Lecture 2: Analysis and Classification of Linear Systems
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- Classifying solutions to matrix ODEs.
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- A couple of motivating examples.
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- Non-Homogenous systems.
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■ Classifying solutions to matrix ODEs.
■ A couple of motivating examples.
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The philosophy: dynamic behavioral analysis is controlled by static behavioral analysis.
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\frac{dx_1}{dt}(t) = a_{1,1}x_1(t) + a_{1,2}x_2(t) + \ldots + a_{1,n}x_n(t),
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This can be written as

$$\frac{d\vec{x}}{dt}(t) = A\vec{x}$$

where $A$ is the matrix of the $a_{i,j}$'s.
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Can we turn our eigen-value analysis of matrices into a qualitative description of the time-limiting behavior of matrix ODEs?
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Can we turn our eigen-value analysis of matrices into a qualitative description of the time-limiting behavior of matrix ODEs?

Yes. And the answer is perhaps the most fundamental application of mathematics to science.
The 1-D Case

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Split into real and imaginary parts: \( \lambda = \lambda_r + i\lambda_i \)

Then

\[ e^{\lambda t} = e^{\lambda_r t + i\lambda_i t} = e^{\lambda_r t} e^{i\lambda_i t}. \]
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**Problem 1** What is \( \lim_{t \to \infty} e^{at} \), qualitatively?
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Problem 1 What is \( \text{Lim} = \lim_{t \to \infty} e^{at} \), qualitatively?

Answer: If \( a < 0 \), \( \text{Lim} = 0 \); if \( a > 0 \), \( \text{Lim} = \infty \); if \( a = 0 \), \( L = 1 \).
In the 1-D case, it boils down to

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Split into real and imaginary parts: \( \lambda = \lambda_{re} + i\lambda_{im} \)

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**Problem 1** What is \( \lim_{t \to \infty} e^{at} \), qualitatively?

**Answer:** If \( a < 0 \), \( \lim = 0 \); if \( a > 0 \), \( \lim = \infty \); if \( a = 0 \), \( L = 1 \).

**Problem 2** What is \( |x(t)| \)?
The 1-D Case

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\[ |x(t)| = \left[ x(t)x(t) \right]^{1/2} \quad \text{– definition of } |x| \]
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Answer:
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|x(t)| = \left[ x(t) \overline{x(t)} \right]^{1/2} \quad \text{definition of } |x|
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= \left[ e^{(\lambda_re + i\lambda_im)t} + (\lambda_re - i\lambda_im)t \right]^{1/2} |x(0)|
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The 1-D Case

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\[ = \left[ e^{2\lambda_{re}t} \right]^{1/2} \left| x(0) \right| \quad – \text{imaginary terms cancel, real terms add} \]
The 1-D Case

Answer:

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\[ = \left[ e^{(\lambda_r + i\lambda_im)t} + (\lambda_r - i\lambda_im)t \right]^{1/2} |x(0)| \]

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Hence,

\[ \lim_{t \to \infty} |x(t)| = \left( \lim_{t \to \infty} e^{\lambda_{re}t} \right) |x(0)| = \begin{cases} 0 & \text{if} \quad \lambda_{re} < 0, \\ \infty & \text{if} \quad \lambda_{re} > 0, \\ |x(0)| & \text{if} \quad \lambda_{re} = 0. \end{cases} \]
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Answer:
\[ |x(t)| \]
\[ = \left[ x(t)x(t) \right]^{1/2} \text{ – definition of } |x| \]
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So the sign of \( \lambda_re \) controls the magnitude of the steady-state;
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Answer:
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So the sign of \( \lambda_re \) controls the magnitude of the steady-state; and there are only three possibilities: contract to 0, blow-up to infinity, or remain the same.
Problem 3  Compute real and imaginary parts of $e^{i\lambda_it}$. 
The 1-D Case

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Answer: Using

$$e^{i\theta} = \cos(\theta) + i\sin(\theta),$$

Real part: $\cos(\lambda mt)$. Imaginary part: $\sin(\lambda mt)$. 
**The 1-D Case**

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Look at Matlab sim: this part rotates around at frequency $\lambda m$. 
The 1-D Case

Problem 3  Compute real and imaginary parts of $e^{i\lambda_{im}t}$.

Answer: Using

$$e^{i\theta} = \cos(\theta) + isin(\theta),$$

Real part: $\cos(\lambda_{im}t)$. Imaginary part: $\sin(\lambda_{im}t)$.

Look at Matlab sim: this part rotates around at frequency $\lambda_{im}$. Aha! So $\lambda_{im}$ controls the angle, as opposed to the magnitude.
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Answer: Using \( e^{i\theta} = \cos(\theta) + isin(\theta) \),

Real part: \( \cos(\lambda_{im}t) \). Imaginary part: \( \sin(\lambda_{im}t) \).

Look at Matlab sim: this part rotates around at frequency \( \lambda_{im} \).
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Thus:

\[ x(t) = e^{\lambda_re^t} \cdot [\cos(\lambda_{im}t) + i\sin(\lambda_{im}t)] \cdot x(0) \]
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**Problem 4** What is this in polar coordinates?
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Problem 4 What is this in polar coordinates?

Answer: \((r(t), \theta(t)) = (e^{\lambda_r t}, \lambda_{im} t)\).
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Hence

\[ x_{t\in[0,\infty]} = \begin{cases} 
\text{Outward Spiral} & \text{if } \lambda_{re} > 0, \quad \lambda_{im} \neq 0 \\
\text{Inward Spiral} & \text{if } \lambda_{re} < 0, \quad \lambda_{im} \neq 0 \\
\text{Periodic Rotation} & \text{if } \lambda_{re} = 0, \quad \lambda_{im} \neq 0 \\
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See Matlab simulation.
Now I say: the “right” definition of growth rate for the above process should yield $\lambda_{re}$. 
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Growth and Stability

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**Answer:**

$$\Lambda = \lim_{t \to \infty} \frac{1}{t} \log[|x(t)|].$$
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The number $\Lambda$ is called the *Lyapunov exponent* of the system.
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$\Lambda > 0$ is **unstable**; $\Lambda < 0$ is **stable**; $\Lambda = 0$ is **neutrally stable**.
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1-D linear systems are characterized by $\Lambda = \lambda_{re}$ – the asymptotic growth rate, and
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1-D linear systems are characterized by $\Lambda = \lambda_{re}$ – the asymptotic growth rate, and $\omega = \lambda_{im}$, the natural frequency.
Special Case: 2D Diagonalizable Systems

After Normal-Form-alization, there are only a couple of different classes of real diagonalizable matrixes in 2D.
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After Normal-Form-alization, there are only a couple of different classes of real diagonalizable matrixes in 2D.

