feedback for midterm

Problem 3: \( d_{k} \) is not the relative metric of \( d_{\infty} \) restriction!
- \( \lim_{k \to \infty} d_{k}(a^{(k)}, L) = 0 \implies \lim_{k \to \infty} d_{\infty}(a^{(k)}, L) = 0 \)
  This implies if \( X \subseteq L' \) is \( L' \)-closed, then \( X \subseteq L' \)-closed (not the other way around!)
- Relation to complete metric space. (will see later)

Problem 4: \( d_{R^{2n} \times R^{n}} = \sqrt{2} d_{R^{2n}} \)
\( B_{R^{2n}} \subseteq B_{R^{2n}} \)
- \( d_{R^{2n}} \leq d_{R^{n} \times R^{n}} \)
\( B_{R^{2n}} \subseteq B_{R^{2n}} \)

Compactness:
Def: An open cover of metric space \( M \) is a collection \( U = \{ U_{\alpha} \} \),
where \( U_{\alpha} \) is open for \( \forall \alpha \in \mathcal{I} \), and \( M = \bigcup_{\alpha \in \mathcal{I}} U_{\alpha} = \bigcup_{\alpha \in \mathcal{I}} U_{\alpha} \)
A subcover of \( U \) is \( U^{*} \subseteq U \) such that \( M = \bigcup_{\alpha \in \mathcal{I}} U_{\alpha} \)

Example: \( M = \mathbb{R}, U = \{(\alpha, \beta): \alpha < \beta \} \)
\( U^{*} = \{(\alpha', \beta'): \alpha < \beta', \alpha, \beta' \in \mathbb{Q} \} \)
\( M = [0,1], U = \bigcup_{n \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} \frac{1}{n}, 1] \)
\( U^{*} = (0, \frac{1}{2}) \bigcup \frac{1}{2}, 1] \)

Def: A metric space \( M \) is compact if every open cover of \( M \)
has a finite subcover.
Example (non-compact): $M_1 = \{(n, n+2), n \in \mathbb{Z}\}$

$M_2 = \{(\frac{1}{n}, 1), n \in \mathbb{N}\}$

To show some $M$ is compact from definition can be very hard! If we restrict to $X \subseteq \mathbb{R}$, we have a simple criterion:

Thm (Heine-Borel): For a closed, bounded interval $[a, b] \subseteq \mathbb{R}$, any open cover has a finite subcover.

Proof: $U = \bigcup_{x \in I} \{a, b\} \subseteq \bigcup_{x \in I} U_x$ (we first restrict to

$U_x \in U$ (each $U_x = (a_x, b_x)$)

Let $X = \{x \in (a, b) \mid (a, x) \subseteq \bigcup_{i=1}^n U_i, \forall i \in U\}$

• If $x \in X$, then $\forall x \in (a, x), x \in X$.

• $(a_1)$ is covered by some open set, so $\exists c \in (a, c)$ s.t. $(a, c) \subseteq X$.

• Let $c = \sup X$, we should $c \in X$, and $c = b$.

Another more useful characterization of compactness:

Thm (Bolzano-Weierstrass) $M$ is compact $\iff$ every sequence in $M$ has a convergent subsequence.

Lemma 1. If $M$ is compact, then every sequence in $M$ has a convergent subsequence.

Proof: Prove by contradiction. Suppose $(a_n) \subset M$ and has no convergent subsequence. $\forall x \in M, \exists \delta > 0$ s.t. $B_{\delta}(x) \cap \{a_n\}$ is finite.

(if not, we can produce $\|a_{n_j}\| \in B_{\delta}(x), \text{ and } a_{n_j} \to x$)

Now take cover of $M: U \subseteq B_{\delta}(x)$. In order to cover $M$ (at least $\{a_n\}$, we need infinitely many sets.

$\square$
Lemma 2: If $M$ is a metric space such that every sequence in a convergent subsequence. Then $\forall \varepsilon > 0$, $\exists$ finitely many points $x_1, \ldots, x_n \in M$ s.t. $M \supseteq B_{\varepsilon}(x_1) \cup B_{\varepsilon}(x_2) \cup \ldots \cup B_{\varepsilon}(x_n)$.

Proof: (Proof by contradiction) Take $\forall x_1 \in M$, and $B_{\varepsilon}(x_1)$. If $B_{\varepsilon}(x_1) = M$, then we are done. Otherwise take $x_2 \in M \setminus B_{\varepsilon}(x_1)$, and if $B_{\varepsilon}(x_1) \cup B_{\varepsilon}(x_2) = M$, we are done. Otherwise take $x_3 \in M \setminus B_{\varepsilon}(x_1) \cup B_{\varepsilon}(x_2)$.

We prove this process must stop in finite steps. Otherwise we get $(x_n)_{n=1}^{\infty}$, and $d(x_i, x_j) > \varepsilon$, $\forall i \neq j$. (critical!) So $(x_n)$ is a sequence that does not have any convergent subsequences. Contradiction!

