Pay-As-Bid vs. First-Price Auctions
Similarities and differences in strategic behavior

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Abstract
Pay-as-bid auctions extend the rules of the well-known first-price auction to the sale of multiple units of the same good. According to a common understanding of the recent literature, strategic incentives in pay-as-bid auctions differ from those in the first-price auctions when bidders have multi-unit demand. I show that each of N symmetrically informed bidders shades his bid for 1 of N shares of a perfectly divisible good in a pay-as-bid auction as if he competed with (N-1)N bidders for one indivisible good in a first-price auction. This analogy carries over to environments where bidders have private information if equilibrium demand schedules are additively separable in the type but breaks otherwise. Whether bidding in pay-as-bid auctions is more complex than in first-price auctions thus depends on the type of uncertainty bidders face.

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JEL classification: D44, D47, D82

1 Introduction

The pay-as-bid auction, also known as discriminatory price auction, is a popular mechanism for allocating assets and commodities worldwide. It extends the rules of the well-known first-price auction to the sale of multiple units of the same good: Bidders submit bidding schedules which specify a price for each unit they demand. Individual demands are then aggregated by the auctioneer to determine the market clearing price above which all bids win. All winners pay “as-they-bid” for all units they won. The pay-as-bid auction is very popular among governments and central banks. It is used to allocate Treasury bonds and implement other operations such as Quantitative Easing on the open market. Outside the financial sector it distributes carbon credits as well as electricity generation in several countries. In total, trillions of dollars are transferred every year using this type of auction. Despite its importance, we know little about strategies used by auction participants. Expect in special circumstances we are even unable to compute best-response strategies. To a large extent the literature focuses on the case of single-unit demand. Assuming that each bidder wants at most one unit is a simplifying assumption that is violated in most real-world applications. A bank bidding in a Treasury auction, for instance, clearly wants

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1For more details see Brenner et al. (2009), Bartolini and Cottarelli (1997), Ghazizadeh et al. (2007), Maurer and Barroso (2011).
more than just a single dollar worth of the offered Treasury bill. With multi-unit demand bidding strategies in pay-as-bid auctions are, according to a common understanding of the recent literature, more complicated than those in its single-unit counterpart, the first-price auction. The reason is that changing one’s n’th bid may affect not only whether the n’th unit is won, but also the bidder’s belief of where the market will clear. To optimize their payoffs, bidders have incentives to shade their bids for each unit differently. This behavior, known as “strategic bid shading” or “demand reduction”, is by design not present in auctions of a single, indivisible object. It is seen as “the key to why the analogy between single-unit and multi-unit auctions does not apply” (Ausubel et al. (2014), p. 1367).

In a simple theoretic framework in which bidders with multi-unit demand compete for shares of a perfectly divisible good, I argue that the complexity of pay-as-bid auctions comes not from demand reduction but more specifically type-dependent demand reduction. In my benchmark model \( N \geq 2 \) risk-neutral bidders are symmetrically informed. They all share the same type, known to them but not the seller, but are uncertain about the total amount of the good that will be for sale. I discover the following analogy between pay-as-bid and first-price auctions: Each of \( N \) symmetrically informed bidders shades his bid in the symmetric equilibrium of the pay-as-bid auction for 1 of \( N \) shares of the perfectly divisible good as if he competed with \((N - 1)N\) bidders for one indivisible good in a canonical first-price auction with independent private types. This analogy might break when bidders are not symmetrically informed but have private information, i.e. types. Whether bidding in pay-as-bid auctions is more complex thus depends on the source of uncertainty bidders face. With private information it can be optimal for bidders of different types to reduce demand in different ways. This suggests that it is not demand reduction (or differential bid shading) per se that makes bidding choices in pay-as-bid auctions more difficult. What gives rise to complicated equilibrium effects seems to be type-dependent demand reduction instead. Such type dependency introduces asymmetric trade-offs not only across units of the good, but also agents. It therewith generates complications that have no equivalent in single-unit auctions.

My findings build on an intuitive bid-representation theorem for pay-as-bid auctions. It characterizes the functional form of the bidding schedule when bidders are symmetrically informed (benchmark model) and - with some limitations - when they have private information (model extension). In future work my theorem might serve as basis to construct equilibrium strategies for other, potentially more general environments, with asymmetrically informed

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2 “Except in the case where bidders have demand for only a single unit of the auctioned commodity, the analysis of multi-unit auctions are [...] more difficult than that of single-unit auctions” (Hortacsu (2011), p. 345).

3 This modeling assumption nicely reflects two common features of real-world pay-as-bid auctions. For one, the amount to be allotted is in some cases, such as Treasury auctions in Germany, Greece, Belgium, Turkey or Sweden (Brenner et al. (2009)), adaptable during the auction. Secondly, the total supply is often shaped by so called “non-competitive” tenders. These are irregular bids in that only a quantity is specified. The price is determined automatically. It is either the average price paid by (regular) bidders or the market clearing price. How many non-competitive tenders will be served is unknown to the (regular) bidders so that the total amount for sale that matters for them is random.
bidders that have multi-unit demand. Computing such equilibrium strategies for pay-as-bid auctions is still an open question in the literature (Hortaçsu and Kastl (2012)). To illustrate how to use my bid-representation theorem to construct equilibria, I conclude the article by deriving an equilibrium in linear bidding strategies. This equilibrium is new to the literature. It is the counterpart to Ausubel et al. (2014)'s linear equilibrium without private information and helps us understand which role private information plays for strategic incentives in pay-as-bid auctions: In the symmetric equilibrium, privately informed agents bid like symmetrically informed agents who all draw the the lowest type, but add a type-specific discount factor.

More generally, my work could be a first step into establishing a more general theoretic connection between bidding in first-price and bidding in pay-as-bid auctions with multi-unit demand. Such a methodological link would increase our poor knowledge of pay-as-bid auctions. We could re-visit the literature on first-price auctions. In contrast to pay-as-bid auctions, first-price auctions have been at the core of auction theory since the very beginning (Vickery (1961)). They have been studied extensively and are well understood. When bidders have independent private values, we know, for instance, that first-price auctions can be revenue equivalent to second-price auctions, and that they are strategically equivalent to the Dutch auction. For the pay-as-bid auction we know much less. We do not know whether, and if so under which conditions, it might be strategically or revenue equivalent to another auction format, for instance the uniform-price auction. It differs from the pay-as-bid auction only in that bidders pay the market clearing price for all units they win, instead of their individual bids. The existing literature has not come to a consensus on which of the two auctions are more efficient or bring higher revenue. By finding conditions that influence the distribution of winning quantities (which is at the center of my analysis) it might be possible to distinguish cases under which either auction format dominates the other and explain why.

Related Literature. Building on the literature of “share auctions”, put forward by Wilson (1979) and further developed most notably by Back and Zender (1993) and Wang and Zender (2002), my analysis of the benchmark model with symmetrically informed bidders is complementary to Pycia and Woodward (2017). In independent work, we derived the functional form of the equilibrium bidding function under the simplifying assumption that bidders are symmetrically informed. Relative to previous studies, such as Wang and Zender (2002) as well as Ausubel et al. (2014), our result is more general in that we neither impose marginal utility to be linear nor total supply to be distributed according to the Pareto distribution. Instead, our theorem holds under a very broad class of utility functions and distributions. It is similar to Holmberg (2009) who studies pay-as-bid procurement auctions with gen-


eral cost functions (here utility functions) and perfectly inelastic demand (here supply). He shows that an equilibrium exists if the hazard rate of demand is monotonically decreasing and bidders have non-decreasing marginal costs. Pycia and Woodward (2017) go one step further in bringing attention to more general sufficient conditions under which equilibrium existence is guaranteed. As such, their work has been acknowledged as the best unique existence result currently available for pay-as-bid auctions (Hortaçsu and McAdams (2018)). Before focusing on the design of pay-as-bid auctions by optimally choosing the distribution of total supply and a reserve price, Pycia and Woodward (2017) show that the equilibrium is symmetric, strictly monotone and differentiable in quantity rather than assuming those properties. Contrary to my work, they do not explain the shape of the bidding function in any detail. This could come from a difference in the way we express the bidding function. Their representation highlights that a “bid for any quantity is a weighted average of the bidder’s marginal values for this and larger quantities, where the weights are independent of the bidder’s marginal values” (p. 4). Mine, instead, underlines the direct connection of bidding behavior to the first-price auction. Therewith I am, to the best of my knowledge, the first to discover this particular linkage. As my main goal is to work out this correspondence rather than to solve the most general model of pay-as-bid auctions, I will make more stringent technical assumptions that simplify the mathematical derivations than I would need to. I invite who is interested in a more general framework with symmetrically informed bidders to consult Pycia and Woodward (2017).

In contrast to Holmberg (2009) and Pycia and Woodward (2017) I make first steps towards an auction environment in which bidders are asymmetrically informed. In a model extension, each bidder draws an independent private type. While the benchmark model with identical bidders is helpful to fix ideas and identify key factors that drive bidding behavior in the multi-unit auction, it is not so useful for evaluating performance. Auctions are typically run to extract individual information from agents, so as to allocate resources to those who benefit the most at the highest price possible. Only a framework with private information allows us to analyze information aggregation and efficiency. Furthermore, the extension towards a framework with independent private values helps to close the gap between theoretic and related empirical work which tends to build on models with private values. Starting with Hortaçsu (2002) researchers have estimated the bidders’ private, marginal willingness to pay in multi-unit auctions (see also Février et al. (2004), Hortaçsu and McAdams (2010), Kastl (2011, 2012), Hortaçsu and Kastl (2012), Cassola et al. (2012), Hortaçsu et al. (2018)). The structural estimation approach is based on an implicit characterization of the bidding function in form of the first-order conditions. For each unit-bid, these necessary conditions have been recognized to capture a similar trade-off to the one in a first-price auction, where bidders trade-off the probability of winning against their gain from it (Kastl (2017)). Unfortunately, these first-order conditions are not informative for a theorist. The reason is that they depend on the distribution of the market clearing price. When bidders have private types it can be defined only implicitly (for any given set of strategies) via market clearing. The econometrician is able to simulate this distribution from the data. The theorist is not. Woodward (2016) nicely reflects the state of the art on pay-as-bid share auctions with pri-
vate types. He proves equilibrium existence in pay-as-bid auctions with private types without specifying the equilibrium bidding function. He shows that bidders might have incentives to “iron”, that is flatten, their bidding functions for small amounts which they are certain to win.

In the remainder of the article, Section 2 sets up the benchmark model with symmetrically informed bidders and states the bid-representation theorem. It builds the basis for the comparison of bidding in the pay-as-bid auction to bidding in the canonical first-price auction (Section 3). I then provide an extension of the main result to an environment with independent private values (Section 4). Before concluding in Section 5, Section 4.1 focuses on a linear example. All proofs are given in the Appendix. Random variables will be highlighted in bold throughout the article.

2 Benchmark Model

\( N \geq 2 \) risk-neutral bidders participate in a pay-as-bid auction. They share the same type \( t \) drawn from some commonly known distribution. It is unknown to the seller. From the perspective of the bidder this common type has no strategic relevance because it is known to all of them. It is fixed at some value \( t \) throughout the analysis. Instead, bidders are uncertain about the total amount of the perfectly divisible good that is for sale, \( Q \). Independent of the bidders’ type, it is drawn from some commonly known, non-degenerated and twice-differentiable distribution \( F_Q(\cdot) \) with bounded support \([0, \overline{Q}] > 0\) and strictly positive density \( f_Q(\cdot) \). Imposing a zero lower bound will simplify the analysis later on. It will rule out that bidders have incentives to iron their bids when they have private types (see Woodward (2016)). In practice the zero lower bound could come from a non-zero probability that the auction is cancelled.

Consuming quantity \( q \) generates utility for each bidder. The marginal utility \( v(q) \) represents the bidder’s true marginal willingness to pay for this amount. \( v(\cdot) \) is strictly decreasing, and twice differentiable. Agents can have a satiation quantity, \( q^s \). This is the amount at which the agent’s marginal valuation turns 0: \( v(q) = 0 \) for \( q \geq q^s \). It is assumed to be large, \( q^s \geq \overline{Q}/N \), for simplicity. If \( q^s \to \infty \), winning some more at a price of zero is always better.

Based on his true marginal willingness to pay each bidder submits a weakly decreasing and differentiable bidding function: \( b_i(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+ \). It is an inverse demand, mapping from the quantity-space into the space of prices. The corresponding demand function maps from prices to quantities. It is denoted by \( x_i(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+ \).

