complex-analytic maps always preserve orientation).

Lemma 4.17. *The action of W on $H^*(K/T, \mathbb{Q})$ is by the regular representation of the Weyl group.*

Proof. Use the Lefschetz fixed point formula: for $w \in W$,

$$\sum_i (-1)^i \operatorname{tr}(w \mid_{H^i(K/T),\mathbb{Q}}) = \#\{\text{fixed points of } w \text{ on } K/T\}.$$

The fixed points on the right have to counted with suitable multiplicities. But in this case, W acts freely on K/T: it is the group of deck transformations for the covering map $K/T \to K/N(T)$. So the character of W acting on $H^*(K/T, \mathbb{Q})$ is

$$\chi(w) = \begin{cases} 0 & w \neq 1 \\ \chi(K/T) = |W| & w = 1 \end{cases}.$$

This is the character of the regular representation. □

Lemma 4.18. *The map $H^*(BK, \mathbb{Q}) \to H^*(BT, \mathbb{Q})$ is injective. In particular, $H^*(BG, \mathbb{Q})$ is concentrated in even degrees.*

Proof. The second assertion is deduced from the first by observing that $T \simeq (S^1)^n$, so $BT = (BS^1)^n = (\mathbb{CP}^\infty)^n$, whose cohomology we know to be concentrated in even degrees.

Let G be the complexification of K. Let $B \subset G$ be a Borel subgroup. We will show that $H^*(BG, \mathbb{Q}) \to H^*(BB, \mathbb{Q})$ is injective, which implies the same for $H^*(BK, \mathbb{Q}) \to H^*(BT, \mathbb{Q})$ since

$$\begin{tikzcd} K \arrow[r] \arrow[d] & T \arrow[d] \\ B \arrow[r] & G \end{tikzcd}$$

is a homotopy equivalence. We have a fiber bundle

$$G/B \to BB \xrightarrow{f} BG.$$

Since f is a fiber bundle whose fibers are closed, oriented real manifolds (in fact, complex manifolds), there is a pushfoward map or "Gysin" map

$$f_* \colon H^i(BB) \to H^{i-2N} BG,$$

where $N = \dim_{\mathbb{C}} G/B$. Geometrically, if we think of H^i classes as being represented by codimension-i submanifolds, the pushforward just takes the image of a manifold. This generically reduces the codimension by the dimension of the fiber. We would be done if we could show that there was a section $BG \to BB$.

There is a vector bundle on BB called T_f, the tangent bundle along the fibers (sometimes called the vertical tangent bundle). This has complex rank N. Then $c_N(T_f) \in H^{2N}(BB, \mathbb{Z})$. We claim that this is "acts like a multisection." For one thing, let's push it down and see what we get in $H^*(BG)$. Indeed, $f_*(c_N(T_f)) \in H^0(BG, \mathbb{Z}) = \mathbb{Z}$. We can figure out what this integer is by restricting to any point in BG. The integer is

$$\int_{G/B} c_N(TG/B) = \chi(G/B) = |W|.$$

The important point is that it is nonzero. Then, for any $y \in H^i(BG, \mathbb{Z})$,

$$f_*(\underbrace{f^*(y) \cdot c_N(T_f)}_{\in H^{i+2N}BB}) \in H^i BG.$$

But $f_*(f^*(y) \cdot c_N(T_f)) = y \cdot f_*(c_N T_f) = y|W|$ by the projection formula. So if $y \neq 0 \in H^*(BG, \mathbb{Q})$ then $f^*(y) \neq 0 \in H^*(BB, \mathbb{Q})$. □

Lemma 4.19. *If $f : X \to Y$ is a Galois covering space (i.e. G acts freely on X, and Y is the quotient X/G), then*

$$H^*(Y, \mathbb{Q}) \cong H^*(X, \mathbb{Q})^G.$$

Proof. The slogan is that there are enough maps in both directions. It is obvious that f^* maps H^*Y into $(H^*X)^G$, since $f \circ g = f$ for all $g \in G$. Since the fibers are oriented, zero-dimensional manifolds, we also have a pushforward (or "transfer homomorphism")

$$f_* : H^*(X, \mathbb{Z}) \to H^*(Y, \mathbb{Z}).$$

Then the projection formula says that

$$f_*(f^*(\alpha)) = f_*(f^*(\alpha) \cdot 1) = \alpha \cdot f_*(1)$$

for $\alpha \in H^*Y$. But $f_*(1) = |G|$, since 1 corresponds to the entire space X, so its pushforward is the entire space Y with multiplicity $|G|$. So f^* is injective on $\mathbb{Q}$-cohomology.

Similarly, $f^*(f_*\beta) = \sum_{g \in G} g(\beta)$. So if β is G-invariant, this is multiplication by $|G|$. That implies that f^* is surjective onto the G-invariants in $\mathbb{Q}$-cohomology. □

Theorem 4.20. *Let K be a compact, connected Lie group. Then $H^*(BK, \mathbb{Q}) \cong H^*(BT, \mathbb{Q})^W$ where T is a maximal torus of K.*

Proof. There's a fibration

$$W \to K/T \to K/N(T)$$

so the preceding Lemma implies that

$$H^*(K/N(T), \mathbb{Q}) \cong H^*(K/T, \mathbb{Q})^W.$$

Since W acts on $H^*(K/T, \mathbb{Q})$ as the regular representation, the space of invariants is just one-dimensional. We deduce that $K/N(T)$ has the $\mathbb{Q}$-cohomology of a point. (Example: for $K = SU(2)$, $K/T \cong S^2$ and $K/N(T) \cong \mathbb{R}P^2$, which, sure enough, has the rational cohomology of a point.) Consider the fibration

$$K/N(T) \to BN(T) \to BK$$

and the associated spectral sequence for $\mathbb{Q}$-cohomology (since K is connected, BK is simply connected, and we can safely apply the Serre spectral sequence) is zero on the E^2 page except in the bottom row (since the fiber has the $\mathbb{Q}$ cohomology of a point).

So we conclude that

$$H^*(BK, \mathbb{Q}) \cong H^*(BN(T), \mathbb{Q}).$$

We have another fibration

$$W \to BT \to BN(T)$$

so by the Lemma again,

$$H^*(BN(T), \mathbb{Q}) \cong H^*(BT, \mathbb{Q})^W.$$

□

Remark 4.21. We have computed $H^*(BT, \mathbb{Q})$ to be a polynomial ring on generators of degree 2. Chevalley showed that for a finite group generated by reflections on a vector space V over a field of characteristic zero, $k[V]^W$ is itself a graded polynomial ring with number of generators equal to $\dim_k V$.

For all simple compact Lie groups, we know the degrees of the generators for the rings of invariants of the Weyl group acting on the Lie algebra of the maximal torus (cf. Bourbaki, Lie groups and Lie algebras, Ch. 4-6).

- $SU(n+1) : 2, 3, \ldots, n+1$,
- $SO(2n+1) : 2, 4, 6, \ldots, 2n$,
- $Sp(n) : 2, 4, 6, \ldots, 2n$,
- $SO(2n) : 2, 4, 6, \ldots, 2n-2; n$,
- $G_2 : 2, 6$,
- $F_4 : 2, 6, 8, 12$,
- $E_6 : 2, 5, 6, 7, 8, 12$,

- $E_7 : 2, 6, 8, 10, 12, 14, 18,$
- $E_8 : 2, 8, 12, 14, 18, 20, 24, 30.$

Exercise 4.1. Compute $H^*(G/B, \mathbb{Q})$ for all compact Lie groups G of rank at most 2. Can you say how W acts on $H^*(K/T, \mathbb{Q})$?

Question. For every simple compact Lie group, the list (above) of the fundamental degrees is symmetric. That is, the list is unchanged if you turn it upside down. Is there any geometric explanation for this?

Finally, we can use our preceding discussion to illuminate Hopf's theorem, stated at the beginning of this chapter. If a compact Lie group K has fundamental degrees $a_1, \ldots, a_r$, where $r = \mathrm{rank}(G)$, then

$$H^*(BK, \mathbb{Q}) \cong \mathbb{Q}[y_{2a_1}, \ldots, y_{2a_r}]$$

so $(BK)_{\mathbb{Q}} \simeq K(\mathbb{Q}, 2a_1) \times \ldots \times K(\mathbb{Q}, 2a_r)$ and

$$K_{\mathbb{Q}} \simeq K(\mathbb{Q}, 2a_1 - 1) \times \ldots \times K(\mathbb{Q}, 2a_r - 1) \simeq S_{\mathbb{Q}}^{2a_1-1} \times \ldots \times S_{\mathbb{Q}}^{2a_r-1}.$$

Chapter 5

Faithfully flat descent

In this chapter, we will explain Grothendieck's theory of faithfully flat descent.

5.1 Faithful flatness

Definition 5.1. A homomorphism $f\colon A \to B$ of commutative rings is *faithfully flat* if an A-linear map $M_1 \to M_2$ is injective if and only if $M_1 \otimes_A B \to M_2 \otimes_A B$ is injective.

Remark 5.2. This definition only depends on B as an A-module, but it seems only to be interesting when B is a ring.

Example 5.1. $\mathbb{Z}[\frac{1}{2}]$ is a flat $\mathbb{Z}$-algebra (localization is exact), but not faithfully flat because there are $\mathbb{Z}$-modules that get killed by tensoring with $\mathbb{Z}[\frac{1}{2}]$, e.g.

$$\mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}] = 0.$$

So a faithfully flat ring homomorphism $A \to B$ is one where B contains "enough information" about A. In the above example, $\operatorname{Spec} \mathbb{Z}[\frac{1}{2}]$ is missing information above the point $(2) \in \operatorname{Spec} \mathbb{Z}$, so there's a module supported at (2) that it does not see.

Lemma 5.3. *Let B be a flat A-algebra. The following are equivalent:*

1. *B is faithfully flat over A,*

2. *$M \otimes_A B = 0 \implies M = 0$ for all A-modules M.*

3. *The homomorphism $M \to M \otimes_A B$ given by $m \mapsto m \otimes 1$ is injective for all A-modules M.*

Proof. (1) $\implies$ (2). The map $M \to 0$ is injective after tensoring with B, hence was already injective before tensoring, so $M = 0$.

(2) $\implies$ (1). We already assumed that B was flat, so we must show that if $M_1 \to M_2$ is an A-module homomorphism such that

$$M_1 \otimes B \to M_2 \otimes B$$

is injective, then the original was injective. (All tensor products are over A.) Let $K = \ker(M_1 \to M_2)$. Then

$$0 \to K \to M_1 \to M_2$$

is exact, so

$$0 \to K \otimes B \to M_1 \otimes B \to M_2 \otimes B$$

is exact. This exhibits $K \otimes B$ as the kernel of $M_1 \otimes B \to M_2 \otimes B$, so it is zero. Then $K = 0$ by (2).

Property (3) is a bit more subtle. It is clear that (3) $\implies$ (2). Now we argue that (1) or (2) implies (3). Here comes the key trick: since B is faithfully flat, it suffices to show that

$$\begin{gathered} M \otimes_A B \to M \otimes_A B \otimes_A B \\ m \otimes b \mapsto m \otimes 1 \otimes b \end{gathered}$$

is injective. Indeed, this map is even split! The trick is to use the multiplication on B: the map back is defined by

$$m \otimes b \otimes c \mapsto m \otimes bc.$$

The composition is obviously the identity. $\square$

Example 5.2. If B is a nonzero algebra over A which is free as an A-module, then B is faithfully flat over A. For example, if k is a field, then every module over k is free, so a k-algebra is faithfully flat over k if and only if it is nonzero.

Lemma 5.4. *Let B be a flat A-algebra. Then B is faithfully flat over A if and only if $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective.*

Example 5.3. $\mathbb{Z}[\frac{1}{2}] \times \mathbb{Z}[\frac{1}{3}]$ is a faithfully flat $\mathbb{Z}$-algebra. Indeed, it is flat over $\mathbb{Z}$ because each factor is, and the map

$$\operatorname{Spec}\left(\mathbb{Z}[\frac{1}{2}] \times \mathbb{Z}[\frac{1}{3}]\right) \to \operatorname{Spec}\mathbb{Z}[\frac{1}{2}] \coprod \operatorname{Spec}\mathbb{Z}[\frac{1}{3}] \to \operatorname{Spec}\mathbb{Z}$$

is surjective.

More generally, for an open cover $X = U_1 \cup \ldots \cup U_r$ of a scheme X, the morphism $U_1 \coprod \ldots \coprod U_r \to X$ is faithfully flat (the inclusion of an open set is a typical example of a flat morphism). Indeed, faithfully flat maps are a kind of generalization of an open cover.

We saw the following consequence of the previous lemma.

Corollary 5.5. *Let B be a faithfully flat A-algebra. Then an A-module M is zero if and only if $M \otimes_A B$ is zero.*

The moral is that a module "being zero" is *local* property in the faithfully flat "topology." If we want to check if a sheaf is zero on a scheme, we can pull it back via a faithfully flat map to check if it is zero - this is analogous to checking "locally" in our analogy of an open cover.

Lemma 5.6. *Let B be faithfully flat over A. An A-module $M_1 \to M_2$ is injective (resp. surjective, an isomorphism) if and only if $M_1 \otimes B \to M_2 \otimes_A B$ is injective (resp. surjective, an isomorphism).*

Proof. The assertion for injectivity is just by definition. What about surjectivity? If $M_1 \to M_2$ is onto, then $M_1 \otimes_A B \to M_2 \otimes_A B$ is onto because the tensor product is right-exact.

