Exercise 17: Let \( \{f_n\} \) be a sequence of measurable functions on \([0, 1]\) with \( |f_n(x)| < \infty \) for a.e. \( x \). Show that there exists a sequence \( c_n \) of positive real numbers such that 

\[
\frac{f_n(x)}{c_n} \to 0 \quad \text{a.e. } x.
\]

Solution: First, notice that if \( f \) is any measurable function on \([0, 1]\) with \( |f(x)| < \infty \) for a.e. \( x \), then 

\[
\lim_{k \to \infty} m(\{x : |f(x)| > k\}) = 0.
\]

Indeed, if we let \( E_k = \{x : |f(x)| > k\} \), then \( E_{k+1} \subseteq E_k \), and the intersection of all \( E_k \) is the set of points where \( |f(x)| = \infty \). Since each \( E_k \) is a subset of \([0, 1]\) and thus has finite measure, it follows that \( \lim_{k \to \infty} m(E_k) \) is the measure of \( \cap E_k \), which by assumption has measure zero. Thus, for each \( n \), we can find a sufficiently large positive real \( c_n \) such that 

\[
m(\{x : |f_n(x)| > c_n/n\}) < 2^{-n}.
\]

Since \( \sum_{n=1}^{\infty} 2^{-n} = 1 < \infty \), by the Borel-Cantelli Lemma (Exercise 16) it follows that the set 

\[
\{x : |f_n(x)| > c_n/n \text{ for infinitely many } n\}
\]

has measure zero. Thus, for almost every \( x \), we have \( |f_n(x)| \leq c_n/n \) for all but finitely many \( n \), and for such \( x \) we have \( \limsup_{n \to \infty} |f_n(x)|/c_n \leq \limsup_{n \to \infty} \frac{1}{n} = 0 \), i.e. \( \lim_{n \to \infty} f_n(x)/c_n = 0 \). ■

Exercise 18: Prove the following assertion: Every measurable function is the limit a.e. of a sequence of continuous functions.

Solution: By Theorem 4.3, a measurable function \( f \) can be approximated by a sequence of step functions \( \{\psi_k\}_{k=1}^{\infty} \) which converges pointwise to \( f(x) \) for almost every \( x \). Recall that a step function is a finite linear combination of characteristic functions of (closed) rectangles. We now show that step functions are approximable by continuous functions. In particular, we show that if \( \psi \) is a step function, then for every \( \epsilon > 0 \) there exists a continuous function \( g \) such that \( m(\{x : g(x) \neq \psi(x)\}) < \epsilon \).

If \( \psi = \chi_R \) is just the characteristic function of a closed rectangle \( R \subseteq \mathbb{R}^n \), we can construct \( g \) as follows. Let \( \varphi : [0, \infty) \to \mathbb{R} \) be a continuous function such that \( \varphi(0) = 1 \) and \( \varphi(t) = 0 \) for \( t \geq 1 \) (e.g. \( \varphi(t) = (1 - t)\chi_{[0,1]}(t) \)), and for some \( r > 0 \), let \( g : \mathbb{R}^n \to \mathbb{R} \) be defined by \( g(x) = \varphi(r \text{dist}(x, R)) \). By construction, \( g \) is continuous (notice that distance functions are continuous), and \( g(x) = 1 \) if
exists a continuous \( g \), from any \( r \) we can pick \( g \) and \( k \) for some \( 0 < r < \infty \) the measure of this set goes to zero (why?), so for any \( \epsilon > 0 \) we can pick \( r \) sufficiently large such that \( g \), as defined above, is a continuous function differing from \( \chi_R \) only on a set of measure less than \( \epsilon \).

For \( \psi = \sum_{k=1}^{N} a_k \chi_{R_k} \), we can construct \( g \) as follows: from the above paragraph, for each \( k \) there exists a continuous \( g_k \) such that \( m(\{g_k \neq \chi_{R_k}\}) < \epsilon/N \). If we let \( g = \sum_{k=1}^{N} a_k g_k \), then \( g \) is continuous, and \( g(x) \neq \psi(x) \) is only possible if \( g_k(x) \neq \chi_{R_k}(x) \) for some \( k \), i.e. only if \( x \) belongs to \( \{g_k \neq \chi_{R_k}\} \) for some \( k \). Thus, 

\[
\{x : g(x) \neq \psi(x)\} \subset \bigcup_{k=1}^{N} \{x : g_k(x) \neq \chi_{R_k}(x)\}.
\]

The latter union has measure at most \( \epsilon \) since each set in the union has measure at most \( \epsilon/N \), so \( \{g \neq \psi\} \) has measure at most \( \epsilon \).