\(^6\)Relying on an idea by Pycia and Woodward (2017) Appendix 4 shows that results extend to distributions with unbounded support in presence of an arbitrarily small but positive reserve price. Since the agents’ true marginal willingness to pay is decreasing by assumption, it will drop below the reserve price at some point. The support of the quantity that will matter for bidding decisions is therefore bounded endogenously. Without positive reserve price and unbounded support, the bidder’s objective functional might not be well-defined because the expectation of the bidder’s total surplus might not exist.
Once all bidders have submitted their individual demands, each market clears at the minimal price for which the aggregate demand of all bidders meets the realized total supply $Q$. If the aggregate demand exactly equals the total supply at the market clearing price $p^c$, each bidder $i$ wins the quantity he demanded at this clearing price: $Q = \sum_i x_i(p^c)$ with $p^c = b_i(q^c_i)$. In that case, all winners pay what they were willing to pay for all units won, abbreviated by $q^c_i \equiv x_i(p^c)$: $\int_0^{q^c_i} b_i(x)dx$. Otherwise, if the aggregate demand at the clearing price is higher than the total supply, bidders have to be rationed according to some tie-breaking rule. In equilibrium no one will have to be rationed because bidding function will be strictly decreasing. This ensures that the market always clears exactly. Which tie breaking rules is used is therefore irrelevant.

From an ex-ante perspective agents do not know how much they will win nor at which price the market will clear. Both the clearing price $p^c$ and the clearing price quantity $q^c_i$ depend on how much there will be for sale $Q$. This amount is random. The adequate solution concept is therefore Bayesian Nash Equilibria. I focus on equilibria in pure-strategies. They consist of a set of bidding functions $\{b^*_i(\cdot)\}_{i=1}^N$ that maximize each bidder $i$’s expected total surplus from winning the ex-ante unknown clearing price quantity $q^c_i$ given all others $j \neq i$ choose $b^*_j(\cdot)$. This total surplus is the difference between the bidder’s total utility from winning the clearing price quantities $\int_0^{q^c_i} v(x)dx$ and his total payments $\int_0^{q^c_i} b_i(x)dx$.

**Definition 1.** A pure-strategy Bayesian Nash Equilibrium (BNE) is a set of bidding functions such that $b^*_i(\cdot) \in \arg \max_{b_i(\cdot)} \mathbb{E} \left[ \int_0^{q^c_i} v(x) - b_i(x)dx \right]$ $\forall i \in N$.

Given the symmetric environment it is natural to restrict attention to symmetric equilibria. In such equilibria bidders share the total supply equally. Later on, it will be convenient to work with the agent’s “equilibrium winning quantity”, instead of the total supply: $q^* = \frac{Q}{N} \in \left[0, \frac{Q}{N}\right] \equiv [0, \bar{q}]$.

Its marginal distribution and density will be denoted by $F_{q^*}(\cdot)$ and $f_{q^*}(\cdot)$.

Having introduced the environment, I turn to the core of the article. I first derive my main result for the benchmark model before generalizing it to an environment with private information.

### 3 Pay-As-Bid vs. First-Price Auctions

My goal is to highlight differences and similarities in bidding strategies between pay-as-bid and first-price auctions. The following bid-representation theorem will serve as basis for the discussion. The Appendix shows that the bidding function is equivalent to Pycia and Woodward (2017)’s Theorem 3.
Theorem 1. Consider distributions of total supply with weakly decreasing hazard rate. There exists a pure-strategy Bayesian Nash equilibrium in which all bidders submit

$$b^*(q) = v(q) - \left( \int_q^{\bar{q}} \left[ \frac{1 - F_{q^*}(x)}{1 - F_{q^*}(q)} \right]^{N-1} \left( -1 \right) \left( \frac{\partial v(x)}{\partial q} \right) dx \right) \text{ on } [0, \bar{q}^*]$$

(1)

and $$b^*(q) = v(q)$$ for $$q \in (\bar{q}^*, \infty)$$. 

The equilibrium exists if total supply is drawn from a distribution with weakly decreasing hazard rate which implies that the distribution of winning quantities $$F_{q^*}$$ has this property. This existence condition is known in the literature (see Holmberg (2009)). It ensures that bidders do not have incentives to deviate from the equilibrium strategy. Technically it is a sufficient condition in the maximization problem that each bidder solves to determine his best reply. Recently, Pycia and Woodward (2017) have derived a weaker condition. They also show that (1) is the only function that can arise in any (not necessarily symmetric) equilibrium on the domain of relevant quantities $$q \in [0, \bar{q}^*]$$. Higher amounts are unachievable. Since no agent ever wins these high amounts nor pays for them, they are out of equilibrium. The bidder’s choice for those high amounts is irrelevant as long as his bidding function is decreasing on the whole domain $$\mathbb{R}^+$$. Here I consider the most natural equilibrium in which the agents bid truthfully for unfeasibly large quantities.

For attainable quantities, the bidding function (1) is surprisingly simple. Because the bidder “pay-as-he-bids” he understates his true marginal willingness to pay for each unit that he might purchase in equilibrium: $$v(q)$$. This is similar to an independent private-value sealed-bid first-price auction, where bidders shade their true types.

The symmetric equilibrium of a canonical first-price auction with $$N \geq 2$$ bidders, each drawing an independent private value $$s \in [0, S]$$ from a common distribution $$F_s(s)$$, is well known. Given his true marginal willingness to pay for the indivisible object, $$v(s) = s$$, the bidder submits

$$\beta^*(s) = v(s) - \left( \int_0^s \left[ \frac{F_s(x)}{F_s(s)} \right]^{N-1} \left( \frac{\partial v(x)}{\partial s} \right) dx \right) \text{ on } [0, S].$$

(1b)

The strategy function maps the agent’s true type into his price offer. Whoever offers the highest price wins the object.

Comparing the bidding functions (1) with (1b) reveals the close similarity between bidding behavior in the pay-as-bid and first-price auction. To see it, however, one must eliminate two differences that distinguish the two functions due to differences in the two set-ups. First,
the uncertainty that bidders face comes from different sources. In the first-price auctions agents have private types. The bidder wins if he has the highest private value: \( s \geq s_j \forall j \neq i \).

In the stylized pay-as-bid auction there are no private types. The equilibrium quantities, each representing a share of the perfectly divisible good, take their place. A bidder now wins \( q \) when the market has not cleared yet: \( Nq \leq Q = Nq^* \). To draw the analogy between both auction formats one must compare the type \( s \) with the equilibrium share \( q^* \) and the corresponding probabilities that determine whether the agent wins or not:

\[
s \leftrightarrow q^* \text{ and } F_s(\cdot) \leftrightarrow 1 - F_{q^*}(\cdot). \tag{2}
\]

Second, the agent’s true valuation for the object is strictly increasing in the first-price auction, while it is strictly decreasing in the pay-as-bid auction. This inverts the bounds of the integrals. In the pay-as-bid auction the integrals go from the realization \( q \) to its largest possible value; in the first-price auction, from the smallest value 0 of the realization to the draw \( s \):

\[
\int_0^s \ldots v'(x)dx \leftrightarrow \int_q^{q^*} \ldots (-1)v'(x)dx. \tag{3}
\]

Bearing (2) and (3) in mind, the two bidding functions differ in one element only: The exponent of the bidding function of the pay-as-bid auction is \( \frac{N-1}{N} \), the one of the first-price auction is \( N - 1 \). In case \( N - 1 \) would equal \( (N - 1)N \) the bidding function of the first-price auction would be analogous to the one in the pay-as-bid auction. This gives rise to the following observation.

**Main Result 1.** In the symmetric equilibrium of the pay-as-bid auction with symmetrically informed bidders, each bidder shades his bid for 1 of \( N \) shares as if he competed with \( (N-1)N \) bidders for 1 indivisible good in a first-price auction with independent private values.

The result is intuitive: In a single good first-price auction, uncertainty over types can be aggregated. A bidder effectively bids against one other bidder whose type is a random variable with the same distribution as the highest order statistic of the common distribution of types. In this sense he chooses his bid given the residual demand curve. Crucially for a first-price auction, he bids as if he pays his bid if and only if he wins. What about a multi-unit pay-as-bid auction? Here he also takes the residual demand of all others as given. In a symmetric equilibrium, he is guaranteed to win \( \frac{Q}{N} \), so he always bids as if he wins \( \frac{Q}{N} \). Now, \( Q \) is uncertain, so we have to think of it slightly differently. More precisely, it is optimal to bid as if he wins the marginal share. In this regard the bidder is playing like in a single item first-price auction “on the margin”. Low supply looks like a high type aggregate opponent; the math doesn’t distinguish where this uncertainty comes from. A natural question to ask is whether this result holds when bidders have private information in both auction formats.
4 Extension: Private Information

I make the following adjustments in the set-up: Each bidder now draws an independent private type from the same, commonly known distribution with twice differentiable distribution on bounded support \([t > 0, \bar{t}]\) and strictly positive density. Bidder type \(t_i\) derives a true marginal value \(v(q, t_i)\) from amount \(q\). As above, it is strictly decreasing and twice differentiable in quantity, plus integrable in the type. If it hits the zero line at some finite satiation quantity it remains zero. Having observed their type, all agents submit a type-dependent bidding function \(b_i(\cdot, t_i) : \mathbb{R}^+ \rightarrow \mathbb{R}^+\). It is weakly decreasing and differentiable in quantity.

With these adaptations, a BNE in pure strategies is defined analogously to Definition 1.

**Theorem 2.** In a symmetric pure-strategy Bayesian Nash equilibrium bidders submit

\[
b^*(q, t_i) = v(q, t_i) - \int_q^{\bar{t}_i} \left[ \frac{1 - F_{q^*_i}(x)}{1 - F_{q^*_i}(q)} \right]^{N-1} (-1) \left( \frac{\partial v(q, t_i)}{\partial q} \right) dx \text{ on } q \in [0, \bar{t}_i] \tag{4}
\]

and \(b^*(q, t_i) = v(q, t_i)\) on \(q \in (\bar{t}_i, \infty)\). This equilibrium exists if

(i) distributions of total supply and types are such that the amount an agent wins in the symmetric equilibrium \(q^*_i\) is drawn from a distribution \(F_{q^*_i}\) with weakly decreasing hazard rate and strictly positive density on support \([0, \bar{t}_i]\) and

(ii) the corresponding demand schedule \(x^*(\cdot, t_i) = b^{-1*}(\cdot, t_i)\) is additively separable in \(t_i\).

In the symmetric equilibrium with private information, agents no longer split the total supply equally, \(q^* = \frac{Q}{N}\). The amount an agent wins now depends on his type: \(q^*_i(t_i)\) abbreviated by \(q^*_i\). The equilibrium bidding function \(4\) has the same shape as function \(1\) without private information. Whether this equilibrium exists depends on the underlying distributions of total supply and types as well as the number of participating bidders. Both determine the distribution of \(i\)'s winning quantity \(F_{q^*_i}(\cdot)\). Its shape in turn will determine whether the bidding function of each type \(4\) assumes an inverse function (the demand function) that is additively separable in the type. Without private types, the existence conditions boil down to the assumption that total supply (and with it the winning equilibrium quantity) is drawn from a distribution with decreasing hazard rate (as in Theorem 1). Determining general conditions on the primitives of the model that guarantee existence of this equilibrium is beyond the scope of this article. The generalized theorem, instead, is meant to underline differences and similarities between pay-as-bid auctions relative to first-price auctions in presence of private information. In line with the previous section it allows me to make the following observation.

**Main Result 2.** In the symmetric equilibrium of the pay-as-bid auction with independent private types, each bidder shades his bid for \(1\) of \(N\) shares as if he competed with \((N - 1)N\) bidders in a first-price auction with independent private values provided the submitted demand function is additively separable in their type and strictly decreasing in price.
My analysis highlights a complication in multi-unit auctions that has, to the best of my knowledge, not yet been made explicit in the literature. Strategizing in pay-as-bid auctions might not be as “simple” as bidding in first-price auctions when agents of different types submit demands with different slopes. Intuitively, a type-dependent slope introduces an asymmetry in incentives not only across prices but also agents with different types. Now type $t_i$ reduces his true demand at price $p$ by a different amount than type $t_j$. In other words, bidders do not only reduce their demand differently across prices but each type does it differently. It seems to be the type-dependency that creates complicated equilibrium effects, not demand reduction per se.

As in [Pycia and Woodward (2017)]’s model without private information, the theorem and with it the main result extend to auctions with reserve prices $R > 0$, where total supply or types may be drawn from distributions with potentially unbounded supports (see Appendix 4). This insight could be valuable for the optimal design of pay-as-bid auctions. For first-price auctions where bidders draw independent private types $s \in [0, S]$ from a common distribution $F_s(s)$ the formula for the optimal reserve-price is well known: $R - \left( \frac{1 - F_S(R)}{F_S(R)} \right) = 0$. For pay-as-bid auctions, we do not know how to set reserve prices optimally. My findings might help to determine an analogous formula for an optimal reserve price in pay-as-bid share auctions in presence of private information.

### 4.1 Linear Example

To conclude I illustrate how Theorem 2 can be used to find equilibria of pay-as-bid share auctions in presence of private information. It is, to the best of my knowledge, new to the literature. My approach of finding it could be used in other set-ups.

