Math 131P PDE I
Winter 2017

Jan 8

Course info:
Lecturer: Xuwen Zhu  xuwenzhu@stanford.edu  383-PF
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Office hour: Tue Thu 4-5:30 PM Thu 3-4:30 PM

Homework: due Monday 12:30 PM

Grading: homework 30% + midterm 25% + final 45%

Midterm: Feb 7 in-class
Final: March 22 8:30-11:30 AM
Notify at least one week before

Textbook: Partial differential equations for scientists and engineers
by Stanley J. Farlow

Useful website:
- Web.stanford.edu/~xuwenzhu/classes/Stanford/2017Winter/131P
- Canvas: grades, online feedback (this is counted into your grade!)

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Today's goal: What is PDE? Why is it useful? Classification

**What is PDE?** Equations containing partial differentials.

Unknown function $\mathcal{U}(t, x, ...)$, a typical setting: $t$: time, $x$: position (could be others!)

Notation of partial differentials:

$$\frac{\partial \mathcal{U}}{\partial x} = \mathcal{U}_x = \partial_x \mathcal{U}, \quad \frac{\partial \mathcal{U}}{\partial t} = \mathcal{U}_t = \partial_t \mathcal{U}, \quad \frac{\partial^2 \mathcal{U}}{\partial x^2} = \mathcal{U}_{xx} = \partial_{xx} \mathcal{U}, \quad \frac{\partial^2 \mathcal{U}}{\partial x \partial t} = \mathcal{U}_{xt} = \partial_{xt} \mathcal{U}$$

Some examples:

**Heat equation:**

1D: $\mathcal{U}(t, x)$: temperature at time $t$ and position $x$

equ.: $\mathcal{U}_t = \mathcal{U}_{xx}$

2D: $\mathcal{U}(t, x, y)$: position $(x, y)$

equ.: $\mathcal{U}_t = \mathcal{U}_{xx} + \mathcal{U}_{yy}$

**Wave equation**

1D: $\mathcal{U}(t, x)$: position of string at time $t$ and position $x$

equ.: $\mathcal{U}_{tt} = \mathcal{U}_{xx}$

2D: $\mathcal{U}_{tt} = \mathcal{U}_{xx} + \mathcal{U}_{yy}$

**Laplace equation**

1D: $\mathcal{U}(x, y, ...)$: does not depend on $t$, equilibrium

equ.: $\mathcal{U}_{xx} = 0$

2D: $\mathcal{U}_{xx} + \mathcal{U}_{yy} = 0$

Other famous equations:

- Schrödinger equation: $i \mathcal{U}_t + \mathcal{U}_{xx} = 0$
- KdV equation: $\mathcal{U}_t + \mathcal{U}_{xxx} + 6 \mathcal{U} \mathcal{U}_x = 0$
- Maxwell's equations (see homework)
- Navier-Stokes equations
- Einstein field equations...

Why PDE is useful: provide models for physical phenomena

electromagnetism, fluid, general relativity, thermal, kinetic...
Goal for this class:

- Learn how to formulate PDEs from physical problems (differential pov.)
- Learn how to solve some PDEs (integral pov.)
- From solutions for 3 types of 2nd order linear equations, learn how to predict the behavior of solutions.

How to solve PDEs (some methods available)

- Explicit solutions
  - Separation of variables, eigenfunction expansion
  - Integral transforms: Fourier, Laplace, Sin/Cos transforms
  - Change of coordinates/variables
- Transform of equations
- Implicit solutions
  - Perturbation for nonlinear problems
- Calculus of variations
- Numerical solutions
  - Finite difference
  - Finite element
  - Spline approximation
  - Integral methods

And a lot more other tools... depends heavily on the type of PDEs.
Classification of PDEs: useful before starting to solve PDEs.