Conversely, suppose that $M_1 \to M_2$ is given such that $M_1 \otimes_A B \to M_2 \otimes_A B$ is surjective. Let $C = \operatorname{coker}(M_1 \to M_2)$. By right-exactness of the tensor product, we have an exact sequence

$$M_1 \to M_2 \to C \to 0$$

Since B is flat over A,

$$M_1 \otimes_A B \to M_2 \otimes_A B \to C \otimes_A B \to 0$$

is exact, so $C \otimes_A B = \operatorname{coker}(M_1 \otimes_A B \to M_2 \otimes_A B) = 0$. Therefore, $C = 0$.

Putting these together, we get the assertion for isomorphism. □

Lemma 5.7. *Let B be a faithfully flat A-algebra. Then an A-module M is flat if and only if $M \otimes_A B$ is flat as a B-module.*

Proof. An easy exercise, similar to the arguments above. □

Again, the moral is that injectivity, surjectivity, and flatness are *local properties* in the faithfully flat "topology."

Remark 5.8. Freeness is *not* a local property. It's not even a local property in the Zariski topology. Indeed, a vector bundle on a scheme X (suppose it is affine) is locally free, but not necessarily trivial.

Lemma 5.9. *Let B be a faithfully flat A-algebra and M an A-module. Then M is finitely generated as an A-module if and only if $M \otimes_A B$ is finitely generated as a B-module.*

Proof. One direction is trivial: if M is finitely generated over A, then obviously $M \otimes_A B$ is finitely generated over B.

The other direction is a bit subtle in that one can't automatically take the generators for $M \otimes_A B$ as generators for M over A. Nonetheless, we will be able to produce a finite generating set. So suppose that $u_1, \dots, u_r$ are generators for $M \otimes_A B$. Each u_i may be written as a *finite sum*

$$u_i = \sum_j m_{ij} \otimes b_{ij}$$

with $m_{ij} \in M$, $b_{ij} \in B$. So we may also say that $M \otimes_A B$ is generated as a B-module by finitely many *decomposable* elements $m_{ij} \otimes b_{ij}$. In fact, it will then be finitely generated by the $m_{ij} \otimes 1$.

The conclusion is that we may pick a set of generators for $M \otimes_A B$ of the special form $\{m_i \otimes 1\}_{i=1}^{s}$. Now, we claim that the m_i's generate M as an A-module. Indeed, consider the A-linear map

$$A^{\oplus s} \to M$$

that takes the generators to the elements $m_1, \dots, m_s$. Then

$$f \otimes 1_B \colon A^{\oplus s} \otimes_A B \to M \otimes_A B$$

is onto by construction. By faithful flatness, the original map was onto as well.

$\square$

A *quasicoherent sheaf* on a scheme X is a sheaf of O_X-modules that on affine open sets $\operatorname{Spec} A$ is the sheaf associated to an A-module. If A is Noetherian, then the sheaves corresponding to finitely generated modules are called coherent sheaves. So the Lemma is saying that coherence can also be checked locally in the faithfully flat topology.

5.2 Faithfully flat descent

The *Amitsur complex* is

$$0 \to M \xrightarrow{f_0} M \otimes_A B \xrightarrow{f_1} M \otimes_A B \otimes_A B$$

where f_0 is defined by $m \mapsto m \otimes 1$, and f_1 is defined by $m \otimes b \mapsto m \otimes b \otimes 1 - m \otimes 1 \otimes b$. This is evidently a complex.

Lemma 5.10. *Let B be a faithfully flat A-algebra, M an A-algebra. Then the Amitsur complex is exact.*

Remark 5.11. There is a way of extending this sequence to a long exact sequence, which is more properly called the Amitsur complex.

The point of this result is to relate modules over A and modules over B, when B is a faithfully flat A-algebra. Most naïvely, one could hope that $M \otimes_A B$ is enough to determine M, but that is not the case. There are certainly cases where M is not free but $M \otimes_A B$ is free. However, the lemma says that M is determined by $M \otimes_A B$ *and* the homomorphism f_1. We will discuss how to think about this in the next lecture.

Proof. We have seen that $M \to M \otimes_A B$ is injective. Let $N = \ker(f_1 \colon M \otimes B \to M \otimes B \otimes B)$. (All tensor products are over A.) Since B is flat over A,

$$N \otimes B = \ker(f_1 \otimes 1_B \colon M \otimes B \otimes B \to M \otimes B \otimes B \otimes B).$$

If we can show that this kernel is $M \otimes B$, then we are done. Indeed, $M \otimes B$ is certainly contained in the kernel, so we would be able to conclude that $(N/M) \otimes B = 0$, hence $N/M = 0$ since B is faithfully flat over A.

We verify this fact by hand.

$$f_1 \otimes 1_B(m \otimes b \otimes c) = m \otimes b \otimes 1 \otimes c - m \otimes 1 \otimes b \otimes c.$$

Suppose that we have an element in the kernel of the map. This is of the form $\sum_i m_i \otimes b_i \otimes c_i$. Then we would have

$$\sum_i m_i \otimes b_i \otimes 1 \otimes c_i = \sum_i m_i \otimes 1 \otimes b_i \otimes c_i \in M \otimes B \otimes B \otimes B.$$

Now, consider the map back

$$\begin{aligned} M \otimes B \otimes B \otimes B &\to M \otimes B \otimes B \\ m \otimes b_1 \otimes b_2 \otimes b_3 &\mapsto m \otimes b_1 \otimes b_2 b_3 \end{aligned}$$

Therefore,

$$\sum_i m_i \otimes b_i \otimes c_i = \sum_i m_i \otimes 1 \otimes b_i c_i$$

so we have shown that any element in $\ker(f_1 \otimes 1_B)$ can be rewritten as $\sum_i m_i \otimes 1 \otimes b_i c_i$, which means precisely that it is in the image of $M \otimes_A B \subset M \otimes_A B \otimes_A B$. Here we view M as a submodule of $M \otimes_A B$ by our map f_0, taking m to $m \otimes 1$. □

What does this mean geometrically? A faithfully flat ring homomorphism $A \to B$ corresponds to a flat surjection of affine schemes, $g \colon X \to S$. So we start with a quasicoherent sheaf $\mathscr{E}$ on S (corresponding to M), and pull it back via g to get a sheaf $\mathscr{F} := g^*\mathscr{E}$ on X (corresponding to $M \otimes_A B$). Now, $B \otimes_A B$ corresponds to $X \times_S X$, so we have a sequence of morphisms

$$X \times_S X \rightrightarrows X \to S.$$

The fiber product has two maps to X, say π_1 and π_2. We have a canonical isomorphism

$$\theta\colon \pi_1^* g^* \mathscr{E} = \pi_2^* g^* \mathscr{E}$$

simply because $g \circ \pi_1 = g \circ \pi_2$. The Lemma shows that $\mathscr{E}$ is determined by $\mathscr{F}$ together with the isomorphism θ. But how, explicitly? Suppose that S and X are affine. Then $\mathscr{E}$ is the quasicoherent sheaf such that

$$H^0(S, \mathscr{E}) = \ker(H^0(X, \mathscr{F}) \xrightarrow{\theta\pi_1^* - \pi_2^*} H^0(X \times_S X, \pi_2^* \mathscr{F})).$$

The isomorphism θ has an extra property worth pointing out. Formally, θ is an isomorphism $\mathscr{F}_x \to \mathscr{F}_y$ for any $x, y \in X$ with the same image in S. For any three points $x, y, z \in X$, we must have

$$\theta_{yz}\theta_{xy} = \theta_{xz}$$

as isomoprhisms $\mathscr{F}_x \to \mathscr{F}_z$. This property is called the *cocycle condition* on θ. This is just a set theoretic description. In more scheme theoretic language,

$$\pi_{23}^*(\theta) \circ \pi_{12}^*(\theta) = \pi_{13}^*(\theta)$$

as isomorphisms $\pi_1^* \mathscr{F} \to \pi_3^* \mathscr{F}$ on $X \times_S X \times_S X$, where the π_{ij} are the projection maps $X \times_S X \times_S X \to X \times_S X$.

Definition 5.12. A morphism of schemes is *faithfully flat* if it is flat and surjective.

Definition 5.13. Given a faithfully flat morphism of schemes $X \to S$, let $\mathscr{F}$ be a quasicoherent sheaf on X. A *descent datum* on F (with respect to $X \to S$) is an isomorphism $\theta\colon \pi_1^* F \to \pi_2^* F$ on $X \times_S X$ satisfying the cocycle condition.

Definition 5.14. A scheme X is *quasicompact* if every open cover of X has a finite subcover. Equivalently, X is a finite union of affine open subsets.

Definition 5.15. A morphism of schemes $f : X \to Y$ is *quasicompact* if the pre-image of any quasicompact set is quasicompact. Equivalently, the pre-image of any affine open subset is quasicompact.

We can now state *Grothendieck's theorem on faithfully flat descent.*

Theorem 5.16 (Grothendieck)**.** *Given $X \to S$ a quasicompact faithfully flat morphism of schemes, pulling back sheaves from S to X gives an equivalence of categories*

$$\{\textit{quasicoherent sheaves on } S\} \leftrightarrow \left\{\substack{\textit{quasicoherent sheaves on } X \\ \textit{with descent datum}}\right\}.$$

Grothendieck's theorem is proved by a straightforward extension of the arguments proving the exactness of the Amitsur complex. See Waterhouse's book for a very readable exposition.

Example 5.4. The theorem fails without quasicompactness. For instance, take $S = \operatorname{Spec} \mathbb{Z}$ and $X = \coprod_{p \text{ prime}} \operatorname{Spec} \mathbb{Z}_{(p)}$. The map $X \to S$ is an infinite disjoint union of open immersions, hence faithfully flat.

Now, consider the $\mathbb{Z}$-modules $M_1 = \mathbb{Z}, M_2 = (\mathbb{Z}/2) \oplus (\mathbb{Z}/3) \oplus (\mathbb{Z}/5) \oplus \dots$. Then one can define a map $\phi \colon g^* M_1 \to g^* M_2$ compatible with the descent data, but which doesn't come from a homomorphism of $\mathbb{Z}$-modules $M_1 \to M_2$. Indeed, on $\mathbb{Z}_{(p)}$ take the map φ to be the obvious one $\mathbb{Z}_{(p)} \to \mathbb{Z}/p$. However, this can't come from a map $\mathbb{Z} \to \mathbb{Z}/2 \oplus \mathbb{Z}/3 \oplus \dots$ because such a map would have to send $1 \mapsto (1, 1, 1, \dots)$, which is not an element of the direct sum.

Example 5.5. This generalizes the construction of a sheaf by gluing. Let $X = \bigcup U_i$. Then $\coprod U_i \to X$ is a faithfully flat morphism of schemes. So a quasicoherent sheaf on X is equivalent to quasicoherent sheaves E_i on each U_i, together with isomorphisms $\theta_{ij} \colon E_i|_{U_i \cap U_j} \cong E_j|_{U_i \cap U_j}$ satisfying the cocycle condition

$$\theta_{ik} = \theta_{jk} \circ \theta_{ij}.$$

Note that in this case, the fiber product just becomes intersections.

5.3 Descent theory and principal bundles

Let G be a group scheme over a field k. That is, G is a scheme over k with morphisms $G \times_k G \to G$ (product), $\operatorname{Spec}(k) \to G$ (identity), and $G \to G$ (inverse) of k-schemes, which satisfy the associativity, identity and inverse axioms.

Definition 5.17. An action of G on a separated scheme X of finite type over k is *free* if the morphism

$$\begin{aligned} G \times_k X &\to X \times_k X \\ (g, x) &\mapsto (x, gx) \end{aligned}$$

is an isomorphism onto a closed subscheme.

Given a free G-action on X, a scheme Y is the *quotient scheme* $Y = X/G$ if Y is of finite type over k, and we have a faithfully flat $X \to Y$ which is constant on G-orbits and such that the map $G \times_k X \to X \times_Y X$ is an isomorphism. In this case, we say that X is a *principal G-bundle* over Y ("in the faithfully flat topology").

Example 5.6. Notice that a principal G-bundle need *not* be locally trivial in the Zariski topology. Consider $G = \mathbb{Z}/2$, acting on $\mathbb{A}^1 - 0$ by $x \mapsto -x$. This action is free, and $(\mathbb{A}^1 - 0)/(\mathbb{Z}/2) \cong \mathbb{A}^1 - 0$. You can check that this is a principal G-bundle. This is not locally trivial, since $\mathbb{A}^1 - 0$ is irreducible, so it doesn't even have any disconnected open subsets.

However, principal G-bundles *are* locally trivial in the faithfully flat topology. Indeed, consider the pullback via the map $X \to Y$.

