Math 172 HW 5 Solutions

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Exercise 2: In analogy to Proposition 2.5, prove that if \( f \) is integrable on \( \mathbb{R}^d \) and \( \delta > 0 \), then \( f(\delta x) \) converges to \( f(x) \) in the \( L^1 \)-norm as \( \delta \to 1 \).

Solution: Let \( g: \mathbb{R}^d \to \mathbb{R} \) be continuous and compactly supported with \( \|f - g\|_{L^1} < \epsilon \). Then

\[
\int_{\mathbb{R}^d} |f(x) - f(\delta x)| \, dx \leq \int_{\mathbb{R}^d} |f(x) - g(x)| \, dx + \int_{\mathbb{R}^d} |g(x) - g(\delta x)| \, dx + \int_{\mathbb{R}^d} |g(\delta x) - f(\delta x)| \, dx.
\]

Since \( \int_{\mathbb{R}^d} |g(\delta x) - f(\delta x)| \, dx = \delta^{-d} \int_{\mathbb{R}^d} |g(x) - f(x)| \, dx < \delta^{-d} \epsilon \), it follows that

\[
\int_{\mathbb{R}^d} |f(x) - f(\delta x)| \, dx < \int_{\mathbb{R}^d} |g(x) - g(\delta x)| \, dx + \epsilon (1 + \delta^{-d}).
\]

Since \( g \) is continuous and compactly supported, we clearly have \( \int_{\mathbb{R}^d} |g(x) - g(\delta x)| \, dx \to 0 \) as \( \delta \to 1 \), and hence

\[
\limsup_{\delta \to 1} \int_{\mathbb{R}^d} |f(x) - f(\delta x)| \, dx \leq 2 \epsilon.
\]

Since \( \epsilon \) was arbitrary, it follows that \( \limsup_{\delta \to 1} \int_{\mathbb{R}^d} |f(x) - f(\delta x)| \, dx = 0 \), i.e. \( \int_{\mathbb{R}^d} |f(x) - f(\delta x)| \, dx \to 0 \) as \( \delta \to 1 \), i.e. \( f(\delta x) \) converges to \( f(x) \) in \( L^1 \) as \( \delta \to 1 \).

Exercise 3: Suppose \( f \) is integrable on \( (-\pi, \pi] \) and extended to \( \mathbb{R} \) by making it periodic of period \( 2\pi \). Show that

\[
\int_{-\pi}^{\pi} f(x) \, dx = \int_I f(x) \, dx,
\]

where \( I \) is any interval in \( \mathbb{R} \) of length \( 2\pi \).

Solution: Notice that for any \( a, b, c \in \mathbb{R} \) we have

\[
\int_a^b f(x) \, dx = \int_{\mathbb{R}} f(x) \chi_{[a,b]}(x) \, dx = \int_{\mathbb{R}} f(x-c) \chi_{[a,b]}(x-c) \, dx = \int_{\mathbb{R}} f(x-c) \chi_{[a+c,b+c]}(x) \, dx = \int_{a+c}^{b+c} f(x-c) \, dx.
\]
Note that the second equality follows from translation invariance. If $I = [a, b]$ is an interval of length $2\pi$, so that $b - a = 2\pi$, then there exists $k \in \mathbb{Z}$ such that $a \leq (2k + 1)\pi \leq b$ (namely, take $k = \lfloor \frac{a - \pi}{2} \rfloor$). We then have
\[
\int_I f(x) \, dx = \int_a^b f(x) \, dx = \int_a^{b-2\pi} f(x) \, dx + \int_{b-2\pi}^{b-4\pi} f(x) \, dx + \cdots + \int_{(2k-1)\pi}^{(2k+1)\pi} f(x) \, dx.
\]

Exercise 12: Show that there are $f \in L^1(\mathbb{R}^d)$ and a sequence $\{f_n\}$ with $f_n \in L^1(\mathbb{R}^d)$ such that
\[
\|f - f_n\|_{L^1} \to 0,
\]
but $f_n(x) \to f(x)$ for no $x$.

Solution: The idea is the following: Let $f_n = \chi_{I_n}$, where $\{I_n\}$ is a sequence of cubes in $\mathbb{R}^d$ such that:

1. $m(I_n) \to 0$
2. Each $x \in \mathbb{R}^d$ belongs to infinitely many $I_n$.

The first condition implies that
\[
\|0 - f_n\|_{L^1} = m(I_n) \to 0
\]
so $f_n$ converges to 0 in $L^1$, while the second condition implies that $f_n(x) = 1$ for infinitely many $n$ for every $x$, so that $f_n(x)$ does not converge to $f(x) = 0$ for any $x$.

So it suffices to construct such a sequence $\{I_n\}$. One possible construction is as follows: for each $n$, let $\{I_{n,k}\}$ be a collection of (closed) cubes of side length $\frac{1}{n}$ covering the cube $[-n,n]^d$, and take the collection to be finite (indeed, $(2n)^d$ cubes are sufficient). Then for each $k$ we have
\[
m(I_{n,k}) = \frac{1}{n^d},
\]
and for every $x$ we have that if $n$ is sufficiently large, then there is some $k$ depending on $n$ such that $x \in I_{n,k}$. The collection $\bigcup_{n,k} \{I_{n,k}\}$ is thus a countable collection of cubes such that every $x \in \mathbb{R}^d$ belongs to infinitely many cubes in the collection, with the property that for every $\epsilon > 0$, only finitely many cubes have volume greater than $\epsilon$ (indeed, if $\frac{1}{N\pi} < \epsilon$, then there are at most $\sum_{n=1}^{N-1} (2n)^d$ cubes of volume greater than $\epsilon$). Taking any enumeration of this countable collection yields the desired sequence of cubes. 

