Research Statement

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1. Overview

My research focuses on geometric topology and quantum topology. The topics I have been working on include: classical and quantum Teichmüller theory, dynamics of mapping class group actions, variational methods for finding geometric structures, and the asymptotic behavior of quantum invariants and its relationship with hyperbolic geometry.

2. Research Summary I: Surfaces

2.1 Dynamics of the mapping class group action. Let \( \Sigma_g \) be the closed surface of genus \( g \). The \( \text{PSL}(2, \mathbb{R}) \)-character variety \( \mathcal{M}(\Sigma_g) \) is the space of conjugacy classes of representations \( \rho : \pi_1(\Sigma_g) \to \text{PSL}(2, \mathbb{R}) \). In [20], Goldman proved that \( \mathcal{M}(\Sigma_g) \) has exactly \( 4g - 3 \) connected components, indexed by the Euler class \( e(\rho) \) which is an integer that satisfies the Milnor-Wood inequality \( 2 - 2g \leq e(\rho) \leq 2g - 2 \). He also showed that the equalities hold if and only if \( \rho \) is Fuchsian, i.e., discrete and faithful representations, hence two connected components of \( \mathcal{M}(\Sigma_g) \) are copies of the Teichmüller space of \( \Sigma_g \), one for each orientation of \( \Sigma_g \). It was known to Fricke and Klein [14] that the action of the mapping class group \( \text{Mod}(\Sigma_g) \) on the Teichmüller space is properly discontinuous. In [22], Goldman conjectured that the \( \text{Mod}(\Sigma_g) \)-action on each of the other components of \( \mathcal{M}(\Sigma_g) \) is ergodic. Closely related to Goldman’s conjecture is a question of Bowditch [6], Question C, that whether for each representation \( \rho \) that is not in the Teichmüller component of \( \mathcal{M}(\Sigma_g) \), there exists a simple closed curve \( \gamma \) on the surface such that \( \rho(\gamma) \) is an elliptic or a parabolic element of \( \text{PSL}(2, \mathbb{R}) \). Recently, Marché-Wolff [40] showed that an affirmative answer to Bowditch’s question implies that Goldman’s conjecture is true, and vice versa.

Bowditch’s question is originally asked for the type-preserving representations of punctured surface group. Let \( \Sigma_{g,n} \) be the punctured surface with genus \( g \) and \( n \) punctures. A representation \( \rho : \pi_1(\Sigma_{g,n}) \to \text{PSL}(2, \mathbb{R}) \) is type-preserving if it sends every peripheral element of \( \pi_1(\Sigma_{g,n}) \) to a parabolic element of \( \text{PSL}(2, \mathbb{R}) \). In [6], Question C asks whether for each non-Fuchsian type-preserving representation \( \rho \), there always exists a non-peripheral simple closed curve \( \gamma \) such that \( \rho(\gamma) \) is an elliptic or a parabolic element in \( \text{PSL}(2, \mathbb{R}) \). In [65], we give a negative answer to this question by considering representations of the four-punctured sphere group.

Theorem 1 (Y.). There are uncountably many non-elementary type-preserving representations \( \rho : \pi_1(\Sigma_{0,4}) \to \text{PSL}(2, \mathbb{R}) \) with relative Euler class \( e(\rho) = \pm 1 \) that send every non-peripheral simple closed curve to a hyperbolic element. In particular, these representations are not Fuchsian.

We also show that the representations in Theorem 1 are the only counterexamples coming from the the four-puncture sphere group, i.e., the answer to Bowditch’s question for Euler class 0 type-preserving representations of \( \pi_1(\Sigma_{0,4}) \) is affirmative.

Theorem 2 (Y.). Every non-elementary type-preserving representation \( \rho : \pi_1(\Sigma_{0,4}) \to \text{PSL}(2, \mathbb{R}) \) with relative Euler class \( e(\rho) = 0 \) sends some non-peripheral simple closed curve to an elliptic or parabolic element.

The main tool we use is Kashaev’s lengths coordinates of the decorated character spaces [25] and a formula of expressing the traces in terms of this coordinates. Using the same tool, we also describe the connected components of the character variety of type-preserving representations of the four-punctured sphere and show that the mapping group class action on each non-Teichmüller component is ergodic, which respectively confirm a conjecture of Kashaev’s [25] and of Goldman’s [22] in this case.

