Part II Relativity - Example Sheet 4

Q1. Massive particle released from rest at infinity outside Schwarzschild BH.

The 'energy' equation:

$$\dot{r}^2 + \frac{h^2}{r^2}\left(1 - \frac{2\mu}{r}\right) - \frac{2\mu c^2}{r} = c^2(k^2 - 1)$$

For radial free-fall, $h = 0$

$$\dot{r}^2 = c^2(k^2 - 1) + \frac{2\mu c^2}{r}$$

As $r \to \infty$, $\dot{r} \to 0$ $\Rightarrow k = 1$

And from the geodesic eqns:

$$\left(1 - \frac{2\mu}{r}\right)\dot{t} = k$$

$$\dot{t} = \left(1 - \frac{2\mu}{r}\right)^{-1}$$

$$\dot{r} = -\left(\frac{2\mu c^2}{r}\right)^{\frac{1}{2}}$$

The 4-velocity:

$$[u^\mu] = \left[\left(1 - \frac{2\mu}{r}\right)^{-1}, -\left(\frac{2\mu c^2}{r}\right)^{\frac{1}{2}}, 0, 0\right]$$

Using the metric, $u_\mu = g_{\mu\nu}u^\nu$

$$[u_\mu] = \left[c^2, \left(\frac{2\mu c^2}{r}\right)^{\frac{1}{2}}\left(1 - \frac{2\mu}{r}\right)^{-1}, 0, 0\right]$$

From the fundamental theorem of calculus, $\frac{d}{dr}\int_a^r f(x)\,dx = f(r)$

$$\left(\frac{2\mu c^2}{r}\right)^{\frac{1}{2}}\left(1 - \frac{2\mu}{r}\right)^{-1} = \frac{d}{dr}\int_\infty^r \left(\frac{2\mu c^2}{\bar{r}}\right)^{\frac{1}{2}}\left(1 - \frac{2\mu}{\bar{r}}\right) d\bar{r}$$

$\therefore u_\mu = c^2 \partial_\mu T$ where $\underline{\underline{T = t + \frac{1}{c}\int_\infty^r \left(\frac{2\mu}{\bar{r}}\right)^{\frac{1}{2}}\left(1 - \frac{2\mu}{\bar{r}}\right)^{-1} d\bar{r}}}$

The Schwarzschild line element:

$$ds^2 = c^2\left(1-\frac{2\mu}{r}\right)dt^2 - \left(1-\frac{2\mu}{r}\right)^{-1}dr^2 - r^2 d\Omega^2$$

$$dT = dt + \frac{1}{c}\left(\frac{2\mu}{r}\right)^{\frac{1}{2}}\left(1-\frac{2\mu}{r}\right)^{-1}dr$$

$$dt = dT - \frac{1}{c}\left(\frac{2\mu}{r}\right)^{\frac{1}{2}}\left(1-\frac{2\mu}{r}\right)^{-1}dr$$

Substituting in to the line element:

$$ds^2 = c^2\left(1-\frac{2\mu}{r}\right)\left[dT - \frac{1}{c}\left(\frac{2\mu}{r}\right)^{\frac{1}{2}}\left(1-\frac{2\mu}{r}\right)^{-1}dr\right]^2 - \left(1-\frac{2\mu}{r}\right)^{-1}dr^2 - r^2 d\Omega^2$$

$$= c^2\left(1-\frac{2\mu}{r}\right)\left[dT^2 - \frac{2}{c}\left(\frac{2\mu}{r}\right)^{\frac{1}{2}}\left(1-\frac{2\mu}{r}\right)^{-1}dT\,dr + \frac{1}{c^2}\left(\frac{2\mu}{r}\right)\left(1-\frac{2\mu}{r}\right)^{-2}dr^2\right]$$
$$- \left(1-\frac{2\mu}{r}\right)^{-1}dr^2 - r^2 d\Omega^2$$

$$= c^2dT^2 - \frac{2\mu c^2}{r}dT^2 - 2\left(\frac{2\mu c^2}{r}\right)^{\frac{1}{2}}dT\,dr + \left(\frac{2\mu}{r}\right)\left(1-\frac{2\mu}{r}\right)^{-1}dr^2$$
$$- \left(1-\frac{2\mu}{r}\right)^{-1}dr^2 - r^2 d\Omega^2$$

$$= c^2dT^2 - \frac{2\mu c^2}{r}dT^2 - 2\left(\frac{2\mu c^2}{r}\right)^{\frac{1}{2}}dT\,dr - \left(1-\frac{2\mu}{r}\right)\left(1-\frac{2\mu}{r}\right)^{-1}dr^2 - r^2 d\Omega^2$$

$$= c^2dT^2 - \frac{2\mu c^2}{r}dT^2 - 2\left(\frac{2\mu c^2}{r}\right)^{\frac{1}{2}}dT\,dr - dr^2 - r^2 d\Omega^2$$

$$= c^2dT^2 - \left(dr + \sqrt{\frac{2\mu c^2}{r}}\,dT\right)^2 - r^2 d\Omega^2$$

Finite at $r = 2\mu$.

If T is constant, $dT = 0$

$$ds^2 = dr^2 - r^2 d\Omega^2$$ → these surfaces are flat (Euclidean metric).

In P-G co-ordinates, the Lagrangian:

$$\mathcal{L} = c^2\dot{T}^2 - \left(\dot{r} + \sqrt{\frac{2\mu c^2}{r}}\,\dot{T}\right)^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\,\dot{\phi}^2$$

Hence, the geodesic equation for T:

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{T}}\right) = \frac{\partial \mathcal{L}}{\partial T}$$

$$\frac{d}{d\tau}\left[2c^2\left(1-\frac{2\mu}{r}\right)\dot{T} - 2\sqrt{\frac{2\mu c^2}{r}}\,\dot{r}\right] = 0$$

$$c^2\left(1-\frac{2\mu}{r}\right)\dot{T} - \sqrt{\frac{2\mu c^2}{r}}\,\dot{r} = \text{const}$$

For observers infalling radially from infinity, $\dot{r} = -\sqrt{\frac{2\mu c^2}{r}}$

$$\therefore\ c^2\left(1-\frac{2\mu}{r}\right)\dot{T} + \frac{2\mu c^2}{r} = \text{const.}$$

This requires $\underline{\underline{\dot{T} = 1}}$ to remove the variable r for it to be constant.