My aim is to construct a linear example. In search for an equilibrium with a linear bidding function, it is natural to assume that the agents’ true marginal willingness to pay is linear: $v(q, t_i) = \max\{t_i - \rho q, 0\}$ with $\rho > 0$. Assuming linear marginal values alone, however, is not enough to generate linear bidding strategies. To see this, recall the bid-representation of Theorem 2. From there we know that the agent’s function depends nontrivially on the distribution of his winning quantity $F_{q^i}^*(\cdot)$. With linear marginal valuations function (4) becomes

$$b^*(q, t_i) = t_i - \rho q - \rho \int_q^{q^i} \frac{1 - F_{q^i}(x)}{1 - F_{q^i}(q)} \frac{N_x}{N} dx. \quad (4)$$

For many distributions the integral, and with it the bidding function, will not be linear in quantity. For an auction environment without private types, [Ausubel et al. (2014)] show that equilibria are linear only if the per-capita supply (here referred to as $i$’s equilibrium

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8 Even though I do not show that this necessary condition is also sufficient, I expect that it is not possible to derive an equilibrium bidding function in the pay-as-bid auction that has the discussed similarities to the one of the first-price auction.
winning quantity, \( q^* \) is drawn from the Generalized Pareto Distribution (GDP). This result extends without complications to an environment with private types where \( q^*_i \) replaces \( q^* \). There are two important differences. First, the distribution of \( i \)'s winning quantity becomes type dependent. More importantly, it is no longer exogenously given by the distribution of total supply but is endogenous. It is now an equilibrium object itself. In the linear example, where demand schedules take the following form

\[
x^*(p, t_i) = a^* + c^*t_i - e^*p \quad \text{with} \quad a^*, c^* \in \mathbb{R}, \ e^* > 0
\]

it depends, by market clearing, on equilibrium coefficient \( c^* \)

\[
q^*_i \equiv \frac{1}{N} \left[ Q - c^* \sum_{j \neq i} t_j + (N - 1)c^*t_i \right].
\]

The winning quantity is a transformed convolution of the independent total supply \( Q \) and \((N - 1)\) iid types, \( t_j \), which are weighted by the equilibrium coefficient, \(-c^*\). As shown in the following corollary, a linear equilibrium exists when \( q^*_i \) follows a Generalized Pareto Distribution. Even though I cannot show that there exist distributions of total supply and types that generate the GDP for \( i \)'s winning quantity, I am optimistic that there are examples. In particular, I conjecture that it is possible to pick suitable Gamma Distributions for total supply and types. My belief comes from the fact that the GDP belongs to the class of Generalized Gamma Convolutions\(^9\), whose elements can be represented as the distribution of the sum of two or more non-constant (not necessarily identically distributed) random variables which are distributed according to a Gamma Distribution\(^10\).

**Corollary 1.** Let \( v(q, t_i) = \max\{t_i - \rho q, 0\} \) with \( \rho > 0 \), and assume \( N > \frac{\rho Q}{\xi} \).

For distributions of total supply and types under which the amount an agent wins in the symmetric equilibrium \( q^*_i \) is drawn from the Generalized Pareto Distribution

\[
F_{q^*_i}(q) = 1 - \left[ \frac{\sigma(\xi, t_i) + \xi q}{\sigma(\xi, t_i)} \right]^{-\frac{1}{\xi}}
\]

with scale parameter

\[
\sigma(\xi, t_i) = -\xi \left[ \frac{N(1 - \xi) - 1}{N(1 - \xi)\rho} \right] (t_i - \xi) - \xi \left( \frac{Q}{N} \right)
\]

\(^9\)This class was introduced by Thorin (1977a,b). It is the smallest class of distributions on \( \mathbb{R}^+ \) that contains Gamma Distributions and is closed with respect to convolution and weak limits. This means that any element of this class is the weak limit of finite convolutions of Gamma Distributions.

\(^10\)Bondesson (1979) showed that distributions that have a density of the form \( f(x) = Cx^{\beta - 1}(1 + cx^\alpha)^{-\gamma}, x > 0, 0 < \alpha \leq 1 \) belong to the class of Generalized Gamma Convolutions. The density of the Generalized Pareto Distribution (with location parameter of 0) is \( f(x) = \frac{1}{\sigma} (1 + \xi x^\sigma)^{-\frac{1}{\xi} + 1}, \) where \( \sigma > 0. \) It can be written as can be written as \( f(x) = (\frac{\theta + 1}{\delta}) (1 + \frac{\theta x}{\delta})^{-\frac{\gamma + 2}{\delta}} \) with \( \xi = \frac{\theta}{\delta + 1} \) and \( \sigma = \frac{\delta}{\delta + 1}. \) Now, setting \( c = \frac{\theta}{\delta} \), \( \alpha = \beta = 1 \) and \( \gamma = \frac{1}{\delta} + 2 \) shows that this density takes the form of a of Generalized Gamma Convolution (Hamedani (2013)).

In statistical terms, one says that the GCC is self-decomposable.
and shape parameter
\[ \xi \in (-\infty, -1] \]
there exists a pure-strategy Bayesian Nash equilibrium in which bidders submit
\[
b^*(q, t_i) = \begin{cases} 
\left( \frac{1}{1-\xi} \right) [t_i - \xi t] - \left( \frac{\rho}{N(1-\xi)-1} \right) [(N-1)q - \xi Q] & \text{for } q \in [0, q^*_i] \\
v(q, t_i) & \text{for } q \in (q^*_i, \infty) \end{cases} \quad \text{with } q^*_i \equiv \left( \frac{\sigma(\xi, t_i)}{-\xi} \right).
\]

The corollary specifies several restrictions on parameters. Before analyzing how agents bid in equilibrium, I explain why.

**Parameter Restrictions.** The first restriction, \( N \geq \frac{Q\rho}{\xi} \), makes sure that the market clears at a non-negative price. In particular, it guarantees that the marginal valuation of the lowest type is non-negative at the highest quantity he might win in equilibrium: \( v(q^*_i, t) \geq 0 \). This in turn ensures that no type will ever submit a bid-price that is negative.\(^{11}\)

The other two conditions restrict the two shape parameters of distribution \( F^{q^*_i} (\cdot) \). For one, \( \xi \) must be weakly smaller than \(-1\) to guarantee that no one has incentives to deviate from the equilibrium. This extra condition is needed because the hazard rate of the Generalized Pareto Distribution with bounded support (\( \xi < 0 \)) is increasing. So far, I have focused on distributions with decreasing hazard rates, where necessary conditions are always sufficient. For distributions with increasing hazard rates they may, or may not be. To avoid creating confusion that would have distracted from my main points I have simply imposed the (unnecessarily strict) condition of decreasing hazard rates in the main body of this article.

Secondly, the scale parameter \( \sigma(\xi, t_i) \) is not just any positive real number but a function of \( \xi \) and \( t_i \). It determines the upper bound of \( q^*_i \)'s support:
\[
q^*_i = \left( \frac{\sigma(\xi, t_i)}{-\xi} \right). \tag{7}
\]

For any fixed types \( t_i \), this upper bound is by definition \( q^*_i \equiv \frac{1}{N} \left[ Q - c^* (n-1) t + (N-1) c^* t_i \right] \) determined by equilibrium coefficient \( c^* \). Since in the equilibrium \( c^* \) is strictly positive, \( i \)'s winning quantity achieves its maximal value when the total supply realizes at its maximum \( Q \) and all other agents draw the minimal type \( t \):
\[
q^*_i \equiv \frac{1}{N} \left[ Q - c^* (n-1) t + c^* (n-1) t_i \right] \quad \text{with } c^* = \left( \frac{N(1-\xi) - 1}{\rho(N-1)(1-\xi)} \right) > 0. \tag{8}
\]

The scale parameter \( \sigma(\xi, t_i) \) must be such that both \(^7\) and \(^8\) hold. Notice that the lowest amount any type may win in equilibrium \( q^*_i \) is always 0 because total supply may be 0.

\(^{11}\)Recall that for large amounts the agent submits his true marginal valuation which is never negative by construction.
Explaining Bidding Behavior. The bidding function is increasing in the type and strictly decreasing in quantity. To derive an intuition for its functional form for relevant quantities \( q \in [0, q^*_i] \), I decompose it into three parts. The first is the true marginal valuation of the lowest type \( v(q, t) \), the second, a type-specific mark-up \( M(t_i) \) and the third a type-independent shading factor \( S(q) \):

\[
b^*(q, t_i) = v(q, t) + M(t_i) - S(q)
\]

with

\[
M(t_i) \equiv \left( \frac{1}{1 - \xi(t_i - t)} \right) \quad \text{and} \quad S(q) \equiv \left( \frac{\overline{Q} - N q}{N(1 - \xi) - 1} \right) (-\xi)\rho \quad \text{for} \quad q \in [0, q^*_i].
\]

Consider first the behavior of the lowest type. Since the distribution of types is common knowledge, the lowest type has no private information. Everybody knows that everyone must at least draw a type of size \( t \). This agent submits his true marginal willingness to pay \( v(q, t) \) in addition to a shading factor \( S(q) \). This factor determines the amount by which his shading differs across quantities (differential bid shading). Such strategic demand reduction is optimal because the true marginal willingness to pay is not constant but strictly decreasing, \( \rho > 0 \). Notice that this factor is independent of the type, which was one of the conditions under which strategizing in pay-as-bid auctions is similar to bidding in first-price auctions. It is strictly positive for \( q = 0 \) and 0 for the highest amount, \( q^*_i \), the lowest type might win.

In contrast, an agent who draws a higher type than the lowest one, values each unit of the good more. He should bid a higher price. If he bid truthfully, he would submit a mark-up of \( (t_i - t) \) for each amount. Since his information is private, however, he does not bid his full extra valuation, but only a fraction \( \left( \frac{1}{1 - \xi} \right) \in (0, \frac{1}{2}] \) of it. This fraction depends on \( \xi \leq -1 \). It assumes its maximal value of \( \frac{1}{2} \) when the winning quantity is uniformly distributed, which is the case for \( \xi = -1 \). As \( \xi \) decreases it approaches 0. Just as the lowest type any other agent shades his bids differently across quantities because the true marginal valuation is strictly decreasing. At the highest winning quantity, the shading factor is strictly positive for any \( t_i > t \). This means that any type higher than the lowest shades the largest amount he might win in the symmetric equilibrium by a type-specific discount in addition to an amount coming from differential bid shading.

Interestingly, behavior of privately informed bidders is similar to behavior of bidders without private information. This is easy to see when comparing my example to previous work by [Ausubel et al., 2014](#) (see Appendix 3.1 for details). They derive the unique linear equilibrium in a pay-as-bid auction in an environment in which agents are only uncertain about the total amount that will be for sale (Proposition 7). As it turns out, agents with private types bid like symmetrically informed agents who all draw the same type \( t \), just adding the type-specific mark-up \( M(t_i) \). All strategic incentives that come from agents having private information are captured by this mark-up.
To close the article I come back to its main theme. Regarding the comparison of bidding in pay-as-bid and first-price auctions, the example illustrates the usefulness of my bid-representation for pay-as-bid auctions. Without it, the similarities of shading behavior across auction formats, summarized in my main results, is difficult to see. This is because bidding functions typically look extremely different, even when the agent’s type in the first-price auction is drawn from the same distribution as i’s winning quantity in the pay-as-bid auction. Those are the two random variables that must be compared to understand the connection (as explained in Section 2 pp. 6 - 8). Under the uniform distribution, for instance

$$\beta^*(s) = \left(\frac{N - 1}{N}\right) s$$  for \(s \in [0, S]\)

$$b^*(q, t_i) = \left(\frac{1}{2}\right) [t_i + t] - \left(\frac{\rho}{2N - 1}\right) [(N - 1)q + Q]$$  for \(q \in [0, q^*_i]\)

where \(q^*_i\) could be normalized to match \(S\), the two bidding functions have not much in common. By comparing them one would not come to the conclusion that in a pay-as-bid auction each bidder shades his bid for 1 of \(N\) shares as if he competed with \((N - 1)N\) bidders in a first-price auction with independent private types (Main Result).

5 Conclusion

Recent literature suggests that strategic incentives in multi-unit auctions differ from those in single-unit auctions when bidders demand more than one unit. It has been shown that bidders shade their bids differentially across quantities when they have multi-unit demand. Such strategic behavior is not present in single-unit auctions and was taken to be the reason for which analogies between single- and multi-unit auctions break down. I refine this view and highlight the importance of the type of uncertainty bidders face. Bidding behavior in a pay-as-bid share auction with symmetrically informed bidders that are uncertain about the total amount for sale, is actually analogous to bidding in the first-price auction: Each of \(N\) bidders shades his bid for 1 of \(N\) shares as if he competed with \((N - 1)N\) bidders for an indivisible good in a first-price auction. This observation can generalize to an environment in which bidders are only ex-ante symmetric, each drawing an iid private type. However, pay-as-bid auctions seem strategically more complex than first-price auctions when agents flatten their bidding functions for small amounts, or shade bids not only differently across quantities, but across types.

