Lecture 5: \( TV = |RT|^2 \).

Thm (Turaev, Walker, Roberts)

Let \( M \) be closed, orientable 3-manifold. Then \( \forall r \geq 3, \)

\[
TV_r(m) = |I_r(m)|^2.
\]

Recall, \((M,T), V,E,F,T = \text{sets of vertices, edges, faces, tetrahedra, } \eta = \frac{-2r}{(A^4 - A^{-2})^2}\)

\[
TV_r(m) = \eta^{-1/n} \leq \prod_{v \in A_r} \prod_{e \in E} \prod_{f \in F} \prod_{\sigma \in T} (\eta - \mu^2)
\]

\( M = M_L, D = D(L) \) standard, \( \sigma = \text{signature of linking matrix} \)

\[
\mu = \frac{A^2 - A^{-2}}{\sqrt{-2r}} \quad (\eta = \mu^2)
\]

\[
I_r(m) = \mu \langle m\omega_r, \ldots, m\omega_T \rangle_D \langle m\omega \rangle_{u^+}^{-\sigma}
\]
Idea:

Triangulation of $M$ → Heegaard splitting of $M$ → Kirby diagram of $M \# (\overline{-M})$

\[ TV = \text{Roberts' invariant } = I_t(M \# (\overline{-M})) \]

Handlebody:

\[ H_g = \{ 0 \text{-handles} \} \cup \{ 1 \text{-handles} \} \]

\[ D^3 \times S^0 \]

\[ D^2 \times D^1 \]

\[ \partial D^2 \times \{ \frac{1}{2} \} \]
Def: A Heegaard splitting of a closed 3-mfld $M$ consists of two handlebodies $H_1 \cong H_2 \cong H_g$ and a homeomorphism $f: \partial H_2 \to \partial H_1$, s.t.

$$M = H_1 \cup_f H_2$$

- $f$ is determined by image of $\beta$-curves.

$\Rightarrow$ handle structure of $M$

$$H_1 = \{0\text{-handles} \} \cup \{1\text{-handles} \}$$

$$H_2 = \{2\text{-handles} \} \cup \{3\text{-handles} \}$$

$$d_i = \# \text{ of } i\text{-handles}$$

$$d_1 = \# \text{ of } \alpha\text{-curves}$$

$$d_2 = \# \text{ of } \beta\text{-curves}.$$
- Heegaard diagram \((S, \alpha, \beta)\)

- Roberts' invariant (chain mail) is constructed by the following steps:

1) Embed \(H_0\) in \(S^3\),

2) Thicken \(\alpha\)- and \(\beta\)-curves along \(\partial H_1\),

3) Push \(\beta\)-curves slightly into \(H_1\) to get a framed link \(L\) in \(S^3\).

4) Definition:

\[
CH_r(m) = \mu^{d_0 + d_3} \langle \mu \omega_r, \ldots, \mu \omega_r \rangle_L
\]
Thm (Roberts).
1) $CH_r(m)$ defines an invariant of $M$, i.e., is independent of the Heegaard splitting and the embedding at $H_1$.
2) $CH_r(m) = TV_r(m)$.
3) $CH_r(m) = |I_r(m)|^2$.

Pf of 1): Any two H.S. are differed by 0-1, 1-2, 2-3 birth and dies, 1- and 2-handle slides.

0-1 (2-3 by duality)

1-2

1-handle slide doesn't change diagram

2-handle slide $\uparrow$

$\text{KM}_2$

Any two embeddings of $H_1$ are differed by

and twist $\rightarrow$

which are composition of $\text{KM}_1$ and $\text{KM}_2$'s.
For proof of 2), we need **Heegaard splitting from a triangulation**. 

\[ H_2 = \text{tubular nbhd at 1-skeleton EUV} \]

\[ H_1 = M \setminus H_2 \]

\[ d_0 = 1T1, \quad d_1 = 1F1, \quad d_2 = 1E1, \quad d_3 = 1V1. \]
\[ L_\varphi = \{ \alpha \text{-curves} \} \cup \{ \beta \text{-curves} \} \] has the property that every \( \alpha \text{-curve} \) encloses 3 \( \beta \text{-curves} \).

Then \( CH_r(m) = \mu^{d_0+d_3} \langle \mu w, \ldots, \mu w \rangle_{L_\varphi} \)

\[
= \mu^{d_0+d_3+d_2-d_1} \sum_{c \in A_r} \prod_{e \in \Theta} \prod_{f \in \Theta} \prod_{o} \leq \prod_{o} \prod_{e} \prod_{f} \prod_{o}
\]

\[
\begin{align*}
(d_0-d_1+d_2-d_3 = 0 & \Rightarrow d_0+d_3+d_2-d_1 = 2d_3 = 21u1) \\
\mu^{d_0+d_3+d_2-d_1} & = \mu^{21u1} = \eta^{-1u1} \leq \prod_{c \in A_r} \prod_{e \in \Theta} \prod_{f \in \Theta} \prod_{o} \leq \eta^{-1u1} = TV_r(m) \end{align*}
\]
Standard Heegaard splitting

$(H_1, H_2, f)$ sit both $H_1$ and $H_2$ have exactly one 0-handle.

Then Prop. $M_{L_s} = M \# (-m)$. 

Then $\text{ch}_r(M) = \mu^2 \langle \mu \omega, \ldots, \mu \omega \rangle_{L_s}^{L_s}$

$= \mu^2 \cdot \mu^{-1} \cdot \text{I}_r(M \# (-m)) \cdot \langle \mu \omega \rangle_{u_+}^{\sigma_{(L_s)}}$

$= \mu^2 \cdot \mu^{-1} \cdot \mu^{-1} |\text{I}_r(M)|^2 \cdot \langle \mu \omega \rangle_{u_+}^{\sigma_{(L_s)}}$

Lemma below $|\text{I}_r(M)|^2$

Lemma 5. $\sigma_{(L_s)} = 0$. 
pf of Prop: By definition, $M_{L_s} = \partial X_{L_s}$, where $X_{L_s}$ is 4-mfd from $B^4$ by attaching 2-handles along $L_s$. Let $X_{L_s}'$ be 4-mfd from $B^4$ by attaching 1-handles to $\alpha$-curves and 2-handles to $\beta$-curves. Then

(i) $\partial X_{L_s} = \partial X_{L_s}'$

(ii) $X_{L_s} = \mathbb{M}^{(2)} \times I$, where $\mathbb{M}^{(2)} = \mathbb{M} \setminus B^3 = 2$-skeleton of $\mathbb{M}$

(iii) $\partial X_{L_s} = \mathbb{M} \# (-\mathbb{M})$

pf of Lemma: The linking matrix $LK(L_s)$ has the form

$$
\begin{pmatrix}
0 & A \\
A^T & 0
\end{pmatrix}
$$

$3 \alpha$, since $\text{lk}(\alpha_i, \alpha_j) = \text{lk}(\beta_i, \beta_j) = 0$

Signature of such matrix $\text{has}$ equals $0$ since it $0 = (v_1, v_2)$ is eigenvector of $\lambda$ (then $-v_1, v_2$ is eigenvector of $\lambda$).