So we have $M = B_{\varepsilon}(x_1) \cup \ldots \cup B_{\varepsilon}(x_n)$.

Lemma 3: For any cover $U$ of $M$ (assume $M$ still satisfies the assumption in Lemma 2), then $\exists \varepsilon > 0$, s.t. $\forall x \in M$, $B_{\varepsilon}(x) \subseteq U$.

Proof: (Proof by contradiction) Suppose $\forall \varepsilon > 0$, $\exists x \in M$, $B_{\varepsilon}(x)$ is not contained in any open set in $U$. Take $\varepsilon = \frac{1}{n}$, we get $x_n$ each time.

$B_{\frac{1}{n}}(x_n) \subseteq U$ for $\forall U \in U$. Take a convergent subsequence $(x_{n_j}) \subseteq (x_n)$, and $\lim_{j \to \infty} x_{n_j} = x$. Since $x \in M$, $x \in U$ well.

$(x_{n_j}) \subseteq (x_n)$, and $\exists S > 0$ s.t. $B_S(x) \subseteq U$. Let $N_k$ be chosen by $d(x_{n_k}, x) < \frac{S}{2}$, $\frac{1}{n_k} < \frac{S}{2}$.

Then $B_{\frac{1}{n_k}}(x_{n_k}) \subseteq B_S(x) \subseteq U$. Contradiction!
Lemma 4: If $M$ satisfies that every convergent sequence has a convergent subsequence, then every open cover of $M$ has a finite subcover.

Proof. By Lemma 3, \( \forall \varepsilon > 0, \exists x \in U \) \ni \( B_\varepsilon(x) \subseteq U_\varepsilon \subseteq U \).

Fix this $\varepsilon$, by Lemma 2, \( \exists \{x_1, \ldots, x_n\} \) s.t. \( M = \bigcup_{i=1}^n B_\varepsilon(x_i) \), \( B_\varepsilon(x_i) \subseteq U_{x_i} \subseteq U \).

So \( M = \bigcup_{i=1}^n U_{x_i} \).

Remark: Compact $\Rightarrow$ sequentially compact.

**Continuous functions on compact space**

**Theorem**: $\mathbb{R}^n$ gives the equivalent topology. Assume any metric is given on $\mathbb{R}^n$.

**Goal**: Show any metric on $\mathbb{R}^n$ gives the equivalent topology.

**Thm**: $X \subset \mathbb{R}^n$ (with relative metric) is compact $\Rightarrow X$ is closed and bounded.

**Proof**: $\Rightarrow$ If $X$ is not closed, i.e. $\exists (a_n) \subset X$, \( \lim_{n \to \infty} a_n = x \in \mathbb{R}^n \) but $x \notin X$. Then take $(a_n)$, it does not have convergent subsequences.

If $X$ is not bounded, take $(a_n)$ with $d(a_n, 0) > n$. Then $(a_n)$ does not have convergent subsequences.

Suppose $X$ is closed and bounded, $(a_n) \subset X$.

- $a(1) = (a_1, a_2, \ldots, a_n)$
- $a(k) = (a_1^{(k)}, a_2^{(k)}, \ldots, a_n^{(k)})$ is bounded and closed in $\mathbb{R}^m$.
- $(a_{1j})_{j=1}^{\infty}$ converges.
- $(a_{2i})_{i=1}^{\infty}$ converges.
repeating this process, (in finite steps!) we get a subseq \((a^{(k)}_{ij...m}) \subset (a^{(k)})\), each component converges, 
\[ \Rightarrow \text{convergent subseq.} \]

Rmk: 
"\(\Rightarrow\)" holds for any compact metric space
"\(\Leftarrow\)" only holds for finite dim space. (\(\ell^\infty\): not true any more!)

Example: \(\mathbb{R}^n, d_E\) \(S_1 = \{ x \in \mathbb{R}^n : d_E(x, 0) = 1 \}\) unit sphere
\(S_1\) is compact \(\Leftarrow\) closed, bounded.

Thm. If \(f : M \to \mathbb{R}\) is continuous on compact \(M\), then
\(f\) is bounded on \(M\), and \(\exists c, d \in M\) s.t. 
\[ f(c) \leq f(x) \leq f(d) \]

Proof:

Steps:
1. \(\forall a \in M, \exists \delta_a > 0\) s.t.
\[ \forall b \in B_{\delta_a}(a) \text{ i.e. } d(a, b) < \delta_a \]
   
   We have 
   \[ |f(b) - f(a)| < 1 \] (Continuity)
   
   So 
   \[ |f(b)| \leq |f(b) - f(a)| + |f(a)| \leq 1 + |f(a)| \]

Step 2: \(M = \bigcup_{a \in M} B_{\delta_a}(a)\) is an open cover.
Compact \(\Rightarrow\) finite subcover \(M = \bigcup_{i=1}^{n} B_{\delta_{a_i}}(a_i)\)

for \(x \in B_{\delta_{a_i}}(a_i)\): 
\[ |f(x)| \leq 1 + |f(a_i)| \]