$\Rightarrow$ T corresponds to the proper time of a massive particle radially free-falling from infinity.

Q2. The Kerr metric:

note this is $2g_{t\varphi}$

$$ds^2 = c^2\left(1 - \frac{2\mu r}{\rho^2}\right)dt^2 + \frac{4\mu a c r \sin^2\theta}{\rho^2}dt\,d\varphi - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2$$
$$- \left(r^2 + a^2 + \frac{2\mu a^2 r \sin^2\theta}{\rho^2}\right)\sin^2\theta\, d\varphi^2$$

Where $\rho^2 = r^2 + a^2\cos^2\theta$
$\Delta = r^2 - 2\mu r + a^2$

For a circular orbit, co-ordinate angular velocity $\frac{d\varphi}{dt} = \frac{\dot\varphi}{\dot t} = \Omega$,

$\dot r = u^r = 0$, $\quad u_r = g_{r\mu}u^\mu = g_{rr}u^r = 0$

In the equatorial plane, $\theta = \frac{\pi}{2}$, $\quad \dot\theta = u^\theta = 0$

Using the form of the geodesic equation, $\dot u_\mu = \frac{1}{2}(\partial_\mu g_{\nu\sigma})u^\nu u^\sigma$

$$\dot u_r = \cancel{\tfrac{1}{2}}(\partial_r g_{\nu\sigma})u^\nu u^\sigma = 0$$

$$(\partial_r g_{tt})(u^t)^2 + 2(\partial_r g_{t\varphi})u^t u^\varphi + (\partial_r g_{\varphi\varphi})(u^\varphi)^2 = 0$$

$$u^\varphi = \dot\varphi = \Omega \dot t = \Omega u^t$$

$$(\partial_r g_{tt})(u^t)^2 + 2(\partial_r g_{t\varphi})\Omega(u^t)^2 + (\partial_r g_{\varphi\varphi})\Omega^2(u^t)^2 = 0$$

$$(\partial_r g_{tt})\left(\frac{1}{\Omega}\right)^2 + 2(\partial_r g_{t\varphi})\left(\frac{1}{\Omega}\right) + (\partial_r g_{\varphi\varphi}) = 0 \quad (*)$$

For the Kerr metric in the equatorial plane,

$$g_{tt} = c^2\left(1 - \frac{2\mu}{r}\right) \qquad (\partial_r g_{tt}) = \frac{2\mu c^2}{r^2}$$

$$g_{t\varphi} = \frac{2\mu a c}{r} \qquad (\partial_r g_{t\varphi}) = -\frac{2\mu a c}{r^2}$$

$$g_{\varphi\varphi} = -\left(r^2 + a^2 + \frac{2\mu a^2}{r}\right) \quad (\partial_r g_{\varphi\varphi}) = -2r + \frac{2\mu a^2}{r^2}$$

Solving the quadratic, ⊛:

$$\frac{1}{\Omega} = \frac{-2(\partial_r g_{t\varphi}) \pm \left[4(\partial_r g_{t\varphi})^2 - 4(\partial_r g_{tt})(\partial_r g_{\varphi\varphi})\right]^{\frac{1}{2}}}{2(\partial_r g_{tt})}$$

$$\frac{1}{\Omega} = \frac{\frac{2\mu a c}{r^2} \pm \left[\left(\frac{2\mu a c}{r^2}\right)^2 - \left(\frac{2\mu c^2}{r^2}\right)\left(\frac{2\mu a^2}{r^2} - 2r\right)\right]^{\frac{1}{2}}}{\frac{2\mu c^2}{r^2}}$$

$$\frac{1}{\Omega} = \frac{\frac{2\mu a c}{r^2} \pm \left[\frac{4\mu^2 a^2 c^2}{r^4} - \frac{4\mu^2 a^2 c^2}{r^4} + \frac{4\mu c^2}{r}\right]^{\frac{1}{2}}}{\frac{2\mu c^2}{r^2}}$$

$$\frac{1}{\Omega} = \frac{\frac{2\mu a c}{r^2} \pm \left[\frac{4\mu c^2}{r}\right]^{\frac{1}{2}}}{\frac{2\mu c^2}{r^2}}$$

$$\frac{1}{\Omega} = \frac{\frac{2\mu a c}{r^2} \pm 2c\mu^{\frac{1}{2}} r^{-\frac{1}{2}}}{\frac{2\mu c^2}{r^2}}$$

$$\frac{1}{\Omega} = \frac{a\mu^{\frac{1}{2}} \pm r^{\frac{3}{2}}}{c\mu^{\frac{1}{2}}}$$

$$\therefore \Omega = \frac{c\mu^{\frac{1}{2}}}{a\mu^{\frac{1}{2}} \pm r^{3/2}}$$

Q3. a) Observer in circular orbit in equatorial plane of Kerr spacetime:

$$\dot{r} = 0 \qquad \dot{\theta} = 0 \qquad \frac{d\varphi}{dt} = \frac{\dot{\varphi}}{\dot{t}} = \Omega$$

4-velocity: $[u^\mu] = (\dot{t}, \dot{r}, \dot{\theta}, \dot{\varphi}) = \dot{t}(1, 0, 0, \Omega)$

