Private False Discovery Rate Control

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Abstract

We provide the first differentially private algorithms for controlling the false discovery rate (FDR) in multiple hypothesis testing, with essentially no loss in power under certain conditions. Our general approach is to adapt a well-known variant of the Benjamini-Hochberg procedure (BHq), making each step differentially private. This destroys the classical proof of FDR control. To prove FDR control of our method we

1. Develop a new proof of the original (non-private) BHq algorithm and its robust variants — a proof requiring only the assumption that the true null test statistics are independent, allowing for arbitrary correlations between the true nulls and false nulls. This assumption is fairly weak compared to those previously shown in the vast literature on this topic, and explains in part the empirical robustness of BHq.

2. Relate the FDR control properties of the differentially private version to the control properties of the non-private version.

We also present a low-distortion “one-shot” differentially private primitive for “top $k$” problems, e.g., “Which are the $k$ most popular hobbies?” (which we apply to: “Which hypotheses have the $k$ most significant $p$-values?”), and use it to get a faster privacy-preserving instantiation of our general approach at little cost in accuracy. The proof of privacy for the one-shot top $k$ algorithm introduces a new technique of independent interest.

1 Introduction

Hypothesis testing is the use of a statistical procedure to assess the validity of a given hypothesis, for example, that a given treatment for a condition is more effective than a placebo. The traditional approach to hypothesis testing is to (1) formulate a pair of null (the treatment and the placebo are equally effective) and alternative (the treatment is more effective than the placebo) hypotheses; (2) devise a test statistic for the null hypothesis, (3) collect data; (4) compute the $p$-value (the probability of observing an effect as or more extreme than the observed effect, were the null hypothesis true; smaller $p$-values are “better evidence” for rejecting the null); and (5) compare the $p$-value to a standard threshold $\alpha$ to determine whether to accept the null hypothesis (conclude there is no interesting effect) or to reject the null hypothesis in favor of the alternative hypothesis.

In the multiple hypothesis testing problem, also known as the multiple comparisons problem, $p$-values are computed for multiple hypotheses, leading to the problem of false discovery: since the
**p-values** are typically defined to be uniform in $(0,1)$ for true nulls, by definition we expect an $\alpha$ fraction of the true nulls to have $p$-values bounded above by $\alpha$.

Multiple hypothesis testing is an enormous problem in practice; for example, in a single genome-wide association study a million SNPs$^1$ may be tested for an association with a given condition. Accordingly, there is a vast literature on the problem of controlling the **false discovery rate** (FDR), which, roughly speaking, is the expected fraction of erroneously rejected hypotheses among all the rejected hypotheses, where the expectation is over the choice of the data and any randomness in the algorithm (see below for a formal definition).

The seminal work of Benjamini and Hochberg [2] and their beautiful “BHq” procedure (Algorithm 1 below) is our starting point. Assuming independence or certain positive correlations of $p$-values [3], this procedure controls FDR at any given nominal level $q$. An extensive literature explores the control capabilities and power of this procedure and its many variants$^2$.

**Differential privacy** [10] is a definition of privacy tailored to statistical data analysis. The goal in a differentially private algorithm is to hide the presence or absence of any individual or small group of individuals, the intuition being that an adversary unable to tell whether or not a given individual is even a member of the dataset surely cannot glean information specific to this individual. To this end, a pair of databases $x,y$ are said to be adjacent (or neighbors) if they differ in the data of just one individual. Then a differentially private algorithm (Definition 2.2) ensures that the behavior on adjacent databases is statistically close, so it is difficult to infer if any particular individual is in the database from the output of the algorithm and any other side information.

**Contributions.** We provide the first differentially private algorithms for controlling the FDR in multiple hypothesis testing. The problem is difficult because the data of a single individual can affect the $p$-values of all hypotheses simultaneously. Our general approach is to adapt the Benjamini-Hochberg step-down procedure, making each step differentially private. This destroys the classical proof of FDR control. To prove FDR control of our method we

1. Develop a new proof of the original (non-private) BHq algorithm and its robust variants – a proof requiring a fairly weak assumption compared to those previously shown in the vast literature on this topic – and

2. Relate the FDR control properties of the differentially private version to the control properties of the non-private version. Power is also argued in relation to the non-private version.

Central to BHq and its variants is the procedure for reporting the experiments with $k$ most significant $p$-values, or known as the top-$k$ problem. To our knowledge, the most accurate approximately private top-$k$ algorithm is a “peeling” procedure, in which one runs a differentially private maximum procedure $k$ times, tuning the “inaccuracy” to roughly $\sqrt{k}/\varepsilon$. Here we present a low-distortion “one-shot” differentially private primitive for “top $k$” with the same dependence on $k$, and use it to get a faster privacy-preserving instantiation of our general approach.$^3$ The proof

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$^1$A **single nucleotide polymorphism**, or SNP, is a location in the DNA in which there is variation among individuals.

$^2$The power of a procedure is its ability to recognize “false nulls” (what a lay person may describe as “true positives”).

$^3$For pure differential privacy, previous work (e.g. [5]) provides a way to avoid peeling by adding Laplace noise of scale $O(k)$. Conceivably, it is much more challenging to get the dependence on $k$ down to $\sqrt{k}$ for approximate differential privacy, which is what we show in this paper.
of differential privacy for our one-shot top $k$ algorithm introduces a new technique of independent interest.

Finally, traditional methods of computation of a $p$-value are not typically differentially private. Our privacy-preserving algorithms for control of the FDR require the $p$-value computation to satisfy a technical condition. We provide such a method for computing $p$-values, ensuring “end to end” differential privacy from the raw data to the set of rejected hypotheses.

### 1.1 Description of Our Approach

**The BHq procedure.** Suppose we are simultaneously testing $m$ null hypotheses $H_1, \ldots, H_m$ and $p_1, \ldots, p_m$ are their corresponding $p$-values. Denote by $R$ the number of rejections (discoveries) made by any procedure and $V$ the number of true null hypotheses that are falsely rejected (false discoveries). The FDR is defined as

$$\text{FDR} := \mathbb{E} \left[ \frac{V}{\max\{R,1\}} \right].$$

Algorithm 1 presents the original BHq procedure for controlling FDR at $q$. The thresholds $\alpha_j^{BH} = qj/m$ for $1 \leq j \leq m$, are known as the *BHq critical values*. The following intuition may demystify the BHq algorithm. In the case that all $m$ null hypotheses are true and their $p$-values are i.i.d. uniform on $(0,1)$, then we expect, approximately, a $qj/m$ fraction of the $p$-values to lie in the interval $[0,qj/m]$. If instead there are at least $j$ many $p$-values in this interval (this is precisely what the condition $p(j) \leq qj/m$ says), then there are “too many” $p$-values in $[0,p(j)]$ for all of them to correspond to null hypotheses: We have $j$ such $p$-values, and should attribute no more than $jq$ of these to the true nulls; so if we reject all these $j$ hypotheses we would expect that at most a $q$ fraction correspond to true nulls.

**Algorithm 1 Step-up BHq procedure**

**Input:** Level $0 < q < 1$ and $p$-values $p_1, \ldots, p_m$ of hypotheses, respectively, $H_1, \ldots, H_m$

**Output:** a set of rejected hypotheses

1. sort the $p$-values in increasing order: $p(1) \leq p(2) \leq \cdots \leq p(m)$
2. for $j = m$ to 1 do
3. if $p(j) > qj/m$ then
4. continue
5. else
6. reject $H_{(1)}, \ldots, H_{(j)}$ and halt
7. end if
8. end for

**Making BHq Private.** There is an extensive literature on, and burgeoning practice of, differential privacy. For now, we need only two facts: (1) Differential privacy is closed under composition, permitting us to bound the cumulative privacy loss over multiple differentially private computations. This permits us to build complex differentially private algorithms from simple differentially private primitives; and (2) we will make use of the well known Report Noisy Max (respectively, Report Noisy Min) primitive, in which appropriately distributed fresh random noise is added to
the result of each of \( m \) computations, and the index of the computation yielding the maximum (respectively, minimum) noisy value is returned. By returning only one index the procedure allows us to pay an accuracy price for a single computation, rather than for all \( m \).\(^4\)

A natural approach to making the BHq procedure differentially private is by repeated use of the Report Noisy Max primitive: Starting with \( j = m \) and decreasing: use Report Noisy Max to find the hypothesis \( H_j \) with the (approximately) largest \( p \)-value; estimate that \( p \)-value and, if the estimate is above (an appropriately more conservative critical value) \( \alpha_j < \alpha^{BH}_j \), accept \( H_j \), remove \( H_j \) from consideration, and repeat. Once an \( H_j \) is found whose \( p \)-value is below the threshold, reject all the remaining hypotheses. The principal difficulty with this approach is that every iteration of the algorithm incurs a privacy loss which can only be mitigated by increasing the magnitude of the noise used by Report Noisy Max. Since each iteration corresponds to the acceptance of a null hypothesis, the procedure is paying in privacy/accuracy precisely for the null hypotheses accepted, which are by definition not the “interesting” ones.