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = J^a_1 \oplus J^b_1; \quad \text{and} \quad \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix} = J^\lambda_1 \oplus J^{\bar{\lambda}}_1.$$
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\end{bmatrix} = J_1^\lambda \oplus J_1^{\overline{\lambda}}.
\]

The class

\[
\begin{bmatrix}
a & 1 \\
0 & a
\end{bmatrix} = J_2^a
\]

is not diagonalizable.
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Further cases:
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Further cases:
- \(a < 0, b < 0\) (real)
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- \(a = 0\) or \(b = 0\) (real)
- \(\lambda_{re} < 0\) (complex)
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- \( \lambda_{re} < 0 \) (complex)
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- \( a = 0 \) or \( b = 0 \) (real)
- \( \lambda_{re} < 0 \) (complex)
- \( \lambda_{re} > 0 \) (complex)
- \( \lambda_{re} = 0 \) (complex)
Stable Nodes

Supposing $a < 0$, $b < 0$, trajectories in the system look like:
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Stable fixed point

converging at exp. rate \( a \)

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Supposing $a < 0$, $b < 0$, trajectories in the system look like:

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if $a > b$
angle tends toward $x$–axis
Supposing $a < 0, b < 0$, trajectories in the system look like:

Note: a) $\text{trace} = a + b < 0$, 

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Stable Nodes

Supposing $a < 0, b < 0$, trajectories in the system look like:

Note: a) $\text{trace} = a + b < 0$, b) $\text{det} = ab > 0$, c) $\text{discriminant} = tr^2 - 4det = (a - b)^2 \geq 0$. 
Unstable Nodes

Supposing $a > 0$, $b > 0$, trajectories in the system look like:
Supposing $a > 0$, $b > 0$, trajectories in the system look like:
Supposing $a > 0, b > 0$, trajectories in the system look like:

Unstable fixed point

if $a > b$

angle tends to $x$–axis
Unstable Nodes

Supposing $a > 0, b > 0$, trajectories in the system look like:

Note: a) trace $= a + b > 0$,
Unstable Nodes

Supposing $a > 0, b > 0$, trajectories in the system look like:

Note: a) $\text{trace} = a + b > 0$, b) $\det = ab > 0$, 
Unstable Nodes

Supposing $a > 0, b > 0$, trajectories in the system look like:

\[ \text{if a > b} \]
\[ \text{angle tends to x-axis} \]

Unstable fixed point

Note: a) trace = $a + b > 0$, b) det = $ab > 0$, c) discriminant = $tr^2 - 4det = (a - b)^2 \geq 0$. 
Saddle Points

Suppose $a > 0, b < 0$ (wlog); trajectories in the system look like:
Saddle Points

Suppose $a > 0, b < 0$ (wlog); trajectories in the system look like:

- converging at exp. rate $b$
- diverging at exp. rate $a$
- Quasi-stable fixed point (Saddle point)
Saddle Points

Suppose $a > 0, b < 0$ (wlog); trajectories in the system look like:
Suppose $a > 0, b < 0$ (wlog); trajectories in the system look like:

Note: a) trace could be anything,
Saddle Points

Suppose $a > 0, b < 0$ (wlog); trajectories in the system look like:

Note: a) trace could be anything, b) $\det = ab < 0$,
Saddle Points

Suppose $a > 0, b < 0$ (wlog); trajectories in the system look like:

```
Suppose \( a > 0, b < 0 \) (wlog); trajectories in the system look like:
```

Note: a) trace could be anything, b) \( \det = ab < 0 \), c) discriminant \( \geq 0 \).
Degenerate Case 1

Suppose $a = 0, b < 0$ (wlog); trajectories in the system look like:
Suppose \( a = 0, b < 0 \) (wlog); trajectories in the system look like:

![Diagram of trajectories converging at an exponential rate]
Suppose \( a = 0, b < 0 \) (wlog); trajectories in the system look like:

Note: a) trace < 0,
Degenerate Case 1

Suppose $a = 0, b < 0$ (wlog); trajectories in the system look like:

Note: a) trace $< 0$, b) $\det = ab = 0$, 

\[ y \]
\[ \downarrow \downarrow \downarrow \]
\[ \text{converging at exp. rate } b \]
\[ \text{Line of Stable Fixed Points} \]
\[ \uparrow \uparrow \uparrow \uparrow \]
\[ x \]
Degenerate Case 1

Suppose $a = 0, b < 0$ (wlog); trajectories in the system look like:

Note: a) trace $< 0$, b) $\det = ab = 0$, c) discriminant $\geq 0$. 
Suppose $a = 0, b > 0$ (wlog); trajectories in the system look like:
Degenerate Case 2

Suppose $a = 0, b > 0$ (wlog); trajectories in the system look like:

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Suppose $a = 0, b > 0$ (wlog); trajectories in the system look like:
Suppose \( a = 0, b > 0 \) (wlog); trajectories in the system look like:

Note: a) trace > 0,
Degenerate Case 2

Suppose $a = 0, b > 0$ (wlog); trajectories in the system look like:

![Diagram showing trajectories](attachment:image.png)

Note: a) trace > 0, b) det = $ab = 0$, 

$\text{x-axis}$ 

$\text{Line of Unstable Fixed Points}$

$\text{y-axis}$
Degenerate Case 2

Suppose $a = 0, b > 0$ (wlog); trajectories in the system look like:

Note: a) trace $> 0$, b) det $= ab = 0$, c) discriminant $\geq 0$. 
Inward Spiral

Suppose $\lambda_{im} > 0$, $\lambda_{re} < 0$; trajectories in the system look like:
Inward Spiral