Definition 5.18. Let X be a G-scheme over k. A *G-equivariant vector bundle $\mathscr{E}$* on X is a vector bundle $\mathscr{E}$ on X together with an isomorphism $\theta\colon \pi_2^* E \to m^* E$ on $G \times_k X$ (that is, θ gives an isomorphism $E_x \cong E_{gx}$ for all $g \in G, x \in X$) which satisfies the cocycle condition:

$$\theta_{(gh,x)} = \theta_{(g,hx)} \circ \theta_{(h,x)} \colon E_x \xrightarrow{\theta_{(h,x)}} E_{hx} \xrightarrow{\theta_{(g,hx)}} E_{ghx}.$$

Theorem 5.19. *Let $X \to Y$ be a principal G-bundle. Then the pullback gives an equivalence of categories from vector bundles on Y to G-equivariant vector bundles on X.*

Proof. This follows immediately from Grothendieck's theorem on descent, using that $X \times_Y X \cong G \times_k X$. □

Corollary 5.20 (Hilbert's Theorem 90)**.** *There is an equivalence of categories between principal* $\mathrm{GL}(n)$*-bundles on a space Y and (algebraic) vector bundles of rank n on Y.*

Proof. We basically just translate our earlier topological identification of this fact into the algebraic setting.

Let's first see how to construct a vector bundle for a $\mathrm{GL}(n)$ bundle. More generally, given a principal G-bundle $X \to Y$ and any representation $\alpha\colon G \to \mathrm{GL}(V)$, we get a vector bundle on Y of rank $\dim V$. Namely, start with the trivial bundle $X \times V \to V$ and α to get a G action via

$$\alpha(g)\colon V_x \to V_{gx}.$$

By Theorem 5.19, this G-equivariant vector bundle on X is the pullback of a vector bundle on Y.

Conversely, given a vector bundle V on Y, define

$$X = \bigcup_{y \in Y} \{\text{isomorphism } \mathbb{A}^n \cong V_Y\}.$$

Since V is Zariski-locally trivial, $X|_U = U \times \mathrm{GL}(n)$.

□

This statement is more nontrivial than it seems. The point is that principal G-bundles are locally trivial in the flat topology but *not* in the Zariski topology, in general. However, this is saying that principal $GL(n)$-bundles *are* locally trivial in the Zariski topology.

Definition 5.21. For a scheme X/k and an affine group scheme G/k,

$$H^1(X, G) := \{\text{isomorphism classes of principal } G\text{-bundles over } X\}.$$

This is a pointed set, but not in general a group. If G is commutative, then it can be given the structure of an abelian group.

Corollary 5.22 (Hilbert's Theorem 90)**.** *For any field k, $H^1(k, \mathrm{GL}(n)) = 1$.*

Proof. This can be identified with the set of isomorphism classes of vector bundles of rank n over $\operatorname{Spec} k$, which is the set of vector spaces of dimension n over k up to isomorphism. Obviously, there is only one such isomorphism class. $\square$

Example 5.7. For a field k, $H^1(k, Sp(2n)_k)$ is the set of isomorphism classes of dimension $2n$ over k with a nondegenerate alternating bilinear form. But any such form is isomorphic to the standard one

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(Exercise: check this is in general.) So $H^1(k, Sp(2n)_k) \cong 1$.

Example 5.8. For a field k, $H^1(k, O(n)_k)$ is the set of isomorphism classes of vector spaces of dimension n over k with a nondegenerate quadratic form, up to isomorphism. This is not trivial in general. For instance, if k does not have characteristic 2 then the determinant map $\det\colon O(n) \to Z/2$ induces

$$H^1(k, O(n)) \to H^1(k, \mathbb{Z}/2) \cong k^*/(k^*)^2,$$

which is in fact surjective for $n \geq 1$. Indeed, $H^1(k, \mathbb{Z}/2)$ parametrizes Galois extensions of k of degree 2.

Here's an explicit description of this map. The group $H^1(k, O(n))$ parametrizes isomorphism classes of quadratic forms of dimension n over k. Given a symmetric bilinear form on V over k, we can choose a basis $e_1, \ldots, e_n$ for V. Thus we get a symmetric $n \times n$ matrix $A = \langle e_i, e_j \rangle \in M_n k$. We can define the discriminant of $(V, \langle , \rangle)$ to be $\det A \in k^*/(k^*)^2$.

This all happened because the group $O(n)$ is disconnected. Can this happen for connected groups?

Definition 5.23. An affine group scheme G/k is *special* if for any field K/k,

$$H^1(K, G_K) = 1.$$

It turns out that G is special if every principal G-bundle is locally trivial in the Zariski topology.

Example 5.9. $\mathrm{GL}(n), \mathrm{SL}(n)$ and $Sp(2n)$ are special. An $\mathrm{SL}(n)$ bundle can be considered as a $\mathrm{GL}(n)$ bundle plus a volume form, and any volume form is equivalent to a standard one, so there is no "flexibility" for $\mathrm{SL}(n)$ bundles. But, for example, disconnected groups are not special, so in particular nontrivial finite groups and $O(n)$ are not special.

Example 5.10. $PGL(2)$ is not special (even though it is connected). For example, $H^1(\mathbb{R}, PGL(2)) \neq 1$. Since $PGL(2)$ is the automorphism group of $\mathbb{P}^1$, this is saying that there are different varieties over $\mathbb{R}$ that are isomorphic to $\mathbb{P}^1$ over $\mathbb{C}$ (a smooth conic in $\mathbb{R}P^2$ need not be isomorphic to $\mathbb{R}P^1$.

Here's a general comment: we often want to classify some kind of "algebraic structures" over a field k. In particular, given an object Y over k, we want to classify the objects over k that becomes isomorphic to Y over $\overline{k}$. These are called *k-forms of Y*. The set of isomorphism classes of k-forms of Y can be identified with the set $H^1(k, G)$, where $G = \mathrm{Aut}(Y)$, viewed as an algebraic group over k.

So we can interpret $H^1(\mathbb{R}, PGL(2))$ as classifying $\mathbb{R}$-forms of any object with automorphism group $PGL(2)$. Well, $\mathbb{P}^1_k$ is such an object. More generally, $\mathrm{Aut}(\mathbb{P}^n_k) = PGL(n+1)_k$. Another example: the automorphism group of the matrix algebra $M_n(k)$ as an associative k-algebra is isomorphic to $PGL(n)$, with $PGL(n)$ acting by conjugation.

So $H^1(k, PGL(2)) = \{$isomorphism classes of smooth plane conics in $\mathbb{P}^2_k\}$. Using the second interpretation, we can also view $H^1(k, PGL(2))$ as the set of isomorphism classes of quaternion algebras over k.

Now note that $\{x^2+y^2+z^2 = 0\}$ is a conic not isomorphic to $\mathbb{P}^1_{\mathbb{R}}$, since it has no real points; so $H^1(\mathbb{R}, PGL(2))$ has at least two elements. Similarly, there is a quaternion algebra over $\mathbb{R}$ not isomorphic to $M_2(\mathbb{R})$, namely Hamilton's quaternions. In fact, the set $H^1(\mathbb{R}, PGL(2))$ has exactly two elements.

Let's flesh out this connection a bit more. Why is $H^1(k, O(V))$ equal to the set of isomorphism classes of n-dimensional vector spaces over k with a nondegenerate symmetric form? (A crucial point is that all such things are isomorphic over $\overline{k}$.) Well, if you have another such object W, then we want to construct a torsor (i.e. principal $O(V)$ bundle). To do this, associate to W the principal $O(V)$-bundle over $\operatorname{Spec} k$ that is $\mathrm{Isom}(V, W)$. (If you go back to the dictionary between vector bundles and principal bundles, you see that this is what comes out.) Now, this is a scheme over k that has no k-rational points (if W is not isomorphic to V over k). To see that this is a principal $O(V)$-bundle, we can base change to the algebraic closure, since the necessary properties are local in the faithfully flat topology.

Conversely, given a principal $O(V)$-bundle E over $\operatorname{Spec} k$, the corresponding vector space W with symmetric form is $(E \times_k V)/O(V)$. (Again, this is what comes out of the dictionary). You can construct this as a scheme using faithfully flat descent: Grothendieck's theorem tells us that an $O(V)$-equivariant vector bundle on E is the same data as a vector bundle over $E/O(V) = \operatorname{Spec}(k)$. In more detail, this is a vector space over

k of dimension n, equipped with a symmetric bilinear form (because V possesses one) that is preserved by $O(V)$.

5.4 References

- Reference: W. Waterhouse, *Introduction to affine group schemes*
- Reference: J.-P. Serre, *Galois cohomology*

Chapter 6

Chow groups and algebraic cycles

The slogan is that Chow groups are an analogue of ordinary homology in algebraic geometry.

Recall that for a scheme X, the *Picard group* $\mathrm{Pic}(X)$ is the group of isomorphism classes of line bundles on X, equivalently $H^1(X, \mathbb{G}_m)$. On the other hand, say for a variety X over a field, the *divisor class group* $\mathrm{Cl}(X)$ is the free abelian group on the set of codimension-one subvarieties of X modulo divisors of nonzero rational functions.

Theorem 6.1. *Let X be a smooth variety over a field k. Then* $\mathrm{Pic}(X) \cong \mathrm{Cl}(X)$.

The isomorphism from $\mathrm{Pic}(X)$ to $\mathrm{Cl}(X)$ is called the first Chern class, since it does what the first Chern class does in topology: to a line bundle, it associates the class of the zero set of a (transverse) section, if one exists.

To generalize this, define the *Chow group* CH_iX for any scheme of finite type X over a field k to be the free abelian group on i-dimensional subvarieties, modulo the relations: for every $(i+1)$-dimensional subvariety $W \subset X$ and every $f \in k(W)^*$, $(f) = 0$. This equivalence relation is called *rational equivalence.*

For $k = \mathbb{C}$, we have a homomorphism $CH_iX \to H_{2i}^{BM}(X, \mathbb{Z})$ where H^{BM} is the Borel-Moore homology ("homology with closed support"). This is equal to $H_{2i}(X, \mathbb{Z})$ if X is compact.

If X is smooth of dimension n over $\mathbb{C}$, Poincaré duality says that

$$H_{2i}^{BM}(X, \mathbb{Z}) \cong H^{2n-2i}(X, \mathbb{Z}).$$

Motivated by this, if X is smooth of dimension n over a field k we define

$$CH^i(X) = CH_{n-i}X.$$

This is a model for the cohomology of X. In particular, for a smooth complex scheme X, we have a homomorphism $CH^iX \to H^{2i}(X,\mathbb{Z})$.

Example 6.1. For a variety X of dimension n,

$$CH_{n-1}(X) = \mathrm{Cl}(X).$$

If X is smooth, $CH^1(X) \cong \mathrm{Pic}(X)$. Chow groups are truly algebro-geometric invariants, not just topological invariants like ordinary cohomology. For example, if X is a smooth projective curve of genus g over $\mathbb{C}$, then we have an exact sequence

$$0 \to J(X)(\mathbb{C}) \to CH^1X \to H^2(X,\mathbb{Z}) \cong \mathbb{Z} \to 0,$$

where $J(X)$ is the Jacobian of the curve (an abelian variety of dimension g).

Example 6.2. For X an elliptic curve over $\mathbb{Q}$, the Jacobian is isomorphic to X itself, and we have

$$0 \to X(\mathbb{Q}) \to CH^1X \to \mathbb{Z} \to 0.$$

The group $X(\mathbb{Q})$ is called the Mordell-Weil group of X. This example already shows that Chow groups are hard to compute, but very interesting.

If X is a smooth scheme over a field k, then CH^*X is a commutative graded ring. The product corresponds to intersection of subvarieties. This is reminiscent of the intersection product for compact, orientable manifolds (coming from the cup product plus Poincaré duality).

6.1 Homotopy invariance for Chow groups

Homotopy invariance for Chow groups is the theorem that for a scheme X and any vector bundle $E \to X$, there is an isomorphism $f^*\colon CH^*X \to CH^*E$. Think about the simple case $E = X \times \mathbb{A}^1$. The homomorphism f^* is just the inverse image.

Why is f^* surjective? Let $Z \subset X \times \mathbb{A}^1$ be a subvariety, which we assume is not contained in the zero section $X \times \{0\}$. Let $\alpha = Z \cap (X \times 0)$ as a cycle. The basic idea is to consider the family of subvarieties obtained by stretching Z by a factor of $\frac{1}{t}$ in the $\mathbb{A}^1$ direction. For $t = 1$, this is the cycle Z, while the "limit" of these cycles at $t = 0$ is $f^*\alpha$. More precisely, if the $\mathbb{A}^1$ coordinate is s then the equations of Z may be written as

$$f_0x + sf_1(x) + \ldots = 0, etc.$$

Replacing these by

$$f_0(x) + stf_1(x) + \ldots = 0, etc$$

gives a variety with the s-coordinates scaled by $\frac{1}{t}$, so anything not on α is pushed off to infinity in the limit. We can view this family of cycles as a rational equivalence from Z to $f^*(\alpha)$, thus proving the surjectivity of f^*. Injectivity of f^* can also be proved by a geometric argument.

To define the product on CH^*X, following Fulton and MacPherson, consider cycles α and β on X. Then we get a cycle $\alpha \otimes \beta$ on $X \times_k X$, and we want to define $\alpha\beta = (\alpha \otimes \beta) \cap \Delta_X$ - this is intuitively the intersection.

To do this formally, observe that associated to $\alpha \otimes \beta$ we get a "normal cone" of $\alpha \otimes \beta$ near the diagonal $X \subset X \times_k X$, which is an explicit algebraic cycle on the normal bundle $N_{X/X\times_k X}$ (viewed as a scheme). By homotopy invariance of Chow groups, this cycle determines an element of CH^*X (though not an explicit cycle on X, in general). This is the product $\alpha\beta$ in CH^*X.