Future work could concentrate on analyzing bidding behavior with private information. With our poor knowledge of how equilibria could look like, we know extremely little about how bidders behave in one of the most commonly used auction formats to allocate assets and commodities for high stakes. The complication arises because equilibrium bidding functions are, except in very rare exceptions, non-linear in quantity. This is an important difference to the other most commonly used multi-unit auction format, the uniform-price auction and might be the reason for which we have a much better understanding of bidding in uniform-price
auctions relative to pay-as-bid auctions. In addition to theoretic value, a complete characterization of bidding strategies in pay-as-bid auctions where bidders are not symmetrically informed would be useful for the related empirical literature (in Industrial Organization) with recent work by Hortaçsu et al. (2018), Allen et al. (2018) and others. These papers estimate the true valuations of bidders in multi-unit auctions. A typical goal is to perform a counterfactual analysis to find out how much could be gained when changing the rules of the auction, for example, by introducing a reserve price. Lacking a one-to-one mapping between (estimated) true valuations and (counterfactual) bidding choices, makes it difficult to achieve this goal. My bid-representation theorem might be useful to determine such mapping.

References


Appendix

1 Proof of Theorem 1

Theorem 1 is equivalent to Theorem 2 when types are drawn from a degenerated distribution. To prove Theorem 1 it suffices to make the following changes in the proof of Theorem 2: First, replace the equilibrium winning quantity \( q^*_i \), its distribution and density by the per-capita supply \( q^* \), its distribution and density. Second, notice that the hazard rate of \( q^* \) is weakly decreasing by the assumption that total supply is drawn from a distribution with decreasing hazard rate. Similarly, the second condition for equilibrium existence of Theorem 1 is always satisfied without private types because \( \frac{\partial x(b(q))}{\partial p} = \left( \frac{\partial b(q)}{\partial q} \right)^{-1} \). I invite who would like to see a full proof for the environment with symmetrically informed bidder, to consult Pycia and Woodward’s proof of Theorem 3 (pp. 46-49).

Theorem 1 in relation to the existing literature (Pycia and Woodward (2017), Holmberg (2006, 2009)):

Pycia and Woodward (2017) derive the following bid-representation

\[
b^*(q) = \int_{Nq}^{y} v\left(\frac{y}{N}\right) dF^{N,q,N}(y) \quad \text{with} \quad F^{q,N}(y) \equiv 1 - \left[ \frac{1 - F_Q(y)}{1 - F_Q(Q)} \right]^{\frac{N-1}{N}} \text{for } Q < \bar{Q}.
\]

(9)

In what follows I show that our bidding functions coincide by re-formulating mine

\[
b^*(q) = v(q) - \int_{q}^{\bar{q}} \left[ \frac{1 - F_{q^*}(x)}{1 - F_{q^*}(q)} \right]^{\frac{N-1}{N}} (-1) \left( \frac{\partial v(x)}{\partial q} \right) dx \quad \text{on } [0, \bar{q}^*] \tag{1}
\]

to match theirs. First integrate the \((-1)\cdot\) integral of function (1) by parts:

\[
\int_{q}^{\bar{q}} \left[ \frac{1 - F_{q^*}(x)}{1 - F_{q^*}(q)} \right]^{\frac{N-1}{N}} v'(x)dx = -v(q) + \int_{q}^{\bar{q}} \left[ \frac{N - 1}{N} \right] \left[ \frac{f_{q^*}(x)}{1 - F_{q^*}(q)} \right] \left[ \frac{1 - F_{q^*}(x)}{1 - F_{q^*}(q)} \right]^{-\frac{1}{N}} v(x)dx
\]

to obtain

\[
b^*(q) = \int_{q}^{\bar{q}} \left[ \frac{N - 1}{N} \right] \left[ \frac{f_{q^*}(x)}{1 - F_{q^*}(q)} \right] \left[ \frac{1 - F_{q^*}(x)}{1 - F_{q^*}(q)} \right]^{-\frac{1}{N}} v(x)dx.
\]

Now change the variable of integration to \( y = xN \) and use the distribution function of the total supply, \( F_Q(y) = F_Q(Nx) = F_{q^*}(x) \) and \( f_Q(Nx) = \frac{1}{N} f_{q^*}(x) \):

\[
b^*(q) = \int_{Nq}^{\bar{q}N} \left[ \frac{N - 1}{N} \right] \left[ \frac{f_Q(y)}{1 - F_Q(Nq)} \right] \left[ \frac{1 - F_Q(y)}{1 - F_Q(Nq)} \right]^{-\frac{1}{N}} v\left(\frac{y}{N}\right) dy.
\]

Note that this is the analogue to Holmberg’s bidding function in a pay-as-bid procurement auction (see Holmberg (2006) equation (6) on page 8 with \( \bar{p} = 0 \)).
Finally use \( \frac{\partial}{\partial y} \left\{ 1 - \left[ \frac{1 - F_Q(y)}{1-F_Q(Nq)} \right] \right\} = \left[ \frac{N-1}{N} \right] \left[ \frac{f_Q(y)}{1-F_Q(Nq)} \right] \left[ \frac{1-F_Q(y)}{1-F_Q(Nq)} \right]^{-\frac{1}{N}} \) and pull \( v \left( \frac{y}{N} \right) \) forward

\[ b^*(q) = \int_{Nq}^{\overline{Q}} v \left( \frac{y}{N} \right) \frac{\partial}{\partial y} \left\{ 1 - \left[ \frac{1 - F_Q(y)}{1-F_Q(Nq)} \right] \right\} dy. \]

To obtain Pycia and Woodward (2017)'s representation (9), it suffices to use their auxiliary distribution function \( F_{Q,N}(y) \).

### 2 Proof of Theorem 2

Throughout the proof I work with the distribution of \( i \)'s clearing price quantity \( q_i^c \). This is the quantity the agent wins at market clearing. When all choose the equilibrium quantities it coincides with the distribution of \( i \)'s equilibrium winning quantity \( q_i^* \), but not otherwise.

**Definition 2.** Define the probability that bidder \( i \) obtains at most quantity \( q \) when submitting \( b_i(\cdot, t_i) \) such that \( b_i(q, t_i) = p \) as

\[ G_i(p, q) \equiv \Pr \left( Q - \sum_{j \neq i} x(p, t_j) \leq q \right). \] (10)

Denote its support by \([0, \overline{q}^*_i]\) and the corresponding density by \( g_i(p, q) \).

Notice that the lower bound of the support is 0 because total supply may realize at \( \overline{Q} = 0 \). The upper bound is endogenous. Since total supply and types are bounded, however, there is a maximal amount that the bidder can win even when submitting a price of 0.

The goal is to show that there exists a pure-strategy BNE in which bidders submit

\[ b^*(q, t_i) = v(q, t_i) - \int_{q}^{\overline{q}^*_i} \left[ 1 - \frac{F_{q_i^*}(x)}{1-F_{q_i^*}(q)} \right]^\frac{N-1}{N} \left( -1 \right) \frac{\partial v(q, t_i)}{\partial q} dx \text{ on } q \in [0, \overline{q}^*_i] \] (11)

and \( b^*(q, t_i) = v(q, t_i) \) on \( q \in (\overline{q}^*_i, \infty) \) given that

1. the distributions of total supply and types are such that the amount an agent wins in the symmetric equilibrium \( q_i^* \) is drawn from a distribution \( F_{q_i^*} \) with weakly decreasing hazard rate and strictly positive density on support \([0, \overline{q}^*_i]\)

2. and the corresponding equilibrium demand schedule \( b^*(\cdot, t_i)^{-1} \) is additively separable in the type on the range of relevant quantities \( q \in [0, \overline{q}^*_i] \).

The second condition restricts the class of functions that I can look for: I want to find an equilibrium of the following form

\[ b^*(q, t_i) = \begin{cases} b(q, t_i) & \text{for } q \in [0, \overline{q}^*_i] \\ b_T(q, t_i) & \text{for } q \in (\overline{q}^*_i, \infty) \end{cases} \] (11)
with
\[ b(q, t_i) = y^{-1}(q - \eta(t_i)) \]  \hspace{1cm} (12)
where \( y(\cdot) \) and \( b_T(\cdot, t_i) \) are twice differentiable and strictly decreasing and \( \overline{q}_i^c \) is the maximal amount the agent can win at market clearing (Definition 2). Assuming (12) guarantees that demand schedules are additively separable in type. To see this, solve \( b(q, t_i) = p \) with \( q = x(p, t_i) \) to obtain
\[ x(p, t_i) = \eta(t_i) + y(p). \]  \hspace{1cm} (13)

2.1 The Core of the Argument

The proof is long and mathematically tedious so that it helps to lay out the main line of argument before carrying it out in all details: Take the perspective of bidder \( i \), fix his type \( t_i \), and let all others \( j \neq i \) choose as in equilibrium. My candidate equilibrium function (11) splits into two parts. Consider the second part first: \( b_T(q, t_i) \) for \( q \in (\overline{q}_i^c, \infty) \). Quantities higher than the maximal amount \( i \) can win at market clearing are unachievable. The agent never wins nor pays higher amounts than \( \overline{q}_i^c \). They are irrelevant. It therefore does not matter which prices the agents bids for these amounts, provided the bidding function is differentiable and decreasing on the whole domain \( \mathbb{R}^+ \). In the equilibrium of the theorem I consider best replies in which the agent just behaves truthfully for irrelevant amounts: \[ b_T(q, t_i) = v(q, t_i) \quad \text{for} \quad q \in (\overline{q}_i^c, \infty) = (\overline{q}_i^c, \infty) \quad \text{in the equilibrium}. \]  \hspace{1cm} (14)
Now consider the first part: \( b(q, t_i) \) for \( q \in [0, \overline{q}_i^c] \). The core of the proof is to show that function (4) is played in the symmetric equilibrium on the domain of relevant quantities \([0, \overline{q}_i^c]\) which in equilibrium become \([0, \overline{q}_i^c]\). I must show that it is optimal for agent \( i \) to choose (4) in responds to all others playing it. He chooses his best reply so as to maximize his expected total surplus subject to two constraints, so-called end-point or boundary conditions. The lower bound may be some arbitrary finite price \( \overline{p} \). The upper bound is given by the bidder’s true marginal willingness to pay, as explained above.

\[ \max_{b_i(\cdot, t_i)} V(b_i(\cdot, t_i)) \quad \text{subject to} \quad b_i(0, t_i) = \overline{p} < \infty \quad \text{and} \quad b_i(\overline{q}_i^c, t_i) = v(\overline{q}_i^c, t_i) \]  \hspace{1cm} (M)
where
\[ V(b_i(\cdot, t_i)) = \int_0^{\overline{q}_i^c} \left[ \int_0^q v(x, t_i) - b_i(x, t_i)dx \right] g_i(q, b_i(q, t_i))dq. \]

First, I derive a necessary condition for equilibria with strictly decreasing and differentiable bidding functions (Lemma 1). Since the agent must choose the same strategy as all others I can solve for the solution explicitly: When imposing symmetry across agents \([0, \overline{q}_i^c]\) of (11) becomes \([0, \overline{q}_i^c]\) and the necessary condition a linear differential equation (Lemma 2). The bidding function (4) of the Theorem is its unique solution (Lemma 3). Three auxiliary lemma then show that this candidate is indeed a BNE (Lemma 4). Auxiliary Lemma 1 verifies that function (4) is strictly decreasing and differentiable, Auxiliary Lemma 2 that the function fulfills the sufficient conditions of a local maximum and Auxiliary Lemma 3 that it is globally maximizing the bidder’s objective functional.

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Electronic copy available at: https://ssrn.com/abstract=2937129
2.2 Full Proof

Throughout the proof I will often drop the superscript * and treat all functions as functions of quantity only since the type is fixed. I denote the first and second derivatives w.r.t. to quantity (or price respectively) by ‘ and “, for instance: \( b'(q, t_i) = \frac{\partial b(q, t_i)}{\partial q} \).

**Lemma 1.** Consider a set of strictly decreasing and differentiable functions \( \{b_i(\cdot, t_i)\}_{i=1}^N \). If they constitutes a BNE it must be that

\[
0 = -\left( \frac{\partial G_i(q, b_i(q, t_i))}{\partial b_i(q, t_i)} \right) [v(q, t_i) - b_i(q, t_i)] - [1 - G_i(q, b_i(q, t_i))] \quad (E)
\]

is satisfied point-wise for all \( q \in [0, \bar{q}_i] \) and all \( i = 1...N \).