Suppose $\lambda_{im} > 0$, $\lambda_{re} < 0$; trajectories in the system look like:

![Inward Spiral Diagram]
Suppose $\lambda_{im} > 0, \lambda_{re} < 0$; trajectories in the system look like:
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Suppose $\lambda_{im} > 0, \lambda_{re} < 0$; trajectories in the system look like:

Note: a) trace $= 2\lambda_{re} < 0$, 

Stable Fixed Point
Suppose $\lambda_{im} > 0$, $\lambda_{re} < 0$; trajectories in the system look like:

Note: a) $\text{trace} = 2\lambda_{re} < 0$, b) $\det = |\lambda|^2 \geq 0$, 

Inward Spiral
Inward Spiral

Suppose $\lambda_{im} > 0, \lambda_{re} < 0$; trajectories in the system look like:

Note: a) $\text{trace} = 2\lambda_{re} < 0$, b) $\text{det} = |\lambda|^2 \geq 0$, c) discriminant $= -4\lambda_{im}^2 < 0$. 
Outward Spiral

Suppose $\lambda_{im} > 0$, $\lambda_{re} > 0$; trajectories in the system look like:
Suppose \( \lambda_{im} > 0, \lambda_{re} > 0 \); trajectories in the system look like:
Suppose $\lambda_{im} > 0, \lambda_{re} > 0$; trajectories in the system look like:

Note: a) $\text{trace} = 2\lambda_{re} > 0,$
Suppose $\lambda_{im} > 0, \lambda_{re} > 0$; trajectories in the system look like:

Note: a) trace = $2\lambda_{re} > 0$, b) det = $|\lambda|^2 \geq 0$, 

Unstable Fixed Point
Outward Spiral

Suppose $\lambda_{im} > 0$, $\lambda_{re} > 0$; trajectories in the system look like:

Note: a) $\text{trace} = 2\lambda_{re} > 0$, b) $\text{det} = |\lambda|^2 \geq 0$, c) discriminant $= -4\lambda_{im}^2 < 0$. 
Suppose $\lambda_{im} > 0$, $\lambda_{re} = 0$; trajectories in the system look like:
Suppose $\lambda_{im} > 0, \lambda_{re} = 0$; trajectories in the system look like:
Suppose $\lambda_{im} > 0, \lambda_{re} = 0$; trajectories in the system look like:

Note: a) trace $= 2\lambda_{re} = 0,$
Pure Rotation

Suppose $\lambda_{im} > 0$, $\lambda_{re} = 0$; trajectories in the system look like:

Note: a) $\text{trace} = 2\lambda_{re} = 0$, b) $\text{det} = |\lambda_{im}|^2 \geq 0$, 

Neutral Fixed Point

$\lambda_{im} > 0$, $\lambda_{re} = 0$; trajectories in the system look like:

Note: a) $\text{trace} = 2\lambda_{re} = 0$, b) $\text{det} = |\lambda_{im}|^2 \geq 0$,
Pure Rotation

Suppose $\lambda_{im} > 0, \lambda_{re} = 0$; trajectories in the system look like:

Note: a) $\text{trace} = 2\lambda_{re} = 0$, b) $\text{det} = |\lambda_{im}|^2 \geq 0$, c) discriminant $=-4\lambda_{im}^2 \leq 0$. 
2-D Special Case

- \( a < 0, b < 0 \) (real) – exponential decay
2-D Special Case

- $a < 0, b < 0$ (real) – exponential decay
- $a > 0, b > 0$ (real) – exponential growth
2-D Special Case

- \( a < 0, b < 0 \) (real) – exponential decay
- \( a > 0, b > 0 \) (real) – exponential growth
- \( a < 0, b > 0 \) or \( a > 0, b < 0 \). (real) – equilibrium chemistry
2-D Special Case

- $a < 0, b < 0$ (real) – exponential decay
- $a > 0, b > 0$ (real) – exponential growth
- $a < 0, b > 0$ or $a > 0, b < 0$. (real) – equilibrium chemistry
- $a = 0$ or $b = 0$ (real)
2-D Special Case

- $a < 0$, $b < 0$ (real) – exponential decay
- $a > 0$, $b > 0$ (real) – exponential growth
- $a < 0$, $b > 0$ or $a > 0$, $b < 0$. (real) – equilibrium chemistry
- $a = 0$ or $b = 0$ (real)
- $\lambda re < 0$ – damped oscillator (complex)
2-D Special Case

- $a < 0, b < 0$ (real) – exponential decay
- $a > 0, b > 0$ (real) – exponential growth
- $a < 0, b > 0$ or $a > 0, b < 0$. (real) – equilibrium chemistry
- $a = 0$ or $b = 0$ (real)
- $\lambda_{re} < 0$ – damped oscillator (complex)
- $\lambda_{re} > 0$ (complex)
2-D Special Case

- $a < 0, b < 0$ (real) – exponential decay
- $a > 0, b > 0$ (real) – exponential growth
- $a < 0, b > 0$ or $a > 0, b < 0$. (real) – equilibrium chemistry
- $a = 0$ or $b = 0$ (real)
- $\lambda_{re} < 0$ – damped oscillator (complex)
- $\lambda_{re} > 0$ (complex)
- $\lambda_{re} = 0$ – harmonic oscillator (complex)
2-D Special Case

Figure 1: Taken from Strogatz p. 137
We’ve not dealt with nilpotency; but it’s *important*. 
Nilpotency

We’ve not dealt with nilpotency; but it’s *important*. For diagonalizable systems, the solutions were of the form:

\[ \vec{y}(t) = \sum_{i=1}^{n} e^{\lambda_i t} y_i(0) \epsilon_i. \]
Nilpotency

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**Problem 6** Compute \( e^{N_2 t} \) where \( N_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \) is the standard 2x2 nilpotent matrix.
We’ve not dealt with nilpotency; but it’s *important*. For diagonalizable systems, the solutions were of the form:

$$\vec{y}(t) = \sum_{i=1}^{n} e^{\lambda_i t} \vec{y}_i(0) \epsilon_i.$$ 

**Problem 6** Compute $e^{N_2 t}$ where $N_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is the standard 2x2 nilpotent matrix.

