
Yes, we speak of things that matter
 With words that must be said
 “Can analysis be worthwhile?”
 “Is the theater really dead?”

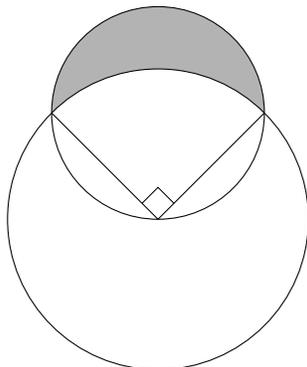
Simon and Garfunkel,
 “The Dangling Conversation”

Though this talk is about Eudoxus, arguably one of the five most important mathematicians before the 17th century, it’s difficult to understand his work outside of its context. So this talk will also cover the state of Greek mathematics before his contributions, and will also discuss how his ideas were carried forward by the next generation, most importantly by Archimedes.

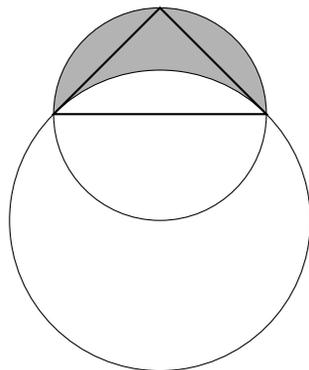
1 Before Eudoxus

To get a feeling for the Greek mathematics that was happening before Eudoxus arrived on the scene, let’s start with what is probably the first instance of truly *Greek* mathematics; as far as we can tell, it is the earliest piece of mathematical knowledge that was not either inherited from the earlier Babylonian tradition, nor a simple extension of their ideas and techniques. Moreover, it is a good example of the hallmarks that would go on to define Greek math: the fairly rigorous proof, the interest in proportion and similarity, and the obsession with finding “surprising” results, rather than only doing math for the sake of applications (to engineering, tax collection, etc.).

This work was done by Hippocrates of Chios (a different Hippocrates from the famous doctor), who is perhaps the first person to really practice Greek mathematics. The result is the determination of the area of the following shaded area, now called the *Lune of Hippocrates*.



Hippocrates showed that the area of this lune is equal to the area of an auxiliary right triangle, drawn below.



This triangle is right-angled since it's embedded in a semicircle, so by the Pythagorean theorem, the total area of the squares on its two legs equals the area of the square on its hypotenuse. Moreover, since this lune was defined by a small circle constructed on a right angle in a large circle, we can see that the unshaded segment of the large circle is similar to the shaded segments of the small circle. Therefore, since the squares on these edges are also similar, the segment takes up the same fraction of the area square in each case. Since the area of the two small squares adds up to the area of the large square, the total area of the two small segments must equal the area of the large segment. Thus, when we remove the large segment from the triangle and add in the two small segments, the area did not change.

One reason why this proof was so remarkable to its Greek audience was that it was the first time a curved area was “measured”, namely shown to have precisely the same area as a rectilinear shape. By this point the Greeks must have known how to precisely compute the areas of simple polygons, and they seem to have also understood that doing the same for the circle—the simplest curved figure—was very difficult, if not impossible. Of course, for practical purposes, they had many good ways of approximating the area of curved figures, but the theoretical task of determining it precisely was still open. As we will see, much of the subsequent work in geometry, following Eudoxus, was based around this same task.

Another important thing to note about this proof is its reliance on proportion and similarity. Crucially, Hippocrates does *not* compute the areas of the segments that are removed from or added to the triangle (indeed, he had no way of doing so). Instead, he simply relates these areas to the areas of squares, and uses the similarity of the figures to show that these areas have the same proportion both for the small and the large segments. He does not know what this proportion is—determining that is the same as determining the area of the segments—but it is a constant, and that is good enough for his proof. Proportion was one of the central mathematical concepts for the Greeks, and almost all non-trivial proofs in Greek mathematics use it in some way.

Why was proportion so important? Perhaps one reason is that the Greeks didn't really have the concept of a number, in the way that we do. They certainly had no concept of a real number, and even their concept of an integer was less robust than ours. Indeed, they didn't really think of 1 as an integer; instead, 1 was the *unit* according to which all other

integers were measured. This is one instantiation of the way in which their arithmetic was thought of in very geometric terms. As another example, even though a very robust and advanced number theory is developed in Euclid’s *Elements*, every number-theoretic proof is accompanied by a geometric diagram (essentially useless for the proof), and every description of addition or subtraction is thought of in terms of concatenating or removing line segments from one another.