**Proof.** In order to derive necessary conditions, I first re-state the objective functional so that it depends on the distribution function of \( i \)'s clearing price quantity instead of its density. Integrating by parts I obtain

\[
\mathcal{V}(b_i(\cdot, t_i)) = \int_0^q [v(x, t_i) - b_i(x, t_i)] dx G_i(q, b_i(q, t_i)) \bigg|_{0}^{\bar{q}_i} - \int_0^\bar{q}_i [v(q, t_i) - b_i(q, t_i)] G_i(q, b_i(q, t_i)) dq.
\]

Since \( G_i(0, b_i(0, t_i)) = 0, G_i(\bar{q}_i, b_i(\bar{q}_i, t_i)) = 1 \) for any function and type, and \( [b_i(0, t_i) - v(0, t_i)] < \infty \) by the end-point condition,

\[
\mathcal{V}(b_i(\cdot, t_i)) = \int_0^\bar{q}_i [v(q, t_i) - b_i(q, t_i)] dq - \int_0^\bar{q}_i [v(q, t_i) - b_i(q, t_i)] G_i(q, b_i(q, t_i)) dq.
\]

To determine the necessary conditions of the following functional

\[
V = \int_0^\bar{q}_i \mathcal{F}(q, b_i(q, t_i)) dq \text{ s.t. } b_i(0, t_i) = \bar{p} < \infty \text{ and } b_i(\bar{q}_i, t_i) = v(\bar{q}_i, t_i) \quad (V)
\]

where \( \mathcal{F}(q, b_i(q, t_i)) \equiv [v(q, t_i) - b_i(q, t_i)] [1 - G_i(q, b_i(q, t_i))] \quad (F) \)

one constructs a class of comparison functions, \( b_i(q, t_i) + \varepsilon \kappa(q) \), around the extremal \( b_i(q, t_i) \). \( V \)'s first variation (the analogue of the first derivative) must be 0 for any variation \( \kappa(q) \). The resulting necessary condition is famously known as Euler-Lagrange Equation. In this special case where \( \mathcal{F} \) does not depend on \( b_i'(q, t_i) \) it simplifies to \( 0 = \frac{\partial \mathcal{F}(q, b_i)}{\partial b_i} \). Given \( \mathcal{F} \)'s functional form:

\[
0 = -\left( \frac{\partial G_i(q, b_i)}{\partial b_i} \right) [v(q, t_i) - b_i] - [1 - G_i(q, b_i)]. \quad (E)
\]

In the standard case, in which \( \mathcal{F} \) depends on the slope of \( b_i(\cdot, t_i) \), the Euler-Equation involves two arbitrary coefficients. To find them, one uses the boundary conditions through which the function must run. Here, where \( \mathcal{F} \) is independent of the functions’ slope, \( 0 = \frac{\partial \mathcal{F}(q, b_i)}{\partial b_i} \) must pass through the end points \((0, \bar{p})\) and \((\bar{q}_i, v(\bar{q}_i, t_i)) \) (Elsgolc (1961) p. 31).
**Lemma 2.** Consider a strictly decreasing and differentiable function $b(\cdot, t_i)$. If it constitutes a symmetric BNE then

$$ [v(q, t_i) - b(q, t_i)] = - \left[ \frac{N}{N - 1} \right] \left[ \frac{1 - F_{q_i}^*(q)}{f_{q_i}(q)} \right] b'(q, t_i) \text{ for } q \in [0, \overline{q}_i^*]. \quad (N) $$

**Proof.** To evaluate the Euler-Lagrange Equation at the guessed, symmetric solution (12), recall that the equilibrium demand function $x(\cdot, t_j)$ is of the following form

$$ x(p, t_j) = \eta(t_j) + y(p). \quad (13) $$

Given that all others choose such strategy, agent $i$’s necessary condition must be satisfied if he himself also plays this strategy, i.e.

$$ b_i(q, t_i) = y^{-1}(q - \eta(t_i)). \quad (12) $$

Imposing this symmetry across agents enables me to simplify the necessary condition. The trick is to re-state the condition using the distribution of $i$’s winning quantity in the symmetric equilibrium $F_{q_i}^*$, instead of $i$’s clearing price quantity. The later determines the probability that $i$ wins at most quantity $q$ when offering some price for this amount. The second is this probability when choosing the equilibrium price.

To do so I first calculate the amount $i$ wins when playing the equilibrium guess (13):

$$ Q = q_i^* + \sum_{j \neq i} x(p^*, t_j) \text{ with } p^* = b_i(q_i^*, t_i) \text{ by market clearing} $$

$$ q_i^* = Q - \sum_{j \neq i} \eta(t_j) - \sum_{j \neq i} y(y^{-1}(q_i^* - \eta(t_i))) \text{ by (12), (13)} $$

$$ \Rightarrow q_i^* = \left[ \frac{1}{N} \right] \left\{ Q - \sum_{j \neq i} \eta(t_j) + (N - 1)\eta(t_i) \right\}. \quad (15) $$

**Definition 3.** The probability that bidder $t_i$ wins at most quantity $q \in [0, \overline{q}_i^*]$ in the symmetric equilibrium is

$$ F_{q_i^*}(q) = \Pr(q_i^* \leq q) \quad \text{with } q_i^* \equiv \left[ \frac{1}{N} \right] \left\{ Q - \sum_{j \neq i} \eta(t_j) + (N - 1)\eta(t_i) \right\}. \quad (15) $$

Denote the corresponding density by $f_{q_i^*}(q)$.

To replace $G_i$ by $F_{q_i^*}$, recall the definition of $i$’s clearing price quantity:

$$ G_i(q, p) \overset{10, 13}{=} \Pr \left( Q - \sum_{j \neq i} \eta(t_j) \leq q + (N - 1)y(p) \right) \text{ for any } (q, p). \quad (23) $$

At $p = b(q, t_i) = y^{-1}(q - \eta(t_i))$ the distributions of the clearing price and winning quantity coincide:

$$ G_i(q, b(q, t_i)) = \Pr(q_i^* \leq q) = F_{q_i^*}(q). \quad (16) $$

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To determine the partial derivative of \( G_i(q, b_i(q, t_i)) \) w.r.t. to price \( b_i = b_i(q, t_i) \), I insert the guessed equilibrium function, apply the chain rule and change the random variable from \( Q - \sum_{j \neq i} \eta(t_j) \) to \( q_i^* \) to obtain

\[
\frac{\partial G_i(q, b(q, t_i))}{\partial b_i} = \left[ \frac{N - 1}{N} \right] f_{q_i^*}(q) y'(b(q, t_i)). \tag{17}
\]

By the definition of an inverse and the chain rule \( y'(p) = \left( \frac{1}{y'(q, t_i)} \right) \), so that

\[
\frac{\partial G_i(q, b(q, t_i))}{\partial b_i} = \left[ \frac{N - 1}{N} \right] f_{q_i^*}(q) \left( \frac{1}{y'(q, t_i)} \right). \tag{18}
\]

With (16) and (18) I can evaluate the Euler-Lagrange equation at the guessed symmetric solution where \( q_i^* = \bar{q}_i^* \). \( [E] \) becomes

\[
0 = - \left[ \frac{N - 1}{N} \right] f_{q_i^*}(q) \left( \frac{1}{y'(q, t_i)} \right) \left[ v(q(t_i)) - b(q(t_i)) \right] - \left[ 1 - F_{q_i^*}(q) \right] \quad \text{for} \ q \in [0, \bar{q}_i^*].
\]

Since \( f_{q_i^*}(\cdot) \) is strictly positive on its support by the assumption that all densities are strictly positive on their respective support, I can rearrange this equation to obtain \( [N] \).

**Lemma 3.** \( \exists ! \) function that fulfills the necessary conditions for a symmetric BNE:

\[
b(q, t_i) = v(q, t_i) - \int_{q}^{\bar{q}_i^*} \left[ \frac{1 - F_{q_i^*}(x)}{1 - F_{q_i^*}(q)} \right] \left( \frac{x}{N} \right) (-1)v'(x, t_i)dx \quad \text{for} \ q \in [0, \bar{q}_i^*]. \tag{4}
\]

**Proof.** The solution to differential equation \( [N] \) for \( q \in [0, \bar{q}_i^*] \) is

\[
b(q, t_i) = \left[ F_{q_i^*}(q) - 1 \right]^{-\left[ \frac{N-1}{N} \right]} \left[ \frac{N - 1}{N} \right] \int_{q_0}^{q} \left[ F_{q_i^*}(x) - 1 \right]^{-\left[ \frac{x}{N} \right]} f_{q_i^*}(x)v(x, t_i)dx \quad \text{with} \ q_0 \in [0, \bar{q}_i^*].
\]

\( C \) is chosen to ensure that the solution passes through \((\bar{q}_i^*, v(q_i^*, t_i))\). Here I will guess

\[
C = - \left[ \frac{N - 1}{N} \right] \int_{q_0}^{\bar{q}_i^*} \left[ F_{q_i^*}(x) - 1 \right]^{-\left[ \frac{x}{N} \right]} f_{q_i^*}(x)v(x, t_i)dx \quad (C')
\]

and verify at the end that the resulting solution indeed goes through the upper end-point condition. Inserting \( [C] \) into the bidding function and simplifying gives the following unique solution

\[
b(q, t_i) = \left[ F_{q_i^*}(q) - 1 \right]^{-\left[ \frac{N-1}{N} \right]} \left[ \frac{N - 1}{N} \right] \int_{q}^{\bar{q}_i^*} \left[ F_{q_i^*}(x) - 1 \right]^{-\left[ \frac{x}{N} \right]} f_{q_i^*}(x)v(x, t_i)dx.
\]

\( ^{11} \) By assumptions all \( j \neq i \) play the guessed equilibrium \( x(p, t_j) = \eta(t_j) + y(p) \). By definition of an inverse, \( x(b(q, t_i)) = y(b(q, t_i)) + \eta(t_j) = q \). By the chain rule \( \left( \frac{\partial y(b(q, t_i))}{\partial p} \right) b'(q, t_i) = 1 \), so that \( \left( \frac{\partial y(b(q, t_i))}{\partial p} \right) = b'(q, t_i)^{-1} \). Differentiating \( x(b(q, t_i)) = y(b(q, t_i)) + \eta(t_j) = q \) a second time gives \( \left( \frac{\partial^2 y(b(q, t_i))}{\partial p^2} \right) b'(q, t_i)^2 + \left( \frac{\partial^2 y(b(q, t_i))}{\partial q \partial p} \right) \left( \frac{\partial^2 b(q, t_i)}{\partial q \partial p} \right) = 0 \). Inserting the last equation to obtain \( \left( \frac{\partial^2 y(b(q, t_i))}{\partial p^2} \right) = - \left( \frac{\partial^2 b(q, t_i)}{\partial q \partial p} \right) b'(q, t_i)^{-3} \).
To simplify the bidding function I integrate its integral by parts and use $F_{q_i^*}(q_i) = 1$.

$$\frac{N - 1}{N} \int_q^{q_i^*} [F_{q_i^*}(x) - 1] \\frac{N}{N} f_{q_i^*}(x) v(x, t_i) dx = [F_{q_i^*}(q) - 1]^{\frac{N - 1}{N}} v(q, t_i) + \int_q^{q_i^*} [F_{q_i^*}(x) - 1]^{\frac{N - 1}{N}} v'(x, t_i) dx$$

The bidding function becomes

$$b(q, t_i) = [F_{q_i^*}(q) - 1]^{-\frac{N - 1}{N}} \left[[F_{q_i^*}(q) - 1]^{\frac{N - 1}{N}} v(q, t_i) + \int_q^{q_i^*} [F_{q_i^*}(x) - 1]^{\frac{N - 1}{N}} v'(x, t_i) dx\right].$$

It simplifies to the bidding function of the theorem

$$b(q, t_i) = v(q, t_i) - \int_q^{q_i^*} \left[\frac{1 - F_{q_i^*}(x)}{1 - F_{q_i^*}(q)}\right]^{\frac{N - 1}{N}} (-1) v'(x, t_i) dx \text{ for } q \in [0, q_i^*]. \quad (4)$$

It remains to show that the function goes through $(q_i^*, v(q_i^*, t_i))$. For this I must show that

$$\lim_{q \to q_i^*} \int_q^{q_i^*} \left[\frac{1 - F_{q_i^*}(x)}{1 - F_{q_i^*}(q)}\right]^{\frac{N - 1}{N}} v'(x, t_i) dx = 0.$$ 

To do so, separate

$$\lim_{q \to q_i^*} \int_q^{q_i^*} \left[\frac{1 - F_{q_i^*}(x)}{1 - F_{q_i^*}(q)}\right]^{\frac{N - 1}{N}} v'(x, t_i) dx = \lim_{q \to q_i^*} \left[1 - F_{q_i^*}(q)\right]^{\frac{N - 1}{N}} \lim_{q \to q_i^*} \int_q^{q_i^*} \left[1 - F_{q_i^*}(x)\right]^{\frac{N - 1}{N}} v'(x, t_i) dx.$$ 

Applying Hospital’s rule in combination with the Fundamental Theorem of calculus:

$$= - \lim_{q \to q_i^*} \left[\frac{N - 1}{N} f_{q_i^*}(q) \frac{1}{N}\right]^{-1} \lim_{q \to q_i^*} \left[1 - F_{q_i^*}(x)\right]^{\frac{N - 1}{N}} v'(x, t_i) q_i^*$$

The first limit is different from zero since $f_{q_i^*}(q) > 0$ on the full support. The second limit is 0 because $F_{q_i^*}(q_i) = 1$ and $v(\cdot, t_i)$ is bounded by 0. Putting both parts together, this shows that the limit is 0. 