**Answer:** $\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$. 
We’ve not dealt with nilpotency; but it’s *important*. For diagonalizable systems, the solutions were of the form:

\[ \vec{y}(t) = \sum_{i=1}^{n} e^{\lambda_i t} \vec{y}_i(0) \vec{e}_i. \]

**Problem 6** Compute \( e^{N_2 t} \) where \( N_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \) is the standard 2x2 nilpotent matrix.

**Answer:** \( \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \). Key point – it has a polynomial (as opposed to exponential) term.
Nilpotency

We’ve not dealt with nilpotency; but it’s important.
For diagonalizable systems, the solutions were of the form:

\[ \vec{y}(t) = \sum_{i=1}^{n} e^{\lambda_i t} \vec{y}_i(0) \epsilon_i. \]

**Problem 6** Compute \( e^{N_2 t} \) where \( N_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \) is the standard 2x2 nilpotent matrix.

Answer: \( \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \). Key point – it has a polynomial (as opposed to exponential) term.

**Problem 7** Now compute \( e^{N_n t} \) where \( N_n \) is the \( n \)-by-\( n \) matrix with 1s on the super-diagonal and zeros elsewhere.
Nilpotency

Answer:

\[
\begin{bmatrix}
1 & t & \frac{1}{2}t^2 & \frac{1}{3!}t^3 & \ldots & \frac{1}{n!}t^n \\
0 & 1 & t & \frac{1}{2}t^2 & \ldots & \frac{1}{(n-1)!}t^{n-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 1
\end{bmatrix}
\]
Nilpotency

Answer:

\[
\begin{bmatrix}
1 & t & \frac{1}{2}t^2 & \frac{1}{3!}t^3 & \ldots & \frac{1}{n!}t^n \\
0 & 1 & t & \frac{1}{2}t^2 & \ldots & \frac{1}{(n-1)!}t^{n-1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & 1
\end{bmatrix}
\]

So, there is growth – but no component grows as fast as exponential.
Nilpotency

Answer:

\[
\begin{bmatrix}
1 & t & \frac{1}{2}t^2 & \frac{1}{3!}t^3 & \ldots & \frac{1}{n!}t^n \\
0 & 1 & t & \frac{1}{2}t^2 & \ldots & \frac{1}{(n-1)!}t^{n-1} \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & 0 & 1
\end{bmatrix}
\]

So, there is growth – but no component grows as fast as exponential.

Writing it out, this says that the solutions of

\[
\frac{d\vec{y}}{dt} = N_n \vec{y}
\]

is

\[
y_i(t) = y_i(0) + y_{i+1}(0)t + \frac{t^2}{2}y_{i+2}(0) + \ldots + \frac{t^{n-i+1}}{n!}y_n(0).
\]
Nilpotency

In the 3-by-3 case, this becomes:
Nilpotency

In the 3-by-3 case, this becomes:

\[ y_1(t) = y_1(0) + y_2(0)t + \frac{1}{2}y_3(0)t^2 \]
In the 3-by-3 case, this becomes:

\[ y_1(t) = y_1(0) + y_2(0)t + \frac{1}{2}y_3(0)t^2 \]

\[ y_2(t) = y_2(0) + y_3(0)t \]

and
Nilpotency

In the 3-by-3 case, this becomes:

\[ y_1(t) = y_1(0) + y_2(0)t + \frac{1}{2}y_3(0)t^2 \]
\[ y_2(t) = y_2(0) + y_3(0)t \]
and
\[ y_3(t) = y_3(0). \]
Nilpotency

In the 3-by-3 case, this becomes:

\[ y_1(t) = y_1(0) + y_2(0)t + \frac{1}{2}y_3(0)t^2 \]

\[ y_2(t) = y_2(0) + y_3(0)t \]

and

\[ y_3(t) = y_3(0). \]

Question: What does this remind you of? (Need a hint? Replace \( y_1 \) with \( x \), \( y_2 \) with \( v \) and \( y_3 \) with \(-g\) or \(a\). See now?)
Nilpotency

In the 3-by-3 case, this becomes:

\[ y_1(t) = y_1(0) + y_2(0)t + \frac{1}{2}y_3(0)t^2 \]

\[ y_2(t) = y_2(0) + y_3(0)t \]

and

\[ y_3(t) = y_3(0). \]

Question: What does this remind you of? (Need a hint? Replace \( y_1 \) with \( x \), \( y_2 \) with \( v \) and \( y_3 \) with \(-g\) or \( a \). See now?)

Of course! Projectile motion from basic physics:

\[ x(t) = x(0) + v(0)t + \frac{1}{2}at^2; \quad v(t) = v(0) + at \]

\[ a(t) = a(0) = a. \]
Nilpotency

In the 3-by-3 case, this becomes:

\[ x(t) = x(0) + v(0)t + \frac{1}{2}at^2; \quad v(t) = v(0) + at \]

\[ a(t) = a(0) = a. \]

Newton’s law

\[ \ddot{x} = a = -g \]

becomes
Nilpotency

In the 3-by-3 case, this becomes:

\[
x(t) = x(0) + v(0)t + \frac{1}{2}at^2; \quad v(t) = v(0) + at
\]

\[a(t) = a(0) = a.\]

Newton's law

\[
\ddot{x} = a = -g
\]

becomes

\[
\begin{bmatrix}
\dot{x} \\
\dot{v} \\
\dot{a}
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
v \\
a
\end{bmatrix}.
\]
Nilpotency

In the 3-by-3 case, this becomes:

\[ x(t) = x(0) + v(0)t + \frac{1}{2}at^2; \quad v(t) = v(0) + at \]

\[ a(t) = a(0) = a. \]

Newton’s law

\[ \ddot{x} = a = -g \]

becomes

\[
\begin{bmatrix}
\dot{x} \\
\dot{v} \\
\dot{a}
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
v \\
a
\end{bmatrix}.
\]

This is why it is important to understand nilpotency.
Problem 8  Compute $e^{J \lambda t}$ where $J \lambda$ was the Jordan-block explored in the previous lecture. Hint: use the fact that the diagonal commutes with the off-diagonal).
Nilpotency

Problem 8  Compute $e^{J^\lambda_n t}$ where $J^\lambda_n$ was the Jordan-block explored in the previous lecture. Hint: use the fact that the diagonal commutes with the off-diagonal).