Instead of numbers, the Greeks used proportions. For instance, they had no “number” called $\sqrt{2}$; instead, they had the proportion between the diagonal and the side of a square. They knew that this “number” was “irrational”, but they didn’t think in those terms; instead, they said that the side and diagonal of a square were *incommensurable*, meaning that it is impossible to find some small line segment that divides both of them, i.e. that fits into both of them an integer number of times.

Their knowledge of proportion came from geometric intuition, together with their fairly robust understanding of proportions of integers (i.e. the theory of the rational numbers). This theory was fairly well-developed because the Greeks were aware that harmony in music was determined by the ratios of small integers; an octave is a ratio of $2 : 1$, a perfect fifth is a ratio of $3 : 2$, a perfect fourth is a ratio of $4 : 3$, etc. Motivated by this to study the ratios of integers, they learned various simple facts. For instance, if $A : B :: C : D$, then we also have the equality of proportions $A : C :: B : D$; in modern terms, this is the fact that if a/b and c/d are the same real number, then $a/c = b/d$. As another example, they knew that if segments with lengths A, B, C, D had the property $A : B :: C : D$, then the rectangles with side lengths A, C and B, D had the same area; in modern terms, this is the fact that if $a/b = c/d$, then $ac = bd$.

However, it is important to note that the Greeks were not always working with ratios of integers. In fact, in the above proof of Hippocrates, there is a similarity of lengths with ratio $\sqrt{2}$, namely the side lengths of the right triangle. Even worse, there is also a proportion between the area of a segment and the area of a square, which we now know is some transcendental real number. So even though all their knowledge of proportion came from ratios of integers, they were constantly working with far more general proportions, and it’s by no means obvious that the same rules apply. In modern terms, we can say that they knew the fact that \mathbb{Q} is an ordered field, and were tacitly assuming that the same is true for \mathbb{R} . In order to make all this rigorous, they needed a theory for general proportions—in modern terms, a theory of the real numbers.

2 Eudoxus: Definitions

As far as we can tell, Eudoxus was the first person to really rise to the challenge of general proportions. His definition, which comes to us as Definition 5 in Book V of the *Elements*, is simultaneously a triumph of abstract mathematical thought and one of the most abstruse and difficult-to-read bits of math I’ve ever encountered.

“Magnitudes are said to be in the same ratio, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first

and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order.” (trans. Thomas Little Heath)

What on earth does this mean? Part of the difficulty is that Greek mathematicians were loath to use symbols in their definition and theorem statements, but we are not so constrained, so let’s introduce some notation. This definition is supposed to tell us when two proportions are equal, i.e. when $a/b = c/d$. It says that whenever we take an “equimultiple” (i.e. multiply by a positive integer) the first and third (i.e. a and c), and when we do the same to b and d (multiplying by some other positive integer), we need to get the same ordering property. In other words, for any two positive integers m, n , we consider the quantities ma, mc, nb, nd . Then the property defining the equality $a/b = c/d$ is the property that if $ma < nb$, then we also have $mc < nd$, while if $ma = nb$, then we also have $mc = nd$, and if $ma > nb$, then we also have $mc > nd$. In other words, we say that $a/b = c/d$, if for any $m, n \in \mathbb{N}$,

$$ma \begin{matrix} \leq \\ \geq \end{matrix} nb \iff mc \begin{matrix} \leq \\ \geq \end{matrix} nd.$$

If we divide both sides of the first expression by bm , and both sides of the second by dm , this becomes

$$\frac{a}{b} \begin{matrix} \leq \\ > \end{matrix} \frac{n}{m} \iff \frac{c}{d} \begin{matrix} \leq \\ > \end{matrix} \frac{n}{m}.$$

In other words, this definition says that a/b and c/d are “the same number” if and only if they have the same order relation with all rational numbers n/m , i.e. if there is no rational number separating them. In other words, Eudoxus’s definition is simply the definition of the real numbers via Dedekind cuts! A real number is uniquely determined by its order relative to the set of rational numbers. Though this definition was formulated more than 2000 years before Dedekind, this should not diminish the significance of Dedekind’s work, particularly because the Greeks did not think in the way I’ve just outlined—to them, the ratio a/b was not a “number”, and they had no theory of “the real numbers”; rather, everything was proportions and properties thereof.