Finally, for any morphism $f\colon X \to Y$ of smooth k-schemes, we have a homomorphism

$$f^*\colon CH^*Y \to CH^*X$$

which is a graded ring homomorphism. If $f\colon X \to Y$ is a proper morphism of schemes, we also have a pushforward homomorphism

$$f_*\colon CH_*X \to CH_*Y.$$

6.2 The basic exact sequence

The *basic exact sequence* for Chow groups says that if X is a scheme of finite type over a field k and $Z \subset X$ is a closed subscheme, then there is an exact sequence

$$CH_iZ \to CH_iX \to CH_i(X - Z) \to 0.$$

The maps are all instances of functoriality between Chow groups: the first map is the pushforward associated to the proper map $Z \to X$. The second comes from the general fact that if $f\colon X \to Y$ is a flat morphism of schemes, then there is a pullback homomorphism $f^*\colon CH_iY \to CH_{i+\dim(X)-\dim(Y)}X$.

For $k = \mathbb{C}$, this sequence maps to the exact sequence

$$\to H_{2i}^{BM}(Z, \mathbb{Z}) \to H_{2i}^{BM}X \to H_{2i}^{BM}(X - Z) \to H_{2i-1}^{BM}(Z) \to \ldots.$$

Note a big difference between the exact sequences for Chow groups and Borel-Moore homology: the homomorphism $CH_iX \to CH_i(X - Z)$ is always surjective. This is because for any subvariety of $X - Z$, its closure in

X is again an algebraic variety, which represents a class in the Chow group. Nothing similar is true in topology. For example, $H_1^{BM}(\mathbb{R}^2 - 0, \mathbb{Z}) \cong \mathbb{Z}$ is generated by the ray from 0 to ∞ on the x-axis, and the closure of this in $\mathbb{R}^2$ is not a cycle (it has nontrivial boundary, namely the origin in $\mathbb{R}^2$).

Example 6.3. If $U \subset \mathbb{A}^n$ is a nonempty open subscheme, then

$$CH_i U = \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq n \end{cases}.$$

This follows from the basic exact sequence plus the computation of $CH_i \mathbb{A}^n$, which you can do by homotopy invariance. This is completely different from what happens in topology: an open subset of Euclidean space (or even a Zariski open subset of $\mathbb{C}^n$) can have a lot of homology.

Example 6.4. If $X \to Y$ is a principal $\mathbb{G}_m$-bundle, then X is the total space of a line bundle L minus the 0-section. Suppose that Y and X are smooth. Then

$$CH^* X \cong CH^* Y/(c_1 L).$$

To see this, use the exact sequence plus homotopy invariance.

Compare this with $H^*(X, \mathbb{Z})$ and $H^*(Y, Z)$ for $k = \mathbb{C}$, where we have the long exact sequence for a circle bundle,

$$\ldots \to H^{i-2} Y \xrightarrow{c_1(L)} H^i Y \to H^i X \to H^{i-1} Y \to \ldots$$

6.3 Classifying spaces in algebraic geometry

Let G be an affine group scheme of finite type over a field k. Then G has a faithful representation

$$G \to GL(V)$$

for some finite dimensional k-vector space V. Why? There's one natural nontrivial action of G, which is on its own ring of functions (the regular representation). This is clearly faithful, and then one argues that there are enough finite dimensional subrepresentations in it.

To replicate what we did for classifying spaces in topology, we need to find varieties that are "nearly contractible" with a free action of G. (It is impossible to find contractible ones.) A good attempt is to go for $V - S$, where V is a representation over k and $S \subset V$ is a Zariski closed subset. If we are lucky, the quotient $(V - S)/G$ is a scheme. (If G is finite, then this is automatic: the quotient of a quasiprojective variety by a finite group is a quasiprojective variety.) Now, if S has high codimension in V, then $V - S$ is "nearly contractible" (at least in topology, its small homotopy groups will vanish). Recall that in topology, removing a set of codimension $i+2$ from a manifold does not affect π_i.

Definition 6.2. We define $CH^iBG = CH^i(V-S)/G$, for any pair (V,S) as above such that $\operatorname{codim}(S \subset V) > i$.

If you have a sequence of such varieties

$$(V_1 \to S_1)/G \to (V_2 - S_2)/G \to \dots$$

of codimension increasing to infinity, we can intuitively think "$BG = \varinjlim (V_i - S_i)/G$."

Theorem 6.3. *CH^iBG is independent of the choice of (V,S).*

Proof. Let (V,S) and (W,T) be two pairs as above, i.e. (V,S) and (W,T) are k-representations of G, S and T are closed subsets, and G acts freely on $V-S$ and $W-T$ and $\operatorname{codim}(V)$ and $\operatorname{codim}(W)$ are greater than i.

We compare both to the direct sum representation. We have a vector bundle

$$\begin{tikzcd} & (V \oplus W - S \times W)/G \arrow[dl] \\ (V-S)/G & \end{tikzcd}$$

with fiber W. (It is not immediately clear that these things actually exist as schemes. This is answered by the theory of faithfully flat descent: the space $(V \oplus W - S \times W)/G$ can be realized as the total space of a vector bundle we construct on $(V-S)/G$.) Now, homotopy invariance of Chow groups implies that

$$CH^*(V-S)/G \cong CH^*((V \oplus W) - (S \times W))/G$$

and likewise for the bundle $((V \oplus W) - (V \times T))/G \to (W-T)/G$, a vector bundle with fiber V.

Next, if X is a scheme and Z is a closed subset of codimension $> i$, then $CH^iX \to CH^i(X-Z)$ is an isomorphism (use the basic exact sequence). Using this, we get the result by comparing both $CH^*(V \oplus W - S \times W)/G$ and $CH^*(W \times V - T \times V)/G)$ to $CH^*((V-S) \times (W-T)/G)$. $\square$

Example 6.5. What is the ring $CH^*B\mathbb{G}_m$? Let $\mathbb{G}_m$ act on $\mathbb{A}^n$ by scalar multiplication. This action is free outside of the origin, and the orbits are the linear through the origin. Then $(\mathbb{A}^n - 0)/\mathbb{G}_m \cong \mathbb{P}^{n-1}$. So

$$CH^*B\mathbb{G}_m = \varprojlim CH^*\mathbb{P}^{n-1}_k = \mathbb{Z}[u]$$

where $|u| = 1$. (We are taking the inverse limit in the category of graded rings, so we can view this ring as $\mathbb{Z}[u]$ rather than the power series ring $\mathbb{Z}[[u]]$.)

Example 6.6. One way to produce suitable representations of G is to take the direct sum of many copies of a faithful representation of G. Let's apply this to compute $CH^*BGL(n)$. Let V be the standard n-dimensional representation of $\mathrm{GL}(V) = \mathrm{GL}(n)$. Let $W = V^{\bigoplus N} = \mathrm{Hom}(\mathbb{C}^N, V)$ for any integer $N > n$. Then $GL(V)$ acts freely on the open subset of surjective linear maps $\mathrm{Surj}(\mathbb{C}^N, V)$. The quotient $\mathrm{Surj}(\mathbb{C}^N, V)/\mathrm{GL}(V) \cong \mathrm{Gr}(N - n, N)$. The codimension of the complement goes to infinity, so

$$CH^*BGL(n) = \varprojlim CH^*Gr(N-n, N).$$

Again, the Chow ring of Grassmanians is the same as the cohomology ring, because $\mathrm{Gr}(N - n, N)$ has an algebraic cell decomposition, so its Chow groups are the free abelian group on the set of cells, as is the ordinary cohomology ring. The conclusion is that

$$CH^*BGL(n) \cong \mathbb{Z}[c_1, \ldots, c_n] \qquad |c_i| = i.$$

Remark 6.4. For a smooth scheme $X/\mathbb{C}$, we have a graded ring homomorphism (the "cycle map")

$$\begin{array}{ccc} CH^iX & \longrightarrow & H^{2i}(X, \mathbb{Z}) \\ \downarrow \approx & & \downarrow \approx \\ CH_{n-i}X & \longrightarrow & H^{BM}_{2n-2i}(X, \mathbb{Z}) \end{array}$$

where $\dim_{\mathbb{C}} X = n$. So if G is an affine algebraic group over $\mathbb{C}$, we have a ring homomorphism

$$CH^*BG \to H^*(BG, \mathbb{Z}).$$

Explicitly, this is the homomorphism $CH^i(V - S)/G \to H^{2i}((V - S)/G, \mathbb{Z})$ for any pair (V, S) as above such that $\operatorname{codim} S > i$.

Example 6.7. What is $CH^1B(\mathbb{Z}/2)$? Let $\mathbb{Z}/2$ act freely on $\mathbb{C}^2$ by ± 1. The action is free on $\mathbb{C}^2 - (0, 0)$ and $\mathbb{C}^2/(\mathbb{Z}/2)$ is isomorphic to $\{(a, b, c) \in \mathbb{A}^3 \colon b^2 = ac\}$ by sending $(x, y) \mapsto (x^2, xy, y^2)$. This is a singular cone in $\mathbb{A}^3$, and removing the origin removes the cone point. When calculating the first Chow group, we may as well throw the cone point back in since it has dimension 0 (see the basic exact sequence if you are uncomfortable). Now, CH_1 of this is $\mathbb{Z}/2$, generated by a line through the origin on the cone. Why is twice this line rationally equivalent to 0? A plane tangent to Y along the line intersects it in a double line. The plane is rationally equivalent to zero, since it is the zero set of a regular function (namely, a linear function on affine 3-space). Restricting to the cone, we get that twice the line is rationally equivalent to 0.

This is the typical example of how a Weil divisor can fail to correspond to a Cartier divisor. The line *cannot* be locally defined by one equation.

6.4 Examples and Computations

Example 6.8. If Q_8 is the quaternion group of order 8, what is $CH^1(BQ_8)$? Now, Q_8 has a faithful 2-dimensional complex representation, by thinking of Q_8 as a subgroup of the unit quaternions S^3. So

$$CH^1 BQ_8 = CH_1(\mathbb{A}^2_{\mathbb{C}}/Q_8).$$

What is $\mathbb{A}^2_{\mathbb{C}}/Q_8$? In this case, one can describe it explicitly as $\mathrm{Spec}\,(\mathbb{C}[x,y]^{Q_8})$, and an explicit model is

$$\mathbb{A}^2_{\mathbb{C}}/Q_8 \cong \{(x,y,z) \in \mathbb{A}^3_{\mathbb{C}} \colon z^2 = xy(x+y)\}.$$

This is called the D_4 singularity. There are only three lines through the origin on this surface, given by $z = 0$ and one of $x = 0$, $y = 0$, or $x+y = 0$. (The real picture of the D_4 singularity looks like three cones meeting at a point.) Then

$$CH^1 BQ_8 = CH_1 Y \cong (\mathbb{Z}/2)^2$$

generated by the three lines through the origin, with the relation that their sum is 0. How can we see these relations geometrically? A plane tangent to the cone alone one of the lines intersects the surface in twice the line, explaining the two-torsion. Furthermore, all three lines lie in the plane $z = 0$, and the intersection of that plane with the surface is precisely the sum of the three lines.

Remark 6.5. For any affine group scheme G, $CH^1(BG) = \mathrm{Hom}(G, \mathbb{G}_m)$. This explains the calculation that $CH^1(BQ_8) \cong (\mathbb{Z}/2)^2$.

Definition 6.6 (Edidin-Graham)**.** Let G be an affine group scheme over a field k, acting on a smooth scheme X over k. Then we can define the *equivariant Chow groups* as

$$CH^i_G X = CH^i(X \times EG/G)$$

using the approximations to EG and BG that we mentioned, as

$$CH^i(X \times (V - S)/G),$$

where G acts freely on $V - S$ and $\mathrm{codim}\, S \subset V) > i$.

Example 6.9. What is $CH^* B(\mathbb{Z}/p)$? (Say the base field is $\mathbb{C}$.) Let W be a faithful 1-dimensional representation of $\mathbb{Z}/p$ and let $V = W^{\bigoplus n}$. Then G acts freely on $V - 0$, so our approximations to $B(\mathbb{Z}/p)$ are $(\mathbb{A}^n - 0)/(\mathbb{Z}/p)$. Here we have $\mathbb{Z}/p \subset \mathbb{G}_m \subset \mathrm{GL}(V)$. We could just mod out by the whole multiplicative group, so we have a kind of "fiber bundle"

$$\mathbb{G}_m \cong \mathbb{G}_m/(\mathbb{Z}/p) \to (\mathbb{A}^n - 0)/(\mathbb{Z}/p) \to (\mathbb{A}^n - 0)/\mathbb{G}_m \cong \mathbb{P}^{n-1}_{\mathbb{C}}.$$

This is the principal $\mathbb{G}_m$-bundle that corresponds to the line bundle $O(p)$ on $\mathbb{P}^{n-1}$ (that is, p times the generator of $\mathrm{Pic}(\mathbb{P}^{n-1}) \cong \mathbb{Z}$). So $CH^*B(\mathbb{Z}/p) \cong CH^*B\mathbb{G}_m/(pc_1)$, where $CH^*B\mathbb{G}_m \cong \mathbb{Z}[c_1]$.