Thus, to obtain our desired sequence of continuous functions, let \( \epsilon_k \) be a sequence of positive real numbers such that \( \sum_{k=1}^{\infty} \epsilon_k < \infty \) (e.g. \( \epsilon_k = 2^{-k} \)), and if \( \{\psi_k\} \) is a sequence of step functions converging to \( f \) pointwise for a.e. \( x \), let \( \{g_k\} \) be a sequence of continuous functions such that \( m(\{g_k \neq \psi_k\}) < \epsilon_k \). Notice that if \( x \) is a point where \( \lim_{k \to \infty} \psi_k(x) = f(x) \), then the only way \( g_k(x) \) can fail to converge to \( f(x) \) is if \( g_k(x) \neq \psi_k(x) \) for infinitely many \( k \). Thus, if we let \( E = \{x : \lim_{k \to \infty} g_k(x) \neq f(x)\} \) (this includes points where the limit doesn’t exist), \( A = \{x : \lim_{k \to \infty} \psi_k(x) \neq f(x)\} \), and \( B = \{x : g_k(x) \neq \psi_k(x) \text{ for infinitely many } k\} \), then the above sentence says that if \( x \notin A \) and \( x \in E \), then \( x \in B \). In other words, \( E \subset A \cup B \). By assumption, we have \( m(A) = 0 \), and by the Borel-Cantelli Lemma, we have \( m(B) = 0 \), so \( E \) is a subset of a measure zero set, and hence is measurable and has measure zero. It follows that \( \lim_{k \to \infty} g_k(x) = f(x) \) outside \( E \), a measure zero set, i.e. \( \lim_{k \to \infty} g_k(x) = f(x) \) for a.e. \( x \).

**Exercise 23:** Suppose \( f(x,y) \) is a function on \( \mathbb{R}^2 \) that is separately continuous: for each fixed variable, \( f \) is continuous in the other variable. Prove that \( f \) is measurable on \( \mathbb{R}^2 \).

**Solution:** The goal is to find a sequence of continuous functions which converges to \( f \) pointwise, since the limit of a sequence of measurable function, if it exists, is measurable (see Property 4, Section 4.1). We define the sequence as follows: for every \( n \), let \( \{x_{n,k}\}_{k \in \mathbb{Z}} \) be a (bi)-sequence of real numbers such that \( k \mapsto x_{n,k} \) is a strictly increasing sequence and \( \lim_{k \to \pm \infty} x_{n,k} = \pm \infty \), and arrange the sequences so that

\[
\text{mesh}(\{x_{n,k}\}_k) := \sup_k |x_{n,k+1} - x_{n,k}| \xrightarrow{n \to \infty} 0.
\]

For example, take \( x_{n,k} = k/n \). Define the approximations \( f_n \) as follows: if \( x = x_{n,k} \) for some \( k \), then for all \( y \) we let \( f_n(x_{n,k},y) = f(x_{n,k},y) \). Otherwise, there exists a unique \( k \) such that \( x_{n,k} < x < x_{n,k+1} \), in which case let

\[
f_n(x,y) = \frac{x_{n,k+1} - x}{x_{n,k+1} - x_{n,k}} f(x_{n,k},y) + \frac{x - x_{n,k}}{x_{n,k+1} - x_{n,k}} f(x_{n,k+1},y).
\]
In other words, for fixed $y$, we define $f_n(x, y)$ on $x_{n,k} \leq x \leq x_{n,k+1}$ as the linear interpolation between $f(x_{n,k}, y)$ and $f(x_{n,k+1}, y)$. I claim that each of these $f_n$ are continuous, and that $f_n \to f$ at every point. To show continuity, let $\epsilon > 0$ and $(x_0, y_0) \in \mathbb{R}^2$, and assume first that $x_{n,k} < x_0 < x_{n,k+1}$ for some $k$. Since $y \to f(x_{n,k}, y)$ is continuous, there exists $\delta_y > 0$ such that $|y - y_0| < \delta_y \implies |f(x_{n,k}, y) - f(x_{n,k}, y_0)| < \epsilon/2$. Similarly, there exists a $\delta'_y > 0$ such that $|y - y_0| < \delta'_y \implies |f(x_{n,k+1}, y) - f(x_{n,k+1}, y_0)| < \epsilon/2$, and we can take $\delta_y = \delta'_y$ by making both small enough. By continuity, we also have that $f(x, n, \cdot)$ and $f(x, n+1, \cdot)$ are both bounded in $|y - y_0| < \delta_y$, so let $M > 0$ satisfy $|f(x, n, y)| < M$ for $j = k, k + 1$ and $|y - y_0| < \delta_y$. If we let