1. Order: highest derivative
   \[ U_t + U_{xx} = 0, \quad iU_t + U_{xx} = 0, \quad U_{xx} + U_{yy} = 0 \Rightarrow 2\text{nd order} \]
   \[ U_t + U_{xx} = 0 \Rightarrow \text{1st order} \]

2. Number of variables:
   \[ U_t + U_{xx} = 0 \text{ (2 variables: } t \text{ and } x) \]
   \[ U_{tt} = U_{xx} + U_{yy} \text{ (3 variables: } t, x, y) \]
   (1 variable? back to ODE)

3. Linearity: \( U \) and all derivatives of \( U \) (\( U_t, U_x, U_{xx}, \text{ etc.}\))
   all appear linear (e.g. \( U_{xx} \) is not linear)
   Non-example: \( UU_t + U_{xx} = 0 \) is nonlinear

4. Homogeneity: only for linear equations
   Is there part that does not contain \( U \)?
   Yes: \text{nonhomogeneous}
   No: \text{homogeneous} \Leftrightarrow \text{if } U \text{ is a solution, then } AU \text{ is also a solution for any number } A.

5. Constant coefficient: only for linear equations
   does the coefficient in front of \( U \) and its derivatives contain variables instead of being pure numbers?
   \( U_t + U_{xx} + 2 = 0 \) \text{ Linear equations} \underleftrightarrow{\text{homogeneous}} \quad \text{\( tU_t + U_{xx} = 0 \)}
   \text{non constant coefficient}
   \( tU_t + U_{xx} = 0 \) \text{\( \Rightarrow \text{inhomogeneous} \quad \text{\( tU_t + U_{xx} = x \)}}
Special classification for 2nd order linear equation

\[ \begin{align*}
A U_{tt} + B U_{xt} + C U_{xx} + D U_t + E U_x + F U &= G \\
(A \sim G \text{ can be non constant; when } G \text{ is not 0, it's inhomogeneous})
\end{align*} \]

Focus on the top order part:

- determinant
- type
- example

\[ \begin{align*}
B^2 - 4AC &= 0 & \text{parabolic} & \text{example} \\
B^2 - 4AC &> 0 & \text{hyperbolic} & U_{tt} - U_{xx} = 0 \text{ (wave)} \\
B^2 - 4AC &< 0 & \text{elliptic} & U_{tt} + U_{xx} = 0 \text{ (laplace)}
\end{align*} \]

Remark: \( U_{xx} \) can be replaced by \( \Delta U = U_{xx} + U_{yy} + U_{zz} + \ldots \) in higher dimensional case

- When \( A, B, C \) are non-constant, the type might change when \( (x,t) \) changes.

Heat equation: derivation, Boundary Conditions for 1D system

Model:

- heat flow only from two ends
- length \( L \) rod, insulated on the side

\[ U(x,t) \text{: temperature of the rod at time } t \text{ and position } x. \]

Initial condition: \( U(x,0) \) boundary condition: \( U(0,t) \) and \( U(L,t) \) (given for all \( t \)).
Example: $u(x,0) = T_0$, $u(0,t) = T_1$, $u(L,t) = T_2$ $T_1 < T_0 < T_2$ \(\text{eqn 6}\)

This process is given by the PDE:

(PDE) $u_t = \alpha^2 u_{xx}$ \(0 < x < L , \ 0 < t \leq \infty\)

(BC) \[
\begin{align*}
U(0,t) &= - \quad U(L,t) &= - \\
\text{for all } t
\end{align*}
\]

(IC) \[
U(x,0) = - \\
\text{for all } x.
\]

First focus on the PDE:

$u_t$: rate of change (deg/sec) \(\alpha^2\): diffusivity constant (m$^2$/sec)

$u_{xx}$: concavity of the temperature profile (deg/m$^2$)

Intuition: the temperature curve wants to flatten out

$u_{xx} < 0$, at mid point $x_0$, $u_t < 0$ (temperature wants to drop)

$u_{xx} > 0$ $\implies$ $u_t > 0$

$u_{xx} = 0$ $\implies$ $u_t = 0$ (temperature wants to remain the same)
To derive the PDE: we will use \(\text{Conservation of energy}\) + \(\text{Fourier's law of cooling}\) focusing on a very small segment.