Answer:

$$e^{(\lambda I_n + N_n)t} = e^{\lambda t} \begin{bmatrix} 1 & t & \frac{1}{2} t^2 & \frac{1}{3!} t^3 & \ldots & \frac{1}{n!} t^n \\ 0 & 1 & t & \frac{1}{2} t^2 & \ldots & \frac{1}{(n-1)!} t^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \end{bmatrix}.$$
Nilpotency

Problem 8 Compute $e^{J^\lambda_n t}$ where $J^\lambda_n$ was the Jordan-block explored in the previous lecture. Hint: use the fact that the diagonal commutes with the off-diagonal).

Answer:

$$e^{(\lambda I_n + N_n)t} = e^{\lambda t} \begin{bmatrix}
1 & t & \frac{1}{2} t^2 & \frac{1}{3!} t^3 & \ldots & \frac{1}{n!} t^n \\
0 & 1 & t & \frac{1}{2} t^2 & \ldots & \frac{1}{(n-1)!} t^{n-1} \\
0 & 0 & \ldots & 0 & 1
\end{bmatrix}.$$

Generally, nilpotent matrices have similar dynamics to diagonalizable ones ... but different growth rates (i.e. some $t^i$ polynomial terms).
Real Eigenvalues

So much for two dimensions.
So much for two dimensions.

Now suppose:

\[
\frac{d\vec{x}}{dt}(t) = A\vec{x}
\]

for some matrix \( A \) with a real eigenbasis.
Real Eigenvalues

So much for two dimensions.

Now suppose:

\[
\frac{d\vec{x}}{dt}(t) = A\vec{x}
\]

for some matrix \( A \) with a real eigenbasis.

**Problem 9** Write \( A = SDS^{-1} \). Find a linear change-of-variables from \( \vec{x} \mapsto \vec{y} \) such that

\[
\frac{d\vec{y}}{dt}(t) = D\vec{y}.
\]
Real Eigenvalues

So much for two dimensions.

Now suppose:

\[ \frac{d\vec{x}}{dt}(t) = A\vec{x} \]

for some matrix \( A \) with a real eigenbasis.

Problem 9 Write \( A = SDS^{-1} \). Find a linear change-of-variables from \( \vec{x} \mapsto \vec{y} \) such that

\[ \frac{d\vec{y}}{dt}(t) = D\vec{y} \]

Answer: \( \vec{y}(t) = S^{-1}\vec{x}(t) \), which works because
Real Eigenvalues

So much for two dimensions.

Now suppose:

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\]

for some matrix $A$ with a real eigenbasis.

**Problem 9** Write $A = SDS^{-1}$. Find a linear change-of-variables from $\vec{x} \mapsto \vec{y}$ such that

\[
\frac{d\vec{y}}{dt}(t) = D\vec{y}.
\]

**Answer:** $\vec{y}(t) = S^{-1}\vec{x}(t)$, which works because

\[
\frac{d\vec{x}}{dt} = \frac{d(S\vec{y})}{dt} = S\frac{d\vec{y}}{dt}.
\]
So much for two dimensions.

Now suppose:

\[ \frac{d\vec{x}}{dt}(t) = A\vec{x} \]

for some matrix \( A \) with a real eigenbasis.

**Problem 9** Write \( A = SDS^{-1} \). Find a linear change-of-variables from \( \vec{x} \leftrightarrow \vec{y} \) such that

\[ \frac{d\vec{y}}{dt}(t) = D\vec{y}. \]

**Answer:** \( \vec{y}(t) = S^{-1}\vec{x}(t) \), which works because

\[ \frac{d\vec{x}}{dt} = \frac{d(S\vec{y})}{dt} = S\frac{d\vec{y}}{dt}. \]

\[ A\vec{x} = SDS^{-1}\vec{x} = SDS^{-1}S\vec{y} = SD\vec{y}. \]
Real Eigenvalues

So much for two dimensions.

Now suppose:

\[
\frac{d\vec{x}}{dt}(t) = A\vec{x}
\]

for some matrix \( A \) with a real eigenbasis.

**Problem 9** Write \( A = SDS^{-1} \). Find a linear change-of-variables from \( \vec{x} \mapsto \vec{y} \) such that

\[
\frac{d\vec{y}}{dt}(t) = D\vec{y}.
\]

**Answer:** \( \vec{y}(t) = S^{-1}\vec{x}(t) \), which works because

\[
\frac{d\vec{x}}{dt} = \frac{d(S\vec{y})}{dt} = S \frac{d\vec{y}}{dt}.
\]

\[
A\vec{x} = SDS^{-1} \vec{x} = SDS^{-1} S\vec{y} = SD\vec{y}.
\]

**Thus** \( S \frac{d\vec{y}}{dt} = SD\vec{y} \Rightarrow \frac{d\vec{y}}{dt} = D\vec{y} \).
Assume $D$ is all real, with diagonal elements $r_i$ in decreasing order.
Assume $D$ is all real, with diagonal elements $r_i$ in decreasing order.

\[
\frac{d\vec{y}}{dt} = \begin{bmatrix}
    r_1 &= r_{max} & 0 & 0 & \ldots & 0 \\
    0 & r_2 & 0 & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \ldots & 0 & r_n &= r_{min}
\end{bmatrix} \vec{y}.
\]
Real Eigenvalues

Assume $D$ is all real, with diagonal elements $r_i$ in decreasing order.

$$\frac{d\vec{y}}{dt} = \begin{bmatrix} r_1 = r_{max} & 0 & 0 & \ldots & 0 \\ 0 & r_2 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & r_n = r_{min} \end{bmatrix} \vec{y}.$$  

This is equivalent to an *uncoupled* set of linear first-order ODEs.

