Let $L_i$ be the $i$-th component of $L$.

$k_i = L_i | S^1 \times S^3$ is the core of $L_i$.

$k_i' = L_i | S^1 \times S^1$ is the parallel copy of $k_i$.

Each $L_i$ has a framing $n_i$ ("number of twists").
Q: What is a 0-forming?

Def: A Seifert surface of a (unframed) link $L \subset S^3$ is a connected, oriented, compact surface $S \subset S^3$ w/ $\partial S = L$.

Eg: $S \sim S$

HW: $S \cong \Sigma_{1,1}$.

Thm (Seifert). Every $L$ has a Seifert surface.

Seifert's algorithm:

- Orient $L$, or pre-resolution.
- Assign $+/-1$ disks according to orientation.
- Glue back together different components by tubes, if any, to get a connected surface.
- $L_i$ is of $0$-framing if it is a collar nbhd of $k_i$ in a Seifert surface $S_i$ of $k_i$.

- $L_i$ is of $n$-framing if the algebraic intersection number

$$i(k_i', S_i) = n.$$ 

- It is independent of choice of $S_i$, since all Seifert surfaces are homologous.

$$H_2(S^3 \setminus N(k_i), \partial N(k_i)) \cong \mathbb{Z}$$ and 

$$[S_i] = 1 \in \mathbb{Z}.$$
Alternative description of $0$-framing:

Consider inclusion $i : \partial N(K_i) \to S^3 \setminus N(K_i)$ and the induced map $i_* : H_i(\partial N(K_i)) \to H_i(S^3 \setminus N(K_i))$. 

\[
\begin{array}{c|c|c}
H_i & H_i(S^3 \setminus N(K_i)) \\
\mathbb{Z}^2 & \mathbb{Z}^2 \\
\mathbb{Z} & \mathbb{Z} \\
\end{array}
\]

The kernel $\ker i_* \cong \mathbb{Z}$ and is generated by the longitude $L \subset N(K_i)$. $L$ bounds a surface $S' \subset S^3 \setminus N(K_i)$ (since $[L] \in \ker i_*$). The framed link $L_i$ determined by $K_i$ and $L$ has framing $0$, and

the Seifert surface $S_i = L \cup L'$.

Eg:

- $0$-framing

(-3)-framing

blackboard framing
A framed link can also be represented by a link diagram with an integer \( n_i \) on each component \( K_i \) (Kirby diagram).

- The **linking number** \( l \) of \( K_i \) and \( K_j \) is
  \[
  l_{K_{ij}} = l(K_i, S_j) = l(K_j, S_i).
  \]

- The **linking matrix** \( L \) of \( L \) is the matrix
  \[
  LK(L) = (l_{K_{ij}}),
  \text{ where } LK_{ii} = n_i \text{ and } LK_{ij} = lk_{ij} \text{ if } i \neq j.
  \]

A framed link \( L \) (or a Kirby diagram \( D \)) determines
1) a 4-manifold \( X_L \) by attaching 2-handles \( B^2 \) along \( L \) and in turn
2) a 3-manifold \( M_L = \partial X_L \).
Notation: \( B^n = D^n \), \( n \in \mathbb{N} \).

4-dimensional 2-handle = \( D^2 \times D^2 \)

\[ \partial (D^2 \times D^2) = S^1 \times D^2 \cup_{s \times s'} D^2 \times S^1 \]

\( \partial B^4 = S^3 \), attaching 2-handle

attaching map \( f: S^1 \times D^2 \to S^3 = \partial B^4 \)

The resulting 4-mfld is \( B^4 \cup_f (D^2 \times D^2) \)

\( f \) is determined by \( f|_{S^1 \times \{0,1\}} \) up to isotopy,

\[ S^1 \times D^2 \]

and \( f(S^1 \times \Gamma_{0,1}) \subset S^3 \) is a framed link.
M: $M_L = \partial X_L$ is the 3-manifold obtained from $S^3$ by doing $n_i$-Dehn surgery along $K_i$.

Recall: $\frac{p}{q}$ - Dehn surgery along $K$.

Let $H = D^2 \times S^1$, meridian $m = \partial D^2 \times \{1\} \subset \partial H$.

In $T = \partial N(K)$, there are curves $m, l$, where $m$ bounds a disk in $N(K)$ and $l$ bounds a surface in $S^3 \setminus N(K)$.

Any $p, q \in \mathbb{Z}$ with $(p, q) = 1$ determines $C_{pq} \subset T$, s.t.

$C_{pq} = p[m] + q[l] \in H_1(T^2)$.

Let $f: T^2 = \partial H \rightarrow \tilde{T} = \partial N(K)$ determined by

$m \mapsto C_{pq}$

Then $M_{K, \frac{p}{q}} \cong HU_f(S^3 \setminus N(K))$. 
Back to 2-hand $B^2 \times B^2$.

$$S^3 = \partial (B^2 \times B^2) = (S^1 \times B^2) \cup_{S^1 \times S^1} (B^2 \times S^1)$$

$B^2 \times S^1$

\[ \neq \]

$\partial (B^4 \cup_f (D^2 \times B^2)) = (\partial B^4 \setminus N(f)) \cup_f (D^2 \times S^1)$

$f : m \mapsto C_{n+1}$.