Using this definition, Eudoxus was able to formally prove various properties of proportions that had been previously taken for granted, such as the ones I mentioned above. In fact, the remainder of Euclid’s Book V (which is likely all based on Eudoxus’s work) goes into extreme length (and tedium) on these “foundational” results, proving not only these useful results, but also a bunch of boring ones, such as the fact that two equal magnitudes have the same proportion to a fixed magnitude. It seems that the point of this whole operation was purely foundational—nobody doubted that all these facts about proportions were true, but Eudoxus wanted to demonstrate that his theory was strong enough to prove all these things rigorously.

It’s also worth pausing for a moment to admire Eudoxus’s definition once more. It is astonishingly abstract; in a culture where everything was thought of in concrete, geometrical terms, it is amazing to see a definition like this, which doesn’t tell you what a ratio *is*, only how to tell if two of them are equal. This is analogous to our modern definition of the real

numbers; we all think of them as “numbers”, but to formally define them, we need to move to a more abstract setting, where they are simply sets of rational numbers modulo some equivalence relation. Moreover, this definition features another level of abstraction, because the criterion for determining if two ratios are equal is an infinite one, which would in theory require checking the order property for every pair of “equimultiples” m and n . The Greeks were often hesitant in their dealings with infinity, and it’s remarkable that Eudoxus came up with a definition which required an infinite check to even use.

By the way, who was Eudoxus? I’ve avoided this question because the answer is that we basically have no idea. He was born in the town of Cnidus (or Knidos) in modern-day Turkey, sometime around 390 BC. He probably studied under Plato in Athens, and even though he was widely respected in his time, none of his writings survive (this is not so uncommon; the vast, *vast* majority of ancient authors whose names we know have no extant writings). He was also an accomplished astronomer, and introduced an important model of the universe that was to remain influential for several centuries. He died sometime in the 330s BC.

Before ending, I want to mention one further definition due to Eudoxus, which will be very important when we consider the *method of exhaustion* that Eudoxus used to compute areas and volumes of various curved shapes. This is recorded as Definition 4 in Book V of the *Elements*:

“Magnitudes are said to have a ratio to one another which are capable, when multiplied, of exceeding one another.” (trans. Thomas Little Heath)

This is what we know as the *Archimedean property* of the reals, though Eudoxus was the first to postulate it, and though he treats it as a definition/axiom rather than as a theorem about the reals (this is because, again, Eudoxus didn’t really define a number system). It asserts that for any pair of magnitudes (i.e. positive reals), some integer multiple of each is larger than the other. Note again that it’s not a theorem about the reals, but rather a restriction on what things we’re allowed to take ratios of; it basically asserts that we’re not allowed to talk about ratios involving infinitesimals or infinities, though it doesn’t rule out their existence.

3 Eudoxus: Theorems

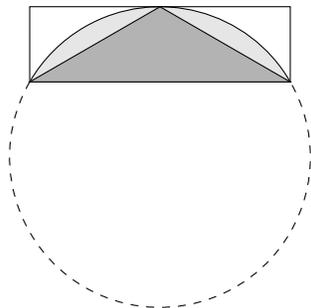
Book VI of the *Elements*, immediately following on the abstract theory of proportions developed in Book V, uses this theory to prove basic results about similar polygons. For instance, Proposition 20 proves the important fact that the areas of similar polygons are in squared proportion to their side lengths. But because Eudoxus’s definition is so unwieldy to work with, only one of these basic results directly uses the definition (namely that triangles with the same height have areas proportional to their bases), and all subsequent results use this fact instead of the direct definition. It is unknown if Eudoxus himself developed this theory of similarity, though he was no doubt aware that all these basic results (which everyone simply took for granted) could be proven in his theoretical framework. Instead, Eudoxus seems

to have focused on some more spectacular results, namely ones about the measurement of curved shapes.