For a massive particle, $g_{\mu\nu} u^\mu u^\nu = c^2$

$$g_{tt}\dot{t}^2 + 2g_{t\varphi}\dot{t}\dot{\varphi} + g_{\varphi\varphi}\dot{\varphi}^2 = c^2$$

$$g_{tt}\dot{t}^2 + 2g_{t\varphi}\Omega\dot{t}^2 + g_{\varphi\varphi}\Omega^2\dot{t}^2 = c^2 \qquad (\text{using } \dot{\varphi} = \Omega\dot{t})$$

$$\dot{t}^2 = \frac{c^2}{g_{tt} + 2\Omega g_{t\varphi} + \Omega^2 g_{\varphi\varphi}}$$

$$u^t = \dot{t} = c\left[c^2\left(1 - \frac{2\mu}{r}\right) + \frac{4\mu a c}{r}\Omega - \left(r^2 + a^2 + \frac{2\mu a^2}{r}\right)\Omega^2\right]^{-\frac{1}{2}}$$

Substituting in metric coefficients, setting $\theta = \frac{\pi}{2}$.

$$u^\varphi = \Omega u^t$$

And, using the metric:

$$u_\mu = g_{\mu\nu} u^\nu$$

$$u_t = g_{t\nu} u^\nu = g_{tt} u^t + g_{t\varphi} u^\varphi$$

$$u_t = u^t (g_{tt} + \Omega g_{t\varphi})$$

$$u_\varphi = g_{\varphi\nu} u^\nu$$

$$u_\varphi = u^\varphi (g_{t\varphi} + \Omega g_{\varphi\varphi})$$

b) For a massive observer, require $g_{\mu\nu}u^\mu u^\nu = c^2$

For a co-ordinate-stationary observer, $[u^\mu] = (\dot{t}, \vec{0})$

$$g_{tt}\dot{t}^2 = c^2$$

Requires $\underline{\underline{g_{tt} > 0}}$

In the equatorial plane of the Kerr metric, $g_{tt} = c^2\left(1 - \frac{2\mu}{r}\right)$

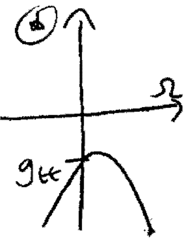
require $\frac{2\mu}{r} < 1$, i.e. $\underline{\underline{r > 2\mu}}$

otherwise, observer cannot be stationary.

For any observer, require u^t to be real

From part (a), this requires $g_{tt} + 2\Omega g_{t\varphi} + \Omega^2 g_{\varphi\varphi} \geq 0$ (⊛) for a circular orbit.

$$\Omega = \frac{g_{t\varphi} \pm \sqrt{g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi}}}{g_{tt}} \quad \text{for limiting case } (⊛ = 0)$$



For real Ω, $g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi} > 0$ to have real roots of quadratic.
If this is less than zero, the sign of the quadratic, ⊛, is the sign of g_{tt} for all Ω, as ⊛ never crosses y-axis.

∴ if $g_{tt} < 0$ (i.e. inside $r = 2\mu$) and $g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi} < 0$, cannot have real $\dot{t}$ for any Ω, so cannot have circular orbit.

so, $g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi} < 0$ for no circular orbits.

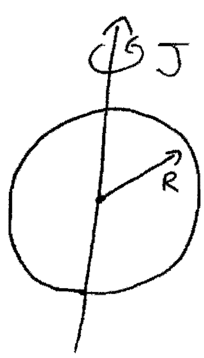
$$\frac{4\mu^2 a^2 c^2}{r^2} + c^2\left(1 - \frac{2\mu}{r}\right)\left(r^2 + a^2 + \frac{2\mu a^2}{r}\right) < 0$$

$$\frac{4\mu^2 a^2}{r^2} + r^2 + a^2 + \frac{2\mu a^2}{r} - 2\mu r - \frac{2\mu a^2}{r} - \frac{4\mu^2 a^2}{r^2} < 0$$

$$\underline{\underline{r^2 + a^2 - 2\mu r < 0}}$$

i.e. $\Delta < 0 \Rightarrow \underline{\underline{r_- < r < r_+}}$

Q4.



Spherical shell mass M

Slowly spinning with angular momentum J.

Outside the shell, it is equivalent to a spinning point mass at the centre, so is described by the Kerr metric. However, shell is <u>slowly</u> spinning, so can take the slowly-rotating limit (i.e. perturbation to the Schwarzschild metric).

$$ds^2_{out} = ds^2_{Schwarzschild} + d\bar{s}^2$$

$$ds^2_{out} = c^2\left(1-\frac{2M}{r}\right)dt^2 - \left(1-\frac{2M}{r}\right)^{-1}dr^2 - r^2 d\theta^2 - r^2\sin^2\theta(d\varphi - \omega dt)^2$$

(applying an <u>axisymmetric</u> perturbation).

Comparing this with the slowly-rotating limit of the Kerr metric (keeping terms to 1^{st} order in a)

$$2r^2\sin^2\theta\,\omega\, d\varphi\, dt = \frac{4GJ}{c^2 r}\sin^2\theta\, d\varphi\, dt \qquad (J = Mac)$$

$$\omega = \frac{2GJ}{c^2 r^3}$$

Inside the shell, by Gauss' theorem, there is no gravity so spacetime must be Minkowski

$$ds^2_{in} = c^2 d\bar{t}^2 - d\bar{r}^2 - r^2 d\theta^2 - r^2\sin^2\theta\, d\bar{\varphi}^2$$

And we must match the boundary conditions on the shell

so $ds_{out}(r=R) = ds_{in}(r=R)$ $(dr = 0$ on the shell$)$

$$\Rightarrow d\bar{\varphi} = d\varphi - \omega dt$$

$$d\bar{t} = \left(1-\frac{2M}{R}\right)^{\frac{1}{2}} dt \qquad \text{at } r = R$$

Inertial frames inside the shell have $d\bar{\varphi} = 0$

$$d\varphi - \omega dt = 0$$

$$\frac{d\varphi}{dt} = \omega$$

$$\frac{d\varphi}{dt} = \frac{2GJ}{c^2 R^3}$$

with respect to observer at rest at infinity.