If, instead of starting with the largest \( p \)-value and considering the values in decreasing order, we were to start with the smallest \( p \)-value and consider the values in increasing order, rejecting hypotheses one by one until we find a \( j \in [m] \) such that \( p(\overline{j}) > \alpha_j \), the “demystification” intuition still applies. This widely-studied variation is called the Benjamini-Hochberg step-down procedure, and hereinafter we call it the step-down BHq (the original BHq is referred to as the step-up BHq in contrast). Their definitions reveal that the step-down variant shall be more conservative than the other. However, as shown in [13], the step-down BHq can assume less stringent critical values than the BHq critical values while still controls FDR, often allowing more discoveries than the step-up BHq. In a different direction, [1] establishes that under a weak assumption on the sparsity the difference between these two BHq’s is asymptotically negligible and both procedures are optimal for Gaussian sequence estimation over a wide range of sparsity classes. This remarkable result implies that, as a pure testing criterion, the FDR concept is fundamentally correct for estimation problems.

**Algorithm 2** Step-down BHq procedure

**Input:** 0 < \( q < 1 \) and \( p \)-values \( p_1, \ldots, p_m \) of hypotheses, respectively, \( H_1, \ldots, H_m \)

**Output:** a set of rejected hypotheses

1. sort the \( p \)-values in increasing order: \( p(1) \leq p(2) \leq \cdots \leq p(m) \)
2. for \( j = 1 \) to \( m \) do
3. if \( p(\overline{j}) \leq qj/m \) then
4. reject \( H(\overline{j}) \)
5. else
6. halt
7. end if
8. end for

If we make the natural modifications to the step-down BHq (using Report Noisy Min now, instead of Report Noisy Max), then we pay a privacy cost only for nulls rejected in favor of the corresponding alternative hypotheses, which by definition are the “interesting” ones. Since the driving application of BHq is to select promising directions for future investigation that have a

\(^4\)The variance of the noise distribution depends on the maximum amount that any single datum can (additively) change the outcome of a computation and the inverse of the privacy price one is willing to pay.
decent chance of panning out, we can view its outcome as advice for allocating resources. Thus, a procedure that finds a relatively small number $k$ of high-quality hypotheses, still achieving FDR control, may be as useful as a procedure that finds a much larger set.

**Proving FDR Control.** A key property of the two BHq procedures is as follows: if $R$ rejections are made, the maximum $p$-value of all the rejected hypotheses is bounded above by $qR/m$. This motivates us to formulate the definition below.

**Definition 1.1.** Given any critical values $\{\alpha_j\}_{j=1}^m$, a multiple testing procedure is said to be adaptive to $\{\alpha_j\}_{j=1}^m$, if either it rejects none or the rejected $p$-values are all bounded above by $\alpha_R$, where the number of rejections $R$ can be arbitrary.

In this paper, we often work with the sequence $\{\alpha_j\}_{j=1}^m$ set to be the BHq critical values $\{\alpha^{BH}_j\}_{j=1}^m$. As motivating examples, both BHq’s in addition obey the property that the rejected hypotheses are contiguous in sorted order; for example, it is impossible that the hypotheses with the smallest and third smallest $p$-values are rejected while the hypothesis with the second smallest $p$-value is accepted. Nevertheless, in general an adaptive procedure does not necessarily have this property as it may skip some smallest $p$-values. For protecting differential privacy, this relaxation is desirable because differentially private algorithms may not reject consecutive minimum $p$-values due to artificial noise introduced. Note that this perfectly matches our “demystification” intuition.

Remarkably, an adaptive procedure with respect to the BHq critical values still controls the FDR. This result also applies to a generalized FDR defined for any positive integer $k$ [14, 15]:

$$\text{FDR}_k := E\left[\frac{V}{R}; V \geq k\right].$$

Built on top of the original FDR, this generalization does not penalize if fewer than $k$ false discoveries are made. Note that it reduces to the original FDR in the case of $k = 1$.

**Theorem 1.** Assume that the test statistics corresponding to the true null hypotheses are jointly independent. Then any procedure adaptive to the BHq critical values $\alpha^{BH}_j = qj/m$ obeys

$$\text{FDR} \leq q \log\left(1/q\right) + Cq,$$

$$\text{FDR}_2 \leq Cq,$$

$$\text{FDR}_k \leq \left(1 + 2/\sqrt{k}\right) q,$$

where $C < 3$ is a universal constant.

One novelty of Theorem 1 lies in the absence of any assumptions about the relationship between the true null test statistics and false null test statistics. In the literature, independence, or some (very stringent) kind of positive dependence [3] between these two groups of test statistics, is necessary for provable FDR control. In a different line, Theorem 1.3 of [3] controls FDR without any assumptions by using the (very stringent) critical values $qj/(m \sum_{i=1}^m \frac{1}{i}) \approx qj/(m \log m), 1 \leq j \leq m$, effectively paying a factor of $\log m$, whereas we pay only a constant factor. This simple independence assumption within the true nulls shall also capture more real life scenarios. As we will see, the additive $q \log(1/q)$ term for the original FDR, i.e. $\text{FDR}_1$, is unavoidable given so few assumptions. Surprisingly, $\text{FDR}_2$ no longer has this dependency, and $\text{FDR}_k$ even approaches $q$ as $k$ grows.
Theorem 1 is the key to proving FDR control even if the procedure is given “noisy” versions of the $p$-values, as happens in our differentially private algorithms. To determine how to add noise to ensure privacy, we study the sensitivity of a $p$-value, a measure of how much the $p$-value can change between adjacent datasets (Section 4.1). For standard statistical tests this change is best measured multiplicatively, rather than additively, as is typically studied in differential privacy. Exploiting multiplicative sensitivity is helpful when the values involved are very small. Since we are interested in the regime where $m$, the number of hypotheses, is much larger than the number of discoveries, the $p$-values we are interested in are quite small: the $j$th BHq critical value is only $qj/m$. We remark that adapting algorithms such as Report Noisy Min and the one-shot top $k$ to incorporate multiplicative sensitivity can be achieved by working with the logarithms of the values involved.

Informally, we say a $p$-value is $\eta$-multiplicatively sensitive truncated at $\nu$ if for any two neighboring databases $D$ and $D'$, the $p$-value computed on them are within multiplicative factor of $e^\eta$ of each other, unless they are smaller than $\nu$ (See Definition 4.1). The justification of the multiplicative sensitivity is provided by the definition of some standard $p$-values under independent bounded statistics. Indeed as we show in Section 4.1, such $p$-values are $\tilde{O}(1/\sqrt{n})$ multiplicatively sensitive. With these results, we can show that our algorithm controls FDR under the bound given in Theorem 1 while having the comparable power to the BHq step-down procedure.

**Theorem 2 (informal).** Suppose $\sqrt{k\eta}/\varepsilon = \tilde{o}(1)$. Given any nominal level $0 < q < 1$, the differentially private BHq algorithms controls the FDR at level $q + o(1)$.

**Theorem 3 (informal).** Again, suppose $\sqrt{k\eta}/\varepsilon = \tilde{o}(1)$. With probability $1 - o(1)$, the differentially private BHq algorithms with target nominal level $q$ makes at least as many discoveries as the BHq step-down procedure truncated at $k$ with nominal level $(1 - o(1))q$.

The One-Shot Top $k$ Mechanism. Consider a database of $n$ binary strings $d_1, d_2, \ldots, d_n$. Each $d_i$ corresponds to an individual user and has length $m$. Let $x_j = \sum d_{ij}$ be the total sum of the the $j$th column. In the reporting top-$k$ problem, we wish to report, privately, some locations $j_1, \ldots, j_k$ such that $x_{j_\ell}$ is close to the $\ell$th smallest element as possible. The peeling mechanism reports and removes the minimum noisy count and repeats on the remaining elements, each time adding fresh noise. Such a mechanism is $(\varepsilon, \delta)$-differentially private if we add $\text{Lap}(\sqrt{k\log(1/\delta)/\varepsilon})$ noise each time.

In contrast, in the one-shot top $k$ mechanism, we add $\tilde{O}(\sqrt{k})$ noise to each value and then report the $k$ locations with the minimum noisy values. Compared to the peeling mechanism, the one-shot mechanism is appealingly simple and much more efficient. But it is surprisingly challenging to prove its privacy. Here we give some intuition.

When there are large gaps between $x_i$’s, the change of one individual can only change the value of each $x_i$ by at most 1, so result of the one-shot algorithm is stable. Hence the privacy is easily guaranteed. In the more difficult case, when there are many similar values (think of the case when all the values are equal), the true top $k$ set can be quite sensitive to the change of input values. But in the one-shot algorithm we add independent symmetric noise, centered at zero, to these

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5 Recall that adjacent datasets differ in the data of just one person.
6 We use $\tilde{O}$ to hide polynomial factors in $\log(m/\delta)/\varepsilon$.
7 In our paper, it is more convenient to consider the minimum-$k$ (or bottom-$k$) elements. But we still call it the top-$k$ problem following the convention.
8 For technical reasons, if we want the values of the computations in addition to their indices we only know how to prove privacy if we add fresh random noise before releasing these values.
values. Speaking intuitively, this yields an (almost) equal chance that on two adjacent input values a noisy value will “go up” or “go down,” leading to cancellation of certain first order terms in the (logarithms of) the probabilities of events and hence a tight control between their ratio.