$\Rightarrow \mathcal{M}_L$ is obtained from $S^3$ by $\eta_1$ - Dehn surgery.
components $k_i \quad 2$-handles basis $\{[H_i]\}$
of $L \quad H_i \quad \otimes H_2(X_L; \mathbb{Z})$

linking matrix $Lk(L)$

intersection forms on $H_2(X_L; \mathbb{Z})$

Thm (Lichorish, Wallace).
Any $M^3 \cong M_L$ for some $L \subset S^3$

(Original statement: any $M^3$ can be obtained from $S^3$ by doing surgery along some $L$ with integer coefficients).

Thm (Kirby, 1978 Invent, "A calculus for framed links in $S^3$$\)
$M_{L_1} \cong M_{L_2}$ if $L_1$ can be obtained from $L_2$
by a sequence of the following Kirby Moves $\text{KM I}$ and $\text{KM II}$.
KMI (Blow up/down): Add or subtract an isolated copy of an unknot at framing \( \pm 1 \).

Notation: \( U_\pm = U \pm 1 \).

KMI (Handle Slid): Replace \( K_i \) by a band sum \( \tilde{K}_i = K_i \# b K_j \) of \( K_i \) and a parallel copy \( K_j \) of \( K_j \), with \( \tilde{n}_i = n_i + n_j \pm 2\ell_k \) (\( \ell \) depends on \( b \)).

Band Sum \( K_0 \#_b K_j \) of \( K_0 \) and \( K_j \).

\( b: K_0 \times I \times I \to S^3 \) embedding s.t. \( b(I \times I \times I) \cap K_i = b(I \times I \times I) \).

Then \( K_0 \#_b K_j = (K_0 \cup K_i - b(I \times I \times I)) \cup b(I \times I \times I) \).
Idea of proof: "\[\text{Km1: } M_{u_\pm} \cong S^3 \Rightarrow M_{L_{0 \cdot u_\pm}} \cong M_L \# S^3 \cong M_L\]

\[f: \quad \begin{array}{c}
\bigcirc \\
\end{array} \quad \quad \xrightarrow{\phi \circ \phi'} \quad \begin{array}{c}
\bigcirc \\
\end{array} \]

\[\exists \phi : T^2 \to T^2 \text{ homotopy } \quad C_{ii} \to \lambda \]

that extends to \[\phi' : H \to H\]

\[\Rightarrow M_{u_\pm} = H_0 U H \cong H_0 \phi \circ \phi' H = S^3\]

\[\text{id} \circ \phi'\]

Km: Consider 4-mfld \(X_{u_\pm}\)

Since \(\partial X_{u_\pm} \cong S^3\), can attach a 4-handle \(B^4\).

Easy to see

0-handle \(U_{u_\pm}\) 2-handle \(U\) 4-handle \(\cong \pm \mathbb{C}P^2\)

\[X_L \cup u_\pm \cong X_L \# (\pm \mathbb{C}P^2) , \text{ Blow up/down.}\]
Similarly, one can see

\[ \bigcirc \quad \leftrightarrow \quad S^2 \times S^2 \quad \leftrightarrow \quad \bigcirc \]

**KM II.**

To see \( \tilde{\nu}_j \), keep track of where \( k'_i \) goes. This can also be seen as follows.

Recall \( \{ k_i \} \leftrightarrow \) basis \( \{ [H_i] \} \) of \( H_2(X, \mathbb{Z}) \)

\[ k_i \leftrightarrow k_i \#_b k'_j \leftrightarrow \ [H_i] \leftrightarrow [H_i] \pm [H_j] \]

Then the intersection form, hence the linking matrix, changes as

\[
\begin{pmatrix}
  \nu_i \\
  \ell_{ij} \\
  \ell_{ij} + n_i \\
\end{pmatrix} \mapsto
\begin{pmatrix}
  \nu_i + \nu_j + 2\ell_{ij} \\
  \ell_{ij} + \nu_i \\
  \ell_{ij} + n_j \\
\end{pmatrix}
\]
Suppose $M_{L_1} \cong M_{L_2} \cong M$.

Step 0: By possible $w_1 \neq cP^2$, $S^2 \times S^2$, $S^2 \times S^2$, we can assume $X_{L_1} \cong X_{L_2} \cong X$.

"Morse theory."

1. $\partial \mathbb{R}$, $\mathbb{S}$, $\mathbb{C}$ can be obtained from $\varnothing$ by KMI and KMI

2. Consider Morse functions $f_i : X_{L_i} \to [0, 1]$ with $f_i(0) = \varnothing$, $f_i(\frac{1}{2}) = S^3$, no critical pts

   $\text{Int} f_i^{-1}(0, \frac{1}{2}) \supset f_i^{-1}(1) = M_{L_i}$

"Cerf theory" $\Rightarrow$ A homotopy $f : X \times [0, 1] \to [0, 1]$ from $f_1$ to $f_2$, sit each $f_t$ is Morse.

For generic $t$, descending disks intersect $S^3$, gives isotopy between $L_1$ and $L_2$; for non-generic $t$, descending disk intersects critical pt of smaller value, that's where handle slide happens.
Critical pts.

Descending disks.

$S^3 \xrightarrow{L^c} B^4$

generic t.

$M$

$S^3 \xrightarrow{} B^4$

non-generic.

$P_1, P_2$