Q5. The cosmological field (Friedmann) equations:

$$\frac{\ddot{R}}{R} + \frac{4\pi G\rho}{3}(1+\epsilon) - \frac{\Lambda c^2}{3} = 0 \qquad (A)$$

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{8\pi G\rho}{3} - \frac{\Lambda c^2}{3} = -\frac{kc^2}{R^2} \qquad (B)$$

In an empty universe, $\rho = 0$
also set $\Lambda = 0$

For $S(\chi) = \sinh\chi$ (as given) $k=-1$

From (B), $\dot{R}^2 = c^2$
$\therefore R = ct$

The FRW metric is the solution to the cosmological field equations.

$$ds^2 = c^2dt^2 - R^2\left(d\chi^2 + S^2(\chi)\,d\Omega^2\right)$$

For our empty universe, $k=-1 \Rightarrow S(\chi) = \sinh\chi$

$$\underline{ds^2 = c^2dt^2 - c^2t^2\left(d\chi^2 + \sinh^2\chi\, d\Omega^2\right)}$$

$k=-1 \to$ hyperbolic (open) spatial surfaces.

Looking at this metric, let $\bar{r} = ct\sinh\chi$
to get term like $\bar{r}^2 d\Omega^2$

$\bar{r} = ct\sinh\chi$
$d\bar{r} = c\sinh\chi\, dt + ct\cosh\chi\, d\chi$

Would also like to get rid of $t\cosh\chi$ term here, so let
$\bar{t} = t\cosh\chi$
$d\bar{t} = \cosh\chi\, dt + t\sinh\chi\, d\chi$

For this to describe Minkowski spacetime, try

$$ds^2 = c^2d\bar{t}^2 - d\bar{r}^2 - \bar{r}^2d\Omega^2$$

$$c^2 d\bar{t}^2 - d\bar{r}^2 - \bar{r}^2 d\Omega^2$$

$$= c^2\left(\cosh^2\chi\, dt^2 + 2t\sinh\chi\cosh\chi\, dt\, d\chi + t^2\sinh^2\chi\, d\chi^2\right)$$
$$- \left(c^2\sinh^2\chi\, dt^2 + 2c^2 t\sinh\chi\cosh\chi\, dt\, d\chi + c^2 t^2\cosh^2\chi\, d\chi^2\right)$$
$$- c^2 t^2 \sinh^2\chi\, d\Omega^2$$

$$= c^2(\cosh^2\chi - \sinh^2\chi)dt^2 - c^2t^2(\cosh^2\chi - \sinh^2\chi)d\chi^2 - c^2t^2\sinh^2\chi\, d\Omega^2$$

$$= c^2dt^2 - c^2t^2(d\chi^2 + \sinh^2\chi\, d\Omega^2)$$

$$\therefore ds^2 = c^2dt^2 - c^2t^2(d\chi^2 + \sinh^2\chi\, d\Omega^2) = c^2 d\bar{t}^2 - d\bar{r}^2 - \bar{r}^2 d\Omega^2$$

$\Rightarrow$ this metric describes Minkowski spacetime

Spatial geometry is flat when $d\bar{t} = 0$
(instantaneous flat space in an expanding universe).

Q6. Particle shot out into FRW spacetime at velocity V_1 wrt comoving observers

The FRW metric:

$$ds^2 = c^2 dt^2 - R^2(t)\left(d\chi^2 + S^2(\chi)\, d\Omega^2\right)$$

Assume for particle trajectory $d\theta = d\varphi = 0$ (radial motion)

Lagrangian: $\mathcal{L} = c^2 \dot{t}^2 - R^2 \dot{\chi}^2$

$\underline{t}$ $\quad \dfrac{d}{d\tau}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{t}}\right) - \dfrac{\partial \mathcal{L}}{\partial t} = 0$

$$\dot{t} = \text{const.}$$

$\underline{\chi}$ $\quad -R^2 \dot{\chi} = \text{const.}$ $\quad (*)$

The 4-velocity of the particle: $\underline{u} = (\dot{t}, \dot{\chi}, \dot{\theta}, \dot{\varphi})$

$$= \dot{t}(1, V, 0, 0) \quad \text{where } V = \frac{d\chi}{dt}$$

In the frame of a comoving (or in fact any) observer, $\underline{u} = (\gamma c, \gamma \vec{v})$ as they make measurements in terms of their proper time, t, and $\dot{t} = \gamma$.

We obtain γv by projecting the particle's 4-velocity into the observer's basis. i.e. we dot it with their χ basis vector.

From the metric, basis vector $\hat{e}_\chi = (0, \frac{1}{R}, 0, 0)$ since $\hat{e}_\chi \cdot \hat{e}_\chi = g_{\mu\nu} \hat{e}_\chi^\mu \hat{e}_\chi^\nu = 1$

$$\therefore \quad \gamma_v V = \underline{u} \cdot \hat{e}_\chi = g_{\mu\nu} u^\mu \hat{e}_\chi^\nu$$

$$= \underline{R\dot{\chi}}$$

$$\gamma_v V = R\dot{\chi}$$

Which, from (*): $R^2\dot{\chi} = \text{const}.$

$$\gamma_v V = \frac{\text{const}}{R}$$

So, comparing the measurements made by observers (1) and (2).

$$\frac{\gamma_{V_1} V_1}{\gamma_{V_2} V_2} = \frac{\frac{\text{const}}{R_1}}{\frac{\text{const}}{R_2}}$$

And the value of the constant is the same along the particle's geodesic

$$\underline{\underline{\frac{\gamma_{V_1} V_1}{\gamma_{V_2} V_2} = \frac{R_2}{R_1}}}$$

For a massive particle, the 4-momentum $\underline{p} = m\underline{u}$

For a photon we instead use the photon's 4-momentum, replacing γV with $\frac{h\nu}{c}$

$$\Rightarrow \underline{\underline{\frac{\nu_1}{\nu_2} = \frac{R_2}{R_1}}}$$

We recover the redshift formula.