Recall that each parabolic element of \( \text{PSL}(2, \mathbb{R}) \) is up to \( \pm I \) conjugate to an upper triangular matrix with trace 2, and its conjugacy class is distinguished by whether the sign of the non-zero off diagonal element is positive or negative. We respectively call the two conjugacy class of parabolic elements the positive and the negative conjugacy classes. For a type-preserving representation \( \rho : \pi_1(\Sigma_{g,n}) \to \text{PSL}(2, \mathbb{R}) \), we denoted by \( s(v) = 1 \) if \( \rho \) sends a peripheral element around this puncture into the positive conjugacy class of parabolic elements, and denoted by \( s(v) = -1 \) if otherwise. Let \( \mathcal{M}(\Sigma_{g,n}) \) be the space of conjugacy classes of type-preserving representations of \( \pi_1(\Sigma_{g,n}) \), and for \( k \) with \( |k| \leq 2g - 2 + n \) and \( s \in \{\pm 1\}^n \) let \( \mathcal{M}_k^s(\Sigma_{g,n}) \) be the subspace of \( \mathcal{M}(\Sigma_{g,n}) \) consisting of representations with relative Euler class \( k \) and signs of the punctures given by \( s \). It is conjectured in [25] that each \( \mathcal{M}_k^s(\Sigma_{g,n}) \), if non-empty, is connected. The following Theorem 3 confirms this conjecture for the four-punctured sphere.
Theorem 3 (Y.) Let $s \in \{ \pm 1 \}^4$. Then

1. $\mathcal{M}_0^s(\Sigma_{0,4})$ is non-empty if and only if $s$ contains exactly two $-1$ and two $1$,
2. $\mathcal{M}_1^s(\Sigma_{0,4})$ is non-empty if and only if $s$ contains at most one $-1$,
3. $\mathcal{M}_{-1}^s(\Sigma_{0,4})$ is non-empty if and only if $s$ contains at most one $1$, and
4. all the non-empty spaces above are connected.

The mapping class group $\text{Mod}(\Sigma_{g,n})$ of $\Sigma_{g,n}$ is the group of relative isotopy classes of orientation preserving self-diffeomorphisms of $\Sigma_{g,n}$ that fix the punctures, which acts on $\mathcal{M}(\Sigma_{g,n})$ preserving the relative Euler classes and the signs of the punctures. Therefore, for any integer $k$ with $|k| \leq 2g - 2 + n$ and for any $s \in \{ \pm 1 \}^s$, $\text{Mod}(\Sigma_{g,n})$ acts on $\mathcal{M}_k^s(\Sigma_{g,n})$. In the spirit of Goldman’s conjecture [22], it is natural to conjecture that this action is ergodic on each non-Teichmüller space. The following Theorem 4 confirms this conjecture for the four-puncture sphere.

Theorem 4 (Y.) The $\text{Mod}(\Sigma_{0,4})$-action on each non-Teichmüller connected component of $\mathcal{M}(\Sigma_{0,4})$ is ergodic.

Theorem 1 and Theorem 4 together indicate that for punctured surfaces, a positive solution of Bowditch’s conjecture does not necessarily imply an affirmative answer to Bowditch’s question, which is in contrast to the case of closed surfaces.

2.2 Decorated Teichmüller space and skein algebras. The Kauffman bracket skein module $K(M)$ of a 3-manifold $M$ is defined by Przytycki [42] and Turaev [57] as an invariant of isotopy classes of framed links in $M$ satisfying the Kauffman bracket skein relation. In the joint paper with Julien Roger [52], we define the skein manifold $\mathcal{S}$ by modulo the corresponding four relations. In [52], a bi-linear map $\{ \} : \mathcal{A}_0(\Sigma) \times \mathcal{A}_0(\Sigma) \to \mathcal{A}_0(\Sigma)$ is defined by $\{ v, \alpha \} = 0$ for a puncture $v$ and a generalized curves $\alpha$ and

$$\{ \alpha, \beta \} = \frac{1}{2} \sum_{p \in \alpha \cap \beta \cap \Sigma} (\alpha_p \beta^+ - \alpha_p \beta^-) + \frac{1}{4} \sum_{v \in \alpha \cap \beta \cap \Sigma} \frac{1}{v} (\alpha_v \beta^+ - \alpha_v \beta^-)$$
for two generalized curve $\alpha$ and $\beta$, where $\alpha \pm \beta$ respectively are the positive and negative resolution of $\alpha$ and $\beta$ at $x \in \Sigma \cup V$. The main results of my joint paper with Roger [52] can be summarized as the following Theorems 5 and 6.

