Volume Conjecture: (Kashaev, Murakami-Murakami)

Let $k$ be a hyperbolic knot ($S^3 \setminus k$ admits a complete hyperbolic metric with finite volume). Then

$$\lim_{n \to +\infty} \frac{2\pi_i}{n} \ln \left| J_n'(k, e^{-\frac{2\pi i}{2(n+1)}}) \right| = Vol_{H^3}(S^3 \setminus k)$$

RM: In other (popular) literatures, people use $q = e^{2\pi i}$ and $V_n(k, q) = J'_{n-1}(k, A)$. Then

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Goal Today:

1. The Figure-8 knot $\bigcirc$ is hyperbolic.
2. Volume conjecture is true for the Figure-8 knot.

Thm (Thurston) The following gives an ideal triangulation of $S^3 \setminus \bigcirc$.

That is, glue $\sigma_1$, $\sigma_2$ along pairs of faces according to color and direction of arrows.

Combinatorics around edges.
In the rest, let $K = \text{Figure-8 knot}$. 

In the Figure, $\uparrow\uparrow$ are the two edges, and the blue curves are intersections of faces w/ $\partial N(K)$. The four faces are 

A: 

B: 

C: 

D: 
A, B, C, D are disjoint in $S^3 \setminus N(K)$.

Small triangles

\[ \text{e.g.} \]

Summary: Keep track of faces and small triangles, this gives "smooth" embedding of $\sigma_1, \sigma_2 (\cong S^2)$ into $S^3 \setminus N(K) \subset S^3$ preserving combinatorics.

By Schönflies Thm (every $C^0$ or PL embedding of $S^2$ in $S^3$ bounds a 3-ball), $S^3 \setminus N(K)$ is the union of $\sigma_1$ and $\sigma_2$. 
**Hyperbolic Structure:**

Let $\sigma_1, \sigma_2$ be copies of regular hyperbolic ideal tetrahedra.

$$H^3 = \mathbb{C} \times \mathbb{R}_+$$

**Regular:** all dihedral angles are $\frac{\pi}{3}$.

- glue faces at $\sigma_1, \sigma_2$ by isometries

$\Rightarrow$ complete hyperbolic w/ finite volume.

**Completeness:**

around edges

$$\frac{\pi}{3} \quad \Sigma \tau = 2\pi$$

around the vertex: tiling of regular $\Delta$ of $\mathbb{H}^2$
Finite volume: volume of hyperbolic ideal tetrahedra with dihedral angles $\alpha, \beta, \gamma$ is

$$V_0(\alpha, \beta, \gamma) = \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma),$$

where $\Lambda(x)$ is the Lobachevskiy function

$$\Lambda(x) = - \int_0^x \ln |2\sin t| \, dt.$$

Functional properties of $\Lambda(x)$.

(i) $\Lambda(x)$ is odd with period $\pi$.

(ii) $\Lambda(2x) = 2\Lambda(x) + 2\Lambda(x + \frac{\pi}{2})$.

$$V_0(\text{regular ideal tetrahedron}) = 3 \Lambda\left(\frac{\pi}{3}\right).$$

$$V_{\text{th}^3}(S^3 \setminus K) = 6 \Lambda\left(\frac{\pi}{3}\right).$$
Habiro's Formula (using $q = A^4$, $V_n(k, q) = J_{n-1}(k, q)$)

$$V_n(k, q) = 1 + \sum_{j=1}^{n-1} \frac{J}{11} \left( q^{\frac{n-k}{2}} - q^{-\frac{n-k}{2}} \right) \left( q^{\frac{n+k}{2}} - q^{-\frac{n+k}{2}} \right)$$

Let $q = e^{\frac{2\pi i}{n}}$. Then

$$V_n(k, e^{\frac{2\pi i}{n}}) = 1 + \sum_{j=1}^{n-1} \frac{J}{11} 4 \sin^2 \frac{k\pi}{n}$$

Let $g_n(j) = \frac{J}{11} 4 \sin^2 \frac{k\pi}{n}$. Then

$$V_n(k, e^{\frac{2\pi i}{n}}) = 1 + \sum_{j=1}^{n-1} g_n(j)$$

Graph of $4 \sin^2 \frac{k\pi}{n}$

![Graph of $4 \sin^2 \frac{k\pi}{n}$](image)
This implies \( g_{n,j} \) is decreasing if \( 0 < j < \frac{n}{6} \) or \( \frac{5n}{6} < j \), and is increasing if \( \frac{n}{6} < j < \frac{5n}{6} \).

Thus, \( g_{n,j} \) achieves the maximum at \( j = \left\lfloor \frac{5n}{6} \right\rfloor \).

\[
\frac{1}{n} \ln g_n \left( \left\lfloor \frac{5n}{6} \right\rfloor \right) \leq \frac{1}{n} \ln |V_n(k, e^{\frac{2\pi\mu}{n}})| \leq \frac{1}{n} \ln g_n \left( \left\lfloor \frac{5n}{6} \right\rfloor \right),
\]

\[
\lim_{n \to \infty} \frac{1}{n} \ln |V_n(k, e^{\frac{2\pi\mu}{n}})|
= \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{\left\lfloor \frac{5n}{6} \right\rfloor} \ln \left( 4\sin^2 \frac{k\pi}{n} \right)
= 2 \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{\left\lfloor \frac{5n}{6} \right\rfloor} \ln 2\sin \frac{k\pi}{n}
= 2 \int_0^{\frac{5\pi}{6}} \ln (2\sin t) \, dt = -\frac{2}{\pi} \Lambda \left( \frac{5\pi}{6} \right),
\]

Functional Prop of \( \Lambda \)

\[
\frac{3}{\pi} \Lambda \left( \frac{\pi}{3} \right) = \frac{\text{Vol} h^3(S^3 \setminus K)}{2\pi}
\]