Q7. The solution to the Einstein field equations for an isotropic and homogeneous Universe is given by the Friedmann equations:

$$\frac{\ddot{R}}{R} + \frac{4\pi G\rho}{3}(1+\epsilon) - \frac{\Lambda c^2}{3} = 0 \qquad (A) \qquad \epsilon = \frac{3p}{\rho c^2}$$

$$\left(\frac{\dot{R}}{R}\right)^2 - \frac{8\pi G\rho}{3} - \frac{\Lambda c^2}{3} = -\frac{kc^2}{R^2} \qquad (B)$$

If $\Lambda = 0$:

For a static solution, $\dot{R} = \ddot{R} = 0$

From (B): $\frac{8\pi G\rho}{3} = \frac{kc^2}{R^2} \Rightarrow \rho = \frac{3kc^2}{8\pi G R^2}$

And (A): $\frac{4\pi G\rho}{3}(1+\epsilon) = 0 \Rightarrow \rho(1+\epsilon) = 0$

For a physically reasonable fluid, $\rho > 0$ and $p \geq 0$ so $\epsilon \geq 0$
$\Rightarrow$ no solution to the above in this case

Now, if $\Lambda \geq 0$:

For a static solution, again set $\dot{R} = \ddot{R} = 0$

From (B): $\frac{8\pi G\rho}{3} + \frac{\Lambda c^2}{3} = \frac{kc^2}{R^2}$

And (A): $\frac{4\pi G\rho}{3}(1+\epsilon) - \frac{\Lambda c^2}{3} = 0 \Rightarrow \Lambda c^2 = 4\pi G\rho(1+\epsilon)$

$$\frac{8\pi G\rho}{3} + \frac{4\pi G\rho}{3}(1+\epsilon) = \frac{kc^2}{R^2}$$

$$\frac{4\pi G\rho}{3}(3+\epsilon) = \frac{kc^2}{R^2}$$

$\Rightarrow$ solution for $k = +1$

Now, perturbing this static solution : $R = R_0 + \delta R$, $\rho = \rho_0 + \delta\rho$
$\dot{R} = \delta\dot{R}$ $\quad \ddot{R} = \delta\ddot{R}$

Ⓐ: $$\frac{\delta\ddot{R}}{R} + \frac{4\pi G}{3}(\rho_0 + \delta\rho)(1+\epsilon) - \frac{\Lambda c^2}{3} = 0$$

$$\delta\ddot{R} = -\frac{4\pi G}{3}(1+\epsilon)(\rho_0 + \delta\rho)(R_0 + \delta R) + \frac{\Lambda c^2}{3}(R_0 + \delta R)$$

$$= -\frac{4\pi G}{3}(1+\epsilon)(\rho_0 R_0 + \rho_0 \delta R + R_0 \delta\rho) + \frac{\Lambda c^2}{3}(R_0 + \delta R)$$
to 1^{st} order

We know that $\frac{\Lambda c^2}{3} = \frac{4\pi G}{3}(1+\epsilon)\rho_0$ from before, so:

$$\delta\ddot{R} = -\frac{4\pi G}{3}(1+\epsilon)(R_0\delta\rho + \rho_0\delta R) + \frac{\Lambda c^2}{3}\delta R$$

From the continuity equation, $\rho \propto R^{-(3+\epsilon)}$
$\therefore \frac{\delta\rho}{\rho_0} = -(3+\epsilon)\frac{\delta R}{R_0}$ (differentiating)

$$\Rightarrow \delta\ddot{R} = -\frac{4\pi G}{3}(1+\epsilon)\left[-(3+\epsilon)\rho_0\delta R + \rho_0\delta R\right] + \frac{\Lambda c^2}{3}\delta R$$

$$= \left[\frac{4\pi G}{3}(1+\epsilon)(2+\epsilon)\rho_0 + \frac{\Lambda c^2}{3}\right]\delta R$$

And using $\frac{4\pi G\rho_0}{3}(1+\epsilon) = \frac{\Lambda c^2}{3}$

$$\underline{\underline{\delta\ddot{R} = \left(1 + \frac{\epsilon}{3}\right)\Lambda c^2\,\delta R}}$$

For $\Lambda > 0$, this is <u>unstable</u>.

Q8. Spherically symmetric body, mass M, rotating with angular momentum $\vec{J}$

Slowly rotating limit of the Kerr metric:

$$ds^2 = c^2\left(1-\frac{2GM}{c^2 r}\right)dt^2 - \left(1-\frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\Omega^2 + \frac{4GJ}{c^2 r}\sin^2\theta \, d\varphi \, dt$$

Binomial expansion (weak field)

$$ds^2 = c^2\left(1-\frac{2GM}{c^2 r}\right)dt^2 - \left(1+\frac{2GM}{c^2 r}\right)^{+1}\left(dx^2+dy^2+dz^2\right) + \frac{4GJ}{c^2 r^3}\left(x\,dy - y\,dx\right)dt$$

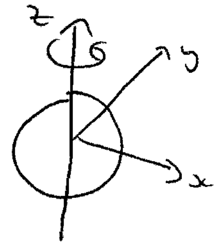
Comparing with the linearised stationary metric:

$$ds^2 = c^2\left(1+\frac{2\Phi}{c^2}\right)dt^2 - \left(1-\frac{2\Phi}{c^2}\right)\left(dx^2+dy^2+dz^2\right) + c\,\frac{2A_i}{c}\,dx^i\,dt$$