**Theorem 5** (Roger-Y.) Let $\Sigma$ be a punctured surface, then $\text{AS}_h(\Sigma)$ is a quantization of $\text{AS}_0(\Sigma)$, i.e.,

(a) $(\text{AS}_h(\Sigma), \cdot )$ is a well defined topologically free associative $\mathbb{C}[[h]]$-algebra,

(b) $(\text{AS}_0(\Sigma), \cdot , \{ , \})$ is a well defined Poisson algebra, and

(c) there exists a $\mathbb{C}$-algebra isomorphism $\Theta : \text{AS}_h(\Sigma)/h\text{AS}_h(\Sigma) \rightarrow \text{AS}_0(\Sigma)$ such that $\{ \alpha, \beta \} = \Theta (\bar{\alpha} \bar{\beta} - \bar{\beta} \bar{\alpha})$ for any $\bar{\alpha} \in \Theta^{-1}(\alpha)$ and $\bar{\beta} \in \Theta^{-1}(\beta)$.

The skein algebra $\text{AS}_h(\Sigma)$ is closely related to the hyperbolic geometry of $\Sigma$ in the following way. For a generalized curve $\alpha$ on $\Sigma$, we defined its curling number $c(\alpha)$ as the number of curls that $\alpha$ contains. For a decorated hyperbolic metric on $\Sigma$ with $r(v)$ the length of the horocycle centered at a puncture $v$, we let $\lambda(\alpha) = 2 \cosh \frac{l(\alpha)}{2}$ if $\alpha$ is a closed curve and $l(\alpha)$ is the length of the unique closed geodesic in the homotopy class of $\alpha$, and let $\lambda(\alpha) = e^{\frac{l(\alpha)}{2}}$ if $\alpha$ is an arc and $l(\alpha)$ is the general length of the unique geodesic arc in the homotopy class of $\alpha$. There is a Poisson bi-vector field $\Omega_{WP}$ on the decorated Teichmüller space $\mathcal{T}^d(\Sigma)$ described by Mondello [41] that extends the Weil-Petersson Poisson structure on the Teichmüller space. We have

**Theorem 6** (Roger-Y.) The map $\Phi : \text{AS}_0(\Sigma) \rightarrow C^\infty(\mathcal{T}^d(\Sigma))$ defined by $\Phi(v) = r(v)$ and $\Phi(l) = (-1)^{c(l)}\lambda(l)$ is a well defined homomorphism of Poisson algebras with respect to $\{ , \}$ on $\text{AS}_0(\Sigma)$ and $\Omega_{WP}$ on $\mathcal{T}^d(\Sigma)$.

As a consequence of Theorem 6, we have the following generalization of Wolpert’s cosine formula [62].

**Corollary 7** (Roger-Y.) For two generalized curves $\alpha$ and $\beta$ on $\Sigma$, we have

$$\Omega_{WP}(l(\alpha), l(\beta)) = \frac{1}{2} \sum_{p \in \alpha \cap \beta \cap S} \cos \theta_p + \frac{1}{4} \sum_{v \in \alpha \cap \beta \cap V} \frac{\theta_\alpha' - \theta_\beta'}{r(v)} ,$$

where $\theta_p$ is the angle from $\alpha$ to $\beta$ at $p$, $\theta_v$ is the length of the horocycle segment from $\alpha$ to $\beta$ and $\theta_\alpha'$ is that from $\beta$ to $\alpha$.

Suppose the decorated hyperbolic surface $\Sigma$ is ideally triangulated, and $E$ is the set of edges. As another consequence of Theorem 2, we have

**Corollary 8** (Roger-Y.) The $\lambda$-length $\lambda(\alpha)$ of any generalized curve $\alpha$ on $\Sigma$ is an explicit Laurent polynomial in $\{ \lambda(e) | e \in E \}$.

2.3 A deformation of Penner’s simplicial coordinates. The decorated Teichmüller space of a punctured surface was introduced by Penner in [44] as a fiber bundle over the Teichmüller space of hyperbolic metrics with cusp ends. To give a cell decomposition of this space, Penner defined the simplicial coordinates $\Psi$ in which the cells can be easily described. The main result of my paper [66] defines a one-parameter family of coordinates $\Psi_h$ of the decorated Teichmüller space that deforms Penner’s simplicial coordinates $\Psi$. As an application, Bowditch-Epstein [7] and Penner’s [44] cell decomposition of the decorated Teichmüller space is reproduced using the $\Psi_h$ coordinates.