On the other hand, this description of $B\mathbb{Z}/p \cong K(\mathbb{Z}/p, 1)$ as an S^1-bundle over $\mathbb{CP}^\infty$ implies that $H^*(B\mathbb{Z}/p, \mathbb{Z}) = \mathbb{Z}[c_1]/(pc_1)$. To see this, one can use the Gysin exact sequence for a sphere bundle that was discussed earlier. So this is another example where

$$CH^*(B\mathbb{Z}/p) \cong H^*(B\mathbb{Z}/p, \mathbb{Z}) = \mathbb{Z}[c_1]/(pc_1).$$

Example 6.10. What is $CH^*B(\mathbb{Z}/p)^r$? Since $B(G \times H) = BG \times BH$ (use a product of the approximating spaces for BG and BH), the answer is $\mathbb{Z}[u_1, \dots, u_r]/(pu_1, \dots, pu_r)$ where $|u_i| = 1$. Here, $CH^*B(\mathbb{Z}/p)^r \to H^*(B(\mathbb{Z}/p)^r, \mathbb{Z})$ is *not* an isomorphism for $r \geq 2$.

Compare: $H^*(B(\mathbb{Z}/p)^r, \mathbb{F}_p) \cong \mathbb{F}_p\langle x_1, \dots, x_r, y_1, \dots, y_r\rangle$ where $|x_i| = 1$ and $|y_j| = 2$, so this is the tensor product of a polynomial ring and an exterior algebra. On the other hand, the Chow ring modulo p is just the polynomial ring on the generators $y_1, \dots, y_r$.

Remark 6.7. It's not known if CH^*BG is a finitely generated $\mathbb{Z}$-algebra. This is not even known for G finite, but we have some more traction in this case. The map $f\colon EG \to BG$ is a finite map, and $f_*f^*\alpha = |G|\alpha$, so $|G|\alpha = 0$ for $\alpha \in CH^*BG$ of positive degree, since $CH^{>0}(EG) = 0$.

Theorem 6.8. *Let G be an affine group scheme over k, with a faithful representation V of dimension n over k. Then*

$$CH^*(\mathrm{GL}(n)/G) \cong CH^*BG/(c_1V, \dots, c_nV).$$

Why is this useful? If you know the Chow ring of BG, then theorem determines the Chow ring of $\mathrm{GL}(n)/G$, which may be of interest. But information also goes in the other direction. The theorem reduces some questions about the Chow ring of BG to questions about one finite-dimensional variety, $\mathrm{GL}(n)/G$. For example:

Corollary 6.9. *CH^*BG is generated as a module over $\mathbb{Z}[c_1V, \dots, c_nV]$ by elements of degree $\leq n^2 - \dim G$.*

Proof. Indeed, the Chow groups of $CH^*(\mathrm{GL}(n)/G)$ are zero in degree bigger than $n^2 - \dim G = \dim \mathrm{GL}(n)/G$. □

Corollary 6.10. *CH^*BG is generated as a ring by elements of degree $\leq \max(n, n^2 - \dim(G))$.*

Example. What is $CH^*BO(n)_{\mathbb{C}}$? Use the obvious faithful representation $O(n) \subset \mathrm{GL}(n)$ and the previous theorem. Now, $\mathrm{GL}(n)$ acts on the symmetric forms $S^2(\mathbb{A}^n) \cong A^{n(n+1)/2}$, thought of as the vector space of all symmetric bilinear forms on $\mathbb{A}^n$. The stabilizer of a certain form is $O(n)$, and in fact all nondegenerate symmetric bilinear forms are $\mathrm{GL}(n)$-equivalent. So $\mathrm{GL}(n)/O(n)$ can be realized as an open subset of $\mathbb{A}^{n(n+1)/2}$. By the basic exact sequence,

$$CH^* \mathrm{GL}(n)/O(n) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & * > 0 \end{cases}.$$

So $CH^*BO(n) = \mathbb{Z}[c_1, \ldots, c_n]/(\text{relations})$.

The representation V of $O(n)$ is self-dual, and in general $c_j(V^*) = (-1)^j c_j(V)$. So the relations include $2c_{2k+1} = 0$. In fact, there are no more relations. One way to see this is that the map

$$\mathbb{Z}[c_1, \ldots, c_n]/(2c_{2k+1} = 0) \hookrightarrow H^*(BO(n), \mathbb{Z})$$

is injective, but the map also factors through $CH^*BO(n)$, so there can be no more relations.

Proof of Theorem. We have a "fibration"

$$GL(n)/G \to BG \to BGL(n)$$

and also (by looping the previous fibration)

$$GL(n) \to GL(n)/G \to BG.$$

So $GL(n)/G$ is "$\mathbb{A}^1$-homotopy equivalent" to the total space of a principal $GL(n)$-bundle on BG. Then the theorem follows from this fact: for a smooth k-variety X and a principal $\mathrm{GL}(n)$-bundle $E \to X$, let V be the corresponding vector bundle on X. Then

$$CH^*E \cong CH^*X/(c_1V, \ldots, c_nV).$$

We have seen this for $n = 1$. That implies an analogous statement for principal $(\mathbb{G}_m)^r$- bundles. Now reduce the case of $\mathrm{GL}(n)$ to the case of a maximal torus, by considering the flag bundle: if $E \to X$ is a principal $\mathrm{GL}(n)$-bundle, you can consider the associated $\mathrm{GL}(n)/B$ bundle $E/B \to X$. $CH^*(E/B)$ is a free module over CH^*X on known generators, and

$$CH^*(E/B) \cong CH^*(E/T)$$

essentially because B is homotopy equivalent to T. Finally, we relate this to CH^*E. □

6.5 References

- W. Fulton, *Intersection theory*.
- B. Totaro, "The Chow ring of a classifying space" (1999).

Chapter 7

Derived categories

7.1 Localization

Derived categories start with a basic idea from homotopy theory: take a category (e.g. topological spaces), and invert a certain set of arrows (e.g. weak homotopy equivalences). (Recall that $f\colon X \to Y$ is a weak homotopy equivalence if $f_*\colon \pi_0(X) \cong \pi_0(Y)$ and $f_*\colon \pi_i(X, x_0) \cong \pi_i(Y, f(x_0))$ for any $x_0 \in X$.) Some "inconvenient" features of topological spaces arise from the fact that there are weak homotopy equivalences which are not homotopy equivalences.

Example 7.1. Any continuous map from a sphere to the Cantor set must be constant, since the latter is totally disconnected. So the map from a discrete space on the same underlying set is a weak homotopy equivalence, but not a homotopy equivalence.

Theorem 7.1. *Inverting weak homotopy equivalences in the category of topological spaces gives a category equivalent to the homotopy category of CW-complexes.*

Now let $\mathcal{C}$ any category and S any set of arrows in $\mathcal{C}$. Then one can always construct a new category $\mathcal{C}[S^{-1}]$, called the *localization* of $\mathcal{C}$ with respect to S, equipped with a functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}[S^{-1}]$ having the property that all arrows in S become isomorphisms in $\mathcal{C}[S^{-1}]$, such that $\mathcal{F}$ is universal with respect to this property.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mathcal{F}"] \arrow[dr, "\mathcal{G}"'] & \mathcal{C}[S^{-1}] \arrow[d, "\exists!"] \\ & \mathcal{D} \end{tikzcd}$$

How do you construct this category? To define $\mathcal{C}[S^{-1}]$, take the objects to be the objects of $\mathcal{C}$. A morphism in $\mathcal{C}[S^{-1}]$ between A and B is given by a diagram

$$A \to A_2 \leftarrow A_3 \to \cdots B$$

where the backwards arrows must lie in S. So one can write a morphism in $\mathcal{C}[S^{-1}]$ as a "word"

$$f_r \dots g_2^{-1} f_2 g_1^{-1} f_1$$

where f_i are morphisms in $\mathcal{C}$ and g_i are morphisms in S. (We allow any sequence of morphisms f and g^{-1}, as long as their domains and codomains fit together as pictured above.) Two words give the same morphism in $\mathcal{C}[S^{-1}]$ if and only if you can get from one to the other by a sequence of the moves: $f_1 f_2 = f_1 \circ f_2$ and $gg^{-1} = g^{-1}g = 1$.

One issue with this is that if $\mathcal{C}$ is a "large category" (a category whose objects do not form a set), then $\mathcal{C}[S^{-1}]$ might not be locally small (meaning that the morphisms from one object to another do not form a set) since we have introduced so many new morphisms. Generally, we want to be working with locally small categories.

Let A be an abelian category, and let $C(A)$ be the category of chain complexes of objects in A

$$\dots A^n \xrightarrow{d_n} A^{n+1} \xrightarrow{d_{n+1}} A^{n+1} \to \dots$$

In homological algebra, we encounter chain complexes when trying to construct homology. Usually, the homology is important but the chain complex itself is somewhat artificial (it might depend on the choice of triangulation, for example, while the homology does not). However, the chain complexes can contain more information than the homology, and the point of derived categories is to extract more information from a chain complex.

Definition 7.2. A *quasi-isomorphism* of chain complexes $f\colon A^\bullet \to B^\bullet$ is a map of chain complexes

$$\begin{tikzcd} \dots \arrow[r] & A^n \arrow[r] \arrow[d] & A^{n+1} \arrow[r] \arrow[d] & \dots \\ \dots \arrow[r] & B^n \arrow[r] & A^{n+1} \arrow[r] & \dots \end{tikzcd}$$

such that the induced homomorphism on homology

$$f_*\colon H^i(A) \to H^i(B)$$

is an isomorphism for all i.

Definition 7.3. The *derived category* $D(A)$ of an abelian category A is the localization of $C(A)$ with respect to the quasi-isomorphisms.

Thus an object of $D(A)$ is a chain complex, but it is not so clear what the morphisms are. One easy first observation is that we have well-defined functors $H^i\colon D(A) \to A$ for all integers i, which are just the homology groups of the chain complex.

7.2 Homological algebra

How do you do actual computations in $D(A)$? You use projective or injective resolutions! For example, suppose that A has enough projectives, that is, any object is the image of some projective. (The category of modules over a ring has this property.) Then for any object M in A, we can consider a projective resolution

$$\ldots \to P_1 \to P_0 \to M \to 0$$

It's actually more useful to drop the M and consider the chain complex

$$\ldots \to P_1 \to P_0 \to 0$$

Now, this complex has nontrivial homology only at H^0, where it is M, so we have a *quasi-isomorphism*

$$\begin{tikzcd} \ldots \arrow[r] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & 0 \\ \ldots \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

So the upshot is that any object M of A is isomorphic in $D(A)$ to any projective resolution of M. More generally, if A has enough projectives, then for any bounded-above complex M in $C(A)$ (i.e. a complex such that $C_i = 0$ for $i \gg 0$), there is a bounded-above complex P of projective modules with a quasi-isomorphism to M. (Note that we are considering cochain complexes, so $M_i := M^{-i}$.)

$$\begin{tikzcd} \ldots \arrow[r] & P_2 \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & 0 \\ \ldots \arrow[r] & M_2 \arrow[r] & M_1 \arrow[r] & M_0 \arrow[r] & 0 \end{tikzcd}$$

Dually, if A has enough injectives, then any bounded below complex has a quasi-isomorphism to a complex of injectives.

Definition 7.4. The *naïve homotopy category* $K(A)$ is the category whose objects are chain complexes over A and morphisms are homotopy classes of chain maps.

Recall that a homotopy $h^\bullet$ between two chain maps $f\colon M^\bullet \to N^\bullet$ consists of morphisms $h^i\colon M^i \to N^{i-1}$ for all integers i such that $dh + hd = f - g$.

The functor $C(A) \to D(A)$ factors through the naïve homotopy category $K(A)$. The point of constructing this category is that it is easier to work with for computations.

Lemma 7.5. *If P is a bounded above complex of projectives in A, then P is* homotopically projective*, meaning that for any quasi-isomorphism $M \to N$ in $C(A)$,*

$$Hom_{K(A)}(P, M) \to Hom_{K(A)}(P, N)$$

is bijective.

Proof. This is the standard diagram chase in homological algebra. To construct an inverse, one uses the lifting property for projective modules. Details are left as an exercise. □

From the definition of the derived category, it is straightforward to check the following statement. (The point is that by the preceding lemma, any formal inverse for a quasi-isomorphism introduced in $D(A)$ can actually be inverted in $K(A)$.)

Lemma 7.6. *If P is a homotopically projective object in $C(A)$, then*

$$Hom_{K(A)}(P, M) \to Hom_{D(A)}(P, M)$$

is bijective for any $M \in C(A)$.

So to compute $\mathrm{Hom}_{D(A)}(X, Y)$ if X is bounded above and A has enough projectives, take a projective resolution P for X. Then

$$\mathrm{Hom}_{D(A)}(X, Y) = \mathrm{Hom}_{D(A)}(P, Y) = \mathrm{Hom}_{K(A)}(P, Y).$$

This is much more concrete than the general description of morphisms in $D(A)$. In particular, one crude thing that we can say is that these morphisms form just a set, if the original category was locally small.

Example 7.2. Given an object M of A, we can think of it as a complex concentrated in degree 0.

$$\cdots \to 0 \to M \to 0 \to \cdots$$

For an integer n, let $M[n]$ be M shifted left by n, that is, $M[n]^i = M^{n+i}$. So for an object M of A, $M[n]$ is a complex concentrated in degree $-n$.