**Lemma 4.** Function (4) is a BNE.

**Proof.** Function (4) is a BNE if (i) it is strictly decreasing and differentiable and (ii) agent $i$ has no profitable deviation from submitting another function than the one that fulfills the necessary conditions of his maximization problem. The proof splits into three auxiliary lemma.

**Auxiliary Lemma 1.** Function (4) is strictly decreasing and twice differentiable.

**Proof.** Twice differentiability of $b(\cdot, t_i)$ follows immediately from the assumptions that all distribution functions and $v(\cdot, t_i)$ are twice differentiable. That $b(\cdot, t_i)$ is strictly decreasing in $q$ can easily be verified by taking the derivative:

$$\frac{\partial b(q, t_i)}{\partial q} < 0 \text{ for } q \in [0, q_i^*] \iff \int_q^{q_i^*} \left[1 - F_{q_i^*}(x)\right]^{\frac{N - 1}{N}} \left(\frac{\partial v(x, t_i)}{\partial q}\right) dx < 0.$$
This always holds given $1 - F_{q_i}(x) > 0$ and $\left( \frac{\partial v(x,t_i)}{\partial q} \right) < 0$ on $x \in [0,q_i^*]$ by assumption. At the boundary point where $b(\overline{q}_i^*,t_i) = v(\overline{q}_i^*,t_i)$, the function is strictly decreasing since the marginal valuation is strictly decreasing in quantity by assumption. □

**Auxiliary Lemma 2.** Function \[\mathcal{F}\] fulfills the sufficient conditions of a local maximum.

**Proof.** Sufficient conditions in variational calculus problems are generally tricky. One needs to show that the second variation (the analogue to the second derivative when maximizing w.r.t. a variable) of functional \[\mathcal{F}\] defined on page 21 is positive for all values of variation $\kappa(q)$ over the interval $0 \leq q \leq \overline{q}_i^*$. Here, where $\mathcal{F}$ is independent of $b_i'(q,t_i)$ the second variation is

$$\delta V_2 = \int_0^{\overline{q}_i^*} \kappa(q)^2 \left[ \frac{\partial^2 \mathcal{F}(q,b_i(q,t_i))}{\partial^2 b_i} \right] dq.$$  

In what follows I show that $\frac{\partial^2 \mathcal{F}(q,b_i)}{\partial^2 b_i}$ evaluated at the solution $b_i(q,t_i) = b(q,t_i)$ is at all relevant points negative:

$$\frac{\partial^2 \mathcal{F}(q,b_i(q,t_i))}{\partial^2 b_i} < 0 \text{ for all } q \in [0,\overline{q}_i^*]. \quad (S)$$

For this I must first specify the functional form of $\frac{\partial^2 \mathcal{F}}{\partial^2 b_i}$. From above we already know

$$\frac{\partial \mathcal{F}(q,b_i(q,t_i))}{\partial b_i} = \left( \frac{\partial G_i(q,b_i(q,t_i))}{\partial b_i} \right) [v(q,t_i) - b_i(q,t_i)] - [1 - G_i(q,b_i(q,t_i))].$$

Taking the partial derivative w.r.t. price $b_i(q,t_i) = b_i$ gives

$$\frac{\partial^2 \mathcal{F}(q,b_i(q,t_i))}{\partial^2 b_i} = 2 \left( \frac{\partial G_i(q,b_i(q,t_i))}{\partial b_i} \right) - \left( \frac{\partial^2 G_i(q,b_i(q,t_i))}{\partial^2 b_i} \right) [v(q,t_i) - b_i(q,t_i)].$$

Analogous to above the sufficient condition simplifies when re-formulating it using the distribution of $i$’s equilibrium winning quantity $q_i^*$. Transform the random variable from $Q - \sum_{j \neq i} \eta(t_j)$ to $q_i^*$, and recall $G_i(q,p) = \text{Pr} \left( Q - \sum_{j \neq i} x(p,t_j) \leq q \right) = \text{Pr} \left( Q - \sum_{j \neq i} \eta(t_j) \leq q + (N - 1)y(p) \right)$ to obtain

$$\frac{\partial G_i(q,b_i(q,t_i))}{\partial b_i} = \left[ \frac{N - 1}{N} \right] f_{q_i^*}(q) y'(b(q,t_i)) \quad (17)$$

and

$$\frac{\partial^2 G_i(q,b_i(q,t_i))}{\partial^2 b_i} = \frac{1}{N^2} \left( \frac{\partial f_{q_i^*}(q)}{\partial q} \right) [N - 1]y''(b(q,t_i))^2 + \frac{1}{N} f_{q_i^*}(q) [N - 1]y''(b(q,t_i)). \quad (19)$$

\[\text{In the standard case, in which } F \equiv F(q,b_i(q,t_i),b_i'(q,t_i)) \text{ depends on the slope of the function one maximizes over, } b_i', \text{ the second variation is known to be} \quad \text{[Kamien and Schwartz (1993), p. 42]}

$$\delta V_2 = \int_0^{\overline{q}_i^*} \left\{ \kappa^2 \frac{\partial^2 F}{\partial^2 b_i} + 2\kappa \left( \frac{\partial^2 F}{\partial b_i \partial b_i'} \right) + (\kappa')^2 \left[ \frac{\partial^2 F}{\partial^2 b_i'} \right] \right\} dq.$$
By definition of an inverse and the chain rule

\[ y'(b(q, t_i)) = \left( \frac{1}{b'(q, t_i)} \right) \]

\[ y''(b(q, t_i)) = -\left( \frac{b''(q, t_i)}{b'(q, t_i)^2} \right). \]

Inserting equations (17), (19), (20), (21) into the sufficient condition (S) and multiplying both sides of the equation by \( \frac{N}{N-1} b'(q, t_i)^2 > 0 \) gives

\[-[v(q, t_i) - b(q, t_i)] \left\{ (N-1) \left[ \frac{\partial f_{q^*}(q)}{\partial q} \right] - f_{q^*}(q) \left[ \frac{b''(q, t_i)}{b'(q, t_i)} \right] \right\} + 2b'(q, t_i)f_{q^*}(q) < 0 \]

for \( q \in [0, \overline{q}_i^*] \). For the boundary point, \( q = \overline{q}_i^* \), we immediately see that this condition holds. The first term drops out since \( v(q, t_i) = b(q, t_i) \) and the second term is strictly negative. To check whether the condition is satisfied for \( q \in [0, \overline{q}_i^*] \), I use (N) to substitute out for \( b'(q, t_i) \) in combination with its first and second derivative to substitute out for \( \left[ \frac{b''(q, t_i)}{b'(q, t_i)} \right] \). The sufficient condition simplifies to

\[ d dx \ln \left[ \frac{1 - F_{q^*}(x)}{f_{q^*}(x)} \right] \bigg|_{x=q} > \left[ \frac{v(q, t_i)}{v(q, t_i) - b(q, t_i)} \right] \] for \( q \in [0, \overline{q}_i^*] \). (S)

The RHS is always negative, because \( [v(q, t_i) - b(q, t_i)] > 0 \) for \( q \in [0, \overline{q}_i^*] \) and \( v(\cdot, t_i) \) is strictly decreasing. The sufficient condition is therefore fulfilled whenever the LHS is weakly positive. This is the case when the winning quantity is drawn from a distribution with weakly decreasing hazard rate, which holds by assumption.

I conclude that the second variation is negative for all admissible functions \( \kappa(\cdot) \). The critical function \( b(\cdot, t_i) \) is a local maximum. \( \Box \)

**Auxiliary Lemma 3.** Function (4) is a global maximum.

**Proof.** In what follows I first show that there is at most one function that satisfies the necessary condition of the agent’s maximization problem (Part 1). From Lemma 3 we know that there is such a function, namely function (4). To prove that this unique local maximum is a global maximum, I verify that it cannot be optimal to choose a function that lies on the boundaries of the function space in the final step (Part 2).

**Part 1.** There is at most one function that satisfies the necessary condition for a symmetric equilibrium.

First, following [Pycia and Woodward (2017)] we know that any best reply must be strictly decreasing on relevant quantities \([0, \overline{q}_1^*] \). In a sequence of lemmas they prove that bidding functions in pay-as-bid auctions in which agents are uncertain about the total supply and all
share the same type are strictly decreasing (pp. 34-37). Their Lemmas 1-4 extend to my environment with private information with minor modifications: Fix a type profile \( t \equiv (t_1, \ldots, t_N) \) and consider agent \( i \). The type-dependent valuation \( v_i(\cdot, t_i) \) and bidding function \( b_i(\cdot, t_i) \) replace \( v_i(\cdot) \) and \( b_i(\cdot) \) in [Pycia and Woodward (2017)]. The clearing price now maps from total supply and the fixed profile of all types into the space of prices: \( p(Q, t) \). The set of relevant quantities is \([0, \pi_i]^c \). Its upper bound is denoted by \( \varphi^i(pQ) \) in [Pycia and Woodward (2017)].

It remains to show that there is at most one function that fulfills the necessary conditions within the class of strictly decreasing functions. For this recall that any such function \( b_i(\cdot, t_i) \) must be such that for any fixed \( q \in [0, \pi_i]^c \),

\[
0 = - \left( \frac{\partial G_i(q, b_i(q, t_i))}{\partial b_i(q, t_i)} \right) \left[ v(q, t_i) - b_i(q, t_i) \right] - [1 - G_i(q, b_i(q, t_i))].
\]

In other words, \( E \) must hold point-wise, for all relevant quantities \( q \in [0, \pi_i]^c \). To show that there is at most one critical function I must show that for any fixed \( q \in [0, \pi_i]^c \) there is at most one bid price \( b_i \equiv b_i(q, t_i) \) that makes the condition bind. Since all candidate functions are strictly decreasing in quantity, I can equivalently show that for any fixed relevant price \( b_i \) there is at most one \( q \) that guarantees \( E \). To show this, I first simplify the condition using the assumption that players \( j \neq i \) choose \( x(p, t_j) = y(p) + \eta(t_j) \). Then I fix some relevant price \( b_i \) and show that the condition is strictly increasing in \( q \) by the assumption that \( i \)'s equilibrium winning quantity has a decreasing hazard rate. This implies that for any relevant \( b_i \) there is at most one \( q \).

To simplify \( E \), recall that for any fixed \( \{q, p\} \) and given \( x(p, t_j) = y(p) + \eta(t_j) \)

\[
G_i(q, p) \equiv \Pr \left( Q - \sum_{j \neq i} \eta(t_j) \leq q + (N - 1)y(p) \right)
\]

\[G_i(q, p) = F_Z(q + (N - 1)y(p)) \text{ with } Z \equiv Q - \sum_{j \neq i} \eta(t_j)
\]

where \( F_Z, f_Z \) denote the CDF and density of auxiliary variable \( Z \). Applying the Chain Rule, condition \( E \) becomes

\[
0 = -f_Z(q + (N - 1)y(b_i))(N - 1)y'(b_i) \left[ v(q, t_i) - b_i \right] - [1 - F_Z(q + (N - 1)y(b_i))]
\]

for any \( \{q, b_i\} \) such that \( z = q + (N - 1)y(b_i) \) lies in the support of \( Z \). By assumption all random variables have strictly positive density on their respective supports so that \( f_Z > 0 \) on its support. I can therefore divide by the density. Furthermore I can divide by \( (N - 1)y'(b_i) \) as \( y'(b_i) < 0 \) and \( N > 1 \). The condition rearranges to

\[
0 = \left( \frac{1 - F_Z(q + (N - 1)y(b_i))}{f_Z(q + (N - 1)y(b_i))} \right) \left( \frac{1}{(N - 1)(-1)y'(b_i)} \right) - [v(q, t_i) - b_i]. \tag{E}
\]
Now, fix some relevant price $b_i$. By assumption $i$’s winning equilibrium quantity, defined $q_i^* = \left[\frac{1}{\lambda} \right] \{Z + (N - 1)v(t_i)\}$ by combining Definition 3 and (24), has a decreasing hazard rate. Since it is a monotone transformation of $Z$, $Z$ must have a decreasing hazard rate as well. It’s inverse is increasing. In condition (E) the inverse hazard rate of $Z$ is evaluated at $z = q + (N - 1)y(b_i)$ which for any fixed $b_i$ is increasing in $q$. Together this implies that $\left[\frac{1-F_Z(q+(N-1)y(b_i))}{F_Z(q+(N-1)y(b_i))}\right]$ is increasing in $q$ for any fixed $b_i$. Since, in addition, $y'(b_i) < 0$ the first term of (E) increases in $q$. The second term of (E) is strictly decreasing in $q$ given $v(\cdot, t_i)$ strictly decreases in $q$. Taken together this shows that the RHS is strictly increasing in $q$. 

Part 2. The optimum lies in the interior of the function space.