We identify $\Phi_g = -\dfrac{GM}{r} = -\dfrac{GM}{|\vec{x}|}$

And since $\vec{J}$ is in the z-direction:

$$\vec{A}_g = -\frac{2G}{c^2 r^3}\,\vec{J}\times\vec{x}$$



The gravitoelectric field:

$$\vec{E}_g = -\vec{\nabla}\Phi_g = -\frac{GM}{|\vec{x}|^3}\,\vec{x}$$

$$= -\frac{GM}{|\vec{x}|^3}\,\hat{\vec{x}}\,|\vec{x}| \qquad \text{(where } \hat{\vec{x}} \text{ is the unit vector).}$$

$$= \underline{\underline{-\frac{GM}{|\vec{x}|^2}\,\hat{\vec{x}}}}$$

And the gravitomagnetic field:

$$\vec{B}_g = \vec{\nabla}\times\vec{A}_g = -\frac{2G}{c^2}\left[\vec{\nabla}\left(\frac{1}{r^3}\right)\times\left(\vec{J}\times\vec{x}\right) + \frac{1}{r^3}\vec{\nabla}\times\left(\vec{J}\times\vec{x}\right)\right]$$

$$= -\frac{2G}{c^2}\left[-\frac{3\vec{x}}{|\vec{x}|^5}\times\left(\vec{J}\times\vec{x}\right) + \frac{1}{|\vec{x}|^3}\vec{\nabla}\times\left(\vec{J}\times\vec{x}\right)\right]$$

$$\vec{B}_g = \frac{-2G}{c^2}\left[-\frac{3}{|\vec{x}|^5}\left[(\vec{x}.\vec{x})\vec{J} - (\vec{x}.\vec{J})\vec{x}\right] + \frac{1}{|\vec{x}|^3}\vec{\nabla}\times(\vec{J}\times\vec{x})\right]$$

$$= \frac{-2G}{c^2}\left[-\frac{3}{|\vec{x}|^5}\left[|\vec{x}|^2\vec{J} - (\vec{J}.\hat{\vec{x}})\hat{\vec{x}}\,|\vec{x}|^2\right] + \frac{1}{|\vec{x}|^3}\vec{\nabla}\times(\vec{J}\times\vec{x})\right]$$

Using the unit vector $\hat{\vec{x}}$

$$= \frac{-2G}{c^2|\vec{x}|^3}\left[-3\left(\vec{J} - (\vec{J}.\hat{\vec{x}})\hat{\vec{x}}\right) + \vec{\nabla}\times(\vec{J}\times\vec{x})\right]$$

$$= \frac{-2G}{c^2|\vec{x}|^3}\left[-3\left(\vec{J} - (\vec{J}.\hat{\vec{x}})\hat{\vec{x}}\right) + (\vec{\nabla}.\vec{x})\vec{J} - (\vec{\nabla}.\vec{J})\vec{x}\right]$$

$$= \frac{-2G}{c^2|\vec{x}|^3}\left[-3\left(\vec{J} - (\vec{J}.\hat{\vec{x}})\hat{\vec{x}}\right) + 3\vec{J} - (\vec{J}.\vec{\nabla})\vec{x}\right]$$

$$= \frac{-2G}{c^2|\vec{x}|^3}\left[-3(\vec{J} - (\vec{J}.\hat{\vec{x}})\hat{\vec{x}} + 3\vec{J} - \left[J_j\,\partial_j\,x_i\right]\right]$$

$$= \frac{-2G}{c^2|\vec{x}|^3}\left[-3(\vec{J} - (\vec{J}.\hat{\vec{x}})\hat{\vec{x}} + 3\vec{J} - \vec{J}\right]$$

as $\partial_j x_i = \delta_{ji}$

$$= \underline{\underline{\frac{2G}{c^2|\vec{x}|^3}\left[\vec{J} - 3(\vec{J}.\hat{\vec{x}})\hat{\vec{x}}\right]}}$$

Q9. The metric in linearised gravity outside a spherically-symmetric point mass:

$$ds^2 = c^2\left(1-\frac{2\mu}{r}\right)dt^2 - \left(1+\frac{2\mu}{r}\right)(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\varphi^2)$$

The Lagrangian: $\mathcal{L} = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$

$$\mathcal{L} = c^2\left(1-\frac{2\mu}{r}\right)\dot{t}^2 - \left(1+\frac{2\mu}{r}\right)(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2)$$

The geodesic equations are obtained from the Euler-Lagrange equation:
(equatorial plane $\rightarrow \theta = \frac{\pi}{2}$)

$$\frac{d}{d\sigma}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right) - \frac{\partial \mathcal{L}}{\partial x^\mu} = 0$$

$\underline{t}$ $\frac{d}{d\sigma}\left[c^2\left(1-\frac{2\mu}{r}\right)2\dot{t}\right] = 0$

$$\left(1-\frac{2\mu}{r}\right)\dot{t} = k \quad \text{(constant)} \qquad (1)$$

$\underline{\varphi}$ $\frac{d}{d\sigma}\left[\left(1+\frac{2\mu}{r}\right)r^2 2\dot{\varphi}\right] = 0$

$$\left(1+\frac{2\mu}{r}\right)r^2\dot{\varphi} = h \quad \text{(constant)} \qquad (2)$$

1st integral: $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \epsilon^2$ $\qquad \epsilon^2 = \begin{cases} c^2 & \text{for massive particles} \\ 0 & \text{for photons} \end{cases}$

$$c^2\left(1-\frac{2\mu}{r}\right)\dot{t}^2 - \left(1+\frac{2\mu}{r}\right)(\dot{r}^2 + r^2\dot{\varphi}^2) = \epsilon^2$$

Substituting for $\dot{t}$ and $\dot{\varphi}$ using (1) and (2):

$$c^2\left(1-\frac{2\mu}{r}\right)k^2\left(1-\frac{2\mu}{r}\right)^{-2} - \left(1+\frac{2\mu}{r}\right)\dot{r}^2 - \left(1+\frac{2\mu}{r}\right)r^2h^2r^{-4}\left(1+\frac{2\mu}{r}\right)^{-2} = \epsilon^2$$

$$c^2k^2 - \left(1-\frac{2\mu}{r}\right)\left(1+\frac{2\mu}{r}\right)\dot{r}^2 - \frac{h^2}{r^2}\left(1-\frac{2\mu}{r}\right)\left(1+\frac{2\mu}{r}\right)^{-1} = \epsilon^2\left(1-\frac{2\mu}{r}\right)$$

