Problem 1: Absorption probabilities

Given $p \in (0, 1)$, we consider a two-state Markov chain with transition matrix

$$
P = \begin{pmatrix} 1 - p & p \\ 0 & 1 \end{pmatrix}
$$

Suppose the chain starts at state 1.

1. What is the probability of hitting state 2 for the first time at time $n$?
2. What is the probability of absorption at state 2? What is the mean time of hitting state 2?

Problem 2: Hitting probabilities

Consider a nearest-neighbor symmetric random walk on the integers 0, 1, 2, 3 with absorption at 0, 3, i.e., once the walker reaches 0 or 3, it stays there. Assuming the chain starts at 2.

1. What is the probability of absorption at 3?
2. What is the mean hitting time of 0?

Problem 3: Simple random walk on $\mathbb{Z}$

Consider a nearest-neighbor random walk on integers, and the probability of moving right is $p \in (0, 1)$. Let $h_i$ be the probability of hitting 0 starting from $i$, and $f_0$ be the probability of returning to 0 after starting from 0.

1. Find the relation between $f_0, h_1, h_{-1}$.
2. Assuming $X_0 = 0$, for any positive integer $n$, compute the probability $\mathbb{P}(X_n = 0)$. Using Stirling’s approximation $n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$ to estimate $\sum_{n=1}^{\infty} \mathbb{P}(X_n = 0)$ and find the condition for the state 0 to be recurrent.

Problem 4: Invariant probability

We have two-state Markov chain which jumps from 1 to 2 with probability $\alpha$ and jumps from 2 to 1 with probability $\beta$. $\alpha, \beta \in (0, 1)$.

1. Find the transition matrix $P$ and the invariant probability by the relation $\pi P = \pi$.
2. Find the $n$-step transition matrix $P^n$ and send $n \to \infty$ to find the invariant probability $\pi$.

Problem 5: MLE for Markov chain

Let $X_n, n = 0, 1, 2, \ldots$, be a two-state Markov chain with transition probabilities

$$
\begin{pmatrix} p & 1 - p \\ 1 - p & p \end{pmatrix}
$$

with $0 < p < 1$. The goal is to estimate the parameter $p \in (0, 1)$ based on the observation. Assuming $X_0$ is at state 1.

1. Find the maximum likelihood estimator $\hat{p}_n = \hat{p}_n(X_1, X_2, \ldots, X_n)$.
2. Show that $\hat{p}_n \to p$ in probability, and construct the 95% confidence interval of $p$. 

Problem 6: Discrete time Ornstein-Uhlenbeck process

Let $X_n, n = 0, 1, 2, \ldots$, be a random walk on the real line defined by

$$X_{n+1} = (1 - \mu)X_n + Z_{n+1}, \quad n = 0, 1, 2, \ldots,$$

with $X_0 = 0$. The random variables $\{Z_n\}_{n \geq 1}$ are i.i.d. Gaussian $N(0, 1)$. This is a discrete time, continuous space (the state space is $\mathbb{R}$) random walk that is mean reverting to zero. The parameter $\mu$ to be estimated is assumed to be in the range $0 < \mu < 1$.

1. Find the transition probability density $P[X_{n+1} = y \mid X_n = x]$.
2. Calculate $E(X_n)$ and $E(X_n^2)$ for all $n \geq 1$. Find the invariant probability for the chain.
3. Find the maximum likelihood estimator $\hat{\mu}_n = \hat{\mu}_n(X_1, X_2, \ldots, X_n)$ for $\mu$.

Problem 7: Coding (extra credit)

Use the basis function from HW 2 problem 5 to generate Poisson random variable $Y(t_i)$ for $t_i, i = 1, \ldots, N$. Let $Y = (Y(t_1), \ldots, Y(t_N))$ and $\theta = (\theta_1, \ldots, \theta_{d-1})$. The goal is to view $\theta$ as a random variable in the Bayesian framework and to sample from the posterior distribution $P(\theta \mid Y)$. We assume the prior distribution of $\theta$ is $N(0, I_{d-1})$, i.e., the multidimensional normal distribution with zero mean and identity covariance matrix. Implement the following algorithm to obtain the Markov chain $\{\alpha_n\}_{n \geq 1}$ which should converge to $P(\theta \mid Y)$. Plot your samples with different starting points and discuss how your results depend on different $N, d, \sigma$:

(i) for each $\alpha_n$, sample from $\beta_n \sim N(0, \sigma I_{d-1})$ for some appropriately chosen $\sigma > 0$, let $\tilde{\alpha}_{n+1} = \alpha_n + \beta_n$;

(ii) if $P(\tilde{\alpha}_{n+1} \mid Y) \geq P(\alpha_n \mid Y)$, accept the sample and let $\alpha_{n+1} = \tilde{\alpha}_{n+1}$; if $P(\tilde{\alpha}_{n+1} \mid Y) < P(\alpha_n \mid Y)$, accept the sample with probability $\frac{P(\tilde{\alpha}_{n+1} \mid Y)}{P(\alpha_n \mid Y)}$, i.e., sample $U \sim U[0, 1]$, if $U < \frac{P(\tilde{\alpha}_{n+1} \mid Y)}{P(\alpha_n \mid Y)}$, let $\alpha_{n+1} = \tilde{\alpha}_{n+1}$ otherwise $\alpha_{n+1} = \alpha_n$.

(iii) write down the transition probability of the Markov chain and check that $P(\theta \mid Y)$ is an invariant probability.

Note: to ensure the convergence of the chain you may want to monitor the acceptance rate along the way and change the $\sigma > 0$ accordingly.