WHEN DOES NON-NEGATIVE MATRIX FACTORIZATION GIVE A CORRECT DECOMPOSITION INTO PARTS?

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NON-NEGATIVE MATRIX FACTORIZATION EXPLAINED

A natural goal in image processing might be to decompose the image into the "parts" that comprise it. One intuitive way to do this is to find a set of basis functions for representing non-negative (image) data. To do this Lee and Seung proposed Non-Negative Matrix Factorization, or NMF.

Consider an image database, *X*, and the following factorization:

$$X = A\Psi$$

where

$$X \text{ is } n \times p \qquad A \text{ is } n \times r$$

$$\Psi \text{ is } r \times p$$

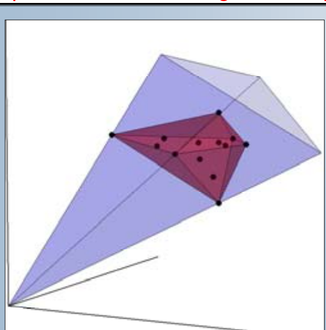
and *n* is the number of images, *p* is the number of pixels in each image, and *r* is the number of parts. Importantly,

$$A \geq 0, \Psi \geq 0$$

Each image is decomposed into a weighted sum of basis images. Since the weights are non-negative, the basis images must represent "parts" in each image. The images are then reconstructed additively.

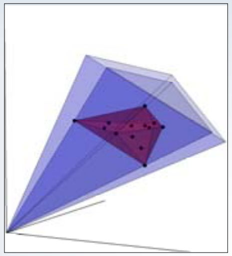
A GEOMETRIC INTERPRETATION

Imagine each image as a point in R^p. Then NMF finds a simplicial cone containing the data points.

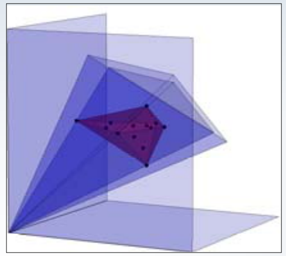


CLAIM: LEE & SEUNG "PARTS DECOMPOSITION" NOT SUPPORTED BY EXISTING ARGUMENTS

- When is the factorization *unique*?
- When does NMF find the *correct* decomposition?

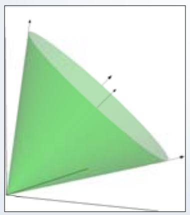


The simplicial cone found by NMF may not be unique!



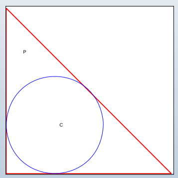
NMF will find a unique simplicial cone if the data "fill out" the positive orthant...

REINTERPRETING NMF USING DUALITY

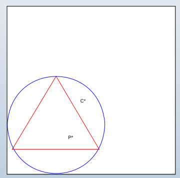


Imagine the image data points filling out the 'ice cream cone', which is tangent to the faces of the positive orthant.

In this case NMF will have a unique solution. To see this consider the dual of the cone. The largest inscribing simplex in the dual space, P*, is exactly the minimal simplex enclosing the cone in the primal space (and in fact the positive orthant, P). Thus the NMF solution is unique.



Primal Space



Dual Space

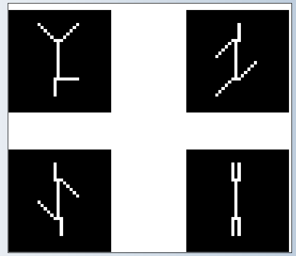
THEOREM

Given a Separable Factorial Articulation Database, NMF will find a unique simplicial hull with r=AP generators, that contains all the data points, and is contained in the intersection of the positive orthant and the linear span of the generators.

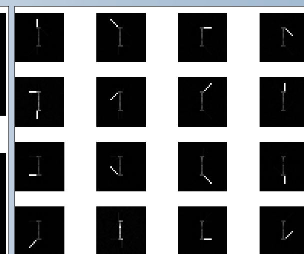
EMPIRICAL RESULTS

We created a Separable Factorial Image Articulation Database from a stick figure with 4 limbs, each limb having 4 possible articulations. For sample images see the left panel below.

Our theoretical results predict that NMF should be able to identify the parts that comprise these images. It does, as evidenced in the right panel.



Sample images from our `Swimmer` database – a Separable Factorial Articulation Database.



The basis images, and parts, identified by NMF.

Definition: Separable Factorial Articulation Families are datasets that satisfy the following criteria:

1. Each image can be represented by the factorization.
2. Existence of a tell-tale pixel for each articulation.
3. The datasets contains all parts in all possible articulations.

If these criteria are satisfied, NMF will be able to find the "parts" of the images.

ONGOING WORK

- How sensitive is NMF to small departures from the Separable Factorial Articulation database definition?
- Does uniqueness depend on the sparsity of the underlying representation?