To be precise, let $(S, T)$ be an ideally triangulated punctured surface with $V$ the set of ideal vertices (or cusps) of $S$. Let $T_c(S)$ be the Teichmüller space of hyperbolic metrics with cusp ends on $S$. Penner [44] defined a decorated hyperbolic metric $(d, r) \in T_c(S) \times \mathbb{R}_{>0}$ on $S$ to be the equivalence class of a hyperbolic metric $d$ in $T_c(S)$ such that each cusp $v$ is associated with a horodisk $B_v$ centered at $v$ so that the length of $\partial B_v$ is $r_v$. The space of decorated hyperbolic metrics $T_c(S) \times \mathbb{R}_{>0}$ is the decorated Teichmüller space. Let $e$ be an edge of $T$. If $a$ and $a'$ are the generalized angles facing $e$, and $b, b', c$ and $c'$ are the generalized angles adjacent to $e$, then Penner’s simplicial coordinates $\Psi : T_c(S) \times \mathbb{R}_{>0} \rightarrow \mathbb{R}^E$ is defined by

$$\Psi(d, r)(e) = \frac{b + c + a}{2} + \frac{b' + c' + a'}{2} .$$

An edge path $(e_1, \ldots, e_n)$ in a triangulation $T$ is an alternating sequence of edges $e_i$ with $e_i \neq e_{i+1}$ for $i = 1, \ldots, n - 1$ and triangles $t_i$ so that adjacent triangles $t_{i-1}$ and $t_i$ share the same edge $e_i$ for any $i = 1, \ldots, n$. 

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An edge loop is an edge path with $t_n = t_0$. In [44], Penner proved that for any vector $z \in \mathbb{R}^E_{\geq 0}$ such that $\sum_{i=1}^{k} z(e_i) > 0$ for any edge loop $(e_1, \ldots, e_k)$, there exists a unique decorated complete hyperbolic metric $(d, r)$ on $S$ so that $\Psi(d, r) = z$. By using a variational principle on decorated ideal triangles, Guo and Luo [24] were able to prove that Penner’s simplicial coordinates $\Psi : T_r(S) \times \mathbb{R}^E_{\geq 0} \to \mathbb{R}^E$ is a smooth embedding with image the convex polytope

$$P(T) = \{ z \in \mathbb{R}^E \mid \sum_{i=1}^{k} z(e_i) > 0, \forall \text{ edge loop } (e_1, t_1, \ldots, e_k, t_k) \}.$$ 

To deform Penner’s simplicial coordinates, we define in [66] for each $h \in \mathbb{R}$ a map $\Psi_h : T_r(S) \times \mathbb{R}^E_{>0} \to \mathbb{R}^E$ by

$$\Psi_h(d, r)(e) = \int_0^{h+t} e^{h^2} dt + \int_0^{\psi(e)} e^{h^2} dt,$$

and the main results of [66] are summarized as

**Theorem 9** (Y.) Suppose that $(S, T)$ is an ideally triangulated punctured surface. Then for all $h \in \mathbb{R}$, the map $\Psi_h : T_r(S) \times \mathbb{R}^E_{>0} \to \mathbb{R}^E$ is a smooth embedding.

**Theorem 10** (Y.) For $h \in \mathbb{R}$ and an ideally triangulated punctured surface $(S, T)$, let $P_h(T)$ be the set of points $z \in \mathbb{R}^E$ such that

(a) $z(e) < 2 \int_0^{+\infty} e^{h^2} dt, \forall e \in E,$

(b) $\sum_{i=1}^{n} z(e_i) > -2 \int_0^{+\infty} e^{h^2} dt, \forall \text{ fundamental edge path } (e_1, \ldots, e_n),$

(c) $\sum_{i=1}^{n} z(e_i) > 0, \forall \text{ edge loop } (e_1, \ldots, e_n).$

Then we have $\Psi_h(T_r(S) \times \mathbb{R}^E_{>0}) = P_h(T)$. Furthermore, if $h > 0$, the image of $\Psi_h$ is the open convex polytope $P(T)$ independent of $h$; and if $h < 0$, then the image $P_h(T)$ is a bounded open convex polytope so that $P_h(T) \subset P_{h'}(T)$ if $h < h'$.