$$
\delta_x = \min(x - x_{n,k}, x_{n,k+1} - x, \frac{\epsilon}{4M} (x_{n,k+1} - x_{n,k})),
$$

then for $|x - x_0| < \delta_x$ and $|y - y_0| < \delta_y$, we have $x_{n,k} < x < x_{n,k+1}$, and

$$
|f_n(x, y) - f_n(x_0, y)| \leq \frac{|x - x_0|}{x_{n,k+1} - x_{n,k}} (|f(x_{n,k}, y)| + |f(x_{n,k+1}, y)|) < \frac{\epsilon/2 (x_{n,k+1} - x_{n,k})}{x_{n,k+1} - x_{n,k}} (M + M) = \frac{\epsilon}{2}.
$$

Since $|y - y_0| < \delta_y$, we have

$$
|f_n(x_0, y) - f_n(x_0, y_0)| = \left| \frac{x_{n,k+1} - x_0}{x_{n,k+1} - x_{n,k}} (f(x_{n,k}, y) - f(x_{n,k}, y_0)) + \frac{x_0 - x_{n,k}}{x_{n,k+1} - x_{n,k}} (f(x_{n,k+1}, y) - f(x_{n,k+1}, y_0)) \right|
$$

$$
\leq \frac{x_{n,k+1} - x_0}{x_{n,k+1} - x_{n,k}} |f(x_{n,k}, y) - f(x_{n,k}, y_0)| + \frac{x_0 - x_{n,k}}{x_{n,k+1} - x_{n,k}} |f(x_{n,k+1}, y) - f(x_{n,k+1}, y_0)|
$$

$$
< \frac{x_{n,k+1} - x_0}{x_{n,k+1} - x_{n,k}} \frac{\epsilon}{2} + \frac{x_0 - x_{n,k}}{x_{n,k+1} - x_{n,k}} \frac{\epsilon}{2} = \frac{\epsilon}{2}
$$

and hence

$$
|f_n(x, y) - f_n(x_0, y_0)| \leq |f_n(x, y) - f_n(x_0, y)| + |f_n(x_0, y) - f_n(x_0, y_0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
$$

Thus, $|x - x_0| < \delta_x$, $|y - y_0| < \delta_y$ implies $|f_n(x, y) - f_n(x_0, y_0)| < \epsilon$, so $f_n$ is continuous at $(x_0, y_0)$. If $x_0 = x_{n,k}$ for some $k$, the proof is similar, except we would need to consider the estimates of $f$ at both $x_{n,k-1}$ and $x_{n,k+1}$ as well. Hence, each $f_n$ is continuous.

To show that $f_n \to f$ everywhere, notice that if $x_{n,k} \leq x \leq x_{n,k+1}$, then

$$
f_n(x, y) - f(x, y) = \frac{x_{n,k+1} - x}{x_{n,k+1} - x_{n,k}} (f(x_{n,k}, y) - f(x, y)) + \frac{x - x_{n,k}}{x_{n,k+1} - x_{n,k}} (f(x_{n,k+1}, y) - f(x, y)).
$$

Thus, if we let $(x_0, y_0) \in \mathbb{R}^2$, with $x_{n,k} \leq x_0 \leq x_{n,k+1}$, then the fact that $x \to f(x, y_0)$ is continuous implies that for every $\epsilon > 0$ there exists $\delta > 0$ such that $|x - x_0| < \delta \implies |f(x, y_0) - f(x_0, y_0)| < \epsilon$. 

3
Since $\text{mesh}(\{x_{n,k}\}_k) \to 0$, for large enough $n$ (say $n > N$) we have $\text{mesh}(\{x_{n,k}\}_k) < \delta$, so that in particular $|x_{n,j} - x_0| < \delta$ and hence $|f(x_{n,j}, y_0) - f(x, y_0)| < \delta$ for $j = k, k+1$. It follows then that for $n > N$, we have

$$|f_n(x_0, y_0) - f(x, y_0)| \leq \frac{x_{n,k+1} - x_0}{x_{n,k+1} - x_{n,k}} |f(x_{n,k}, y_0) - f(x, y_0)| + \frac{x_0 - x_{n,k}}{x_{n,k+1} - x_{n,k}} |f(x_{n,k+1}, y_0) - f(x, y_0)|$$

$$\leq \frac{x_{n,k+1} - x_0}{x_{n,k+1} - x_{n,k}} \epsilon + \frac{x_0 - x_{n,k}}{x_{n,k+1} - x_{n,k}} \epsilon = \epsilon.$$

This implies that $\lim_{n \to \infty} f_n(x_0, y_0) = f(x_0, y_0)$, as desired. ■