**Problem 10** What are they?
Real Eigenvalues

Assume $D$ is all real, with diagonal elements $r_i$ in decreasing order.

\[
\frac{d\vec{y}}{dt} = \begin{bmatrix}
  r_1 = r_{\text{max}} & 0 & 0 & \ldots & 0 \\
  0 & r_2 & 0 & \ldots & 0 \\
  & \vdots & & \ddots & \vdots \\
  0 & 0 & \ldots & 0 & r_n = r_{\text{min}}
\end{bmatrix} \vec{y}.
\]

This is equivalent to an *uncoupled* set of linear first-order ODEs.

**Problem 10** What are they?

**Answer:** \[
\frac{dy_i}{dt}(t) = r_i y(t)
\]
Thus (as you’ve seen with Johan)
Thus (as you’ve seen with Johan)

\[ \vec{y}(t) = \begin{bmatrix} e^{r_1 t} & 0 & 0 & \ldots & 0 \\ 0 & e^{r_2 t} & 0 & \ldots & 0 \\ \vdots \\ 0 & 0 & \ldots & 0 & e^{r_n t} \end{bmatrix} \vec{y}. \]
Real Eigenvalues

Thus (as you’ve seen with Johan)

\[
\vec{y}(t) = \begin{bmatrix}
e^{r_1 t} & 0 & 0 & \cdots & 0 \\
0 & e^{r_2 t} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & e^{r_n t}
\end{bmatrix}\vec{y}.
\]

Or, written out:

\[
\vec{y}(t) = \sum_{i=1}^{n} e^{r_i t} y_i(0) \epsilon_i.
\]
Thus (as you’ve seen with Johan)

\[
\vec{y}(t) = \begin{bmatrix}
e^{r_1 t} & 0 & 0 & \ldots & 0 \\
0 & e^{r_2 t} & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & e^{r_n t}
\end{bmatrix} \vec{y}.
\]

Therefore:

\[
|\vec{y}(t)| = (e^{2r_1 t} + e^{2r_2 t} + \ldots + e^{2r_n t})^{1/2} |y(0)|.
\]
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Therefore:

\[ |\vec{y}(t)| = (e^{2r_1 t} + e^{2r_2 t} + \ldots + e^{2r_n t})^{1/2} |\vec{y}(0)|. \]

**Problem 11** What is \( \lim_{t \to \infty} |\vec{y}(t)| \)?
Thus (as you’ve seen with Johan)

\[
\vec{y}(t) = \begin{bmatrix}
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\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & e^{r_n t}
\end{bmatrix} \vec{y}.
\]

Therefore:

\[
|\vec{y}(t)| = (e^{2r_1 t} + e^{2r_2 t} + \ldots + e^{2r_n t})^{1/2} |y(0)|.
\]

**Problem 11** What is \( \lim_{t \to \infty} |\vec{y}(t)| \)?

**Answer:**

\[
\lim_{t \to \infty} |\vec{y}(t)| = \begin{cases} 
\infty & \text{if } r_i > 0 \text{ for some } i \\
m|y(0)| & m = \text{number of } r_i \text{ equal to 0}
\end{cases}
\]
Now, let’s compute the Lyapunov exponent:
Real Eigenvalues

\[ \Lambda = \lim_{t \to \infty} \frac{1}{t} \log [||y(t)||] = \lim_{t \to \infty} \frac{1}{2t} \log \left[ \sum_{i=1}^{n} e^{2r_i t} \right]. \]
Real Eigenvalues

\[ \lambda = \lim_{t \to \infty} \frac{1}{t} \log[|\vec{y}(t)|] = \lim_{t \to \infty} \frac{1}{2t} \log \left[ \sum_{i=1}^{n} e^{2r_i t} \right]. \]

Now, pull out the \( r_{max} \) terms in the sum term:
Real Eigenvalues

\[ \Lambda = \lim_{t \to \infty} \frac{1}{t} \log[|\vec{y}(t)|] = \lim_{t \to \infty} \frac{1}{2t} \log \left[ \sum_{i=1}^{n} e^{2r_{i}t} \right]. \]

Now, pull out the \( r_{\text{max}} \) terms in the sum term:

\[ \sum_{i=1}^{n} e^{2r_{i}t} = e^{2r_{\text{max}}t} \left( N + \sum_{i>n_{1}} e^{2(r_{i} - r_{\text{max}})t} \right) \]

where \( N \) is the dimension of the \( r_{\text{max}} \) eigenspace.
Real Eigenvalues

\[ \Lambda = \lim_{t \to \infty} \frac{1}{t} \log \left[ |\mathbf{y}(t)| \right] = \lim_{t \to \infty} \frac{1}{2t} \log \left[ \sum_{i=1}^{n} e^{2r_i t} \right]. \]

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\[ \frac{1}{2t} \log \left[ \sum_{i=1}^{n} e^{2r_i t} \right] = r_{\text{max}} + \frac{1}{2t} \log \left[ N + \sum_{i>n_1}^{n} e^{2(r_i - r_{\text{max}})t} \right], \]
Real Eigenvalues

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and since \( r_i < r_{\text{max}} \), the RHS \( \to \frac{1}{2t} \log(N) \to 0 \) as \( t \to \infty \).
Real Eigenvalues

\[ \Lambda = \lim_{t \to \infty} \frac{1}{t} \log(|\vec{y}(t)|) = \lim_{t \to \infty} \frac{1}{2t} \log \left( \sum_{i=1}^{n} e^{2r_{i}t} \right) . \]

Now, pull out the \( r_{\text{max}} \) terms in the sum term:

\[ \sum_{i=1}^{n} e^{2r_{i}t} = e^{2r_{\text{max}}t} \left( N + \sum_{i>n_{1}}^{n} e^{2(r_{i}-r_{\text{max}})t} \right) \]

where \( N \) is the dimension of the \( r_{\text{max}} \) eigenspace.

\[ \frac{1}{2t} \log \left( \sum_{i=1}^{n} e^{2r_{i}t} \right) = r_{\text{max}} + \frac{1}{2t} \log \left( N + \sum_{i>n_{1}}^{n} e^{2(r_{i}-r_{\text{max}})t} \right), \]

and since \( r_{i} < r_{\text{max}} \), the RHS \( \to \frac{1}{2t} \log(N) \to 0 \) as \( t \to \infty \).

