MATH 131 - HOMEWORK 2

Exercise 1

First of all, notice that we are considering $m, n$ natural numbers greater than 0. Using the identity (the so-called prosthaphaeresis identities) suggested by the hint we obtain

\[
\int_0^L \sin\left(\frac{\pi mx}{L}\right) \sin\left(\frac{\pi nx}{L}\right) dx = \frac{1}{2} \int_0^L \left(\cos \frac{\pi (m-n)x}{L} - \cos \frac{\pi (m+n)x}{L}\right) dx
\]

Notice that one has always

\[
\int_0^L \left(\cos \frac{\pi (m+n)x}{L}\right) dx = \frac{1}{\pi (m+n)} [\sin \pi (m+n)] = 0
\]

as \(\sin x = 0\) on every multiple of \(\pi\). If \(m \neq n\), the same argument leads to

\[
\int_0^L \left(\cos \frac{\pi (m-n)x}{L}\right) dx = \frac{1}{\pi (m-n)} [\sin \pi (m-n)] = 0,
\]

hence for \(m \neq n\) we obtained the result. If \(m = n\), the integral reduces to

\[
\frac{1}{2} \int_0^L dx = \frac{L}{2}
\]

as desired.
Exercise 2

Recall the integration by part formula

\[ \int_a^b f'g = f(b)g(b) - f(a)g(a) - \int_a^b f'g. \]

The choice of \( f = x, g = -\frac{\cos(m\pi x)}{m\pi} \), which implies \( f' = 1, g' = \sin(m\pi x) \) leads to

\[ \int_0^1 x \sin(m\pi x) = -\frac{\cos(m\pi)}{m\pi} + \int_0^1 \frac{\cos(m\pi x)}{m\pi} \]

Notice that the latter integral is zero, because a primitive is \( \frac{\sin(m\pi x)}{m^2 \pi^2} \) which is zero at 0 and 1 for all values of \( m \). The first term is indeed \( -\frac{(-1)^m}{m\pi} \), which shows the result.
Exercise 3

The Fourier expansion of $\phi(x)$ is, by definition,

$$\sum_{n=1}^{+\infty} c_n \sin(n\pi x),$$

where $c_n$ is given by

$$c_n = 2 \int_0^1 \phi(x) \sin(n\pi x) dx.$$

A simple computation shows that

$$c_n = 2 \int_0^1 \sin(n\pi x) = \frac{2}{n\pi} [-\cos(n\pi) + 1].$$

Notice that if $n$ is even, than $\cos(n\pi) = 1$ so $c_n$ vanishes. If $n$ is odd, then $c_n = \frac{4}{n\pi}$.

Therefore, one obtains that

$$\phi(x) = \sum_{n=0}^{+\infty} \frac{4}{(2k+1)\pi} \sin((2k+1)\pi x), \quad x \in (0, 1)$$

Here there is the graph of the first three terms of the series.
Exercise 4

Using the Fourier series, the solution $u(x, t)$ to the heat equation with Dirichlet conditions reads as

$$u(x, t) = \sum_{n=1}^{+\infty} c_n e^{-n^2 \pi^2 t} \sin(n \pi x).$$

If we compare this with the initial condition and using the fact that $\{\sin(n \pi x)\}$ constitutes a basis, one obtains

$$c_n = \hat{\phi}_n$$

where $\hat{\phi}_n$ denotes the $n$-th Fourier coefficient of $\phi(x) := u(x, 0) = 1$. From the previous exercise, we obtain that the solution is

$$u(x, t) = \sum_{n=0}^{+\infty} \frac{4e^{-(2n+1)^2 \pi^2 t}}{(2n + 1) \pi} \sin((2n + 1) \pi x)$$
Exercise 5

- If we plug in function $u$ of the form suggested in the PDE, we obtain
  \[ X(x)T'(t) = X''(X)T(t) \]
  This implies, by separating the variables, that there exists $\lambda \in \mathbb{R}$ such that
  \[ \frac{X''(x)}{X(x)} = \lambda = \frac{T'(t)}{T(t)} \]
  Therefore, $T(t) = Ae^{\lambda t}$ for some constant $A$, while if $k^2 = \lambda > 0$ one has
  \[ X(x) = Ae^{kx} + Be^{-kx}, \]
  while for $\lambda < 0$, $\lambda = -k^2$,
  \[ X(x) = A\sin(kx) + B\cos(kx) \]
  Finally, $k = 0$ leads to solution of type $X(x) = Ax + B$.

- If one wants the derivative in $x$ to be 0 at both 0, 1, the case $\lambda > 0$ leads to solution, as one would need
  \[ Ak - Bk = 0, \quad Ake^k - Bke^{-k} = 0 \]
  which leads to $A = B$ and $A = 0$ or $k = 0$. Therefore, we must have $\lambda < 0$, one needs
  \[ Ak = 0, \sin(k) = 0 \]
  which leads to $A = 0$ and $k_n = n\pi$. Finally, the case $\lambda = 0$ leads to $A = 0$ (i.e., constant solution).

