MATH 131P HOMEWORK 3

DUE JAN 29 12:30PM

(1) (20 pt) Look at the system given by Lesson 8 problem 3.
   (a) Solve the PDE by direct separation of variables.
   (b) Use the transformation $u(x, t) = e^{-t}w(x, t)$ to solve the PDE.
   (c) Compare the two solutions you obtained. (They should match!)

(2) (10 pt) Lesson 8, problem 1.

(3) (30 pt) Follow the steps below to solve this PDE:

\[ u_t = u_{xx} - t \cos(\pi x) + e^{-t}x, \quad 0 \leq x \leq 1, \quad t \geq 0 \]
\[ u_x(0, t) = -e^{-t} \]
\[ u_x(1, t) = -e^{-t} \]
\[ u(x, 0) = 1 - x \]

(a) Find the “steady state”
\[ S(x, t) = c(t)x \]

such that the boundary conditions become homogeneous, i.e. the solution can be written as $u(x, t) = w(x, t) + S(x, t)$ and the boundary conditions for $w(x, t)$ is
\[ w_x(0, t) = w_x(1, t) = 0. \]

(b) Write out the PDE and initial condition satisfied by $w(x, t)$. It should be the form of
\[ w_t = w_{xx} - f(x, t), \quad 0 \leq x \leq 1, \quad t \geq 0 \]
\[ w_x(0, t) = 0 \]
\[ w_x(1, t) = 0 \]
\[ w(x, 0) = \phi(x). \]

(c) We need to use eigenfunction expansion to solve this PDE. First, use the homogeneous PDE and boundary conditions
\[ w_t = w_{xx}, \quad 0 \leq x \leq 1, \quad t \geq 0 \]
\[ w_x(0, t) = 0 \]
\[ w_x(1, t) = 0 \]

to work out the basis $X_n(x), n = 0, 1, \ldots$. You can cite from previous homework and/or textbook.
(d) For the $f(x, t)$ you get from (b), decompose using $X_n(x)$:

$$f(x, t) = \sum_{n=0}^{\infty} f_n(t)X_n(x).$$

Then decompose the initial condition $\phi(x)$ (again from (b)):

$$\phi(x) = \sum_{n=0}^{\infty} A_nX_n(x).$$

(e) Write the solution as

$$w(x, t) = \sum_{n=0}^{\infty} T_n(t)X_n(x),$$

put into the system with initial condition

$$\left(\sum T_n(t)X_n(x)\right)_t = \left(\sum T_n(t)X_n(x)\right)_{xx} - \sum f_n(t)X_n(x)
\sum T_n(0)X_n(x) = \phi(x) = \sum A_nX_n(x).$$

Match the coefficients for each $X_n(x)$ to get equations for $T_n(t)$ (for each $n$, it should be an ODE with a given initial condition.)

(f) For each $n$, solve for $T_n(t)$.

(g) Put together $T_n, X_n$ from (f) and $S(x, t)$ from (a), write down explicitly the solution $u(x, t)$ as

$$u(x, t) = S(x, t) + \sum T_n(t)X_n(x).$$

Remark: you can check if your answer solves the original PDE by putting it back to the system.

(4) (20 pt) Lesson 9, problem 1. (Note: The solution to the PDE system is already given. Please plot the solution for the second question.)

(5) (20 pt) Lesson 9, problem 2.