Corollary 7.7. *For objects M and N of A,*

$$Hom_{D(A)}(M, N[j]) = \mathrm{Ext}^j_A(M, N).$$

Proof. Actually, one should really *define* Ext to be the derived functors of Hom, but let's compare this with the usual definition.

The usual definition of $\mathrm{Ext}^j_A(M, N)$ is that one takes a projective resolution of M, applies $\mathrm{Hom}(-, N)$, and then takes the H_j of this complex. This is exactly the same as taking Hom to $N[j]$ in $K(A)$. $\square$

In particular, since a projective resolution lives "to the left" of M, the Ext groups vanish for $j < 0$. The intuition is that morphisms in $D(A)$ only go to the left.

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0 \\
0 \arrow[r] & N \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

Remark 7.8. Here is a simple example showing that having a quasi-isomorphism of chain complexes $M \to N$ does not imply that there is a quasi-isomorphism $N \to M$. Let A be an abelian category with enough projectives. Let M be an object of A, and let $P \to M$ be a projective resolution. Then we get a quasi-isomorphism

$$\begin{tikzcd}
\dots \arrow[r] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & 0 \\
\dots \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

When is there a quasi-isomorphism $M \to P$? It turns out that this happens if and only if M is projective. Indeed, suppose we have a chain map $M \to P$ that induces an isomorphism on cohomology. That gives a morphism $M \to P_0$ that splits the surjection from P_0. So M is a summand of P_0, and hence is projective. (For example, you could let A be the category of abelian groups and $M = \mathbb{Z}/2$, which is not projective.)

Categories of interest (e.g. the category of sheaves of abelian groups on a topological space) often do not have enough projectives. But these categories *do* have enough injectives. We therefore explain how to compute in an abelian category using injective resolutions. One could say that results on injective resolutions follow from the corresponding results on projective resolutions by reversing the arrows (since the axioms for an abelian category are self-dual). For clarity, I will write out in detail how this works.

Definition 7.9. An object M of a category A is *injective* if for every injection $N_1 \to N_2$ in A, every homomorphism $N_1 \to M$ extends to a homomorphism $N_2 \to M$.

We say that a category A has *enough injectives* if every object A is a subobject of an injective object. If A has enough injectives, then every bounded-below complex M in $C(A)$ has an injective resolution, i.e. a quasi-isomorphism

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & M^0 \arrow[r] \arrow[d] & M^1 \arrow[r] \arrow[d] & \dots \\ 0 \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & \dots \end{tikzcd}$$

For any complex M and bounded-below complex N, if $N \to I$ is an injective resolution then

$$\mathrm{Hom}_{D(A)}(M, N) \cong \mathrm{Hom}_{K(A)}(M, I').$$

Why? Of course, $\mathrm{Hom}_{D(A)}(M, N) = \mathrm{Hom}_{D(A)}(M, I)$. Now, a morphism in the derived category from M to I is by definition given by a diagram

$$M \to M_1 \to \ldots \leftarrow M_{r-1} \to M_r \leftarrow I$$

where the backwards arrows are quasi-isomorphisms. Given that, we want to produce a chain map $M \to I$, up to chain homotopy. Since I is homotopically injective, for any quasi-isomorphism $M \to N$ the induced map

$$\mathrm{Hom}_{K(A)}(N, I) \to \mathrm{Hom}_{K(A)}(M, I)$$

is bijective. That allows us to "reverse the backwards arrows," up to chain homotopy.

Parallel to the discussion for projectives, we can see that

$$\mathrm{Hom}_{D(A)}(M, N[j]) \cong \mathrm{Ext}^j_A(M, N)$$

by replacing N with an injective resolution.

7.3 Triangulated categories

Definition 7.10. For a complex M in $C(A)$ and $n \in \mathbb{Z}$, $M[n]$ is the complex $M[n]^i = M^{n+i}$, with the differential on $M[n]$ equal to $(-1)^n$ times the differential on M.

Definition 7.11. Let $f: M \to N$ be a chain map. The *cone* of f is the complex $M[1] \oplus N$ as graded objects of A, with differential

$$d_{\mathrm{cone}(f)} = \begin{pmatrix} -d_M & 0 \\ f & d_N \end{pmatrix}.$$

We then have a short exact sequence of complexes:

$$0 \to N \to \mathrm{cone}(f) \to M[1] \to 0$$

which induces a long exact sequence of cohomology groups

$$\ldots \to H^i M \to H^i N \to H^i(\mathrm{cone}(f)) \to H^{i+1} M \to \ldots$$

The topological analogue: given a map $f\colon X \to Y$ of pointed topological spaces, the *homotopy cofiber* of f is constructed as follows: make f a cofibration by changing Y to something homotopy equivalent, and then the cofiber is Y/X. So you get a cofibration

$$X \to Y \to Y/X$$

giving a long exact sequence on (co)homology. You could continue this process: the cofiber of $Y \to Y/X$ is ΣX, so this continues to

$$X \to Y \to Y/X \to \Sigma X \to \Sigma Y \to \Sigma(Y/X) \to \Sigma^2 X \to \cdots .$$

Definition 7.12. A *distinguished triangle* in $D(A)$ is a sequence of maps in $D(A)$

$$X \to Y \to Z \to X[1]$$

which is isomorphic in $D(A)$ to the triangle

$$M \to N \to \mathrm{cone}(f) \to M[1]$$

for some chain map $f : M \to N$.

In the derived category $D(A)$, distinguished triangles are the analogue of *both* fiber sequences and cofiber sequences.

Example 7.3. Any short exact sequence in A

$$0 \to X \to Y \to Z \to 0$$

gives a distinguished triangle in $D(A)$

$$X \to Y \to Z \to X[1].$$

Indeed, the cone of $f\colon X \to Y$ is the complex

$$0 \to X \xrightarrow{f} Y \to 0$$

which has a natural quasi-isomorphism to Z:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & Z \arrow[r] & 0 \end{tikzcd}$$

The morphism $Z \to X[1]$ in $D(A)$ is the element of $\mathrm{Ext}^1_A(Z, X)$ corresponding to this exact sequence.

Lemma 7.13. *A distinguished triangle in $D(A)$ gives a long exact sequence*

$$\ldots \to H^i X \to H^i Y \to H^i Z \to H^{i+1} X \to \ldots$$

Lemma 7.14. *Every morphism $f\colon X \to Y$ in $D(A)$ fits into a distinguished triangle $X \to Y \to Z \to X[1]$. The object Z is well-defined up to isomorphism, but not up to unique isomorphism. It is called a cone of f.*

So any morphism $X \to Y$ in $D(A)$ gives a sequence of distinguished triangles

$$X \to Y \to Z \to X[1] \to Y[1] \to Z[1] \to \ldots$$

Notice that we can also define the fiber of $f\colon X \to Y$ to be $Z[-1]$. So we get a sequence of distinguished triangles in both directions:

$$\ldots \to X[-1] \to Y[-1] \to Z[-1] \to X \to Y \to Z \to \ldots$$

Thus, in $D(A)$, fiber sequences and cofiber sequences are the same. Such a category is called a *triangulated category*. So here $\Omega\Sigma X \cong X$ and $\Sigma\Omega X \cong X$, interpreting $\Sigma X = X[1]$ and $\Omega X = X[-1]$.

Another example of a triangulated category is the *stable homotopy category*. For example, a finite spectrum is a symbol $\Sigma^a X$ where $a \in \mathbb{Z}$ and X is a finite CW-complex with a base point. In other words, we pretend that every space is a suspension. We then define

$$\mathrm{Hom}_{\mathrm{stable}}(\Sigma^a X, \Sigma^b Y) = \varinjlim_{c \geq 0} [\Sigma^{a+c} X, \Sigma^{b+c} Y].$$

Definition 7.15. Let M be a complex in $C(A)$. Let $\tau_{\leq 0} M$ be the following complex, which has cohomology groups

$$H^i(\tau_{\leq 0} M) = \begin{cases} H^i(M) & i \leq 0 \\ 0 & i > 0 \end{cases}.$$

$$\begin{tikzcd}
\ldots \arrow[r] & M^{-2} \arrow[r] \arrow[d] & M^{-1} \arrow[r] \arrow[d] & \ker d^0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \ldots \\
\ldots \arrow[r] & M^{-2} \arrow[r] & M^{-1} \arrow[r] & M^0 \arrow[r] & M^1 \arrow[r] & \ldots
\end{tikzcd}$$

We see from this construction that there is a natural map $\tau_{\leq 0} M \to M$.

Likewise, we can define a complex $\tau_{\geq 0} M$ whose cohomology is that of M restricted to degrees ≥ 0, and we have a natural map $M \to \tau_{\geq 0} M$.

Corollary 7.16. *If $M \in D(A)$ has $H^i M = 0$ for $i \neq 0$, then M is isomorphic in $D(A)$ to the object $H^0(M)[0]$.*

Think of $D(A)$ as an analogue of the homotopy category in topology, and H^* as an analogue of homotopy groups. Then this corollary is analogous to the statement that an Eilenberg-Maclane space $K(G,n)$ is determined up to isomorphism in the homotopy category by the group G.

Proof. Think of M as a complex. The map

$$\tau_{\leq 0}M \to M$$

is a quasi-isomorphism. So M is isomorphic in $D(A)$ to the complex $N = \tau_{\leq 0}M$. Also, the map $N \to \tau_{\geq 0}N = W$ is a quasi-isomorphism. So we have that W is a complex in degree 0 only, and

$$H^0(W) \cong H^0(M),$$

so W is the complex $H^0[M][0]$. □

Lemma 7.17. *An object X in $D(A)$ with $H^j(X) = 0$ for $j \neq 0, r$ (where $r > 0$) is determined up to isomorphism by H^0X, H^rX, and an element of $\mathrm{Ext}_A^{r+1}(H^rX, H^0X)$.*

Proof. Using the truncation functors, we have a morphism in $D(A)$,

$$f\colon \tau_{\leq 0}X \to X.$$

Here $\tau_{\leq 0}X \cong (H^0X)[0]$ in $D(A)$, by the previous lemma. Let Y be a cone of f. By the long exact sequence of cohomology for a distinguished triangle, Y is isomorphic to $(H^rX)[-r]$ in $D(A)$. We have a distinguished triangle:

$$\tau_{\leq 0}X \to X \xrightarrow{f} Y \xrightarrow{g} (\tau_{\leq 0}X)[1].$$

So we have realized X as the fiber of g, where g is an element of $\mathrm{Ext}_A^{r+1}(H^rX, H^0X)$. Conversely, given any objects H^0X and H^rX in A and an element $g \in \mathrm{Ext}_A^{r+1}(H^rX, H^0X)$, we can produce an object X in $D(A)$ by taking the fiber of the corresponding map $(H^rX)[-r] \to (H^0X)[1]$. □

7.4 Sheaf cohomology

Let X be a topological space, and let A be the abelian category of sheaves of abelian groups on X. One way to define $H^*(X, \mathscr{E})$ for a sheaf $\mathscr{E}$ is to take the right derived functors of the global sections, i.e. choose an injective resolutions $\mathscr{E} \to I^\bullet$ and then set

$$H^*(X, \mathscr{E}) = H^*(0 \to H^0(X, I^0) \to H^0(X, I^1) \to \dots)$$

Let $f\colon X \to Y$ be a continuous map. For a sheaf E on X, the *direct image sheaf* f_*E is the sheaf $(f_*E)(U) = E(f^{-1}(U))$ for open subsets U of Y.

More generally, we can define the *higher direct image* functor

$$Rf_*\colon D^*(X) \to D^*(Y)$$

(where $D^*(X)$ is the category of bounded-below complexes of sheaves on X) by: given a bounded below complex E on X, let $E \to I$ be a quasi-isomorphism to a complex of injective sheaves on X, and set

$$Rf_*(E) = (0 \to f_*I^0 \to f_*I^1 \to \dots)$$

The cohomology sheaves of Rf_*E are $H^i(Rf_*E) = R^if_*E$, the sheaf associated to the presheaf $U \mapsto H^i(f^{-1}(U), E)$.

Example 7.4. If $f\colon X \to Y$ is a fibration, then the sheaves $R^if_*(\mathbb{Z}_X)$ are locally constant with fiber $H^i(F,\mathbb{Z})$, corresponding to the action of $\pi_1 Y$ on $H^i(F,\mathbb{Z})$.

Example 7.5. Consider maps $X \xrightarrow{f} Y \xrightarrow{g} Z$. Then $R(g \circ f)_* \cong Rg_* \circ Rf_*$. Consider the case $X \to Y \to *$. Then the isomorphism says that

$$H^i(X, \mathscr{E}) \cong H^i(Y, Rf_*\mathscr{E}).$$

This "encodes" the whole Leray spectral sequence

$$E_2^{ij} = H^i(Y, R^jf_*\mathscr{E}) \implies H^{i+j}(X, \mathscr{E}).$$

7.5 References

- S. Gelfand and Y. Manin, *Methods of homological algebra*
- C. Weibel, *An introduction to homological algebra*
- M. Kashiwara and P. Schapira, *Categories and sheaves*

Chapter 8

Stacks

8.1 First thoughts

Stacks come up when trying to describe quotient spaces by group actions. If G acts freely on X, then there is no problem taking the quotient as a *topological* space (and we get a principal G-bundle). What if it is not free? We want an "interesting" geometric object X/G, but this can produce "singularities." For instance, if X is a manifold, then the naive quotient space X/G is typically not a manifold. If one works with the naïve topological space X/G, there are problems; for instance, vector bundles on X/G are not equivalent to G-equivariant vector bundles on X.