1. Net change of heat in this segment = net flux across 2 ends + total heat generated inside.

\[
\frac{d}{dt} \int_x^{x+\Delta x} \rho \ C \ A \ u(s, t) \, ds = \frac{d}{dt} \int_x^{x+\Delta x} \rho \ C \ A \ \frac{d}{dt} u(s, t) \, ds
\]

\(C:\) thermal capacity, \(\rho:\) density, \(A:\) across area \((\rho A \, ds = dm)\) in the infinitesimal mass.

2. Flux across \(x+\Delta x\) section + flux across \(x\) section.

\[= kA \ u_x(x+\Delta x, t) - kA \ u_x(x, t)\]

\(k:\) thermal conductivity.

Fourier's law of cooling: heat flow = \(kA(\text{normal derivative of temperature profile})\)

3. \(= \int_x^{x+\Delta x} f(s, t) \, ds\) \(f(s, t):\) a heat generating source inside.

Combine 1 \(\sim\) 3:

\[
CPA \int_x^{x+\Delta x} u_t(s, t) \, ds = kA \left[ \frac{u_x(x+\Delta x, t) - u_x(x, t)}{\Delta x} \right] + \int_x^{x+\Delta x} f(s, t) \, ds
\]

let \(\Delta x \to 0\),

\[
CPA \ u_t(x, t) = kA \ u_{xx}(x, t) + A \ f(x, t)
\]

let \(\alpha^2 = \frac{k}{\rho C_p}\), cancel out \(A\),

\(u_t = \alpha^2 u_{xx} + F(x, t)\) when \(F = \frac{1}{\rho C_p} f\).
Boundary Conditions

1. Dirichlet BC: Fix boundary values.
   \[ u(x,t) = g(x,t) \] for any \( x \) on the boundary.

1D: \( u(0,t) = g_1(t) \), \( u(L,t) = g_2(t) \)

2D: If we have a disk
   \[ u(R,\theta,t) = \text{cos}\theta \sin\theta \] for example.

(Think of the example before: \( u(0,t) = T_1 \), \( u(L,t) = T_2 \))

2. Neumann BC: Specify the normal derivative at the boundary
   or \( \frac{\partial u}{\partial n} \) (heat flow by Fourier's Law of Cooling)

1D: \( u_x(0,t) = g_1(t) \), \( u_x(L,t) = g_2(t) \)

2D: \( \frac{\partial u}{\partial n} = \frac{\partial u}{\partial r} (\theta,t) = g(\theta,t) \)

Important BC: Insulation: \( \frac{\partial u}{\partial n} = 0 \)
mixed BC (Robin BC)

\[ ID : \begin{cases} \alpha U(x=0, t) + \beta U(0, t) = g_1(t) \\ \gamma U(L, t) + \delta U(L, t) = g_2(t) \end{cases} \]

Physics model: two ends are in contact with some other material of given temperature.

Fourier's law of cooling

\[ k A \frac{du}{dx}(x=0, t) = h A(u(0, t) - g_1(t)) \Rightarrow \text{rearrange to get expression above} \]

\[ -k A \frac{du}{dx}(x=L, t) = h A(U(L, t) - g_2(t)) \]

Remark: when \( g_1(t), g_2(t) \neq 0 \), we have inhomogeneous BC.

Initial condition: initial temperature, usually just given by a fixed function \( g(x) \).

Steady state: a solution to the heat equation satisfying \( U_t = 0 \), i.e., \( U \) does not change with time any more.

In the simplest case: \( U_t = \alpha^2 U_{xx} \Rightarrow U_{xx} = 0 \).

\[ \Rightarrow U(x, t) = C_1 x + C_2 \], a straight line profile.

\[ U(x) \]