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Part I: Multiresolution Representation of $S^3$-Valued Data via Midpoint-Interpolating Refinement

Part II: Multiresolution Representation of $\mathbb{R}^2$-Valued Data via Midpoint-Interpolating Refinement
1. New Data Types Everywhere!
2. Is there a Multiscale Representation?
3. Properties of Our Construction
4. We develop tools to provide Multiresolution Analysis and "wavelets" in Symmetric Spaces
   - Coefficient Decay
   - Smoothness
   - Convergence
5. Applications: Noise Removal, Compression
Figure 1: 28 K-helix Observations

Synthetic Data as a Function of Time: $\mathcal{S}O(3)$
Figure 2: Final 30 seconds of USAR Flight 427, 1994

Data Recorder Data

Data as a Function of Time: Flight 50(3)
Figure 3: HIBall Tracker Device and LED Ceiling

HIBall Motion Tracker

Data as a Function of Time: $\text{SO}^{(3)}$
In this talk we develop multiscale methods that work for all called Symmetric Spaces. Actually these data live in a class of Riemannian Manifolds. Note that $SO(3)$ data lives in a Riemannian manifold. Unifying viewpoint.
Our methods are analogous to those developed for real-valued data.

- Suppressing noise-gerated phenomena
- Represent data approximately

2. Noise Removal (e.g. Wavelet Thresholding)

- Using few coefficients/bits
- Represent data approximately

1. Data Compression (e.g. JPEG-2000)

**Why a Multiscale Representation?**
Building a Multiscale Representation

1. Midpoint Pyramid
2. Nonlinear Midpoint-Interpolating Scheme
3. Nonlinear Multiresolution Analysis
4. Nonlinear "Wavelet" Transform
convert from/to local coordinate system

- local coordinate system for manifold: "Exp" and "Log" to
- Geodesics well defined

Key Facts about Symmetric Spaces: I
Key Facts about Symmetric Spaces

• Space symmetric about midpoint

• Midpoints well defined
3. Wavelet Coefficient = Residual:

\[ c_{j',k} = \sum_{k'} a_{j'+1,2k'} \]

Linear:

Here \( \text{AIRpred} = \text{Coarse-to-Fine Prediction Operator} \)

\[ \forall j, k : a_{j',k} = 0 \quad \forall j < j' = j + 1 \]

2. Predict Fine-Scale Averages:

\[ \forall j, k \sum_{k'} a_{j'+1,2k'} = \forall j, k \]

1. Form Pyramid of Averages over dyadic intervals:

How to build Classical Wavelets
Average Interpolation for Real Data
local coordinates = \text{log} T
\quad (f, g)_{m, n} = \text{log} T_{m, n}
(\mathcal{A}^T m, n)

3. Wavelet Coefficient = Residual:
\[ [((\mathcal{V}^T m, n)\text{Red})\mathcal{D} x, p]_{m, n} = \mathcal{M} \]

Nonlinear:
Heavy \text{ Midpoint} = \text{Coarse-to-Fine Prediction Operator}

(\mathcal{D} x, p)_{m+1, 2^n}
\quad \text{Midpoints} = \mathcal{M}

2. Predict Fine-Scale Midpoint:
\[ m, n_{\text{Midpoint}} = \mathcal{M} \]

1. Form Pyramid of Midpoints over dyadic intervals:

Space

How to build \text{"Wavelets" for Symmetric}
The nonlinear refinements exhibit the same convergence and smoothness as their linear counterparts.