Hence, \( \Lambda = r_{\text{max}} \); the largest eigenvalue controls growth rate.
Real Eigenvalues

Consider:
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\[(e^{r_1 t} y_1(0), e^{r_2 t} y_2(0), \ldots, e^{r_n t} y_n(0))\]
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Now, normalize by \(e^{r_1 t}\); this gives
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Now, normalize by \( e^{r_1 t} \); this gives

\[ (y_1(0), y_2(0), \ldots, y_N(0), e^{(r_{N+1} - r_{\max}) t}, \ldots, e^{(r_n - r_{\max}) t}). \]
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Question: As \(t \to \infty\), what does this become?
Real Eigenvalues

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Question: As \(t \to \infty\), what does this become?

Answer: \((y_1(0), y_2(0), \ldots, y_N(0), 0, 0, \ldots, 0)\), since \(r_i - r_{max} < 0\).
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This is the projection of \(\vec{y}(0)\) onto the \(r_{max}\)-eigenspace.
Real Eigenvalues

- $x$-projection
  $(1,1) \rightarrow (1,0)$

- $y$-projection
  $(1,1) \rightarrow (0,1)$
Real Eigenvalues
Real Eigenvalues

Consider:

\[(e^{r_1 t}y_1(0), e^{r_2 t}y_2(0), \ldots, e^{r_n t}y_n(0))\]

Now, normalize by \(e^{r_1 t}\); this gives

\[(y_1(0), y_2(0), \ldots, y_N(0), e^{(r_{N+1}-r_{max})t}, \ldots, e^{(r_n-r_{max})t}).\]

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This is the projection of \(\vec{y}'(0)\) onto the \(r_{max}\)-eigenspace.
Real Eigenvalues

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Now, normalize by \(e^{r_1 t}\); this gives

\[(y_1(0), y_2(0), \ldots, y_N(0), e^{(r_{N+1} - r_{\text{max}}) t}, \ldots, e^{(r_n - r_{\text{max}}) t}).\]

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\(r_i - r_{\text{max}} < 0\).

This is the projection of \(\vec{y}'(0)\) onto the \(r_{\text{max}}\)-eigenspace. 
Thus, \(r_{\text{max}}\) not only controls growth rate, but vectors started 
anywhere converge to the \(r_{\text{max}}\)-eigenspace exponentially quickly.
Let me rephrase that *extremely important* comment:
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The largest eigenvalue, $r_{max}$ controls the asymptotic growth rate, as well as direction, of the system.
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Hence, asymptotically, the system behaves like the system restricted to the \( r_{\text{max}} \) eigen-space:

\[
\frac{d\vec{y}}{dt} \sim \begin{bmatrix}
0 & 1 & \cdots & 0 \\
0 & 0 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
r_{\text{max}} \\
0 \\
\vdots \\
0
\end{bmatrix} \vec{y}.
\]
Let me rephrase that *extremely important* comment:

The largest eigenvalue, $r_{max}$ controls the asymptotic growth rate, as well as direction, of the system.

Hence, asymptotically, the system behaves like the system *restricted* to the $r_{max}$ eigen-space:

$$\frac{d\vec{y}}{dt} \sim \begin{bmatrix} r_{max} I_N & 0_{N,n-N} \\ 0_{n-N,N} & 0_{n-N} \end{bmatrix} \vec{y}.$$ 

But lo! In the long-run, all real-eigenvalue systems thus behave as if they were several independent copies of 1-D systems with growth (or decay) rate $r_{max}$. 
Complex Eigenvalues

Recall real matrices only have non-real eigenvalues in conjugate pairs.
Complex Eigenvalues

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**Problem 12** Let

\[ A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}. \]

Compute \( e^{At} \). (Hint: use diagonal form found in previous lecture.)
Recall real matrices only have non-real eigenvalues in conjugate pairs.

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**Answer:**

\[ e^{at} \begin{bmatrix} \cos(bt) + i\sin(bt) & 0 \\ 0 & \cos(bt) - i\sin(bt) \end{bmatrix}. \]
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This is a pair of *out of phase waves* with *natural frequency* \( b \) – spirals going to zero if \( a < 0 \), to \( \infty \) if \( a > 0 \), and periodic if \( a = 0 \).
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(See Matlab simulation)
Now, what if we have multiple 2x2 non-real blocks? (Think back to the $r_1, \ldots, r_k, c_1, \overline{c_1}, c_2, \overline{c_2}, \ldots, c_m, \overline{c_m}$ listing of eigenvalues.)
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\[
\begin{bmatrix}
exp\left(\begin{bmatrix}
c_1 & 0 \\
0 & \overline{c_1}
\end{bmatrix}\right) & 0 & \ldots & 0 \\
0 & exp\left(\begin{bmatrix}
c_2 & 0 \\
0 & \overline{c_2}
\end{bmatrix}\right) & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & exp\left(\begin{bmatrix}
c_m & 0 \\
0 & \overline{c_m}
\end{bmatrix}\right)
\end{bmatrix}
\]
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The general solution is:

\[
\begin{align*}
&= e^{a_1 t} \begin{bmatrix} \cos(b_1 t) + isin(b_1 t) & 0 \\ 0 & \cos(b_1 t) - isin(b_1 t) \end{bmatrix} \\
&= e^{a_2 t} \begin{bmatrix} \cos(b_2 t) + isin(b_2 t) & 0 \\ 0 & \cos(b_2 t) - isin(b_2 t) \end{bmatrix} \\
&= e^{a_m t} \begin{bmatrix} \cos(b_m t) + isin(b_m t) & 0 \\ 0 & \cos(b_m t) - isin(b_m t) \end{bmatrix}
\end{align*}
\]
Now, what if we have multiple 2x2 non-real blocks? (Think back to the \( r_1, \ldots, r_k, c_1, c_1, c_2, c_2, \ldots, c_m, c_m \) listing of eigenvalues.)