### 3. Research Summary II: 3-manifolds

#### 3.1 Volume conjectures for Reshetikhin-Turaev and Turaev-Viro invariants.

In a recent joint paper with Qingtao Chen [13], we extend the Turaev-Viro invariants [59], which are originally only defined for closed 3-manifolds, to 3-manifolds with boundary. The main focus of [13] is on the values at the root $q = e^{\frac{\pi \sqrt{-1}}{r}}$ instead of the usually considered root $q = e^{\frac{\pi}{2r}}$, and on its asymptotic behavior. We propose the following

**Conjecture 1** (Chen-Y.) Let $M$ be a hyperbolic 3-manifold (closed, cusped or with geodesic boundary) and let $Vol(M)$ be its hyperbolic volume. Then for $r$ running over all odd integers and for $q = e^{\frac{\pi \sqrt{-1}}{r}}$

$$\lim_{r \to +\infty} \frac{2\pi}{r-2} \log (TV_r(M; q)) = Vol(M).$$

This conjecture is supported by plenty of promising numerical calculations. For closed $M$, we also consider the Reshetikhin-Turaev invariants [49, 50, 51], and propose the following

**Conjecture 2** (Chen-Y.) Let $M$ be a closed oriented hyperbolic 3-manifold and let $RT_r(M; q)$ be its Reshetikhin-Turaev invariants. Then for $r$ running over all odd integers and $q = e^{\frac{\pi \sqrt{-1}}{r}}$, and with a suitable choice of the arguments,

$$\lim_{r \to +\infty} \frac{4\pi}{r-2} \log (RT_r(M; q)) \equiv Vol(M) + \sqrt{-1}CS(M) \pmod{\sqrt{-1}\pi^2\mathbb{Z}},$$

where $CS(M)$ is the usual Chern-Simons invariants of $M$ multiplied by $2\pi^2$.

Conjecture 2 is quite unexpected since it was previously widely believed that the only interesting values of the Reshetikhin-Turaev invariants are at the root $q = e^{\frac{\pi}{2r}}$. Comparing Conjecture 2 and Witten’s Asymptotic Expansion Conjecture (cf. [53]) which predicts that $RT_r(M; q)$ grows polynomially in $r$ at $q = e^{\frac{\pi}{2r}}$, one sees...
that values at a different root of unity have a rather different asymptotic behavior, and could potentially provide more topological and geometric information of the manifold.

Recently, joint with Detcherry and Kalfagianni, we find a relationship between the Turaev-Viro invariant of a link complement and values of the colored Jones polynomial of the link, using which we prove Conjecture 1 for certain cases.

**Theorem 11** (Detcherry-Kalfagianni-Y.) Let \( L \) be a link in \( S^3 \) with \( k \) components. Then
\[
TV_r(S^3 \setminus L, q) = \eta_r \sum_n |J_n(L, q)|^2,
\]
where \( \eta_r \) is a quantity depending only polynomially on \( r \), the sum is over \( n = (n_1, \ldots, n_k) \) such that \( 0 \leq n_i \leq r-2 \) for \( i = 1, \ldots, k \), and \( J_n(L, q) \) is the \( n \)-th colored Jones polynomial of \( L \).

**Theorem 12** (Detcherry-Kalfagianni-Y.) Conjecture 1 is true for the complement of the Figure-eight knot, the Whitehead link and Borromean rings.

Some work in progress is made for the case of Whitehead chains. We hope Theorem 11 can relate Conjectures 1 and 2 to the famous Volume Conjecture for the colored Jones polynomials.