- The fundamental solution, therefore, is such that $X_n(x) = \cos(n\pi x)$, where $n$ is also allowed to take value 0 (corresponding to the case $X(x) = 1$), and correspondingly $T_n(t) = e^{-n^2\pi^2 t}$. Therefore, we obtain
  \[ u_n(x, t) = \cos(n\pi x)e^{-n^2\pi^2 t} \]

- If we look for a solution of the type
  \[ u(x, t) = \sum_{n=0}^{+\infty} A_n u_n(x, t), \]
  the initial condition says $A_n = 0$ unless $A_0 = 2, A_2 = 1$. In other words, the solution looks like
  \[ u(x, t) = 2 + \cos(2\pi x)e^{-4\pi^2 t}. \]
  When $t \to +\infty$, the second term approaches zero, and therefore the solution converges to 2 which is indeed the steady-state solution.
Exercise 6

Assume that $h \neq -1$, otherwise the technique illustrated to transform BC’s to homogeneous ones does not work. As shown in Lesson 6 of the book, we want to find the steady-state part of the solution

$$S(x) = A(1 - x) + Bx$$

where are able to state that $S(x,t)$ independent from time because the right hand side of the BCs are independent from time. Plugging $S(x)$ into the BCs we obtain

$$A = 1, \quad -A + B + hB = 1$$

As $h \neq -1$, we obtain $A = 1, B = \frac{2}{1+h}$. Therefore, we have that

$$S(x) = 1 - x + \frac{2}{1+h} x = 1 + \frac{1-h}{1+h} x.$$

Now, if we consider

$$v(x,t) := u(x,t) - S(x) = u(x,t) - (1 + \frac{1-h}{1+h} x),$$

one has that $v(x,t)$ satisfies the heat equation with ICs

$$v(0,t) = 0, \quad v_x(1,t) + hv(1,t) = 0$$

and BCs

$$v(x,0) = \sin(\pi x) + x - (1 + \frac{1-h}{1+h} x).$$

By separation of variables, we want to find $\lambda, A, B$ such that

$$e^{-\lambda^2 t} \left( A \sin(\lambda x) + B \cos(\lambda x) \right)$$

satisfies the homogenous BCs. A direct comparison shows that one must have

$$B = 0, \quad \lambda \cos \lambda + h \sin \lambda = 0.$$

Notice that the latter can be expressed, for $h \neq 0$, by $\tan(\lambda) = -\frac{\lambda}{h}$. Now we distinguish two cases:

- $h \neq 0$:

  In this case, there is a countable family of $\lambda_k$ satisfying $\tan(\lambda_k) = -\frac{\lambda_k}{h}$ (just draw a picture!). Therefore, Sturm-Liouville Theory guarantees us
that the family of corresponding eigenfunctions $A_k \sin(\lambda_k x)$ are orthogonal to each other, and we can expand $v(x, 0)$ in terms of it, obtaining

$$
\int_0^1 v(x, 0) \sin \lambda_k x \, dx = A_k \int_0^1 \sin^2(\lambda_k x) = A_k \frac{\lambda_k - \sin(\lambda_k) \cos(\lambda_k)}{2\lambda_k}
$$

which leads to

$$
A_k = \frac{2\lambda_k \int_0^1 v(x, 0) \sin \lambda_k x \, dx}{\lambda_k - \sin(\lambda_k) \cos(\lambda_k)}
$$

and therefore

$$
u(x, t) = \sum_{n=1}^{+\infty} A_k e^{-(\lambda_k + 1/2)^2 \alpha^2 t} \sin((k + 1/2)\pi x).
$$

• $h = 0$:

In this case, the need $\lambda \cos \lambda = 0$. Notice that $\lambda = 0$ is not acceptable (it provides the zero solution, which is not an eigenfunctions). Therefore, one has that $\lambda_k = k\pi + \frac{1}{2}\pi$ where $k$ is an arbitrary integer. Combining the two terms $\lambda = (k + \frac{1}{2})\pi$ and $\lambda = (-k - \frac{1}{2})\pi$, we only need to consider

$$
v(x, t) = \sum_{k=0}^{+\infty} e^{-(2k+1)^2 \pi^2 t} A_k \sin((k + 1/2)\pi x).
$$

Using the IC for $v(x, t)$, we want to find $A_k$ such that

$$
\sin(\pi x) + x - (1 + \frac{1}{1+h})x = \sum_{k=0}^{+\infty} A_k \sin((k + 1/2)\pi x)
$$

As in the previous exercise, this is just a matter of integration by parts, which leads to

$$
A_k = \frac{1}{(k - 1/2)(k + 3/2)\pi} + \frac{4}{(1+h)((k + 1/2)\pi)} - \frac{(-1)^k - 1}{(k + 1/2)\pi}
$$

and, again,

$$
u(x, t) = \sum_{k=1}^{+\infty} A_k e^{-(\lambda_k^2 + 1/2)^2 \alpha^2 t} \sin(\lambda_k x) + 1 + x.
$$