**Key Empirical Finding**
Figure 4: Refinement of a Kronecker Sequence

Convergence
Figure 5: First and Second Differences
Figure 6: Coefficient Decay for Piecewise Geodesic

Coefficient Decay
Figure 7: SO(3) Data from Two Flights

Flight 1

Flight 2
Figure 8: Flight Data Reconstruction from 20 detail coefficients, 16 midpoints.
Figure 16: Flight Data Reconstruction from 20 detail coefficients, Flight 2 de-noised

Figure 16: Flight Data Reconstruction from 20 detail coefficients, Flight 2 original

Compression Results: Flight 2
Figure 1: Reconstruction Error as the Number of Coefficients Increases

Figure 2
- Well-behaved Approximation Order
- Well-behaved for thresholding, tree expansions
- Fast Algorithms for Synthesis
- Fast Algorithms for Analysis

Properties

- Wavelets for Symmetric Space Valued Data
- Wavelet Decomposition
- Multiresolution Analysis
- Refinement Schemes
- Midpoint Pyramids

Can Construct for Symmetric Space Data:

Results
January 14, 2003

David Donoho, Jonathon Kaplan

Morphologies

Part II: Multiscale Representation of
Fundamental Questions

Accurate to desired precision

Respect Underlying Dynamics

What are the sparsest representations that:

- Multiscale in space?
- Multiscale in time?
- Multiscale in time?

Does there exist a multiscale representation of the family

\[ (x(x)) \frac{d}{dx} \]

\[ \text{warpings': } x \leftrightarrow (x) \frac{d}{dx} \frac{d}{dx} \]

\[ \text{Given evolving family of diffeomorphisms of a space} \]
Hamiltonian — The Group of Symplectic Diffeomorphisms Preserving

— The Group of Area-Preserving Diffeomorphisms

Ambition: transfer such tools to:

principle

Technology is general, covers any finite-dimensional Lie group in

various scales and locations.

wavelet coefficients, decomposing motion into discretely, at

Example: \( \mathcal{O}_S \in \mathcal{C} \) the-valued multi-scale analysis gives

representations of the group-valued functions of time

Existing work (e.g., Victoria Stodden) on multi-scale

Thus morphology is a group-valued function of time

\(((x)\phi)\phi = (x)(\phi \circ \phi)\)

Program
Outline of Our Algorithm
Area-Preserving Diffeomorphism case

- Notation \( t_{j,k} = k/2^j \) Dyadic times
- \( \varphi_{j,k} = \phi_{t_{j,k}}(x) \)
- Apply multiscale analysis for Lie-Valued Data
- Get “wavelet coefficients in time”.
- These are divergence-free vector fields \( v^{j,k}(x) \)
- Apply Divergence-Free Wavelet Transform to each \( v^{j,k} \)
- Get “wavelet coefficients” in space \( \alpha_{l,m}^{j,k} \), \( l = \) spatial scale, \( m = \) spatial location
Support of functional $\gamma^m$ is a warped tube of trajectories

Wavelets themselves

Wavelt's themselfes

Roughly speaking – course corrections for the how
time and space.

expectations are based on immediately coarse scales in both
trajectories

expectations from expected
detection of those trajectories

[1] $\gamma^m, [t, t, \gamma + 1]$ during time interval $I^m, t_0$; $t_1$

studies trajectories that started in spatial dyadic box $O^m$ 

Wavelet Coefficient $\gamma^m, I^m$

Interpretation
Infinite-dimensional algebraic and geometric definitions of \( \text{Exp} \)

Conceptual issues: requires careful attention to difference between

\( N \times N \times N \) array \( \mathcal{H} \) is an \( \mathcal{N} \) (assuming \( \mathcal{N} \) and \( \mathcal{N} \) coefficients, \( \mathcal{N} \) geodesically, and initial headdings vary smoothly, \( \mathcal{N} \))

Answer to the first question is automatically yes, if particles move

- are the reconstructions smooth?
- is it stable under thresholding?
- compression of \( \mathcal{N} \) does it give better compression than e.g. standard wavelet

Practical person wants to know:

**Technical Issues**
(Hamiltonian Systems)

- Representing solutions to PDE (Euler Equation)
- Compressive Movies

Possible insights for possible developments for Hamiltonian diffeomorphisms preserving a Hamiltonian.

Analogous developments possible for symplectic ideally-adapted (curved) wavelets.

If it all works...