To capture this intuition, we consider the “bad” events, which have large probability bias between two neighboring inputs. Those bad events can be shown to happen when the sum of some dependent random variables deviates from its mean. The dependencies among the random variables prevent us from applying a concentration bound directly. To deal with this difficulty, we first partition the event spaces to remove the dependence. Then we apply a coupling technique to pair up the partitions for the two neighboring inputs. For each pair we apply a concentration bound to bound the probability of bad events. The technique appears to be quite general and might be useful in other settings.

2 Preliminaries on Differential Privacy

We revisit some basic concepts in differential privacy.

**Definition 2.1.** Data sets $D, D'$ are said to be neighbors, or adjacent, if one is obtained by removing or adding a single data item.

Differential privacy, now sometimes called pure differential privacy, was defined and first constructed in [10]. The relaxation defined next is sometimes referred to as approximate differential privacy.

**Definition 2.2** (Differential privacy [10, 9]). A randomized mechanism $M$ is $(\varepsilon, \delta)$-differentially private if for all adjacent $x, y$, and for any event $S$: $\mathbb{P}_D[S] \leq e^{\varepsilon} \mathbb{P}_{D'}[S] + \delta$. Pure differential privacy is the special case of approximate differential privacy in which $\delta = 0$.

Denote by $\text{Lap}(\lambda)$ the Laplace distribution with scale $\lambda > 0$, whose probability density function reads

$$f_{\text{Lap}}(z) = \frac{1}{2\lambda} e^{-|z|/\lambda}.$$

**Definition 2.3** (Sensitivity of a function). Let $f$ be a function mapping databases to $\mathbb{R}^k$. The sensitivity of $f$, denoted $\Delta f$, is the maximum over all pairs $D, D'$ of adjacent datasets of $\|f(D) - f(D')\|$.

We will make heavy use of the following two lemmas on differential privacy.

**Lemma 2.4** (Laplace Mechanism [10]). Let $f$ be a function mapping databases to $\mathbb{R}^k$. The mechanism that, on database $D$, adds independent random draws from $\text{Lap}((\Delta f)/\varepsilon)$ to each of the $k$ components of $f(D)$, is $(\varepsilon, 0)$-differentially private.

**Lemma 2.5** (Advanced Composition [12]). For all $\varepsilon, \delta, \delta' \geq 0$, the class of $(\varepsilon, \delta')$-differentially private mechanisms satisfies $(\sqrt{2k \ln(1/\delta)} \varepsilon + k\varepsilon(e^\varepsilon - 1)/2, k\delta' + \delta)$-differential privacy under $k$-fold adaptive composition.
Algorithm 3 Report Noisy Min

Input: \( m \) values \( x_1, \ldots, x_m \)

Output: \( j \)

1: for \( i = 1 \) to \( m \) do
2: \( y_i = x_i + g_i \) where \( g_i \) is sampled i.i.d. from \( \text{Lap}(2/\varepsilon) \)
3: end for
4: return \( (i_j = \text{argmin}_i y_i, x_{ij} + g') \), where \( g' \) is fresh random noise sampled from \( \text{Lap}(2/\varepsilon) \)

2.1 Reporting top-\( k \) elements

Consider the problem of privately reporting the minimum \( k \) locations of \( m \) values \( x_1, \ldots, x_m \). Here two input values \((x_1, \ldots, x_m)\) and \((x'_1, \ldots, x'_m)\) are called adjacent if \( |x_i - x'_i| \leq 1 \) for all \( 1 \leq i \leq m \). When \( k = 1 \), this is solved by Algorithm 3 stated below.

A standard result in differential privacy

**Lemma 2.6.** Algorithm 3 is \((\varepsilon, 0)\)-differentially private.

Notably, directly reporting \( y_j \) shall violate pure privacy. That is why we need to add fresh random noise to \( x_j \). For the top-\( k \) problem, one can apply the above process \( k \) times, each time removing the output element and applying the algorithm to the remaining elements, hence called the peeling mechanism. By composition theorem, it is immediate such a mechanism is \((\varepsilon, 0)\)-differentially private if we add noise \( \text{Lap}(k/\varepsilon) \) at each step. Or one can add \( \text{Lap}(\sqrt{k \log(1/\delta)}/\varepsilon) \) noise to get \((\varepsilon, \delta)\)-differential privacy by applying the advanced composition theorem [11].

**Theorem 4.** The peeling mechanism is \((\varepsilon, \delta)\)-differentially private. Assume \( k = o(m) \), then with probability \( 1 - o(1) \), for every \( 1 \leq j \leq k \),

\[
x_{ij} - x_{(j)} \leq (1 + o(1)) \frac{\sqrt{k \log(1/\delta) \log m}}{\varepsilon}.
\]

One naturally asks if we can avoid peeling by adding the noise \( \text{Lap}(\tilde{O}(\sqrt{k}) \) once and then reporting the \( k \)-minimum elements. Such a one-shot mechanism would be much more efficient and more natural. One major contribution of this paper is to show indeed the one-shot mechanism is private.

3 FDR Control of Adaptive Procedures

We now study the FDR control of the adaptive procedures introduced in Definition 1.1 and then prove Theorem 1. The class of adaptive procedures includes both the step-up and step-down BHq’s. This serves as the first step in proving FDR control properties of our differentially private algorithm.

The “natural” way of making the step-down BHq differentially private is to add sufficient noise to each \( p \)-value to ensure privacy and then run this procedure on the noisy \( p \)-values. But working with noisy \( p \)-values completely destroys some crucial properties for proving FDR control of step-down BHq. In particular, the \( j \)th most significant noisy \( p \)-value may not necessarily correspond to the \( j \)th most significant true \( p \)-value. As a result, it is possible that this (in fact any) procedure may be comparing this noisy \( p \)-value to the “wrong” critical values. In addition, some hypotheses
with small \( p \)-values may even not be rejected due to the noise, so unlike in the non-private BHq and its variants, there may be gaps in the sequence of \( p \)-values of the rejected hypotheses.

In the following, we show that even under the above unfavorable conditions, the FDR is still controlled for any procedure adaptive to the BHq critical values \( \alpha_j^{BH} \). This provides theoretical justification for the robustness of BHq observed in a wide range of empirical studies. While our goal is to show that the step-down BHq still controls the FDR when the \( p \)-values of the input are randomly perturbed, it is more convenient and general to consider the procedures which relax the condition of BHq but on unperturbed (true) \( p \)-values.

To this end, the class of adaptive procedures, including both BHq and BHq_{sd} as special examples, is characterized in Definition 1.1. Compared with BHq_{sd}, an adaptive procedure only requires all the rejections below a critical value that depends on the number of rejections, but not that each \( p \)-value is below the respective critical value. Further, it allows to reject only a subset of \( p \)-values below some threshold. Surprisingly, even so an adaptive procedure still controls the FDR as stated in Theorem 1. An novel element in Theorem 1 is that it only assumes independence within the true null test statistics. Compared with other assumptions in the literature, this new assumption is particularly simple and has the potential to capture many real life examples.

In the original proof of FDR control by BHq [2], it assumes (i) the true null statistics are independent, and (ii) the false null statistics are independent from the true null statistics. Later, [3] proposed a slightly weaker sufficient condition for FDR control called positive regression dependency on subsets, which, roughly, amounts to saying that every null statistic is positively dependent on all true null statistics. In sharp contrast, our theorem makes no assumption about the correlation structure between the false null statistics and true null statistics, at a mere cost of a constant multiplying the nominal level. To see an example that only satisfies our assumption, take i.i.d. uniform random variables on \((0, 1)\) as the true null \( p \)-values, and let all the false null \( p \)-values be \( 1 \) minus the median of the true null \( p \)-values.

We consider the bounds on FDR, FDR_2 and on FDR_k separately. Here we outline the proof for FDR and FDR_2 and leave FDR_k case in the supplemented full version. In the following we prove Theorem 1 in two parts. First we show the claimed bounds for FDR and FDR_2. For FDR_k, we actually consider a related quantity

\[
FDR^k := E \left[ \frac{V}{R}; R \geq k \right].
\]

Since \( V \leq R \), clearly we have that \( FDR_k \leq FDR^k \). We will show (3.3) holds even for FDR^k.

### 3.1 Controlling FDR and FDR_2

We will prove the result by constructing the most “adversarial” set of \( p \)-values. Imagine the following game for a powerful adversary \( A \) who are informed of all the \( m_1 \) false null hypotheses and can even set \( p \)-values for them. The remaining \( p \)-values, which are all from the true nulls, are then drawn from i.i.d. \( U(0, 1) \). Then \( A \) can pick out a subset \( S \) of \( p \)-values with the only requirement that those \( p \)-values are upper bounded by \( \alpha_{|S|} \) for the critical values \( \alpha_j = jq/m \). \( A \)'s payoff is then the ratio of the true nulls (i.e. FDR) in \( S \). The expected payoff of \( A \) would be the upper bound on FDR for any adaptive procedure. If we require \( A \) only receive payoff when he includes at least \( k \) true nulls in \( S \), then the corresponding payoff is an upper bound of FDR_k.