(the 'energy' equation)

$$\dot{r} = \frac{dr}{d\tau} = \frac{dr}{d\varphi}\frac{d\varphi}{d\tau} = \frac{dr}{d\varphi}\dot{\varphi}$$

$$\dot{r} = \frac{dr}{d\varphi}\frac{h}{r^2}\left(1+\frac{2\mu}{r}\right)^{-1}$$

Substituting in to the 'energy' equation:

$$c^2k^2 - \left(1-\frac{2\mu}{r}\right)\left(1+\frac{2\mu}{r}\right)^{-1}\left(\frac{h}{r^2}\frac{dr}{d\varphi}\right)^2 - \frac{h^2}{r^2}\left(1-\frac{2\mu}{r}\right)\left(1+\frac{2\mu}{r}\right)^{-1} = \epsilon^2\left(1-\frac{2\mu}{r}\right)$$

Let $u \equiv \frac{1}{r}$ $\qquad \frac{du}{dr} = -\frac{1}{r^2} = -u^2$

$$\frac{dr}{d\varphi} = \frac{du}{d\varphi}\frac{dr}{du} = -u^{-2}\frac{du}{d\varphi}$$

$$c^2k^2 - (1-2\mu u)(1+2\mu u)^{-1}h^2u^4u^{-4}\left(\frac{du}{d\varphi}\right)^2 - h^2u^2(1-2\mu u)(1+2\mu u)^{-1} = \epsilon^2(1-2\mu u)$$

$$h^2\frac{1-2\mu u}{1+2\mu u}\left[\left(\frac{du}{d\varphi}\right)^2 + u^2\right] = c^2k^2 - \epsilon^2(1-2\mu u)$$

$$\Rightarrow \left(\frac{du}{d\varphi}\right)^2 + u^2 = \frac{1}{h^2}\left[c^2k^2 - \epsilon^2(1-2\mu u)\right]\frac{1+2\mu u}{1-2\mu u}$$

(the 'shape' equation)

For photons, $\epsilon = 0$

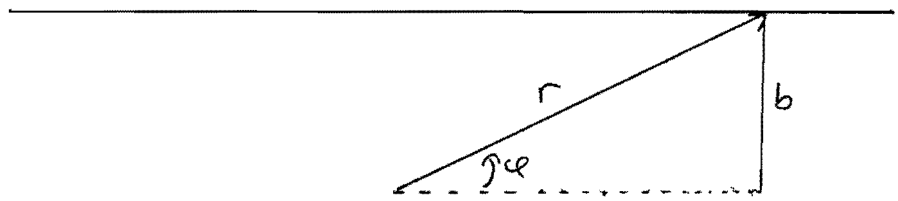
$$\left(\frac{du}{d\varphi}\right)^2 + u^2 = \frac{c^2k^2}{h^2}\frac{1+2\mu u}{1-2\mu u}$$

And for a weak field, $\mu u \ll 1$ $\left(\frac{GM}{c^2r} \ll 1\right)$, so can take binomial expansion of RHS:

$$\left(\frac{du}{d\varphi}\right)^2 + u^2 \sim \frac{c^2k^2}{h^2}(1+2\mu u)(1+2\mu u + \dots)$$

$$\left(\frac{du}{d\varphi}\right)^2 + u^2 \sim \frac{c^2k^2}{h^2}(1+4\mu u) \quad \text{to 1st order}$$

With no mass, light path is straight line with impact parameter b:



$$r = \frac{b}{\sin\varphi} \qquad u = \frac{\sin\varphi}{b}$$

The 'shape' equation for a photon orbit in linearised gravity:

$$\left(\frac{du}{d\varphi}\right)^2 + u^2 \sim \frac{c^2k^2}{h^2}(1+4\mu u)$$

With no mass ($\mu = 0$), substituting in $u = \frac{\sin\varphi}{b}$

$$\frac{\cos^2\varphi + \sin^2\varphi}{b^2} = \frac{c^2k^2}{h^2}$$

We identify the impact parameter, $b = \frac{h}{ck}$

Differentiating the 'shape' equation wrt φ:

$$2\frac{du}{d\varphi}\frac{d^2u}{d\varphi^2} + 2u\frac{du}{d\varphi} = \frac{4\mu}{b^2}\frac{du}{d\varphi}$$

$$\frac{d^2u}{d\varphi^2} + u = \frac{2\mu}{b^2} \qquad (*)$$

Now, taking a small perturbation, Δu, to the solution for no mass:

$$u = \frac{\sin\varphi}{b} + \Delta u$$

Substituting into $(*)$:

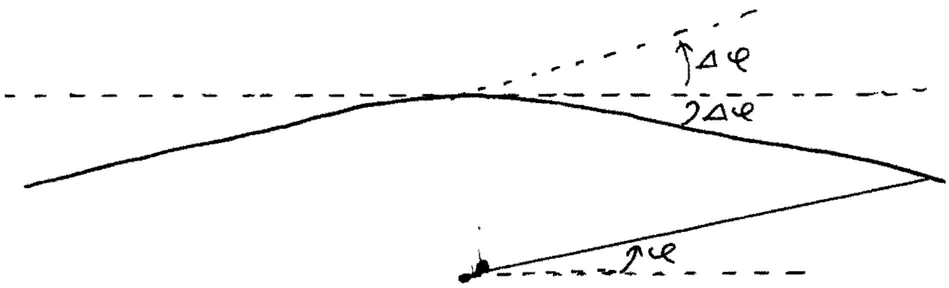
$$-\frac{\sin\varphi}{b} + \frac{d^2\Delta u}{d\varphi^2} + \frac{\sin\varphi}{b} + \Delta u = \frac{2\mu}{b^2}$$

$$\frac{d^2\Delta u}{d\varphi^2} + \Delta u = \frac{2\mu}{b^2}$$

Solution to this:

$$\Delta u = \frac{2\mu}{b^2}$$

so, $$u = \frac{\sin\varphi}{b} + \frac{2\mu}{b^2}$$



To get the deflection, $r \to \infty$ $(u \to 0)$

Without mass, $\varphi = 0$ at $r = \infty$, so for small deflection, φ is small, so $\sin\varphi \sim \varphi$:

$$0 = \frac{\varphi}{b} + \frac{2\mu}{b^2}$$

$$\underline{\varphi = -\frac{2\mu}{b}}$$

The total deflection (both sides of the mass) is twice this

$$\underline{\underline{2\varphi = \frac{4\mu}{b}}}$$

(Same deflection as obtained using full Schwarzschild metric).