Step 3: \(R = \max_{i=1}^{n} \left\{ 1 + |f(a_i)| \right\}\) then 
\[ |f(x)| \leq R, \quad \forall x \in M. \]

Step 4: Since \(\{f(x) : x \in M\}\) is bounded above/below, take
\[ y = \sup \{ f(x) \}, \quad z = \inf \{ f(x) \}\] we prove they can be attained.
suppose \( f(x_i) \rightarrow y, \ x_i \in M \).
\( (x_i) \) has a convergent subseq \( (x_{i_j}) \). (Compactness)
\( x_{i_j} \xrightarrow{j \to \infty} \ a \in M \)
\( \xrightarrow{\text{Continuity}} \)
\( f(x_{i_j}) \xrightarrow{j \to \infty} f(a) \).

So \( y = f(a) \). Similarly we have \( z = f(c) \).

So \( f(c) \leq f(x) \leq f(d) \)

\( \Box \).

Application: equivalence of norms on \( \mathbb{R}^d \).

Def: Let \( V \) be a vector space. A norm is \( \| \cdot \| : V \rightarrow [0, \infty) \)

s.t.
(i) \( \| x \| = 0 \Leftrightarrow x = 0 \)
(ii) \( \| cx \| = |c| \| x \| \quad \forall c \in \mathbb{R}, x \in V \)
(iii) \( \| x + y \| \leq \| x \| + \| y \| \).

Relation to metric: if \( \| \cdot \| \) is a norm, then \( d(x, y) = \| x - y \| \) gives a metric.

Example: \( \| x \|_{d_2} = \sqrt{\sum \| x_i \|^2} \)
\( \| x \|_{d_1} = d(f(x), a) \)

Recall: If we fix a point \( a \in M \), \( d(\cdot, a) \) is a continuous function on \( M \).

Similarly: \( \| \cdot \| : (\mathbb{R}^n, d_2) \rightarrow \mathbb{R} \) is also continuous (actually requires a proof). (Theorem)

Thm: For any norm \( \| \cdot \| \) on \( \mathbb{R}^n \), there are two constants \( M, m \in \mathbb{R} \)

s.t. \( m \| x \| \leq \| x \| \leq M \| x \|, \ \forall x \in \mathbb{R}^n \).

Proof: \( f: \| \cdot \| : S_1 \rightarrow \mathbb{R} \) where \( S_1 \) is the unit sphere with \( \{ \| x \| = 1 \} \).
\( f(c) \leq f(x) \leq f(d) \) for some \( c, d \in S_1 \).
Let \( f(c) = m, \ f(d) = M \) note \( m, M > 0 \).

\[
m \leq \frac{f(x)}{\|x\|} \leq M \quad \text{for any} \quad \|x\| = 1.
\]

**Scaling**: \( m\|x\| \leq \|x\|d \leq M\|x\| \quad \forall x \in \mathbb{R}^d \quad \square \)

**Corollary**: The topology given by \( d \) and \( d_E \) are equivalent:

open set in \( d \Longleftrightarrow \) open set in \( d_E \).

**Proof**: \( \quad \)

**Rmk**: Not true in infinite dimensional space!

Unit sphere is compact \( \iff \) finite dim.

\[
\|x\| = S \quad \rightarrow \quad \mathbb{R} \quad \rightarrow \quad \text{achieve max and min}
\]

Continuously Closed-bounded \( \rightarrow \) compact in \( \mathbb{R}^n \)

**Why** \( \|\cdot\| \) is continuous on \( \left( \mathbb{R}^n, d_E \right) \):

\[
\forall x \in \mathbb{R}^n = (x_1, \ldots, x_n) \quad \tilde{e}_i = (0, 0, \ldots, 1, 0, \ldots) \quad \text{triangle inequality} \quad \text{Cauchy-Schwarz}
\]

\[
\|x\|_d = \|\sum_{i=1}^{n} x_i \tilde{e}_i\|_d \leq \sum_{i=1}^{n} \|x_i\| \|\tilde{e}_i\|_d \leq \sqrt{\sum_{i=1}^{n} x_i^2} \cdot \sqrt{\sum_{i=1}^{n} 1^2} = R \cdot \|x\|_d \quad \left( R = \sqrt{\sum_{i=1}^{n} \|\tilde{e}_i\|_d^2} \right)
\]

Continuity of \( \|\cdot\| \): \( \forall \varepsilon > 0, \ \exists \delta = \frac{\varepsilon}{R} \) s.t.

\[
d_E(\tilde{a}, \tilde{b}) < \delta \quad \Rightarrow \quad \|\tilde{a} - \tilde{b}\|_d \leq R \cdot \|\tilde{a} - \tilde{b}\|_d = R \cdot \|\tilde{a} - \tilde{b}\|_d < \varepsilon.
\]

\[
|f(a) - f(b)|
\]

\( f(x) = \frac{1}{1 + \|x\|^2} \)