First how should \( A \) set the \( p \)-values of alternatives? 0! because this way \( A \) can include any number of alternatives in \( S \) to push up the size of \( S \) so raise up the critical values but without
wasting any “space” for including more true nulls. With this we can reduce bounding FDR to bounding the expected value of a random variable.

We now present the rigorous argument below. Denote by \( \mathcal{M}_0 \), with cardinality \( m_0 \), the set of true null hypotheses, and \( \mathcal{M}_1 \), with cardinality \( m_1 \), the set of false nulls. Define \( \text{FDP}_k = V / (R \vee 1) \) for \( V \geq k \) and 0 otherwise. Given a realization of \( p_1, \ldots, p_m \), we would like to obtain a tight upper bound on \( V / (R \vee 1) \) with the constraint that the maximum of the rejected \( p \)-values is no larger than \( \alpha_R \). With this in mind, call \((p_{i1}, p_{i2}, \ldots, p_{iR})\) the rejected \( p \)-values, among which \( V \) of them are from the \( m_0 \) many true null \( p \)-values. Hence, denoting by \( p_{01}, \ldots, p_{0m_0} \) the true null \( p \)-values, we see

\[
P^0(V) = \max_{1 \leq i \leq R} p_{ij} \leq \alpha_R.
\]

Taking the BHq critical value \( \alpha_R = qR/m \) and rearranging this inequality yield \( R \geq \lceil mp^0(V)/q \rceil \), which also makes use of the additional information that \( R \) is an integer. As a consequence, we get \( V / (R \vee 1) \leq V / \lceil mp^0(V)/q \rceil \). Hence, it follows that

\[
\text{FDP}_k \leq \max_{k \leq j \leq m_0} \frac{j}{\lceil mp^0(j)/q \rceil}.
\]  (3.1)

We pause here to point out that this upper bound is tight for the class of procedures adaptive to the BHq critical values. Let \( j^* \) be the index where the maximization of (3.1) is attained. In constructing the least “favorable” set of \( p \)-values and the most “adversarial” procedure, we set \( p_j = 0 \) whenever \( j \in \mathcal{M}_1 \), and let the procedure reject the \( j^* \) smallest true null \( p \)-values and \( \lceil mp^0(j)/q \rceil - j^* \) false null \( p \)-values (which are all zero). It is easy to verify the adaptivity of this case.

Recognizing that the true null \( p \)-values are independent and stochastically no smaller than uniform on \((0, 1)\), we may assume the true null \( p \)-values are i.i.d. uniform on \((0, 1)\) in taking expectations of both sides of (3.1). In addition, it is easy to see that increasing \( m_0 \) to \( m \) only makes this inequality more likely to hold. Hence, we have

\[
\text{FDR}_k \leq \mathbb{E} \left[ \max_{k \leq j \leq m} \frac{j}{mU(j)/q} \right],
\]  (3.2)

where, as earlier, \( U(1) \leq U(2) \cdots \leq U(m) \) are the order statistics of i.i.d. uniform random variables \( U_1, \ldots, U_m \) on \((0, 1)\).

Taking \( k = 2 \) in (3.2), we proceed to obtain the bound on FDR2 in the following lemma. We note that because of some technical issues, the following analysis applies to the case of \( k = 2 \). For general \( k \), we use a different technique to bound \( \text{FDR}_k \). See Section 3.2.

**Lemma 3.1.** There exists an absolute constant \( C \) such that

\[
\mathbb{E} \left[ \max_{2 \leq j \leq m} \frac{qj}{mU(j)} \right] \leq Cq.
\]

To bound the above expectation, we use a well-known representation for the uniform order statistics (e.g. see [6]): \( (U(1), \ldots, U(m)) \overset{d}{=} (T_1/T_{m+1}, \ldots, T_m/T_{m+1}) \), where \( T_j = \xi_1 + \cdots + \xi_j \), and \( \xi_1, \ldots, \xi_{m+1} \) are i.i.d. exponential random variables. Denote by \( W_j = jT_{m+1}/T_j \). Then Lemma 3.1 is equivalent to bounding \( \mathbb{E} (\max_{2 \leq j \leq m} W_j/m) \leq C \).
Intuitively the maximum is more likely to be realized by smaller $j$’s as increasing $j$ would increase the concentration of $T_j/j = \sum_{i=1}^{l} \xi_i/j$. Then above expectation can be bounded by considering a few terms of $W_j$ for small $j$’s. Indeed this intuition can be made rigorous by observing that $W_1,\ldots,W_{m+1}$ is a *backward submartingale*, and hence we can use the well known technique to bound $\mathbb{E}\max_{2\leq j \leq m} W_j$ by some statistics of $W_2$, which we can then estimate. We present the details as below.

**Lemma 3.2.** With $T_j,W_j$ defined as above, we have that $W_1,\ldots,W_{m+1}$ is a backward submartingale with respect to the filtration (or conditional on the “history”) $\mathcal{F}_j = \sigma(T_j,T_{j+1},\ldots,T_{m+1})$ for $j = 1,\ldots,m+1$, i.e. $\mathbb{E}(W_j|\mathcal{F}_{j+1}) \geq W_{j+1}$, for $j = 1,\ldots,m$.

Now we apply Lemma 3.2 to prove Lemma 3.1.

**Proof of Lemma 3.1.** Using the exponential random variables representation, it suffices to prove the inequality in which $\mathbb{E}\left[\max_{2\leq j \leq m} \frac{q j}{mU(j)}\right]$ is replaced by

$$\frac{q}{m} \mathbb{E}\left[\max_{2\leq j \leq m} \frac{j T_{m+1}}{T_j}\right] = \frac{q}{m} \mathbb{E}\left[\max_{2\leq j \leq m} \frac{W_j}{m}\right],$$

where all the notations are followed from Lemma 3.2. Given that $W_j/m$ is a backward submartingale by Lemma 3.2, we can apply Theorem 5.4.4 in [7], which concludes

$$\mathbb{E}\left(\max_{2\leq j \leq m} W_j/m\right) \leq (1 - e^{-1})^{-1} \left[1 + \mathbb{E}\left(\frac{W_2}{m} \log \frac{W_2}{m}; \frac{W_2}{m} \geq 1\right)\right] = (1 - e^{-1})^{-1} \left[1 + \mathbb{E}\left(\frac{2}{mU(2)} \log \frac{2}{mU(2)}; \frac{2}{mU(2)} \geq 1\right)\right].$$

A bit of analysis reveals that the RHS of the last display is bounded. Hence, to complete the proof it suffices to show that the RHS of the above display is uniformly bounded for all $m$. The fact that $U(2)$ is distributed as Beta$(2, m - 1)$ allows us to evaluate the expectation in the last display as

$$\mathbb{E}\left(\frac{2}{mU(2)} \log \frac{2}{mU(2)}; \frac{2}{mU(2)} \geq 1\right) = \int_0^2 u (1-u)^{m-2} \frac{2}{B(2, m-1)} \frac{2}{mu} \log \frac{2}{mu} \, du \leq \int_0^2 \frac{2}{v} \log \frac{2}{v} \, dv,$$

which is finite and independent of $m$. 

Provided the bound on $\text{FDR}_2$, we can easily obtain the bound on $\text{FDR}$. Taking $k = 1$ in (3.2), we get

$$\text{FDR} \leq \mathbb{E}\left[\max_{2\leq j \leq m} \frac{j}{mU(j)/q}\right] + \mathbb{E}\left[\min\left\{\frac{1}{mU(1)/q}, 1\right\}\right],$$

It is easy to show the second term is bounded by $q \log(1/q)$. The lemma below controls the second term in the above display.

**Lemma 3.3.**

$$\mathbb{E}\left[\min\left\{\frac{1}{mU(1)/q}, 1\right\}\right] \leq q \log \frac{1}{q} + C_0 q,$$

where $C_0 = 1 + 2/\sqrt{e} \leq 2.3$.  

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This proves the bound for FDR in Theorem 1. Without a lower bound on the number of discoveries, we suffer an extra additive term of $q \log(1/q)$. Here comes an explanation how does this term emerge. Let $p^0_{\min}$ be the smallest $p$-value from all the $m_0$ true null hypotheses and, to be the most adversarial, all the false null $p$-values are set to zero. Then an adversary can reject $R - 1$ false null hypotheses, and then the true null hypothesis with $p$-value $p^0_{\min}$, where $R = \lceil mp^0_{\min}/q \rceil$. For large $m_0$, $E := m_0 p^0_{\min}$ is asymptotically distributed as exponential with mean 1. If $m_0 \approx m$, observe that

$$\frac{1}{R} \approx \frac{q}{mp^0_{\min}} = \frac{qm_0}{mE} \approx \frac{q}{E}.$$

The partial expectation of the last term $q/E$ on $[q, \infty)$ is approximately

$$\int_q^\infty \frac{q}{u} e^{-u} du = q \log \frac{1}{q} + O(q).$$

This justifies the logarithmic term $\log \frac{1}{q}$.