The general solution is:

\[
\begin{align*}
\mathbf{x}(t) &= e^{a_1 t} \begin{bmatrix} \cos(b_1 t) + i\sin(b_1 t) & 0 \\
0 & \cos(b_1 t) - i\sin(b_1 t) \end{bmatrix} + \mathbf{x}_2(t) + \mathbf{x}_m(t) \\
\mathbf{x}_2(t) &= e^{a_2 t} \begin{bmatrix} \cos(b_2 t) + i\sin(b_2 t) & 0 \\
0 & \cos(b_2 t) - i\sin(b_2 t) \end{bmatrix} \\
\mathbf{x}_m(t) &= e^{a_m t} \begin{bmatrix} \cos(b_m t) + i\sin(b_m t) & 0 \\
0 & \cos(b_m t) - i\sin(b_m t) \end{bmatrix}
\end{align*}
\]

where \( a_i = Re(c_i) \) and \( b_i = Im(c_i) \).
Now, by arguments precisely like in the real case, the eigenvalues with highest real part, $a_{max}$, completely win out asymptotically.
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$$e^{a_{max}t} \left( \begin{bmatrix} e^{ib_1t} & 0 \\ 0 & e^{-ib_1t} \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} e^{ib_Nt} & 0 \\ 0 & e^{-ib_Nt} \end{bmatrix} \right)$$
Complex Eigenvalues

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where $N$ is the number of different conjugate-pairs with highest real part.
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Just as before, these systems also get asymptotically projected onto the eigenspace associated with maximum real part eigenvalue; and growth rate is $a_{max}$. 
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Just as before, these systems also get asymptotically projected onto the eigenspace associated with maximum real part eigenvalue; and growth rate is $a_{max}$.

What’s left to know: When is there a natural frequency?
Figure 2: A torus. (Donut!)
Natural Frequencies

Figure 3: The two loops of a torus.
Figure 4: Torus = square with both wrap-arounds.
Figure 5: Equally-marked edges are identified.
Figure 6: First make a tube.
Natural Frequencies

Figure 7: Then connect up the tube to make a donut.
Now, suppose you drew a line on the torus-square.
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It eventually could return to 0 and become periodic.
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Natural Frequencies

Figure 8: Trajectory viewed in actual torus view.
Question: Assuming each square side has length 1, what is the condition on the slope $r$ of the line for its trajectory to become periodic?
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Figure 9: Slope is 2.
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Answer: The trajectory is periodic IFF $r$ is rational, i.e. $r \in \mathbb{Q}$. 
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Answer: The trajectory is \textit{ergodic} – that is, it comes arbitrary close to any point in the square, filling up the space. (See Matlab simulation).
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Now we apply this to the ODE system.
Recall our system of interest:
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\[ e^{a_{\text{max}} t} \left( \left[ \begin{array}{cc} e^{ib_1 t} & 0 \\ 0 & e^{-ib_1 t} \end{array} \right] \oplus \cdots \oplus \left[ \begin{array}{cc} e^{ib_N t} & 0 \\ 0 & e^{-ib_N t} \end{array} \right] \right) \]
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\( a_{\text{max}} \) was largest real-part of eigenvalues and \( b_1,\ldots,N \) are (positive) imaginary parts of the various eigenvalues \( c_i \) with \( \Re(\lambda) = a_{\text{max}} \).
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Normalizing by \( e^{a_{\text{max}} t} \) gives

\[ \left( \left[ e^{i b_1 t} \ 0 \right] \oplus \cdots \oplus \left[ e^{i b_N t} \ 0 \right] \right) \]

**Problem 13** What “space” does this (normalized) trajectory happen on? (Don’t answer this yet, unless you’re sure.)
Natural Frequencies

Each of the $c^{ibt}$ terms is a periodic trajectory on the circle.
Natural Frequencies

Each of the $e^{ibt}$ terms is a periodic trajectory on the circle.

**Problem 14** What is the period of the trajectory? I.e., what is the smallest positive $T$ for which $e^{ibt}(t+T) = e^{ibt}$?
Natural Frequencies

Each of the $e^{ibt}$ terms is a periodic trajectory on the circle.

**Problem 14** What is the period of the trajectory? I.e., what is the smallest positive $T$ for which $e^{ibt}(t+T) = e^{ibt}$?

**Answer:** $T_i = 2\pi/b_i$, since $e^{ibt}(t+2\pi/b_i) = e^{ibt+2\pi i} = e^{ibt}$. 
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Now, think of the torus as Circle $\times$ Circle.
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![Diagram of a periodic trajectory on a circle](image-url)
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Now, answer problem 13, on what spaces does the trajectory

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\left(\begin{bmatrix} e^{ib_1 t} & 0 \\ 0 & e^{-ib_1 t} \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} e^{ib_N t} & 0 \\ 0 & e^{-ib_N t} \end{bmatrix}\right) 
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Trajectory 1 has a natural frequency IFF the numbers \(b_i\) are rationally related – that is, if \(b_i/b_j\) is a rational number for all \(i, j\).
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Trajectory 1 has a natural frequency IFF the numbers \(b_i\) are rationally related – that is, if \(b_i/b_j\) is a rational number for all \(i, j\). Otherwise, the system hits all points (is ergodic) in at least one dimension.
Putting it All Together

We now have a complete description of all the behaviors of a diagonal linear system.
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- Each complex eigenvalue pair consists of two out-of-phase spiral or standing waves.
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- The system grows with Lyapunov exponent given by $a_{max}$.
- Each real eigenvalue consists of exponential growth or decay.
- Each complex eigenvalue pair consists of two out-of-phase spiral or standing waves.
- The system possesses an overall natural frequency only when the imaginary parts of eigenvectors of $A^{Top}$ are rationally related; and is ergodic in some dimension(s) otherwise.
Inhomogenous Equations

So far we’ve dealt with what are known as “homogenous equations":
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**Problem 15** Write the kinematics equations from before as a 2x2 in-homogenous equation.
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**Answer:** \[
\begin{bmatrix}
\dot{x} \\
\dot{v}
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
x \\
v
\end{bmatrix} + \begin{bmatrix}
0 \\
a
\end{bmatrix}.
\]
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Suppose the equation is

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**Problem 16** Solve for $\vec{g}$ that makes the above guess work.
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$$\vec{x}(t) = e^{tA}\vec{x}(0) + \int_0^t e^{(t-s)A}\vec{f}(s)ds.$$
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Homogenous solution + correction, known as "Variation of Constants" formula.