If X is a scheme and G a group, then we want to produce a stack $[X/G]$ such that sheaves on $[X/G]$ are the same as G-equivariant sheaves on X.

Example 8.1. For a group G acting on a point, the stack $[\text{point}/G]$ is called *the classifying stack* BG. We want sheaves on BG to be equivalent to representations of G. Topologically, taking the quotient of a point by G is just a point, so the stack must remember additional data. A first (not very accurate) approximation is to think of the stack $[X/G]$ as the quotient space together with the information of the stabilizer groups of G at ccrresponding points of X. So the stack BG is a point together with the information of the group G.

This is related to an issue that comes up in studying moduli spaces. Informally, these are objects that parametrize algebro-geometric objects. More precisely, they represent functors of interest.

Example 8.2. Consider the functor φ_g: **Schemes** $\to$ **Sets** sending X to the set of isomorphism classes of smooth projective morphisms $f\colon Y \to X$ where the fibers of f are curves of genus g.

If you had a "moduli space of curves of genus g," M_g, together with a universal family of genus g curves over M_g, then the functor would be represented by M_g, i.e.

$$\mathrm{Hom}_{\mathrm{Sch}}(X, M_g) = \varphi_g(X).$$

This is not true as stated: there is no "fine moduli space of curves of genus g" (which is a scheme). The problem is that some curves have nontrivial automorphisms.

Example 8.3. If $G = \mathbb{Z}/2$ acts on a curve D of genus g, then we can produce a nontrivial family of curves over X where all the fibers are isomorphic to D. Indeed, let $E \to X$ be any principal $\mathbb{Z}/2$-bundle of schemes. Then consider the family of curves over X, $(E \times D)/(\mathbb{Z}/2)$. It has fibers all isomorphic to D, but it is typically not a trivial bundle. However, the corresponding morphism to M_g would have to be constant (since all the fibers are isomorphic). This illustrates why the presence of automorphisms is problematic for moduli spaces.

Recall that a groupoid is a category in which all morphisms are isomorphisms. Roughly, a stack will give a contravariant functor **Schemes** $\to$ **Groupoids** generalizing the way a scheme X gives a contravariant functor **Schemes** $\to$ **Sets** by $Y \mapsto \mathrm{Hom}(Y, X)$.

Example 8.4. The stack BG for a group scheme G/k is the functor taking a scheme X to the groupoid of principal G-bundles over X. This is not literally a contravariant functor from schemes to groupoids, because the composition of pullbacks is not the pullback of the composition (they are naturally isomorphic). We will therefore emphasize a slightly different framework, that of *fibered categories*.

8.2 Fibered categories

Let $\mathcal{C}$ be a category. A category F over C is a category F with a functor ρ_F to C. Write $\xi \mapsto U$ to mean that ξ is an object of F, and it maps to X under ρ_F. (Think of C as the category of schemes, and F as the category of principal G-bundles over schemes.)

Definition 8.1. Let F be a category over C. An arrow $\varphi\colon \xi \to \eta$ in F is *cartesian* if for any arrow $\zeta \to \eta$ in F and any arrow $Z \to U$ that makes the the diagram in C commute,

$$\begin{tikzcd}
\zeta \arrow[d] \arrow[dr, dotted, "\exists!"'] \arrow[drr] & & \\
Z \arrow[dr] \arrow[drr] & \xi \arrow[r, "\varphi"] \arrow[d] & \eta \arrow[d] \\
& U \arrow[r] & V
\end{tikzcd}$$

there is a unique arrow $\zeta \to \xi$ in F which maps to $Z \to U$ in C and which makes the triangle in F commute.

Example 8.5. Let C be the category of schemes and F the category of vector bundles on schemes. Morphisms in F are maps of vector bundles over a morphism of schemes. Equivalently, for vector bundles ξ over U and η over V, we can view a morphism $\xi \to \eta$ in F as a morphism $f \colon U \to V$ of schemes together with a homomorphism $\xi \to f^*(\eta)$ of vector bundles over U.

$$\begin{tikzcd}
\xi \arrow[r] \arrow[d] & \eta \arrow[d] \\
U \arrow[r] & V.
\end{tikzcd}$$

An arrow $\xi \to \eta$ in F is cartesian if and only if the corresponding homomorphism $\xi \to f^*\eta$ of vector bundles on U is an isomorphism. In other words, ξ is the pullback of η by the morphism $U \to V$ associated to $\xi \to \eta$.

You can define a scheme X to be a functor $R \mapsto X(R)$, from commutative rings to sets, satisfying some conditions. Equivalently, one could define the category of affine schemes to be $\mathbf{CommRing}^{\mathrm{op}}$, and view a scheme as a contravariant functor from affine schemes to sets.

Our construction of stacks will be analogous to the second approach. Roughly speaking, stacks will be contravariant functors from schemes to groupoids.

Lemma 8.2. *Let F be a category over a category C. For any morphism $U \to V$ in C and any object η of F over V, a cartesian arrow $\xi \to \eta$ over $U \to V$ is unique up to unique isomorphism, if it exists.*

$$\begin{tikzcd}
\xi \arrow[r, dotted] \arrow[d, dotted] & \eta \arrow[d] \\
U \arrow[r] & V
\end{tikzcd}$$

Proof. Suppose that we have two different arrows $\xi_1 \to \eta$ and $\xi_2 \to \eta$ over $U \to V$. Regard ξ_1, ξ_2 over the triangle

$$\begin{tikzcd}
\xi_1 \arrow[d] \arrow[dr, dotted, "\exists!"'] \arrow[drr] & & \\
U \arrow[dr, "="'] \arrow[drr] & \xi_2 \arrow[r, "\varphi"] \arrow[d] & \eta \arrow[d] \\
& U \arrow[r] & V
\end{tikzcd}$$

we get a *unique* map $\xi_1 \to \xi_2$. Similar, we get a unique morphism $\xi_2 \to \xi_1$. By the usual argument, uniqueness implies that both compositions must be the identity.

□

Definition 8.3. Given a morphism $U \to V$ and an object η in F, ζ is a *pullback* of η over $U \to V$ if $\zeta \to \eta$ is a cartesian arrow over $U \to V$.

Definition 8.4. A *fibered category* F over a category C is a category F over C such that for any morphism $U \to V$ in C and any object η over V, a pullback of η over U exists.

Example 8.6. Our prototypical example is $C =$ category of schemes, $F =$ category of vector bundles over schemes. More generally, we could replace schemes by topological spaces.

Example 8.7. Another example is $C =$ category of schemes/topological spaces, G a given k-group scheme/topological group, and F the category of principal G-bundles over schemes/topological spaces.

Now let's try to relate fibered categories F over C to the rough idea of a contravariant functor from C to the category of categories.

Definition 8.5. Let F be a fibered category over C. For an object U of C, we define the *fiber* of F over U, denoted $F(U)$, to be the subcategory of F consisting of objects over U, with morphisms being morphisms of F over 1_U.

Definition 8.6. A *cleavage* of a fibered category F over C is a class K of cartesian morphisms in F such that for each morphism $U \to V$ in C and each object η in $F(V)$, there is a unique morphism in K with target η lying over $U \to V$.

The point is basically to pick a cartesian morphisms from each equivalence class.The axiom of choice says that we can choose a cleavage for any fibered category. Given a cleavage of F, a morphism $U \to V$ in C determines a functor

$$f^* \colon F(V) \to F(U).$$

Indeed, send η to the (unique) element of the cleavage that is the pullback of η over $U \to V$. For a composition, we have

$$(g \circ f)^* \cong f^* \circ g^*.$$

Summary: roughly, a fibered category F over C gives a contravariant functor $U \mapsto F(U)$.

Definition 8.7. A *category fibered in groupoids* (CFG) is a fibered category F over a category C such that for any object U of C, $F(U)$ is a groupoid.

Example 8.8. Let C be the category of schemes/topological spaces, F the category of principal G-bundles over k-schemes, is a CFG. Indeed, the category of principal G-bundles over a base is a groupoid.

Example 8.9. Let C be the category of schemes, F the category of vector bundles over schemes, with morphism being any linear map $\xi \to f^*(\eta)$. This category is not fibered in groupoids, but if we define F_2 instead to be the category of vector bundles whose morphisms are isomorphisms $\xi \to f^*(\eta)$, then this is a CFG.

Example 8.10. A special case of a fibered category is a *category fibered in sets* over C. A category fibered in sets over C is equivalent to a contravariant functor $C \to \mathbf{Set}$. Given such a functor α, define an object of F to be a pair (U, x) where U is an object of C and $x \in \alpha(U)$. A morphism $(U, x) \to (V, y)$ is a morphism $U \to V$ such that $x = \alpha(f)(y) \in \alpha(U)$.

8.3 Grothendieck Topologies

When trying to define cohomology theories in algebraic geometry, one is confronted with the problem that the Zariski topology is just too coarse. (For instance, non-empty open subsets are dense in any irreducible scheme.) When trying to construct the ℓ-adic cohomology, Grothendieck realized that we don't really *need* to use the classical apparatus of open sets, and that it better to work with the notion of open covering.

Definition 8.8. Let C be a category. A *Grothendieck topology* on C is a collection of sets of morphisms $\{U_i \to U\}$ called *coverings*, such that

1. for any isomorphism $U \to V$, $\{U \to V\}$ is a covering of V.
2. If $\{U_i \to U\}$ is a covering of U and $V \to U$ is a morphism in C, then the fiber products $V \times_U U_i$ exist in C and $\{V \times_U U_i \to V\}$ is a covering of V.
3. If $\{U_i \to U\}$ is a covering, and for each i we have a covering $\{V_{ij} \to U_i\}$, then $\{V_{ij} \to U\}$ is a covering.

Example 8.11. Let X be a topological space, C be the category of open subsets of X (morphisms are inclusions). Define a covering $\{U_i \to U\}$ to be a collection of open sets whose union is U. This is a Grothendieck topology, and the traditional notion of covering.

Example 8.12. The fpqc (flat) topology on the category of schemes over S: a collection of arrows $\{U_i \to U\}$ is a covering if the morphism $\coprod U_i \to S$ is faithfully flat, and every affine subset of U is the image of some quasi-compact open subset. For example, any Zariski open covering is an fpqc covering.

Definition 8.9. A *site* is a category C with a Grothendieck topology.

For a site C, a *sheaf* F is a contravariant functor $C \to \mathbf{Set}$ such that

1. (Identity axiom) For every covering $\{U_i \to U\}$ and two sections $a, b \in F(U)$ which agree in $F(U_i)$ for all i, then $a = b$.
2. (Sheaf axiom) For any covering $\{U_i \to U\}$ and any $a_i \in F(U_i)$ agreeing in $F(U_i \times_U U_j)$ for all i, j, there is a section $a \in F(U)$ that restricts to all a_i.

Theorem 8.10 (Grothendieck)**.** *Let S be a scheme. A representable functor on $\boldsymbol{Sch}/S$ is a sheaf in the flat topology.*

Proof. To unravel the statement, given a scheme $Y \to S$ we consider the contravariant functor $X \mapsto \mathrm{Hom}_S(X, Y)$. This is a functor from schemes over S to sets, and the claim is that this is a sheaf in the flat topology.

For simplicity, we just consider a flat covering which consists of a single morphism. The theorem says that for any surjective flat quasicompact morphism $X' \to X$ over S, and any morphism $X' \to Y$ over S such that the two compositions $X' \times_X X' \to X' \to Y$ are equal, there is a map from $X \to Y$ inducing $X' \to Y$.

$$\begin{tikzcd}
X' \times_X X' \arrow[d] \arrow[dr] & \\
X' \arrow[r] \arrow[d] & Y \\
X \arrow[ur, dotted] &
\end{tikzcd}$$

We may reduce to the case where Y is affine. This is a map to affine space with some conditions, so it reduces to the case where $Y = \mathbb{A}^n$, in which case mapping to Y is just a bunch of maps to $\mathbb{A}^1$. So we may reduce to the case $Y = \mathbb{A}^1$.

That is, we're given a regular function $f \in \mathcal{O}(X')$ such that the two pullbacks to $X' \times_X X'$ are equal, and we want to deduce that f is pulled back from X.

Let's think about what this means algebraically. Reducing to the case where B is affine, we have a faithfully flat A-algebra B, and we want to know that if $f \in B$ satisfies $1 \otimes f = f \otimes 1$ in $B \otimes_A B$, then f comes from A. This is a special case of the exactness of the Amitsur complex

$$0 \to A \to B \to B \otimes_A B$$

which we saw when doing descent theory. $\square$

If F is a fibered category over a category C with a Grothendieck topology, the functor $U \to F(U)$ is (roughly) a presheaf of categories on C. When studying stacks, we will want to stipulate that it be a sheaf. Recall that we are thinking of this as a generalization of the "functor of points" definition for schemes.

Definition 8.11. A *stack* F over a site C is a fibered category over C such that for each covering $\{U_i \to U\}$ in C, the functor $F(U) \to F(\{U_i \to U\})$ is an equivalence of categories.