### 3.2 Controlling FDR

FDR control comes naturally for studies where a large number of rejections are expected to be made. Provided with this side information, we can improve the constant $C$ in (1.1) to 1 asymptotically for $R \gg 1$. As mentioned earlier, we will provide an upper bound on FDR$^k$.

**Theorem 5.** Assume joint independence of the true null test statistics. Then any procedure adaptive to the BH$q$ critical values obeys

$$\text{FDR}^k \leq \left(1 + 2/\sqrt{qk}\right) q. \quad (3.3)$$

**Proof of Theorem 5.** Similar to the argument for bounding FDR and FDR$_2$, it suffices to consider the most adversarial scenario, where it always holds that

$$V \leq \# \{i \in \mathcal{N}_0 : p_i \leq \frac{qR}{m} \},$$

which gives an upper bound on FDP:

$$\text{FDP} \leq \max_{R \leq j \leq m} \# \{i \in \mathcal{N}_0 : p_i \leq qj/m \}.$$\quad (3.4)

Similar to what has been argued previously, the above inequality still holds if all the null $p$-values are $m$ i.i.d. uniform on $(0, 1)$. This observation leads to the proof of (3.3) by showing the following inequality, again by applying tools from Martingale theory

$$\mathbb{E} \left[ \max_{k \leq j \leq m} \# \{1 \leq i \leq m : U_i \leq \frac{qj}{m} \} \right] \leq \left(1 + 2/\sqrt{qk}\right) q.$$

Denote by as usual $V_j = \# \{1 \leq i \leq m : U_i \leq \frac{qj}{m} \}$ and $W_j = \frac{V_j}{j}$. Conditional on $W_{j+1}$, $V_{j+1}$ of $U_i$ are uniformly distributed on $[0, q(j+1)/m]$. Hence the conditional expectation of $V_j$ given by $W_{j+1}$ is

$$\mathbb{E}(V_j|W_{j+1}) = \frac{V_{j+1} \frac{qj}{m}}{\frac{q(j+1)}{m}} = \frac{jV_{j+1}}{j+1}.$$
which is equivalent to
\[ \mathbb{E}(W_j|W_{j+1}) = W_{j+1}. \]
Thus, \((W_j - q)_+\) is a backward submartingale, which allows us to use the martingale \(l^2\) maximum inequality:
\[
\mathbb{E} \left[ \max_{k \leq j \leq m} (W_j - q)_+ \right]^2 \leq \left( \frac{2}{2 - 1} \right)^2 \mathbb{E}(W_k - q)^2 \leq 4\mathbb{E}(W_k - q)^2 = \frac{4q(1 - qk/m)}{k} < \frac{4q}{k}.
\]
Last, Jensen’s inequality gives
\[
\mathbb{E} \left[ \max_{k \leq j \leq m} W_j \right] \leq q + \mathbb{E} \left[ \max_{k \leq j \leq m} (W_j - q)_+ \right] \leq q + \sqrt{\mathbb{E} \left[ \max_{k \leq j \leq m} (W_j - q)_+ \right]^2} \leq q + 2\sqrt{\frac{q}{k}}.
\]

4 Private Adaptive Procedures

One alternative interpretation of Theorem 1 is that BH\(_{q_{\text{sd}}}\) is robust with respect to small perturbation of \(p\)-values. That is if we add small enough noise to the \(p\)-values and then apply BH\(_{q_{\text{sd}}}\) procedure, the FDR can still be controlled within the bound given in the theorem. It turns out the additive sensitivity of \(p\)-values might be large, but the relative change is much smaller.

Now the important question is how to add noise. In the literature, often additive noise is used to guarantee privacy. But this would not work well for \(p\)-values as BH\(_{q_{\text{sd}}}\) considers the \(p\)-values in the range of \(O(1/m)\) but a simple analysis shows that the sensitivity of some standard \(p\)-values can be as large as \(1/\sqrt{n}\), which can be much larger than \(1/m\). It turns out that while the additive sensitivity might be large, the relative change is much smaller. This motivates us to consider multiplicative sensitivity. Hence we will add multiplicative noise (or additive noise to the logarithm of \(p\)-values). Together with the private top-\(k\) algorithm, this gives us the private FDR algorithm.

For the ease of presentation, we assume that we are given an upper bound \(k\) of the number of rejections. The parameter \(k\) should be comparable to the number of true rejections and small compared to \(m\). It is easy to adapt our algorithm to the case when \(k\) is not given, for example, by the standard doubling trick. This only incurs an extra logarithmic factor in our bound.

4.1 Sensitivity of \(p\)-values

Assume the input to our private FDR algorithm is an \(m\)-tuple of \(p\)-values \(p = (p_1, \ldots, p_m)\) obtained by running \(m\) statistical tests on a dataset \(D\). Motivated by the normal approximation (or related \(\chi^2\) approximation) frequently used in computing \(p\)-values, the change in a \(p\)-value caused by the addition or deletion of the data of one individual is best measured multiplicatively; however, when the \(p\)-value is very small the relative change can be very large. This gives rise to the following definition of (truncated) multiplicative neighborhood.

**Definition 4.1 \(((\eta, \nu)\text{-neighbors})\).** Tuples \(p = (p_1, \ldots, p_m), p' = (p'_1, \ldots, p'_m)\) are \((\eta, \nu)\text{-neighbors}\) if, for all \(1 \leq i \leq m\), either \(p_i, p'_i < \nu\), or \(e^{-\eta}p_i \leq p'_i \leq e^{\eta}p_i\).
The privacy of our FDR algorithm will be defined with respect to such neighborhood. Next we will explain that some standard \( p \)-values computed on neighboring databases are indeed \((\eta, \nu)\)-neighbors for small \( \eta \) and \( \nu \). We will give an intuitive explanation using Gaussian approximation but omit the proof details. Consider a database consisting of the records of \( n \) people and a hypothesis \( H \) to be tested, where each person contributes a statistic \( t_i, i = 1, \ldots, n \) with \( |t_i| \leq B \) (for example, \( t_i \) is the number of minor alleles for a given SNP.) In many interesting cases, the sufficient statistic for testing the hypothesis is \( T = t_1 + \cdots + t_n \), and under the null hypothesis \( H \), each \( t_i \) has mean \( \mu \) and variance \( \sigma^2 \). Then \((T - n\mu)/\sqrt{n}\sigma)\) is asymptotically distributed as standard normal variable. Assuming \( T \) tends to be larger under the alternative hypothesis, we can approximately compute \( p \)-value \( p(T) = \Phi(- (T - n\mu)/\sqrt{n}\sigma) \), where \( \Phi \) is the cumulative distribution function of standard Gaussian. Consider a neighboring database where person \( i \) is replaced, so \( T' = T_t - t_i \). Writing \( p' = p(T') \) and invoking that \( \Phi(-x) \approx \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \) for large \( x > 0 \), we can show that \( p(T) \) is \((\eta, \nu)\) multiplicative sensitive for \( \eta \approx B\sqrt{2 log(1/\nu)/n/\sigma} = O(\sqrt{log(1/\nu)/n/\sigma}) \).

### 4.2 A private FDR algorithm

To achieve privacy with respect to \((\eta, \mu)\)-neighborhood, we apply \( M_{\text{Peel}} \) with properly scaled noise to the logarithm of the input \( p \)-values (see Step 1 of Algorithm 4). Denote by \( \Delta_k = (1 + o(1)) \sqrt{k \log(1/\delta) \log m/\varepsilon} \) the accuracy bound provided by \( M_{\text{Peel}} \). Then we run BHqsd with cutoff values set as \( \alpha'_j = \log(\alpha_j + \nu) + \eta \Delta_k \). The details are described in Algorithm 4.

#### Algorithm 4 Differentially private FDR-controlling algorithm

**Input:** \((\eta, \nu)\)-sensitive \( p \)-values \( p_1, \ldots, p_m \) and \( k \geq 1 \) and \( \varepsilon, \delta \)

**Output:** a set of up to \( k \) rejected hypotheses

1. for each \( 1 \leq i \leq m \), set \( x_i = \log(\max\{p_i, \nu\}) \)
2. apply \( M_{\text{Peel}} \) to \( x_1, \ldots, x_m \) with noise Lap\((\eta \sqrt{k \log(1/\delta) / \varepsilon})\) to obtain \((i_1, y_1), \ldots, (i_k, y_k)\)
3. apply BHqsd to \( y_1, \ldots, y_k \) with cutoffs \( \alpha'_j \) for \( 1 \leq j \leq k \) and reject the corresponding hypotheses

Our main result is that this algorithm is differentially private and controls FDR, while maintaining a descent power. In particular, the second and third conclusions serve as, respectively, formal versions of Theorems 2 and 3.