To prove that the unique local maximum is a global maximum, I must verify that it cannot be optimal to choose a function that lies on the boundaries of the function space. Corner solutions in variational calculus problems (or more generally optimal control theory) are points at which the functional (or Hamiltonian) is not differentiable. By assumption any function on which the functional depends is differentiable. There are no such points and with it no corner solutions. This completes the proof of Theorem 2.

3 Proof of Corollary 1

Let $v(q, t_i) = \max\{t_i - \rho q, 0\}$ with $\rho > 0$, assume $N > \frac{2\sigma}{\xi}$, and consider distributions of total supply on support $[0, \bar{Q}] > 0$ and types on $[\xi > 0, \bar{t}]$ under which $q_i^*$ is drawn from the Generalized Pareto Distribution $F_{q_i^*}(q) = 1 - \left[\frac{\sigma(\xi, t_i) + \xi q}{\sigma(\xi, t_i)}\right]^{-\frac{1}{\xi}}$ with scale parameter $\sigma(\xi, t_i) = -\xi \left(\frac{N(1-\xi)-1}{N(1-\xi)\rho}\right)(t_i - \xi) - \xi \left(\frac{\bar{Q}}{N}\right)$ and shape parameter $\xi \in (-\infty, -1]$. The goal is to show that

$$b(q, t_i) = \begin{cases} \left(\frac{1}{1-\xi}\right) [t_i - \xi t] - \left(\frac{\rho}{N(1-\xi)-1}\right) [(n-1)q - \xi \bar{Q}] & \text{for } q \in \left[0, -\left(\frac{\sigma(\xi, t_i)}{\xi}\right)\right] \\ v(q, t_i) & \text{for } q \in \left(0, -\left(\frac{\sigma(\xi, t_i)}{\xi}\right)\right), \infty \end{cases}$$

(26)

is a symmetric equilibrium. The proof is split into two parts, summarized in two lemmas.

The proof follows a guess and verify approach. I guess that there is such an equilibrium and compute how much an agent wins $q_i^*$ when all play this strategy. With this I can use Theorem 2 to compute the bidding function that an agent submits when $q_i^*$ follows the assumed Generalized Pareto Distribution. I verify that this function coincides with the guess (Lemma 3). If the hazard rate of $q_i^*$ was decreasing, this would conclude the proof. Yet, the hazard rate of the Generalized Pareto Distribution with bounded support ($\xi < 0$) is increasing. The theorem does not apply directly. What remains to show is that the agent has indeed no profitable deviation because the necessary conditions which defined his bidding function are...
also sufficient (Lemma 6).

**Lemma 5.** Function (26) fulfills the necessary conditions of a symmetric BNE.

**Proof.** Assume all play equilibrium guess (26). By Theorem 2 a candidate function for a symmetric equilibrium takes the following form

$$b(q, t_i) = t_i - \rho q - \rho \int_q^{q_i} \left[ \frac{1 - F_{q_i}(x)}{1 - F_{q_i}(q)} \right] \frac{N-1}{N} dx$$

on \([0, q_i^\ast]\).

To verify that the guess is a valid equilibrium candidate we must compute (i) the winning quantities and (ii) their support.

(i) To calculate how much an agent wins in equilibrium \(q_i^\ast\), first, invert \(b(\cdot, t_i)\) to determine the corresponding demand functions

$$x(p, t_i) = y(p) + \eta(t_i)$$

with \(y(p) \equiv a^* - e^*p\) and \(\eta(t_i) \equiv c^*t_i\) and equilibrium parameters

$$c^* = \frac{e^*}{1 - \xi} = \left(\frac{N(1 - \xi) - 1}{\rho(N - 1)(1 - \xi)}\right)$$

$$e^* = \left(\frac{N(1 - \xi) - 1}{\rho(N - 1)}\right)$$

$$a^* = \left(\frac{\xi}{N - 1}\right) \left[ q - \left(\frac{N(1 - \xi) - 1}{\rho(1 - \xi)}\right) t \right]$$

Now sum over all agents to obtain the market clearing price

$$Q = x(p^e, t_i) + \sum_{j \neq i} x(p^e, t_j)$$

\(Na^* + c^* t_i + c^* \sum_{j \neq i} t_j - Ne^*p^e\)

\(\Rightarrow p^e = \frac{a^*}{e^*} + \frac{c^*}{Ne^*} \left[ t_i + \sum_{j \neq i} t_j \right] - \frac{Q}{N}\)

and evaluate \(i\)’s submitted demand at the clearing price:

$$q^*(t_i) \equiv x(p^e, t_i) = \frac{1}{N} \left[ Q - c^* \sum_{j \neq i} t_j + (N - 1)c^*t_i \right].$$

In the main text I call this amount \(q_i^e\) for notational ease.

(ii) The support \([q_i^*(t_i), \bar{q}_i^*(t_i)]\) is given by the support of the total supply and the types. Since \(c^* > 0\)

$$q_i^*(t_i) = \max \left\{ 0, \frac{1}{N} \left[ 0 - c^*(N - 1)[t - t_i] \right] \right\}$$

$$\bar{q}_i^*(t_i) = \frac{1}{N} \left[ Q - c^*(N - 1)[t - t_i] \right]$$

with

$$c^* = \left(\frac{N(1 - \xi) - 1}{\rho(N - 1)(1 - \xi)}\right)$$
Now, from basic statistics we know that the support of a standard Generalized Pareto Distribution with bounded support (i.e. with $\xi < 0$), location parameter $\alpha \in \mathbb{R}$, and scale parameter $\sigma > 0$ is $[\alpha, \alpha - \left(\frac{\xi}{\sigma}\right)]$. In my framework, where total supply may be 0, the lower bound of $i$’s winning quantity, and with it the location parameter, is 0. To see this, consider the lowest winning quantity the highest type can achieve: $q^*(i) = q^*(\bar{t}) = 0 - c^*(N - 1)[\bar{t} - \bar{t}] = 0$. Since quantity is bounded below by 0, all lower types also win 0 in the worst case scenario: $q^*(t_i) = 0$ for all $t_i$. The upper-bound of the support is, for all $t_i$, given by the scale parameter $\left(\frac{\sigma(q,t_i)}{\xi}\right)$. For any fixed $\xi$, it is easy to verify that the hazard rate is now weakly increasing.

Therefore the bidding function never drops below 0. 

\[ b(q, t_i) = t_i - \rho q - \rho \int_q^{q^*(t_i)} \left[ \frac{1 - F_{q^*(x)}}{1 - F_{q^*(q)}} \right] \frac{N-1}{N} dx \]

\[ b(q, t_i) = \left( \frac{1}{1 - \xi} \right) [t_i - \xi t] - \left( \frac{\rho}{N(1 - \xi) - 1} \right) [(n - 1)q - \xi Q]. \]

The solution coincides with the guess $q^*(t_i) = q^*(t_i) = \xi + \left( \frac{t_i - t}{(\xi - 1)N} \right) - \frac{Q}{N}q \geq 0 \forall t_i.$

Therefore the bidding function never drops below 0. 

\[ \text{Lemma 6. Function (26) is a BNE.} \]

\[ \text{Proof. The proof is analogous to the proof of Lemma 1 in the proof of the theorem. Here I only replace the parts which rely on the assumption that the distribution of } i \text{'s winning quantity has a decreasing hazard rate (Auxiliary Lemma 2 and Part 1 of Lemma 3). The hazard rate is now weakly increasing.} \]

\[ \text{Auxiliary Lemma 4 (Analogous to Auxiliary Lemma 2). Function (26) fulfills the sufficient conditions of a local maximum on } q \in [0, q^*_i] = [0, -\left(\frac{\sigma(q,t_i)}{\xi}\right)]. \]

\[ \text{Proof. Recall the sufficient condition for a local maximum from page 26:} \]

\[ -[v(q, t_i) - b(q, t_i)] \left\{ (N - 1) \left[ \frac{\partial f_{q^*(q)}}{\partial q} \right] - f_{q^*(q)} \left[ \frac{b'(q, t_i)}{b'(q, t_i)} \right] \right\} + 2b'(q, t_i)f_{q^*(q)} < 0. \]

\[ \text{Notice that this is the easiest case, at it eliminates type-dependence of the lower bound of } q^*(t_i). \]
It is always satisfied when the hazard rate is decreasing, and might hold when the hazard rate is increasing. Inserting all functional forms of the corollary, the sufficient condition becomes

\[
\left( \frac{(N - 1)N\rho^{2}(1 - \xi)}{(N(1 - \xi) - 1)(-\xi)} \right) \left( \frac{1}{f_{2}(t_{i})} \right)^{-\frac{1}{\xi}} f_{1}(q, t_{i})^{-\left[\frac{1+\xi}{\xi}\right]}(-2 + N(1 + \xi)) < 0 \quad (32)
\]

\(\forall q \in [0, \bar{q}^{*}(t_{i})], t_{i}\) with

\[
f_{1}(q, t_{i}) \equiv -(t_{i} - t - (1 - \xi)(\rho \bar{Q} - N(\rho q - t_{i} + t))) \quad \text{decreasing in } q \text{ and increasing in } t_{i}
\]

\[
f_{2}(t_{i}) \equiv (t_{i} - t)(N(1 - \xi) - 1) + \bar{Q}\rho(1 - \xi) \quad \text{increases in } t_{i}.
\]

In what follows I show that this condition is fulfilled for \(\xi \leq -1\). To see this I simplify it. The first term is strictly positive since \(\xi < 0\), \(N \geq 2\), \(\rho > 0\), \(\bar{Q} > 0\).

\[
\left( \frac{(N - 1)N\rho^{2}(1 - \xi)}{(N(1 - \xi) - 1)(-\xi)} \right) > 0
\]

Also the second term is strictly positive for all types, \(\left( \frac{1}{f_{2}(t_{i})} \right)^{-\frac{1}{\xi}} > 0\). This is because

\[
f_{2}(t) > 0 \Rightarrow f_{2}(t_{i}) > 0 \forall t_{i} \text{ since } f_{2}(\cdot) \text{ is increasing in } t_{i}.
\]

Canceling the two terms that are strictly positive, the sufficient condition becomes

\[
f_{1}(q, t_{i})^{-\left[\frac{1+\xi}{\xi}\right]}(-2 + N(1 + \xi)) < 0 \forall q \in [0, \bar{q}^{*}(t_{i})], t_{i}.
\]

I must show that this inequality holds for all types and all relevant quantities. This is not straightforward because the LHS may increase or decrease in quantity and type, depending on the sign of the exponent \(-\left[\frac{1+\xi}{\xi}\right]\) and \((-2 + N(1 + \xi))\). The remainder of this section shows that it holds for \(\xi \in (-\infty, -1]\).\(^{15}\)

If \(\xi = -1\), the sufficient condition is always satisfied. If \(\xi < -1\) the exponent is negative, i.e. \(-\left[\frac{1+\xi}{\xi}\right] < 0\). \(\xi < -1\) further implies that \((-2 + N(1 + \xi)) < 0\), so that the sufficient condition simplifies to

\[
f_{1}(q, t_{i})^{-\left[\frac{1+\xi}{\xi}\right]} > 0 \forall q \in [0, \bar{q}^{*}(t_{i})], t_{i}.
\]

Since \(f_{1}(q, t_{i})^{-\left[\frac{1+\xi}{\xi}\right]}\) is increasing in \(q\) and decreasing in \(t_{i}\), it is fulfilled for all \(q \in [0, \bar{q}^{*}(t_{i})], t_{i}\) when:

\[
f_{1}(0, \bar{t}) > 0 \iff \bar{t} > \bar{t} + (\bar{Q}\rho + N(\bar{t} - t)(\xi - 1)) \iff \bar{Q}\rho > -\left(\frac{(\bar{t} - t)(N(1 - \xi) - 1)}{1 - \xi}\right).
\]

This is always satisfied given \(\bar{Q}, \rho > 0, N \geq 2, \xi \leq -1\).

**Auxiliary Lemma 5** (Analogous to Part 1 of Lemma 3). There is a unique function that satisfies the necessary condition for a symmetric equilibrium.