### 3.2 Hyperbolic cone metrics and angle structures.
Let \( M \) be a compact 3-manifold with boundary consisting of surfaces of negative Euler characteristic and let \( T \) be an ideal triangulation of \( M \). A hyperbolic cone metric on \( (M, T) \) is a metric \( d \) on \( M \) so that the restriction of \( d \) on each tetrahedron is isometric to a hyperideal tetrahedron in the 3-dimensional hyperbolic space \( \mathbb{H}^3 \). The combinatorial curvature of a hyperbolic cone metric \( d \) assigns each edge \( 2\pi \) less than the sum of the dihedral angles at this edge. Since a hyperbolic tetrahedron is determined by its six edge lengths, we can identify a hyperbolic cone metric on \( M \) with the set of all hyperbolic cone metrics on \( (M, T) \), the combinatorial curvature map \( K : \mathcal{L}(M, T) \to \mathbb{R}^E \) assigns each edge \( 2\pi \) less than the sum of the dihedral angles at this edge. Since a hyperbolic tetrahedron is determined by its six edge lengths, we can identify a hyperbolic cone metric on \( (M, T) \) with its edge lengths function \( l : E \to \mathbb{R} \), where \( E \) is the set of edges in \( T \). Denoting by \( \mathcal{L}(M, T) \subset \mathbb{R}^E \) the set of all hyperbolic cone metrics on \( (M, T) \), the combinatorial curvature map \( K : \mathcal{L}(M, T) \to \mathbb{R}^E \) sends a hyperbolic cone metric \( l \) to its combinatorial curvature \( K(l) \in \mathbb{R}^E \). One of the main problems in geometry is to understand the curvature map \( K \). Earlier works include [13] by Hodgson and Kerckhoff and [61] by Weiss which proved a general infinitesimal rigidity theorem for the hyperbolic cone metrics of cone angles at most \( 2\pi \) and [34] by Luo which showed that the combinatorial curvature map \( K : \mathcal{L}(M, T) \to \mathbb{R}^E \) is a local diffeomorphism. The main result in the joint paper [37] with Feng Luo is the following rigidity theorem.

**Theorem 13** (Luo-Y.) For an ideally triangulated 3-manifold \( (M, T) \) with boundary consisting of surfaces of negative Euler characteristic, the combinatorial curvature map \( K : \mathcal{L}(M, T) \to \mathbb{R}^E \) is a smooth embedding. In particular, a hyperbolic cone metric on \( (M, T) \) is determined by its combinatorial curvature.

The key idea in the proof is based on the observation obtained in [34] that the combinatorial curvature map \( K \) is the gradient of the co-volume function \( W \) which is locally strictly convex on \( \mathcal{L}(M, T) \subset \mathbb{R}^E_0 \). By studying the degeneration of hyperideal tetrahedra, we show that the co-volume function \( W \) can be extended to a \( C^1 \)-smooth convex function in the convex set \( \mathbb{R}^E_0 \), called the extended co-volume function, which implies Theorem 1. Using the extended co-volume function as a tool, we reveal a relationship between hyperbolic cone metrics and semi-angle structures. Recall that an angle structure on \( (M, T) \) makes each individual tetrahedron in \( T \) a hyperideal tetrahedron in \( \mathbb{H}^3 \) so that the sum of dihedral angles around each edge is \( 2\pi \). Let \( \mathcal{A}(M, T) \) be the space of all angle structures of \( (M, T) \) considered as a subspace of \( (\mathbb{R}^6)^T \). By definition, \( \mathcal{A}(M, T) \) is a bounded open convex polytope. The volume function \( Vol : \mathcal{A}(M, T) \to \mathbb{R} \) which sends an angle structure to its volume is known to be smooth and strictly convex by the work of Schlenker [54]. A theorem of Rivin [48] shows that the volume function \( Vol \) can be extended continuously to the compact closure \( \overline{\mathcal{A}(M, T)} \) of \( \mathcal{A}(M, T) \) in \( (\mathbb{R}^6)^T \). We call an element in \( \overline{\mathcal{A}(M, T)} \), which assigns each corner a non-negative real number, a semi-angle structure.

**Theorem 14** (Luo-Y.) Let \( (M, T) \) be an ideally triangulated 3-manifold with boundary consisting of surfaces of negative Euler characteristic.

(a) If the extended dihedral angles \( a(l) \) of \( l \in \mathbb{R}^E_0 \) form a semi-angle structure, then \( a(l) \) achieves the maximum volume on \( \mathcal{A}(M, T) \).

(b) Conversely, if \( \mathcal{A}(M, T) \neq \emptyset \) and \( a \in \overline{\mathcal{A}(M, T)} \) achieves the maximum volume, then there exists an \( l \in \mathbb{R}^E_0 \) so that \( a(l) = a \).
It is known due to Kojima [26] that every compact hyperbolic 3-manifold with totally geodesic boundary admits a geometric ideal triangulation so that each tetrahedron is either hyperideal or flat hyperideal whose dihedral angles are 0 and \( \pi \). One consequence of Theorem 12 (a) is

**Theorem 15 (Luo-Y.)** Let \( M \) be a compact hyperbolic 3-manifold with totally geodesic boundary which admits a geometric ideal triangulation \( T \) so that each tetrahedron is either hyperideal or flat hyperideal. Then the maximum volume on \( \mathcal{A}(M, T) \) is equal to the hyperbolic volume of \( M \).