**Theorem 6.** Algorithm 4 obeys:

(a) It is \((\varepsilon, \delta)\)-differentially private;

(b) If BHqsd rejects \( k' \leq k \) hypotheses, Algorithm 4 rejects at least \( k' \) hypotheses with probability \( 1 - o(1) \);

(c) Suppose \( \nu = o(1/m) \), then the FDR of Algorithm 4 satisfies the bounds in Theorem 1 with \( q \) replaced by \( e^{\theta \Delta_k} (1 + o(1)) q \).

**Proof.** (a) For two \((\eta, \nu)\) neighbor \( p, p' \), by definition for each \( i \), either \( p_i, p'_i \leq \nu \) or \( e^{-\eta} p_i, e^{\eta} p'_i \). In either case, we have that \( |x'_i - x_i| \leq \eta \). Hence the privacy of Algorithm 4 follows from the privacy of \( M_{\text{Peel}} \).
(b) By Theorem 4, we have that with probability $1 - o(1)$, $y_j \leq x(j) + \eta \Delta_k$. Hence if for $1 \leq j \leq k'$, $p(j) \leq \alpha_j$, then we have $x(j) \leq \log(\alpha_j + \nu)$. Thus $y_j \leq \log(\alpha_j + \nu) + \eta \Delta_k = \alpha'_j$. Hence Algorithm 4 rejects at least $k'$ hypotheses as well.

(c) Suppose that the algorithm rejects the set of hypotheses $t_1, \ldots, t_{k'}$, then we have that $y_{t_j} \leq \alpha'_j$. By Theorem 4 with high probability the corresponding $p$-value is bounded by $e^{O(\eta \Delta_k)(\alpha_j + \nu)}$. By that $\alpha_j = j q / m$, Algorithm 4 is an adaptive procedure with respect to the cutoff

$$(q' / m, 2q' / m, \ldots, kq' / m),$$

where $q' = e^{O(\eta \Delta_k)(1 + o(1))q}$. Then the bounds in Theorem 1 apply by plugging in $e^{O(\eta \Delta_k)(1+ o(1))q}$ in place of $q$.

\[ \square \]

For the binary database with independent statistics, take $\eta = O(\sqrt{\log m/n})$ and $\nu = 1/m^2$. Then, from the discussion in Section 4.1 we get

**Corollary 4.2.** Algorithm 4 is $(\varepsilon, \delta)$-differentially private. In addition, if $k \ll n / \log^{O(1)}(m)$, with probability $1 - o(1)$, it rejects at least the same amount of hypotheses as in Algorithm 2 and controls FDR at $q \log(1 / q) + C(1 + o(1))q$, FDR$_2$ at $C(1 + o(1))q$, and FDR$_k$ at $(1 + o(1))q$.

One drawback of the above algorithm is that it needs to run the peeling algorithm which takes $\tilde{O}(km)$ time. This can be expensive for large $k$. In the next section, we show the privacy of one-shot algorithm $\mathcal{M}_{\text{os}}$. By replacing $\mathcal{M}_{\text{Ped}}$ with $\mathcal{M}_{\text{os}}$ in Algorithm 4, we can obtain the Fast Private FDR algorithm which has essentially the same quality bound but with running time $\tilde{O}(k + m)$.

## 5 One-shot mechanism for reporting top-$k$

In the one-shot mechanism $\mathcal{M}_{\text{os}}$, we add noise $\text{Lap}(\lambda)$ once to each value and report the indices of the minimum-$k$ noisy values (Algorithm 5). Here the input to our algorithms is $m$ counts $x = (x_1, \ldots, x_m)$ which are the marginals of database $D$. So two set of counts $x$ and $x'$, representing respectively the marginals of two neighboring databases $D$ and $D'$, satisfy $\|x - x'\|_\infty \leq 1$.

Using a coupling argument, it is easy to show by setting $\lambda = O(k / \varepsilon)$, the one-shot mechanism is $(\varepsilon, 0)$-differentially private.

**Lemma 5.1.** Algorithm 5 is $(\varepsilon, 0)$-differentially private if we set $\lambda = 2k / \varepsilon$.

\textbf{Proof of Lemma 5.1.} Let $x = (x_1, \ldots, x_m)$ and $x' = (x'_1, \ldots, x'_m)$ be adjacent, i.e., $\|x - x'\|_\infty \leq 1$. Let $S$ be an arbitrary $k$-subset of $\{1, \ldots, m\}$. Let $\mathcal{G}$ consist of all $(g_1, \ldots, g_m)$ such that $\mathcal{M}_{\text{os}}$ on $x$ reports $S$. Similarly we have $\mathcal{G}'$ for $x'$. It is clear to see that $\mathcal{G} - 2 \cdot 1_S \subseteq \mathcal{G}'$, where $1_S \in \{0, 1\}^m$ satisfies $1_S(i) = 1$ if and only if $i \in S$. Hence we get

$$\mathbb{P}(\mathcal{M}_{\text{os}}(x) = S) = \int_{\mathcal{G}} \frac{1}{2^m \lambda^m} e^{-\frac{\|g\|_1}{\lambda}} \, dg \geq \int_{\mathcal{G}' + 2 \cdot 1_S} \frac{1}{2^m \lambda^m} e^{-\frac{\|g\|_1}{\lambda}} \, dg$$

$$= \int_{\mathcal{G}'} \frac{1}{2^m \lambda^m} e^{-\frac{\|g + 2 \cdot 1_S\|_1}{\lambda}} \, dg \geq \int_{\mathcal{G}'} \frac{e^{-2 \lambda / \lambda}}{2^m \lambda^m} e^{-\frac{\|g\|_1}{\lambda}} \, dg = e^{-\varepsilon \mathbb{P}(\mathcal{M}_{\text{os}}(x') = S)}.$$

On the opposite side, we have $\mathbb{P}(\mathcal{M}_{\text{os}}(x) = S) \leq e^{\varepsilon \mathbb{P}(\mathcal{M}_{\text{os}}(x') = S)}$, which completes the proof since $S$ is arbitrary.

\[ \square \]
Our goal is to reduce the dependence on $k$ to $\sqrt{k}$ for approximate differential privacy.

**Theorem 7.** Take $\varepsilon \leq \log(m/\delta)$. There exists universal constant $C > 0$ such that if we set $\lambda = C\sqrt{k}\log(m/\delta)/\varepsilon$, then $\mathcal{M}^{os}$ is $(\varepsilon, \delta)$-differentially private. In addition, with probability $1-o(1)$, for every $1 \leq j \leq k$, $x_{i_j} - x_{(j)} = O(\sqrt{k}\log(m/\delta)\log m/\varepsilon)$.

Unlike in the peeling approach, in which an ordered set is returned, in $\mathcal{M}^{os}$, only a subset of $k$ elements, but not their ordering, is returned. The privacy proof crucially depends on this. If we would also like to report the values, we can report the noisy values by adding random noise freshly sampled from $\text{Lap}(O(\sqrt{k}\log(1/\delta)/\varepsilon))$ to each value of those $k$ elements.

**Algorithm 5** The one-shot procedure $\mathcal{M}^{os}$ for privately reporting minimum-$k$ elements

**Input:** $m$ values $x_1, \ldots, x_m$, $k \geq 1$, $\lambda$

**Output:** $i_1, \ldots, i_k$

1: for $i = 1$ to $m$ do
2: set $y_i = x_i + g_i$ where $g_i$ is sampled i.i.d. from $\text{Lap}(\lambda)$
3: end for
4: sort $y_1, \ldots, y_m$ from low to high, $y(1) \leq y(2) \leq \cdots \leq y(m)$
5: return the set $\{(1), \ldots, (k)\}$

In Theorem 7, the quality bound on $x_{i_j} - x_{(j)}$ is an immediate consequence from the properties of exponential random variables. The claim on the privacy part is equivalent to the lemma below.

**Lemma 5.2.** For any $k$-subset $S$ of $\{0, 1\}^m$ and any adjacent $x, x'$, we have

$$\Pr(\mathcal{M}^{os}(x) \in S) \leq e^{\varepsilon}\Pr(\mathcal{M}^{os}(x') \in S) + \delta.$$  

We divide the event space by fixing the $k$th smallest noisy element, say $j$, together with the noise value, say $g_j$. For each partition, whether an element $i \neq j$ is selected by $\mathcal{M}^{os}$ only depends on whether $x_i + g_i \leq x_j + g_j$, which happens with probability $q_i = G((x_j + g_j - x_i)/\lambda)$. Here $G$ denotes the cumulative distribution function of the standard Laplace distribution. Hence, we consider the following mechanism $\mathcal{M}$ instead: given $(q_1, \ldots, q_m)$ where $0 < q_i < 1$, output a subset of indices where each $i$ is included with probability $q_i$. In the following, we will first understand the sensitivity of $q_i$ dependent on the change of $x_i$ and then show that $\mathcal{M}$ is “private” with respect to the corresponding sensitivity on $q$.

**Lemma 5.3.** For any $z, z'$, where $|z' - z| \leq 1, |G(z') - G(z)| \leq 2|z' - z|G(z)(1 - G(z))$.

**Definition 5.4** $(c$-closeness for vectors). For two vectors $q = (q_1, \ldots, q_m)$ and $q' = (q'_1, \ldots, q'_m)$, we say $q, q'$ are $c$-close if for each $1 \leq i \leq m$, $|q_i - q'_i| \leq cq_i(1 - q_i)$.