■
Exercise 22: Prove that if \( f \in L^1(\mathbb{R}^d) \) and

\[
\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i x \cdot \xi} \, dx,
\]
then \( \hat{f}(\xi) \to 0 \) as \( |\xi| \to \infty \). (This is the Riemann-Lebesque lemma.)

**Solution:** Per the hint, notice that for a fixed nonzero \( \xi \in \mathbb{R}^d \), if \( \xi' = \frac{\xi}{|\xi|} \), then

\[
e^{-2\pi i \xi} = e^{-2\pi i(x-\xi') \cdot \xi - 2\pi i \xi' \cdot \xi} = e^{-2\pi i(x-\xi') \cdot \xi} e^{2\pi i \frac{\xi}{|\xi|} \xi'} = e^{-2\pi i(x-\xi') \cdot \xi} e^{-\pi i} = e^{-2\pi i(x-\xi') \xi}.
\]

It follows that

\[
\int_{\mathbb{R}} f(x-\xi')e^{-2\pi i x \cdot \xi} \, dx = -\int_{\mathbb{R}} f(x-\xi')e^{-2\pi i (x-\xi') \cdot \xi} \, dx = -\int_{\mathbb{R}} f(x)e^{-2\pi i x \cdot \xi} \, dx
\]

where in the last equality we used translation invariance, and hence

\[
\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i x \cdot \xi} \, dx
\]

\[
= \frac{1}{2} \left( \int_{\mathbb{R}^d} f(x)e^{-2\pi i x \cdot \xi} \, dx - \left( -\int_{\mathbb{R}^d} f(x)e^{-2\pi i x \cdot \xi} \, dx \right) \right)
\]

\[
= \frac{1}{2} \left( \int_{\mathbb{R}^d} f(x)e^{-2\pi i x \cdot \xi} \, dx - \int_{\mathbb{R}^d} f(x-\xi')e^{-2\pi i x \cdot \xi} \, dx \right) = \frac{1}{2} \int_{\mathbb{R}^d} (f(x) - f(x - \xi'))e^{-2\pi i x \cdot \xi} \, dx.
\]

As \( |\xi| \to \infty \), we have \( \xi' \to 0 \), and by Proposition 2.5 the integral on the right-hand side above goes to zero as \( \xi' \to 0 \). Hence, \( \hat{f}(\xi) \to 0 \) as \( |\xi| \to \infty \).

Exercise 24: Consider the convolution

\[
(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y) \, dy.
\]

(a) Show that \( f * g \) is uniformly continuous when \( f \) is integrable and \( g \) bounded.

**Solution:** Let \( C \) satisfy \( \sup_{y \in \mathbb{R}^d} |g(y)| < C \). For \( x, x' \in \mathbb{R}^d \), we have

\[
|f * g)(x) - (f * g)(x')| = \int_{\mathbb{R}^d} |f(x-y) - f(x'-y))g(y) \, dy| \leq C \int_{\mathbb{R}^d} |f(x-y) - f(x'-y)| \, dy.
\]

Note that

\[
\int_{\mathbb{R}^d} |f(x-y) - f(x'-y)| \, dy = \int_{\mathbb{R}^d} |f(-y) - f(x'-x - y)| \, dy = \int_{\mathbb{R}^d} |f(y) - f(x'-x + y)| \, dy.
\]
By Proposition 2.5, the above integral goes to zero as $|x' - x| \to 0$, since $f$ is integrable. In particular, for any $\epsilon > 0$, there exists $\delta > 0$ such that $|x' - x| < \delta$ implies the above integral is at most $\epsilon/C$, in which case

$$|x' - x| < \delta \implies |(f * g)(x) - (f * g)(x')| \leq C \frac{\epsilon}{C} = \epsilon.$$

This shows that $f * g$ is uniformly continuous. \hfill \blacksquare

(b) If in addition $g$ is integrable, prove that $(f * g)(x) \to 0$ as $|x| \to \infty$.

**Solution:** By Exercise 2.6(b) (on last week’s homework), it is enough to show that $f * g$ is integrable, since part (a) of this exercise establishes that $f * g$ is uniformly continuous\(^1\). Since

$$\int_{\mathbb{R}^d} |(f * g)(x)| \, dx = \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} f(x - y)g(y) \, dy \right| \, dx$$

$$\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x - y)||g(y)| \, dx \, dy$$

$$= \int_{\mathbb{R}^d} |g(y)| \int_{\mathbb{R}^d} |f(x - y)| \, dx \, dy$$

$$= \int_{\mathbb{R}^d} |g(y)| \|f\|_{L^1} \, dy = \|f\|_{L^1} \|g\|_{L^1} < \infty$$

it follows that $f * g$ is integrable, as desired. \hfill \blacksquare

\(^{1}\)The result in last week’s homework applied to functions on $\mathbb{R}$; however similar reasoning works here.