One of the key ingredients in the proof of Theorem 12 is that the Fenchel dual function of the extended co-volume function maximizes the volume on the space of semi-angle structures. Another key ingredient in the proof is the uniqueness of the maximum volume semi-angle structure

**Theorem 16 (Luo-Y.)** Let \((M, T)\) be an ideally triangulated 3-manifold with boundary consisting of surfaces of negative Euler characteristic. If \( \mathcal{A}(M, T) \neq \emptyset \), then there exists a unique \( a \in \mathcal{A}(M, T) \) that achieves the maximum volume.

Analogous results for hyperbolic 3-manifolds with cusp ends and with mixed boundary components were obtained in [38].

3.3 Hyperbolic gluing equation of closed 3-manifolds. In his Princeton notes [56], Thurston introduced a system of algebraic equations (called the hyperbolic gluing equations) to construct hyperbolic metrics on orientable 3-manifolds with torus cusps. He used solutions to the hyperbolic gluing equations to produce a complete hyperbolic metric on the figure-eight knot complement in the early stages of formulating his geometrization conjecture. On a closed, triangulated, oriented 3-manifold \( M \), the hyperbolic gluing equations can be defined in the same way: We assign to each edge of each oriented tetrahedron in the triangulation a shape parameter \( z \in \mathbb{C} \setminus \{0, 1\} \), such that

1. opposite edges of each tetrahedron have the same shape parameter;
2. the three shape parameters assigned to the three pairs of opposite edges in each tetrahedron are \( z, \frac{1}{1-z} \) and \( \frac{z-1}{z} \) subject to an orientation convention; and
3. for each edge \( e \) in \( M \), if \( z_1, ..., z_k \) are the shape parameters assigned to the tetrahedron edges identified with \( e \), then we have \( \prod_{i=1}^{k} z_i = 1 \).

The equations above are termed the hyperbolic gluing equations, and the set of all solutions is the parameter space \( \mathcal{P}(M) \). In the joint work with Luo and Tillmann [56], we observe that, just as in the case of hyperbolic manifolds with cusps, the hyperbolic structure on a closed hyperbolic 3-manifold can be recovered from any solution of the hyperbolic gluing equations with maximal volume. The proof makes a crucial use of Thurston’s spinning construction from [56].

**Theorem 17 (Luo-Tillmann-Y.)** Let \( M \) be a closed, oriented, triangulated, hyperbolic 3-manifold with the property that each edge in \( M \) is not a null-homotopic loop. Then there exists \( Z_\infty \in \mathcal{P}(M) \) such that \( Vol(Z_\infty) = Vol(M) \). Moreover, for any such \( Z_\infty \), the associated holonomy representation \( \rho_\infty : \pi_1(M) \to PSL(2; \mathbb{C}) \) is discrete and faithful, and \( M \) is isometric with \( \mathbb{H}^3/\rho_\infty(\pi_1(M)) \).

4. Further Directions

4.1 Asymptotic of TQFTs and the Volume Conjecture. A \((2 + 1)\)-topological quantum field theory (TQFT) assigns to every oriented closed surface a finite dimensional vector space and to every equivalence class of cobordisms between two surfaces a linear map between the two assigned vector spaces, which satisfies certain naturality properties. We call two cobordisms equivalent if they are homeomorphic relative to the boundary. In particular, since every closed 3-manifold can be considered as a cobordism between the empty surface and itself, a TQFT defines an invariant of closed 3-manifolds. The Reshetikhin-Turaev and the Turaev-Viro invariants are induced by the Reshetikhin-Turaev and Turaev-Viro TQFTs. The asymptotic behavior of the above TQFTs has been successfully studied by \([11] [39]\). In particular, in [39], Marché and Narimannejad showed that as \( r \) increases, the Reshetikhin-Turaev TQFT vector space at the root of unity \( q = e^{\frac{2\pi r}{r}} \) converges to the Hilbert space of the \( L^2 \) functions on the \( SU(2) \)-character variety. We hope the techniques in [39] could be used to study the asymptotic behavior of the Reshetikhin-Turaev and the Turaev-Viro TQFTs at the root of unity \( q = e^{\frac{2\pi \sqrt{r}}{r}} \) and to find its relationship with the \( PSL(2; \mathbb{C}) \)-character variety, and hence 3-dimensional hyperbolic geometry. Since one of the consequences
of the naturality properties of a TQFT is that the invariant of a mapping torus of a surface equals the trace of the linear map assigned to the corresponding mapping cylinder, we hope the study of the asymptotics of TQFTs will be a step of toward solving Conjecture 1 and 2, at least for the pseudo-Anosov mapping tori. Then for a general hyperbolic 3-manifold, we study the behavior of the invariants under finite covering, and recall Agol’s solution to the Virtually Fibered Conjecture that every hyperbolic 3-manifold is finitely covered by a pseudo-Anosov mapping torus.