Here $F(\{U_i \to U\})$ is the category of objects $A_i \in F(U_i)$ for every i, together with descent data: that is, for every pair i, j, we are given an isomorphism

$$\varphi_{ij} \colon p_2^* A_j \to p_1^* A_i \in F(U_i \times_U U_j)$$

satisfying the cocycle condition:

$$p_{13}^* \varphi_{ik} = p_{12}^* \varphi_{ij} \circ p_{23}^* \varphi_{jk}$$

as isomorphisms $p_3^* A_k \to p_1^* A_i$.

Thus, informally, a stack is a sheaf of categories on the site C.

Example 8.13. Let S be a scheme. The fibered category QCoh$/S$ of quasicoherent sheaves on the category of S-schemes is a stack on Sch$/S$, in the fpqc topology. This is essentially Grothendieck's theorem on descent for quasicoherent sheaves.

Example 8.14. For a group scheme G over a field k, the stack BG over Sch$/k$ (for the fpqc topology) is the fibered category of principal G-bundles over k-schemes.

8.4 Algebraic spaces

An algebraic space is a generalization of a scheme - it is a stack where points have no automorphisms (so it is intermediate between schemes and stacks). We'll define an algebraic space over a scheme S to be a certain type of contravariant sheaf of sets on **Sch**$/S$. For an algebraic space X and a scheme Y over S, we think of the set $X(Y)$ as the set of morphisms from Y to X.

8.4.1 The fppf topology

Definition 8.12. Let X be a scheme. An *fppf covering* is a set of morphisms of schemes, $f_i \colon U_i \to X$ such that each f_i is flat and locally of finite presentation, and $\bigcup_i f(U_i) = X$.

So this demands slightly more than fpqc. (A morphism of finite presentation is automatically quasicompact.)

In particular, we can view any scheme Y/S as a sheaf on the category of schemes over S (for the fppf topology).

We can define an algebraic space to be the "quotient of a scheme by an étale equivalence relation." That is, an algebraic space Y can be defined by a scheme U/S (an "atlas" for Y) together with an étale equivalence relation R on U. (Once these properties are defined, we will have a morphism $U \to Y$ which is étale and surjective. If $R = U \times_Y U$, then Y is the quotient of U by the equivalence relation R.) A special case of taking the quotient by an étale equivalence relation is taking the quotient by a *free* action of a finite group.

Formally, an equivalence relation R on a scheme U/S is a scheme with a monomorphism

$$R \to U \times_S U$$

such that for any scheme T, the image of

$$R(T) \to U(T) \times U(T)$$

is an equivalence relation on the set $U(T)$. We could also say: R is a groupoid over U, and we have morphisms

$$R \times_U R \to R$$

satisfying associativity, etc.

Definition 8.13. An equivalence relation R on U is *étale* if both projections $R \to U$ are étale.

The *quotient sheaf* U/R is the sheaf on $\mathbf{Sch}/S$ which is the sheafification of the obvious presheaf $U(T)/R(T)$ (the set of equivalence classes for the equivalence relation $R(T)$ on $U(T)$). An *algebraic space* is a sheaf of sets on $(\mathrm{Sch}/S)_{fppf}$ defined in this way.

To explain this "sheafification," a morphism from a scheme T to the algebraic space U/R is given by an fppf covering $\{T_i \to T\}$ and morphisms $f_i \colon T_i \to U$ such that for all i, j, $f_i|_{T_i \times_T T_j}$ and $f_j|_{T_i \times_T T_j}$ differ by a morphism

$T_i \times_T T_j \to R$, i.e. there is a map to R whose projections give the f_i, f_j. (It would be enough to consider étale coverings of T here.)

Most geometric notions for schemes generalize naturally to algebraic spaces (smooth, proper, ...). For instance, every algebraic space Y has an étale surjective morphism from a scheme U. We can extend the definition of all "local properties" of schemes to algebraic spaces, by saying that Y has a given property if and only if U does.

Example 8.15. We give some examples of algebraic spaces that are not schemes. Let Y be the quotient of $\mathbb{A}^1$ modulo the étale equivalence relation $x \sim -x$ if $x \neq 0$.

This algebraic space comes with a morphism $Y \to \mathbb{A}^1$, sending $x \mapsto x^2$, which is an isomorphism over $\mathbb{A}^1 - \{0\}$, bijective, and both Y and $\mathbb{A}^1$ are smooth over k. But f is not étale at 0 (so f is not an isomorphism). Indeed, at 0 it looks like the map $x \mapsto x^2$.

Remark 8.14. For a scheme X and a finite group G acting freely on X, X/G is an algebraic space. Indeed, the two morphisms $G \times_k X \to X$, $(g, x) \mapsto x$ and $(g, x) \mapsto gx$, form an etale equivalence relation on X. If X is quasiprojective, then X/G is a quasiprojective scheme. For X not quasiprojective, X/G need not be a scheme, as Hironaka showed.

Example 8.16. (This is also due to Hironaka. See Hartshorne, *Algebraic Geometry*, Appendix B, or the Stacks Project's chapter on Examples.) There is a smooth proper algebraic space X of dimension 3 over $\mathbb{C}$ containing a curve isomorphic to $\mathbb{P}^1$ in X such that every divisor $D \subset X$ has intersection number $D \cdot \mathbb{P}^1 = 0$. (Hironaka gives an explicit construction to show this, but it happens in many examples. For example, when you resolve a cubic 3-fold with a nodal singularity by replacing the singular point with a $\mathbb{P}^1$, you get an algebraic space with this property.) I claim that X cannot be a scheme. If it were, a point $p \in \mathbb{P}^1$ would have an affine open neighborhood U. On U, we could choose a regular function f vanishing at p but not on all of $\mathbb{P}^1$. If D is one irreducible component of $\{f = 0\}$ through p, take its closure in X. Then D is a surface in X with $D \cdot \mathbb{P}^1 > 0$.

8.5 Stacks

We've mentioned groupids before, but let's now recall the definition more formally. A groupoid consists of

- two sets: U (the set of objects) and R (the set of arrows),
- maps $s, t\colon R \rightrightarrows U$ (associating to an arrow the "source" and "target") and a morphism $R \times_U R \to R$ which gives the composition of morphisms

- an identity: $U \to R$
- an inverse: $R \to R$ (reversing source and target)
- associativity, identity, and inverse axioms.

Let S be a scheme. A *smooth groupoid* over S is a scheme $U \to S$ and a scheme $R \to S$ with morphisms $R \xrightarrow{(s,t)} U \times_S U$, $\mathrm{id}\colon U \to R$, $\mathrm{inv}\colon R \to R$ satisfying the same identities, such that the source and target maps $R \rightrightarrows U$ are both smooth morphisms.

This generalizes the notion of an étale equivalence relation in two ways. An étale map is a smooth map of relative dimension zero, so we are allowing positive relative dimension (previously we were taking what looked like a covering space quotient; now we allow bigger quotients). Furthermore, in defining an étale equivalence relation, we required the map $R \to U \times_S U$ to be a monomorphism, and we make no such requirement here.

Example 8.17. Let G be a smooth group scheme over a field k, acting on a scheme X/k. Then $G \times X \rightrightarrows X$ with the two maps $(g,x) \mapsto x$ and $(g,x) \mapsto gx$ is a smooth groupoid.

Definition 8.15. Let $R \rightrightarrows U$ be a smooth groupoid over a base scheme S. The *quotient stack* $[U/R]$ is the fibered category in groupoids over $\mathbf{Sch}/S_{fppf}$ which is the "stackification" of the fibered category

$$T \mapsto (U(T), R(T) \rightrightarrows U(T)).$$

Remark 8.16. We have actually only specified the fibers in the fibered category.

So what is "stackification?" It's basically sheafification. The point is that to give a map to $[U/R]$ from T, we should really only have to specify compatible maps from to U from a cover of T. That is, for a scheme T a morphism $T \to [U/R]$ is given by an fppf covering $\{T_i \to T\}$ and morphisms $a_i\colon T_i \to U$ together with morphisms $\varphi_{ij}\colon T_i \times_T T_j \to R$ such that

$$\varphi_{ij}(a_i|_{T_i \times_T T_j}) = a_j|_{T_i \times_T T_j}$$

and the cocycle condition holds:

$$\varphi_{jk} \cdot \varphi_{ij} = \varphi_{ik}\colon T_i \times_T \times T_j \times_T T_k \to R.$$

where $\cdot$ is the product on R. (We are abusing notation and identifying these maps with their pullbacks to the triple product.)

Definition 8.17. An *algebraic stack* over a scheme S is a stack fibered in groupoids over $(\mathrm{Sch}/S)_{fppf}$ that is equivalent to $[U/R]$ for some scheme $U \to S$ and some smooth groupoid $R \rightrightarrows U$.

Example 8.18. For a smooth k-group scheme G acting on a k-scheme X, we have a quotient stack $[X/G]$, the quotient of X by the smooth groupoid $G \times X \rightrightarrows X$. For $X = \operatorname{Spec} k$, $[\operatorname{Spec} k/G]$ is called BG, the *classifying stack* of G.

For a scheme X over S, a morphism $X \to BG$ is equivalent to a principal G-bundle over X. Indeed, to specify such a morphism we have to give a flat covering $\{U_i \to X\}$, plus morphisms a_i from U_i to a point (this part is trivial), plus morphisms from the fiber products $U_i \times_X U_j \to G$ which satisfy the cocycle condition. This is the standard way to describe a principal G-bundle in terms of transition functions.

More generally, for any scheme Y over S, a morphism

$$Y \to [X/G]$$

is equivalent to a principal G-bundle E over Y and a G-equivariant morphism $E \to X$.

Any property of schemes which is "local in the smooth topology" immediately generalizes from schemes to stacks. By definition of an algebraic stack, for every algebraic stack X/S, there is a scheme U/S and a smooth surjective morphism $U \to X$. We then define X to be smooth over k, etc. if U has that property.

Example 8.19. Most moduli spaces in algebraic geometry can be viewed as algebraic stacks. For example, the moduli space of smooth, projective curves of genus $g \geq 2$ is more properly considered as a stack, since some curves have nontrivial automorphisms. The stack M_g is smooth (over any base field). The associated "coarse moduli space" has quotient singularities.

For any curve X of genus $g \geq 2$, the tangent space to the stack $[M_g]$ at the point corresponding to X is $H^1(X, TX)$. In fact, deformation theory identifies $H^1(X, TX)$ with isomorphism classes of smooth projective morphisms $\mathscr{X} \to \operatorname{Spec} k[t]/(t^2)$ with the fiber over $\{t = 0\}$ having a given isomorphism to X (these are "first-order deformations of X").

Note that $H^1(X, TX) = 3g - 3$ for every curve X of genus $g \geq 2$ (e.g. by Riemann-Roch), which tells us that $\dim M_g = 3g - 3$.

Example 8.20. The moduli stack of elliptic curves over $\mathbb{C}$ is the quotient stack $\mathbb{H}/SL(2, \mathbb{Z})$. The corresponding coarse moduli space is just $\mathbb{A}^1_{\mathbb{C}}$. This isomorphism is called the j-invariant.

What is the difference between the stack and the coarse moduli space? The extra information comes in describing the automorphisms. Most elliptic curves have automorphism group $\mathbb{Z}/2$ (inverse in the group law), but one (up to isomorphism over $\mathbb{C}$) has automorphism group $\mathbb{Z}/4$ and one has automorphism group $\mathbb{Z}/6$. These correspond to the the fact that the group

$\{\pm 1\}$ in $SL(2, \mathbb{Z})$ acts trivially on the whole upper half plane, whereas a subgroup of order 4 or 6 in $SL(2, \mathbb{Z})$ has one fixed point.

The fact that the moduli stack of elliptic curves is a quotient $\mathbb{H}/\operatorname{SL}(2, \mathbb{Z})$ means that this moduli stack has a Kähler metric with curvature -1. This has some interesting geometric consequences: for instance, this implies that any family of elliptic curves over $\mathbb{P}^1$, or $\mathbb{A}^1$, has all fibers isomorphic. Indeed, such a family would give a morphism $\mathbb{A}^1 \to [\mathbb{H}/\operatorname{SL}(2, \mathbb{Z})]$. The universal cover of this latter space is $\mathbb{H}$, and since $\mathbb{A}^1$ is simply-connected, this lifts to a map $\mathbb{A}^1 \to \mathbb{H}$. Since $\mathbb{H}$ is isomorphic to the unit disc as a complex manifold, Liouville's theorem implies that any such analytic map is constant.

On the other hand, the fact that the coarse moduli space of elliptic curves is isomorphic to the affine line (where the natural Kähler metric has curvature 0 rather than -1) also has geometric consequences. This is related to the fact that there are infinitely many elliptic curves over $\mathbb{Q}$. Indeed, the affine line has infinitely many rational points, and one can check that there is an elliptic curve over $\mathbb{Q}$ with any given j-invariant.

By contrast, for big enough positive integers n, the coarse moduli space of elliptic curves with a point of order n has genus at least 2. Faltings showed that every curve of genus at least 2 over $\mathbb{Q}$ has only finitely many rational points. (The special case here was proved earlier by Mazur.) So for all sufficiently large n, there are only finitely many elliptic curves over Q with a point of order n, up to isomorphism over $\overline{\mathbb{Q}}$.

Conclusion: we should try to understand the geometric properties of moduli stacks, but also of the associated coarse moduli spaces.

8.6 References

- A. Vistoli, *Notes on Grothendieck topologies, fibered categories, and descent theory* (2005).
- Knutson, *Algebraic spaces.*
- *Stacks project*, on the web.