---

\(^{15}\)For \(\xi \in (-1, 0)\) I could neither show that the sufficient condition holds nor that it cannot hold. Therefore my proof therefore does not contradict Ausubel et al. (2014) who allow for \(\xi < \frac{N - 1}{N}\).
Proof. The proof is analogous to the proof of Lemma 3. I show that there is exactly one $q$ for any fixed relevant price $b_i \equiv b_i(q, t_i)$ that guarantees that the necessary condition is fulfilled. With decreasing hazard rate this was the case because

\[ 0 = \left[ \frac{1 - F_Z(q + (N-1)y(b_i))}{F_Z(q + (N-1)y(b_i))} \right] \left( \frac{1}{(N-1)(-1)y'(b_i)} \right) - [v(q, t_i) - b_i] \]  

(E)

was strictly increasing in quantity for any fixed price $b_i$. Now it is strictly increasing. To show this, I will, as above go via the hazard rate of $i$’s winning quantity, which is drawn from a GPD by assumption. It assumes the following hazard rate

\[ \left[ \frac{1 - F_{q^*}(q)}{f_{q^*}(q)} \right] = \sigma(\xi, \varepsilon) + q\xi. \]

By definition $Z = Nq_i^* - (N-1)\eta(t_i)$ so that $F_Z(z) = F_{q_i^*}(q)$ and $f_Z(z) = Nf_{q_i^*}(q)$ for any $z$ in the support of $Z$. Therefore

\[ \left[ \frac{1 - F_Z(z)}{f_Z(z)} \right] = \frac{1}{N}[\sigma(\xi, \varepsilon) + z\xi] \text{ for any } z \text{ in the support of } Z. \]  

(33)

In particular at realization $z = q + (N-1)y(b_i)$

\[ \left[ \frac{1 - F_Z(q + (N-1)y(b_i))}{f_Z(q + (N-1)y(b_i))} \right] = \frac{1}{N}[\sigma(\xi, \varepsilon) + [q + (N-1)y(b_i)]\xi]. \]  

(34)

By assumption all other player’s than $i$ play the equilibrium guess. We thus know that agents $j \neq i$ choose $y(b_i) = a^* - e^*b_i$ with equilibrium coefficients as defined above. In addition the true marginal willingness to pay is linear $v(q, t_i) = t_i - \rho q$. Taken all together, condition (E) becomes

\[ 0 = \frac{1}{N}[\sigma(\xi, \varepsilon) + [q + (N-1)[a^* - e^*b_i]]\xi] \left( \frac{1}{(N-1)(-1)(-e^*)} \right) - [t_i - \rho q - b_i]. \]

Taking the derivative w.r.t. $q$, and simplifying one can show that the RHS is strictly increasing in $q$ given $N \geq 2, \xi \leq -1, \rho > 0$ and $\sigma(\xi, t_i)$ as defined in the corollary. Therefore, there is at most one quantity $q$ for this price $b_i$ that makes the condition bind.

This completes the proof of Corollary 1. \[\square\]

3.1 Corollary 1 vs. Proposition 7 in [Ausubel et al., 2014]

The Corollary relates to Ausubel et al. (2014)’s Proposition 7. They derive the unique linear equilibrium in a pay-as-bid auction in an environment without private types, where bid offers may drop below 0 in equilibrium. In their set-up, all bidders draw the same type, here called $t$, and are uncertain about the total supply $Q$. Both are drawn from a joint distribution $F(Q, t)$, which is commonly known and has non-degenerate support. Different to my framework with independent private types, the single type of all agents and total supply may be correlated. With slight adaptation to my framework, their proposition reads as follows:
Proposition 7 (Ausubel et al. (2014)) Let per-capita supply for any value be distributed according to the Generalized Pareto distribution with $\alpha = 0$, i.e. $F_q^*(q\mid \xi) = 1 - \left(\frac{\sigma + \xi q}{\sigma}\right)^{-\frac{1}{\xi}}$. In the unique linear equilibrium, the strategy of bidder $i$ is

$$b(q, t) = t - \left(\frac{\rho}{N(1 - \xi) - 1}\right) \left[(N - 1)q + N\sigma\right] \text{ for } \xi < \frac{N - 1}{N}. \tag{35}$$

Notice that I impose a stricter bound on $\xi \leq -1$. This comes from a difference in how we verify sufficient conditions. Ausubel et al. (2014) rely on the Maximum Theorem for compact intervals according to which a maximum (and a minimum) exists. I instead derive sufficient conditions. Ausubel et al. (2014) rely on the Maximum Theorem for compact intervals according to which a maximum (and a minimum) exists. I instead derive sufficient conditions that guarantee that the critical function is a maximum. To compare their result to mine not that the support of $q^* \equiv \frac{Q}{N}$ is bounded when $\xi < 0$: $q \in \left[0, \frac{Q}{N}\right] = \left[0, -\frac{\sigma}{\xi}\right]$. Using $\frac{Q}{N} = -\frac{\sigma}{\xi}$ their bidding function becomes

$$b(q, t) = t - \left(\frac{\rho}{N(1 - \xi) - 1}\right) \left[(N - 1)q - \xi \frac{Q}{N}\right] \text{ for } \xi \leq -1. \tag{35}$$

It is identical to the function of my Corollary given all draw the lowest type $t_i = t$.

4 Extension: Reserve Price

The main result extends to auctions with reserve prices with distributions that may have unbounded support. The following extension is stated for distributions with unbounded support. The case of bounded support is analogous.

Theorem 2b. Consider distributions of total supply and types such that the amount an agent wins in the symmetric equilibrium $q^*_R$ is drawn from distribution $F_{q^*_R}$ with weakly decreasing hazard rate on $[0, \infty)$ and strictly positive density on $[0, \bar{q}^R]$, with $v(q^R, t_i) = R$.

There exists a pure-strategy Bayesian Nash equilibrium in which bidders submit

$$b^*(q, t_i) = v(q, t_i) - \int_{q}^{\bar{q}^R} \left[1 - F_{q^*_R}(x)\right]^{\frac{N-1}{N}} (-1) \left(\frac{\partial v(q, t_i)}{\partial q}\right) dx \quad \text{for } q \in [0, \bar{q}^R] \tag{1'}$$

provided the resulting equilibrium demand schedule is additively separable in the type on $[0, R]$, and $b^*(q, t_i) = v(q, t_i)$ for $q \in (\bar{q}^R, \infty)$.

The analogy with the first-price auction carries over. To see this, note that an agent of type $s$, drawn iid from $F_s(s)$ in a first-price auction with reserve price $R > 0$ chooses

$$\beta^*(s) = v(s) - \int_{s^R}^{s} \left[\frac{F(x)}{F(s)}\right]^{N-1} \left(\frac{\partial v(x)}{\partial s}\right) dx \quad \text{for } s \in [s^R, \bar{s}] \tag{1'R}$$

where $v(s) = s$ and $v(s^R) = R$, and 0 otherwise. Comparing the bidding function in the first-price auction to the one in the pay-as-bid auction in the way I illustrate in the main text, one can derive the main result for this extension. One only needs to replace $\bar{q}^*_R$ by $\bar{q}^R / 0$ by $s^R$. Both bounds are now determined by the reserve price instead of the upper/lower bound of the distribution.
Proof. The proof is analogous to the proof of Theorem 2. Here I only highlight what changes. The support in Definition 2 is now \([0, \overline{q}^R]\), no longer \([0, \overline{q}^*_i]\). The difference is that the upper bound of the support is now exogenously given by the reserve price \(R\). The bidder chooses this function so as to maximize his expected total surplus.

\[
\max_{b_i(\cdot, t_i)} \mathcal{V}(b_i(\cdot, t_i)) \text{ s.t. } b_i(0, t_i) = \overline{p} < \infty \text{ and } b_i(\overline{q}^R, t_i) = R \tag{M}
\]

where

\[
\mathcal{V}(b_i(\cdot, t_i)) = \int_0^{\overline{q}^R} \left(\int_0^q v(x, t_i) - b_i(x, t_i)dx\right) g_i(q, b_i(q, t_i))dq.
\]

The derivation of the Euler-Lagrange equation goes through without problems. One obtains the following necessary condition

\[
0 = -\left(\frac{\partial G_i(q, b_i)}{\partial b_i}\right) [v(q, t_i) - b_i] - \left[1 - G_i(q, b_i)\right] \text{ for } q \in [0, \overline{q}^R] \tag{E}
\]

together with the end-point condition: \(b_i(\overline{q}^R, t_i) = R\). Analogous to above, the next step is to evaluate the Euler-Lagrange Equation at the symmetric solution. The difference now is that \(i^*\)'s winning quantity \(q^{R*}_i\) has unbounded support \([0, \infty)\). If there was no positive reserve price \(R > 0\), one would need to deal with improper integrals. Showing that they converge is not trivial. With a positive reserve price, however, the distribution that actually matters for the agent is bounded. Following Pycia and Woodward (2017) I construct the following auxiliary distribution

\[
F^{R}_q(q) = \begin{cases} F^{R}_q(q) & \text{for } q < \overline{q}^R \\ 1 & \text{for } q \geq \overline{q}^R. \end{cases}
\]

With it the Euler-Lagrange Equation can be expressed exactly in the way it was expressed above \((N)\). The difference is that it now holds for \(q \in [0, \overline{q}^R]\) instead of \(q \in [0, \overline{q}^*_i]\).

\[
[v(q, t_i) - b(q, t_i)] = -\left[1 - \frac{F^{R}_q(q)}{f^{R}_q(q)}\right] b'(q, t_i) \text{ for } q \in [0, \overline{q}^R] \tag{N^R}
\]

Analogous to above one can show that a function which fulfills the necessary condition is a global maximum. The solution to this differential equation for \(q \in [0, \overline{q}^R]\) is

\[
b(q, t_i) = [F^{R}_q(q) - 1]^{-\left[\frac{N - 1}{N}\right]} \left[C + \left[\frac{N - 1}{N}\right] \int_{q_0}^q [F^{R}_q(x) - 1]^{-\left[\frac{1}{N}\right]} f^{R}_q(x)v(x, t_i)dx\right] \text{ with } q_0 \in [0, \overline{q}^R].
\]

The trick now is to choose \(C\) so that the solution goes through the end-point, i.e. \(b(\overline{q}^R, t_i) = R\). For this approach \(\overline{q}^R\) from the left and solve

\[
R = \lim_{y \to \overline{q}^R} [F^{R}_q(y) - 1]^{-\left[\frac{N - 1}{N}\right]} \left[C + \left[\frac{N - 1}{N}\right] \lim_{y \to \overline{q}^R} \int_{q_0}^y [F^{R}_q(x) - 1]^{-\left[\frac{1}{N}\right]} f^{R}_q(x)v(x, t_i)dx\right].
\]

By definition \(\lim_{y \to \overline{q}^R} F^{R}_q(y) = F(\overline{q}^R)\) so that

\[
C = [F^{R}_q(\overline{q}^R) - 1]^{-\left[\frac{N - 1}{N}\right]} - \left[\frac{N - 1}{N}\right] \int_{q_0}^{\overline{q}^R} [F^{R}_q(x) - 1]^{-\left[\frac{1}{N}\right]} f^{R}_q(x)v(x, t_i)dx.
\]
Inserting $C$ into the bidding function and simplifying gives

$$b(q, t_i) = \left[F_{q^*_i}(q) - 1\right]^{-\frac{N-1}{N}} \left[R[F_{q^*_i}(\bar{q}^R) - 1]\left[1 - \frac{1}{N}\right] - \frac{N-1}{N}\right] \int_{q}^{\bar{q}^R} \left[F_{q^*_i}(x) - 1\right]^{-\frac{N-1}{N}} f_{q^*_i}(x) v(x, t_i) dx .$$

Integrate by parts to simplify the function

$$\frac{N-1}{N} \int_{q}^{\bar{q}^R} \left[F_{q^*_i}(x) - 1\right]^{-\frac{N-1}{N}} f_{q^*_i}(x) v(x, t_i) dx = -\left[F_{q^*_i}(x) - 1\right]^{-\frac{N-1}{N}} v(x, t_i) \bigg|_{q}^{\bar{q}^R} + \int_{q}^{\bar{q}^R} \left[F_{q^*_i}(x) - 1\right]^{-\frac{N-1}{N}} v'(x, t_i) dx$$

Different to above $F_{q^*_i}(\bar{q}^R) \neq 1$, but strictly smaller.

$$= -\left[F_{q^*_i}(\bar{q}^R) - 1\right]^{-\frac{N-1}{N}} v(\bar{q}^R, t_i)$$

$$+ \left[F_{q^*_i}(q) - 1\right]^{-\frac{N-1}{N}} v(q, t_i) + \int_{q}^{\bar{q}^R} \left[F_{q^*_i}(x) - 1\right]^{-\frac{N-1}{N}} v'(x, t_i) dx$$

The bidding function becomes for $q \in [0, \bar{q}^R]$

$$b(q, t_i) = \mathcal{M}(q, R) + v(q, t_i) - \int_{q}^{\bar{q}^R} \left[1 - F_{q^*_i}(x)\right]^{-\frac{N-1}{N}} (-1) v'(x, t_i) dx$$

with mark-up

$$\mathcal{M}(q, R) \equiv [R - v(\bar{q}^R, t_i)] \left[1 - F_{q^*_i}(\bar{q}^R)\right]^{-\frac{N-1}{N}} = 0 \text{ by definition of } \bar{q}^R = v^{-1}(R, t_i).$$

We are left with

$$b(q, t_i) = v(q, t_i) - \int_{q}^{\bar{q}^R} \left[1 - F_{q^*_i}(x)\right]^{-\frac{N-1}{N}} (-1) v'(x, t_i) dx \text{ for } q \in [0, \bar{q}^R]$$

where I extended the domain to include the point $\bar{q}^R$.

To complete the proof one can show as above that the solution fulfills the properties that I have assumed to derive it. \[\square\]