4.2 Goldman’s Conjecture. In [21], Goldman proved that for a surface $\Sigma$, the action of the mapping class group on the $SU(2)$-character variety of $\Sigma$ is ergodic with respect the Goldman measure, and he proposed the following

**Conjecture 3** (Goldman) The action of any Johnson subgroup $J_n(\Sigma)$ of the mapping class group on the $SU(2)$-character variety of $\Sigma$ is ergodic.

We hope to solve this conjecture using techniques from the study of quantum objects. The link between the two seemingly different area is given by the isomorphisms between the Kauffman bracket skein module $K(\Sigma)$ at $A = -1$ and the ring of the regular functions on the $SU(2)$-character variety of $\Sigma$. In [9], Bullock, Frohman and Kania-Bartoszyńska defined an inner product on $K(\Sigma)$ called the Yang-Mills inner product, and in the case that $A = -1$, they showed that the completion $\widetilde{K}_1(\Sigma)$ of the Kauffman bracket skein module under the Yang-Mills inner product is isomorphic to the space of $L^2$-functions on the $SU(2)$-character variety of $\Sigma$, with respect to the Goldman measure. The relationship between the ergodicity of mapping class group action on the $SU(2)$-character variety and that on the skein module then comes from the fact that a group action is ergodic if and only if all the invariant $L^2$-functions are constant. Therefore, Goldman’s ergodicity result [21] can be rephrased in terms of skein modules as that the only mapping class group invariant element of $\widetilde{K}_1(\Sigma)$ is, up to a scalar, the empty link, and Conjecture 3 can be rephrased as the following

**Conjecture 4** The only $J_n(\Sigma)$-invariant element of $\widetilde{K}_1(\Sigma)$ is up to a scalar the empty link.

Recently, using the shadow world formula for the Yang-Mills inner product developed by Frohman and Kania-Bartoszyńska [15], we made some progress toward the solution to Conjecture 4.

On the other hand, Kashaev’s length coordinates has proved an useful tool to study the $PLS(2, \mathbb{R})$-character variety of the four-puncture sphere [65]. It is natural to attempt to solve the general case using this method. Recently, we also found some relationship between Costantino-Martelli’s construction [11] and mapping class group action on the $PSL(2, \mathbb{R})$-character varieties, which might shed light on Goldman’s conjecture on the ergodicity of this action on the non-Teichmüller components.

4.3 Casson’s Conjectures. Let $M$ be a hyperbolic 3-manifold with cusp ends or totally geodesic boundary and let $T$ be any ideal triangulation of $M$. If the space of angle structures $A(M, T) \neq \emptyset$, then $\max \{Vol(a) | a \in A(M, T) \} \leq Vol_{g^3}(M)$, where $Vol_{g^3}(M)$ is the hyperbolic volume of $M$.

It is proved by Casson-Lackenby [28] and Rivin [46] that when $T$ is a geometric triangulation, i.e., each tetrahedron of $T$ is ideal or hyperideal, then the conjecture is true with $\max Vol(a) = Vol_{g^3}(M)$. The work of Luo [35] and Theorem 13 proved that Casson’s conjecture holds if $T$ is a geometric triangulation with flat tetrahedra. Theorem 12 seems to shed new light on this conjecture in the general case. One related problem is the conjecture by Stoker on the rigidity of hyperbolic polyhedra, and I am interested in seeing if our method in [37] will shed new light on it.

4.4 Quantization of the decorated Teichmüller space. Combining Theorems 5 and 6, it is tempting to interpret $\mathcal{A}S_n(\Sigma)$ as a quantization of the decorated Teichmüller space. This would require restricting the range of the homomorphism $\Phi$ to a carefully chosen sub-algebra of $C^\infty(T^d(\Sigma))$ so that the map becomes surjective and possibly an isomorphism if the following conjecture holds.

**Conjecture** The homomorphism $\Phi : K_0(\Sigma) \rightarrow C^\infty(T^d(\Sigma))$ is injective.

**References**