The following is the crucial lemma whose proof requires much technical work.

**Lemma 5.5.** Assume $\varepsilon \leq \log(1/\delta)$ and $k \geq \log(1/\delta)$. There exists constant $C_1 > 0$ such that if $q$ and $q'$ are $c$-close with $c \leq \frac{\varepsilon}{C_1\sqrt{k}\log(1/\delta)}$, then for any set $S$ of $k$-subsets of $\{0, 1\}^m$, $\Pr(\mathcal{M}(q) \in S) \leq e^{\varepsilon}\Pr(\mathcal{M}(q') \in S) + \delta$.

We first show how these two lemmata imply Lemma 5.2 and then prove Lemma 5.5.
Proof of Lemma 5.2. If \( k \leq \log(m/\delta) \), then by Lemma 5.1, the mechanism is \((\varepsilon, 0)\) private. Assume \( k \geq \log(m/\delta) \). Denote by \( I_k \) the random variable of the \( k \)th smallest element in terms of the noisy count. For any given \( I_k = j \) and the noise \( g_j = g \), we have that \( \mathbb{P}(i \in \mathcal{M}^\text{os}(x)) = G((x_j + g - x_i)/\lambda) := q_i \). Set \( q = q(g) = (q_i) \) for \( 1 \leq i \leq m \) and \( i \neq j \). Write \( S_j = \{s/\{j\} : s \in S \text{ and } j \in s\} \). Then

\[
\mathbb{P}(\mathcal{M}^\text{os}(x) \in S, I_k = j|g_j = g) = \mathbb{P}(M(q) \in S_j).
\]

Since \( \|x - x'\|_{\infty} \leq 1 \), we have that for any \( i, |(x_j + g - x_i)/\lambda - (x'_j + g - x'_i)/\lambda| \leq 2/\lambda \). By Lemma 5.3, \( q \) and \( q' \) are \( 4/\lambda \)-close. Set \( \lambda \geq 4C_1 \sqrt{k \log(m/\delta)}/\varepsilon \), then \( q \) and \( q' \) are \( C_1 \sqrt{\varepsilon k \log(m/\delta)} \)-close. Applying Lemma 5.5 to \( q \) and \( q' \) with parameters \( \varepsilon \) and \( \delta/m \), we have

\[
\mathbb{P}(\mathcal{M}^\text{os}(x) \in S, I_k = j|g_j = g) \leq e^\varepsilon \mathbb{P}(\mathcal{M}^\text{os}(x') \in S, I_k = j|g_j = g) + \delta/m.
\]

\( \square \)

Now we prove Lemma 5.5. Since \( S \) consists of \( k \)-sets, we first show that if \( \sum_i q_i \gg k \), then \( \mathbb{P}(M(q) \in S) \) is small. This can be done by applying the standard concentration bound. Write \( K = (1 + 2\sqrt{\log(m/\delta)/k})k \leq 3k \) (recall we assume that \( k \geq \log(m/\delta) \)).

**Lemma 5.6.** Let \( Z_1, \ldots, Z_m \) be \( m \) independent Bernoulli random variables with \( \mathbb{P}(Z_i = 1) = q_i \). Suppose

\[
\sum_{i=1}^m q_i \geq (1 + t)k,
\]

for any \( t > 0 \). Then

\[
\mathbb{P}(\sum_i Z_i \leq k) \leq \exp\left(- (1 + t) h(t/(1 + t))k \right),
\]

where \( h(u) = (1 + u) \log(1 + u) - u \). Consequently, by setting \( t = 2\sqrt{\log(m/\delta)/k} \), we have that if \( \sum_{i=1}^m q_i \geq K \), then \( \mathbb{P}(\sum_i Z_i \leq k) \leq \delta \).

The above lemma follows immediately from the classical Bennett’s inequality stated as follows.

**Lemma 5.7** (Bennett’s inequality, [4]). Let \( Z_1, \ldots, Z_n \) be independent random variables with all means being zero. In addition, assume \( |Z_i| \leq a \) almost surely for all \( i \). Denoting by

\[
\sigma^2 = \sum_{i=1}^n \text{Var}(Z_i),
\]

we have

\[
\mathbb{P}\left(\sum_{i=1}^n Z_i > t\right) \leq \exp\left(- \frac{\sigma^2 h(at/\sigma^2)}{a^2}\right)
\]

for any \( t \geq 0 \), where \( h(u) = (1 + u) \log(1 + u) - u \).

With Lemma 5.6, we only need to consider the case of \( \sum_i q_i \leq K \). But this is a more difficult case. We first represent a set \( s \subseteq \{1, \ldots, m\} \) by a binary vector \( z \in \{0, 1\}^m \). Write \( \mathbb{P}_q(z) \) as

\[
\mathbb{P}_q(z) = \prod_{i: z_i = 1} q_i \prod_{i: z_i = 0} (1 - q_i).
\]
Now our goal is to show that for any $S$ consisting of weight $k$ vectors in $\{0,1\}^m$,
\[
\sum_{z \in S} P_q(z) \leq e^\varepsilon \sum_{z \in S} P_{q'}(z) + \delta.
\]
Denote by
\[
S^* = \{ z : P_q(z) \geq e^\varepsilon P_{q'}(z) \}.
\]
The proof of Lemma 5.5 would be completed by showing
\[
\sum_{z \in S^*} P_q(z) \leq \delta.
\]
By (5.1), $x \in S^*$ if and only if that
\[
\prod_{i : z_i = 1} q_i \prod_{i : z_i = 0} (1 - q_i) \geq e^\varepsilon \prod_{i : z_i = 1} q'_i \prod_{i : z_i = 0} (1 - q'_i).
\]
(5.2)
Write $\Delta_i = q'_i - q_i$. By the $c$-closeness of $q$ and $q'$, we have that $|\Delta_i| \leq c q_i(1 - q_i)$. Taking logarithm of the two sides and rearranging, (5.2) is equivalent to
\[
\sum_{i : z_i = 1} \log(1 + \Delta_i/q_i) + \sum_{i : z_i = 0} \log(1 - \Delta_i/(1 - q_i)) \leq -\varepsilon.
\]
To bound $\sum_{z \in S^*} P_q(z)$, consider the independent 0,1 random variables $Z_1, \ldots, Z_m$, where for each $i$, $Z_i = 1$ with probability $q_i$. Define
\[
\zeta_i = Z_i \log(1 + \Delta_i/q_i) + (1 - Z_i) \log(1 - \Delta_i/(1 - q_i)).
\]
Let $I$ be the indicator function. Then we have
\[
\sum_{z \in S^*} P_q(z) = \sum_z I \left( \sum_i \zeta_i \leq -\varepsilon \right) P_q(z) = P \left( \sum_i \zeta_i \leq -\varepsilon \right),
\]
where the last probability is over the distribution of $Z_1, \ldots, Z_m$.

The rest of the proof is to prove that $P(\sum_i \zeta_i \leq -\varepsilon) \leq \delta$. It is easy to check that $\zeta_1 + \ldots + \zeta_m$ has mean
\[
\sum_{i=1}^m (q_i \log(1 + \Delta_i/q_i) + (1 - q_i) \log(1 - \Delta_i/(1 - q_i)))
\]
and variance
\[
\sigma^2 = \sum_{i=1}^m q_i (1 - q_i) \log^2 \frac{1 + \Delta_i/q_i}{1 - \Delta_i/(1 - q_i)}.
\]
Before applying Bennett’s inequality, we check the ranges of the centered random variables $\zeta_i - q_i \log(1 + \Delta_i/q_i) - (1 - q_i) \log(1 - \Delta_i/(1 - q_i))$, which are bounded in absolute value by
\[
\max_{1 \leq i \leq m} \left| \log \frac{1 + \Delta_i/q_i}{1 - \Delta_i/(1 - q_i)} \right|.
\]
Then Bennett’s inequality asserts that for any \( t \geq 0 \),

\[
\sum_{i=1}^{m} \zeta_i \geq \sum_{i=1}^{m} \left( q_i \log(1 + \Delta_i/q_i) + (1 - q_i) \log(1 - \Delta_i/(1 - q_i)) \right) - t
\]

with probability at least

\[
1 - \exp\left( -\sigma^2 h(A_t/\sigma^2)/A^2 \right),
\]

where \( A = \max_{1 \leq i \leq m} |\log \frac{1+\Delta_i/q_i}{1-\Delta_i/(1-q_i)}| \). Then, taking

\[
t = \varepsilon + \sum_{i=1}^{m} \left( q_i \log(1 + \Delta_i/q_i) + (1 - q_i) \log(1 - \Delta_i/(1 - q_i)) \right),
\]

Bennett’s inequality concludes that

\[
P\left( \sum_i \zeta_i \leq -\varepsilon \right) \leq \exp\left( -\sigma^2 h(A_t/\sigma^2)/A^2 \right).
\]

Hence, the case of \( \sum_i q_i \leq K \) can be established by proving

\[
\frac{\sigma^2 h(A_t/\sigma^2)}{A^2} \geq \log \frac{1}{\delta}.
\] (5.3)

The proof of (5.3) is deferred to the appendix. This proves Theorem 7 via first completing the proof of Lemma 5.5.

