They are due Tuesday April 18th in class.

**Problem 1: Independence**

Let \((X_1, X_2)\) be multivariate normal distribution \(N(\mu, \Sigma)\) where \(\mu = (\mu_1, \mu_2)\) and \(\Sigma\) is a 2 \(\times\) 2 positive definite matrix. Assume that \(X_1\) and \(X_2\) are uncorrelated, i.e.,

\[
\Sigma_{12} = \mathbb{E}[(X_1 - \mu_1)(X_2 - \mu_2)] = 0
\]

Show that \(X_1\) and \(X_2\) are independent, i.e.,

\[
\mathbb{P}(X_1 \leq x_1, X_2 \leq x_2) = \mathbb{P}(X_1 \leq x_1) \mathbb{P}(X_2 \leq x_2)
\]

Now let \(Y \sim U([0, 1])\) be uniformly distributed in the interval \([0, 1]\). Consider the random variables,

\[
Y_1 = \sin(2\pi Y) \\
Y_2 = \cos(2\pi Y)
\]

Show that \(Y_1\) and \(Y_2\) are uncorrelated. Are they independent?

**Problem 2: Linear transformation of Gaussian**

Let \((X_1, \ldots, X_n)\) be a multivariate normal distribution \(N(\mu, \Sigma)\), where \(\mu = (\mu_1, \ldots, \mu_n)\) and \(\Sigma\) is a \(n \times n\) positive definite covariance matrix. For any invertible matrix \(A \in \mathbb{R}^{n \times n}\) and vector \(b \in \mathbb{R}^n\), define

\[
Y = AX + b
\]

(1) Find the distribution of \(Y\).

(2) Suppose you can sample from standard normal distribution \(N(0, 1)\), how do you generate samples of random vectors from \(N(\mu, \Sigma)\)?

**Problem 3: Empirical variance**

Let \(X_1, X_2, \ldots, X_n\) be independent identically distributed random variables with mean \(\mu\) and variance \(\sigma^2\). Assume they also have finite fourth moment. Let \(\bar{X}_n = (X_1 + X_2 + \cdots + X_n)/n\) be the sample mean and

\[
\sigma_n^2 = \frac{1}{n-1} \sum_{j=1}^{n} (X_j - \bar{X}_n)^2
\]

be the sample variance.

(1) Compute \(\mathbb{E}(\sigma_n^2)\).

(2) the LLN tells \(\bar{X}_n \to \mu\) as \(n \to \infty\), what is the limit of \(\sigma_n^2\)? Try to write a formal proof.

**Problem 4: Poisson approximation**

Let \(X_1^{(n)}, X_2^{(n)}, \ldots, X_n^{(n)}\) be independent Bernoulli random variables with \(P(X_j^{(n)} = 1) = p_n\) and \(P(X_j^{(n)} = 0) = 1 - p_n\). That is, for each \(n\) they are independent and identically distributed but the probability of “success” \(p_n\) depends on \(n\). Let \(Z_n = X_1^{(n)} + X_2^{(n)} + \cdots + X_n^{(n)}\). Suppose that \(p_n = \lambda/n\) with \(\lambda\) a positive number. Show that \(Z_n\) converges in distribution to a Poisson random variable \(Z\), where

\[
\mathbb{P}(Z = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots
\]

(Hint: compute the characteristic function of \(Z_n\) using the fact that \(\mathbb{E}[e^{i\alpha X + Y}] = \mathbb{E}[e^{i\alpha X}] \mathbb{E}[e^{i\alpha Y}]\) if \(X, Y\) are independent, and show that the characteristic function of \(Z_n\) converges to that of \(Z\).)
Problem 5: Estimating volatility

The Brownian motion process $B_t, 0 \leq t \leq T$, is a family of random variables with the following property:

Let $\{t_i\}_{i=0}^N$ be any partition of the interval $[0, T]$ with $t_0 = 0, t_N = T$ and $t_i - t_{i-1} = T/N = \Delta_N$. Then $\{B_{t_i} - B_{t_{i-1}}\}_{i=1}^N$ are i.i.d. normal random variables with mean 0 and variance $\Delta_N$.

Consider the process $X_t = \sigma B_t$ with $\sigma > 0$ representing the volatility. We observe $\{X_{t_i}\}_{i=0}^N$ and want to estimate the volatility $\sigma$. Define $\Delta_i X = X_{t_i} - X_{t_{i-1}}$. Show that

$$\frac{1}{\sqrt{\Delta_N}} \left( \sum_{i=1}^N |\Delta_i X|^2 - T\sigma^2 \right) \overset{\text{weakly}}{\Rightarrow} \text{Normal}(0, 2T\sigma^4)$$

as $N \to \infty$, $\Delta_N \to 0$, $N\Delta_N = T$. This also means $\frac{1}{T} \sum_{i=1}^N |\Delta_i X|^2$ is a consistent estimate of $\sigma^2$. (Hint: you may want to directly apply CLT and also compute the value of $E[Y^4]$ when $Y \sim \text{Normal}(0, 1)$)

Problem 6: Monte Carlo integration

Let $f(x) = e^{-x^3}$.

(1) Compute $\int_0^1 f(x)dx$ using Monte Carlo. Compare with the true value (you may need to use some numerical integration package or just simply ask WolframAlpha).

(2) Do the simulation many times, and plot the histogram of the errors. Rescale your error in a way so that your new histogram looks like a bell-shape curve $\frac{1}{\sqrt{2\pi}} e^{-x^2}$ (you may need WolframAlpha again to compute some variance).

(3) Construct $\alpha-$confidence intervals for your Monte Carlo result for $\alpha = 50\%, 75\%, 95\%$. How do you check (again by simulation) that the interval you constructed is indeed the $\alpha-$confidence interval? Is your way of checking becoming harder as $\alpha$ increases?