Q10. In linearised GR, the Riemann tensor is given by

$$R_{\sigma\mu\nu\rho} = \tfrac{1}{2}(\partial_\nu \partial_\mu h_{\sigma\rho} + \partial_\rho \partial_\sigma h_{\mu\nu} - \partial_\nu \partial_\sigma h_{\mu\rho} - \partial_\rho \partial_\mu h_{\sigma\nu})$$

Where the metric is $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ for small $h_{\mu\nu}$.

For a plane gravity wave, $h_{\mu\nu} = A_{\mu\nu} e^{ik_\lambda x^\lambda}$

$$\partial_\sigma h_{\mu\nu} = ik_\sigma h_{\mu\nu}$$

$$\partial_\rho \partial_\sigma h_{\mu\nu} = -k_\rho k_\sigma h_{\mu\nu}$$

$$\therefore R_{\sigma\mu\nu\rho} = \tfrac{1}{2}(k_\nu k_\sigma h_{\mu\rho} + k_\rho k_\mu h_{\sigma\nu} - k_\nu k_\mu h_{\sigma\rho} - k_\rho k_\sigma h_{\mu\nu})$$

The Ricci tensor:

$$R_{\mu\nu} = R^\rho{}_{\mu\nu\rho} = \tfrac{1}{2}(k_\nu k^\rho h_{\mu\rho} + k_\rho k_\mu h^\rho{}_\nu - k_\nu k_\mu h^\rho{}_\rho - k_\rho k^\rho h_{\mu\nu})$$

$k^2 = k_\rho k^\rho$ $\qquad$ $k_\rho h^\rho{}_\nu = k^\rho h_{\rho\nu}$

$$R_{\mu\nu} = \tfrac{1}{2}(k_\nu k^\rho h_{\mu\rho} + k_\mu k^\rho h_{\rho\nu} - k_\mu k_\nu h^\rho_\rho - k^2 h_{\mu\nu})$$

$$= \tfrac{1}{2}(k_\nu W_\mu + k_\mu W_\nu - k^2 h_{\mu\nu})$$

where $W_\mu = k^\rho h_{\mu\rho} - \tfrac{1}{2} k_\mu h$ $\qquad (h = h^\rho_\rho)$

And the Ricci scalar:

$$R = R^\mu_\mu = \tfrac{1}{2}(k^\mu k^\rho h_{\mu\rho} + k^\mu k^\rho h_{\rho\mu} - k^\mu k_\mu h^\rho_\rho - k^2 h^\mu_\mu)$$

$$= 0$$

So, the Einstein field equation requires

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = -\kappa T_{\mu\nu} \qquad T_{\mu\nu} = 0 \text{ (empty space)}$$

$$R_{\mu\nu} = 0$$

$$\therefore k^2 h_{\mu\nu} = k_\nu W_\mu + k_\mu W_\nu \qquad (*)$$

If $k^2 \neq 0$,

$$h_{\mu\nu} = \frac{1}{k^2}\left(k_\nu W_\mu + k_\mu W_\nu\right)$$

In this case, the Riemann tensor:

$$R_{\sigma\mu\nu\rho} = \frac{1}{2k^2}\Big(k_\nu k_\sigma(k_\mu W_\rho + k_\rho W_\mu) + k_\rho k_\mu(k_\sigma W_\nu + k_\nu W_\sigma) - k_\nu k_\mu(k_\sigma W_\rho + k_\rho W_\sigma) - k_\rho k_\sigma(k_\mu W_\nu + k_\nu W_\mu)\Big)$$

$$= \frac{1}{2k^2}\Big(k_\nu k_\sigma(k_\mu W_\rho + k_\rho W_\mu) + k_\rho k_\mu(k_\sigma W_\nu + k_\nu W_\sigma) - k_\nu k_\sigma(k_\mu W_\rho + k_\rho W_\mu) - k_\rho k_\mu(k_\sigma W_\nu + k_\nu W_\sigma)\Big)$$

(relabelling indices)

$$= \underline{\underline{0}}$$

$\Rightarrow$ <u>flat</u> spacetime $\rightarrow$ not a physical wave
$\rightarrow$ just oscillation of co-ordinate system.

If $k = 0$, to satisfy (b),

$$k_\mu W_\nu + k_\nu W_\mu = 0 \quad \Rightarrow \quad W_\mu = 0$$

$$\therefore k^\rho h_{\mu\rho} = \frac{1}{2} k_\mu h \quad \text{(from definition of } W_\mu\text{)} \quad (f)$$

$$R_{\sigma\mu\nu\rho}k^\rho = \frac{1}{2}\left(k_\nu k_\sigma h_{\mu\rho}k^\rho + k_\rho k_\mu h_{\sigma\nu}k^\rho - k_\nu k_\mu h_{\sigma\rho}k^\rho - k_\rho k_\sigma h_{\mu\nu}k^\rho\right)$$

$$= \frac{1}{2}\left(k_\nu k_\sigma h_{\mu\rho}k^\rho + k^2 k_\mu h_{\sigma\nu} - k_\nu k_\mu h_{\sigma\rho}k^\rho - k^2 k_\sigma h_{\mu\nu}\right)$$

$k^2 = 0$:

$$= \frac{1}{2}\left(k_\nu k_\sigma h_{\mu\rho}k^\rho - k_\nu k_\mu h_{\sigma\rho}k^\rho\right)$$

using (f):

$$= \frac{1}{2}\left(k_\nu k_\sigma\left(\tfrac{1}{2}k_\mu h\right) - k_\nu k_\mu\left(\tfrac{1}{2}k_\sigma h\right)\right)$$

$$= \underline{\underline{0}}$$