6 Discussion

In addition to all the usual scientific reasons for investigating FDR control, our interest in the problem was piqued by the following observation. As suggested by the Fundamental Law of Information Recovery, exposure of data to a query causes some amount of privacy loss that eventually accumulates until privacy is lost. This leads to the notion of a privacy budget, in which a limit on what is deemed to be a reasonable amount of privacy loss is established, and access to the data is terminated once the cumulative loss reaches this amount. This denial of access can be difficult to accept.

In a sense the multiple comparisons problem describes a different way in which data can be “used up,” to wit, in accuracy. This connection between multiple hypothesis testing and differential privacy provided the seeds for an exciting recent result showing that differential privacy protects against false discoveries due to adaptive data analysis [8]. Although often confounded, adaptivity is a different source of false discovery than multiple hypothesis testing; resilience to adaptivity does not address the problem studied in the current paper.

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References


A Technical Proofs

Proof of Lemma 3.2. The proof is similar to Example 5.6.1 in [7]. By scaling, assume that $\xi_i$ are exponential random variables with parameter 1, i.e, $E\xi_i = 1$. Note that $W_j$ is measurable with respect to $\mathcal{F}_j$. In the proof, we first consider the conditional expectation $E(W_j^{-1}|\mathcal{F}_{j+1})$, then return to $E(W_j|\mathcal{F}_{j+1})$ by applying Jensen’s inequality. Specifically, note that

$$E\left[\frac{\xi_l}{jT_{m+1}}|\mathcal{F}_{j+1}\right] = \frac{1}{jT_{m+1}}E(\xi_l|\mathcal{F}_{j+1}),$$

since $T_{m+1}$ is measurable in $\mathcal{F}_{j+1}$. Next, observe that by symmetry we get

$$E(\xi_l|\mathcal{F}_{j+1}) = E(\xi_k|\mathcal{F}_{j+1})$$

for any $l, k \leq j + 1$. Combining the last two displays gives

$$E\left[\frac{T_j}{jT_{m+1}}|\mathcal{F}_{j+1}\right] = E\left[\frac{T_j}{(j+1)T_{m+1}}|\mathcal{F}_{j+1}\right] + \sum_{l=1}^{j} E\left[\frac{\xi_l}{j(j+1)T_{m+1}}|\mathcal{F}_{j+1}\right]$$

$$= E\left[\frac{T_j}{(j+1)T_{m+1}}|\mathcal{F}_{j+1}\right] + \sum_{l=1}^{j} E\left[\frac{\xi_{l+1}}{j(j+1)T_{m+1}}|\mathcal{F}_{j+1}\right]$$

$$= E\left[\frac{T_{j+1}}{(j+1)T_{m+1}}|\mathcal{F}_{j+1}\right] = \frac{T_{j+1}}{(j+1)T_{m+1}}.$$ 

To complete, note that Jensen’s inequality asserts that

$$E(W_j|\mathcal{F}_{j+1}) \geq \frac{1}{E(W_j^{-1}|\mathcal{F}_{j+1})} = \frac{(j+1)T_{m+1}}{T_{j+1}} = W_{j+1},$$

as desired. □

Proof of Lemma 3.3. The order statistic $U_{(1)}$ follows from Beta$(1, m)$, whose density is $m(1-x)^{m-1}$ at the value $x$. Thus,

$$E\left[\min \left\{ \frac{1}{mU_{(1)}/q}, 1 \right\}\right] \leq E\left[\min \left\{ \frac{q}{mU_{(1)}}, 1 \right\}\right]$$

$$= \mathbb{P}(U_{(1)} < q/m) + \int_{\frac{q}{m}}^{1} \frac{q(1-x)^{m-1}}{x} \, dx.$$ 

In this display, the term $\mathbb{P}(U_{(1)} < q/m)$ is simply equal to $1 - (1 - q/m)^m \leq q$, and the integral satisfies

$$\int_{\frac{q}{m}}^{1} \frac{q(1-x)^{m-1}}{x} \, dx \leq q \int_{\frac{q}{m}}^{1} \frac{(1-y/m)^{m-1}}{y} \, dy$$

$$\leq q \int_{\frac{q}{m}}^{1} \frac{1}{y} \, dy + q \int_{1}^{m} e^{-y/2} \, dy$$

$$\leq q \log \frac{1}{q} + \frac{2q}{\sqrt{e}}.$$ 

□
Proof of Lemma 5.6. Bennett’s inequality asserts

\[ P\left(\sum_{i} Z_i \leq k\right) = P\left(\sum_{i} Z_i - \sum_{i} q_i \leq -tk\right) \leq e^{-\sigma^2 h(tk/\sigma^2)}, \]

where \( \sigma^2 = \sum_{i=1}^{m} q_i (1-q_i) \leq (1+t)k \). Invoking the monotonically decreasing property of \( \sigma^2 h(tk/\sigma^2) \) as a function of \( \sigma^2 \), we get

\[ P\left(\sum_{i} Z_i \leq k\right) \leq e^{-\sigma^2 h(tk/\sigma^2)} \leq \exp\left(-(1+t)h(t/(1+t))k\right). \]

\( \square \)

We now aim to bound \( t, A \), and \( \sigma \) using the fact \( |\Delta_i| \leq cq_i(1-q_i) \). First, using \( u - u^2 \leq \log(1 + u) \leq u \) for \( |u| \leq 1/2 \), we have that

\[
\begin{align*}
\sum_{i=1}^{m} (q_i \log(1 + \Delta_i/q_i) + (1 - q_i) \log(1 - \Delta_i/(1 - q_i))) & \\
& \leq \sum_{i} |q_i(\Delta_i/q_i + (\Delta_i/q_i)^2) + (1 - q_i)(-\Delta_i/(1 - q_i) + (\Delta_i/(1 - q_i))^2)| \\
& = \sum_{i} (q_i(\Delta_i/q_i)^2 + (1 - q_i)(\Delta_i/(1 - q_i))^2) \\
& = \sum_{i} (q_i(\Delta_i/q_i)^2 + (1 - q_i)(\Delta_i/(1 - q_i))^2) \\
& \leq \left(c^2 \sum_{i} q_i(1-q_i)^2 + q_i(1-q_i)^2\right) \text{ by } |\Delta_i| \leq cq_i(1-q_i) \\
& \leq c^2 \sum_{i} q_i \leq O(c^2k) \text{ by } \sum_i q_i \leq K \leq 3k.
\end{align*}
\]

Hence, we get

\[ |t - \varepsilon| = O(c^2k) \tag{A.1} \]

and

\[
\left|\log \frac{1 + \Delta_i/q_i}{1 - \Delta_i/(1 - q_i)}\right| \leq \left|\log \frac{1 + c(1-q_i)}{1 - cq_i}\right| \leq \frac{c}{1 - cq_i} = O(c).
\]

Combining these results gives

\[ A = O(c). \tag{A.2} \]

In a similar way, we have

\[
\sigma^2 = \sum_{i=1}^{m} q_i(1-q_i) \log^2 \frac{1 + \Delta_i/q_i}{1 - \Delta_i/(1 - q_i)} \\
\leq \sum_{i} q_i(1-q_i)O(c^2) \\
= O(c^2k). \tag{A.3}
\]
Since \( uh(a/u) \) is a decreasing function in \( u \), from (A.3) it follows that for any sufficiently large \( C_2 \)
\[
\sigma^2 h(At/\sigma^2) \geq C_2 c^2 k h(At/(C_2 c^2 k)),
\]
which, together with (A.1) and (A.2), gives
\[
At/(C_2 c^2 k) = O(c(\epsilon + c^2 k)/(C_2 c^2 k)) = \frac{1}{C_2} O(\epsilon/(ck + \epsilon)).
\]
For any sufficiently large \( C_1 \), set \( c = \epsilon/(C_1 \sqrt{k \log(1/\delta)}) \) and choose a sufficiently large constant \( C_2 \). Recognizing \( k \geq \log(1/\delta) \), we have
\[
At/(C_2 c^2 k) \leq 0.5.
\]
Making use of the fact that \( h(u) = \Omega(u^2) \) for \( u \leq 0.5 \), we have that
\[
\frac{\sigma^2 h(At/\sigma^2)}{A^2} \geq C_2 c^2 k h(At/(C_2 c^2 k))/A^2 = \Omega(t^2/(c^2 k)).
\]
Owing to \( \epsilon \leq \log(1/\delta) \) and \( c = \epsilon/(C_1 \sqrt{k \log(1/\delta)}) \), we set \( C_1 \) large enough such that \( t = \Omega(\epsilon) \). Consequently, we get
\[
\frac{\sigma^2 h(At/\sigma^2)}{A^2} = \Omega(\epsilon^2/(c^2 k)) = \Omega(\log(1/\delta)).
\]
This proves (5.3) and completes the proof of Lemma 5.5. Thus, Theorem 7 is proved.