To do this, Eudoxus developed the so-called *method of exhaustion*; he was likely not the first person to come up with this method (it seems that someone named Antiphon did something like it before Eudoxus), but he was certainly the first person to make it rigorous. Essentially, the method of exhaustion is what we'd consider an epsilon-delta approach to computing a limit, though of course the Greeks did not think of it like this. To understand the method, let's see it in action, through Proposition 2 of Book XII of the *Elements*. It asserts that the ratio of areas of two circles is equal to the square of the ratio of the lengths of their diameters. Though this fact was surely understood by earlier mathematicians, Eudoxus was probably the first person to prove it rigorously.

The proof in Euclid is quite long and tedious, but the idea is very simple. We have two circles C_1, C_2 , with areas a_1, a_2 and diameters d_1, d_2 , and we wish to show that $a_1/a_2 = (d_1/d_2)^2$ (recall that this notation, and indeed this way of thinking about real numbers, is extremely anachronistic and non-Greek!). We suppose for contradiction that this is not the case. Then we may find some area x so that $a_1/x = (d_1/d_2)^2$, where $x \neq a_2$. So either $x < a_2$ or else $x > a_2$. Let's suppose at first that it's the first.

Now, we will inscribe a sequence of polygons inside C_2 , and we will consider the area between each polygon and the circle. We begin with a simple geometric observation: if we have the area between some chord of a circle and the circumference, and we bisect the arc, then the area of the resulting triangle is at least half the area of the segment we started with. To see this, draw a tangent to the circle at the bisection point, and complete it to a rectangle; then this rectangle contains the entire segment, and its area is exactly twice the area of the triangle.



Now, we begin by inscribing a square in C_2 . Next, we bisect each arc between its vertices to inscribe an octagon. Next, we again bisect each arc to inscribe a 16-gon, etc. At each step, each segment between the polygon and the circle loses at least half its area, so the total area between the polygon and the circle shrinks by at least half at each step. By a consequence of the Archimedean property proved earlier in the *Elements* (Proposition X.1), at some point this remainder area must be less than $a_2 - x$ (recall that we assumed $x < a_2$). Let P_2 be the polygon at this step, with area p_2 . Then we have that $a_2 - p_2 < a_2 - x$, or equivalently $p_2 > x$. Inside C_1 we repeat the exact same construction to obtain a similar polygon P_1 with

area p_1 . Since we've already proved the relevant property for similar polygons (their area scales quadratically with their length), we know that

$$\frac{p_1}{p_2} = \left(\frac{d_1}{d_2}\right)^2 = \frac{a_1}{x}$$

where the last equality is our definition of x . Rearranging, we get that

$$\frac{x}{p_2} = \frac{a_1}{p_1}$$

However, $a_1 > p_1$, since P_1 is inscribed in C_1 . Thus, $x > p_2$, which is a contradiction. Finally, we need to deal with the case where $x > a_2$. However, by reversing the roles of the two circles, this actually follows from the previous argument, and we are done.

This argument is a relatively tame example of the method of exhaustion, which can often get unwieldy (and which leads to unintended aptness of the name). As this example suggests, every proof by exhaustion basically consists of two parts. The first is of the “epsilon-delta” type, which basically says “We wish to prove two things are equal. By contradiction, if not, one must be larger than the other. We tackle the two cases separately, but in each one, the fact of inequality means they need to differ by some finite ε , and we will use this to derive a contradiction.” The second part actually deals with the problem at hand, and involves showing that some operation, if done enough times, will lead to an error smaller than any fixed ε , which is used to derive the above contradiction. In this case, it was the simple geometric operation of bisecting a circular segment and observing that the resulting triangle takes up at least half the area. In other proofs involving the method of exhaustion, this step is often far more involved, and uses much more geometry.

In modern terms, the method of exhaustion generally amounts to computing some integral by evaluating a sequence of Riemann sums. The evaluation of the Riemann sum is the second step above, and it is where most of the ideas are—for each problem one tackles with this method, one needs a different approach to evaluate the sum. The first step is essentially purely mechanical, but it is repeated in detail in all of these proofs, which is part of what makes them so unwieldy. In modern terms, this amounts to checking some simple convergence of a sequence. The development of calculus in the 17th century finally made the method of exhaustion obsolete, and one of the things that this development did was make this step less cumbersome: by proving general theorems about limits of sequences, one can avoid having to deal directly with the ε s and δ s in each and every proof. As an example, suppose that in every evaluation of a limit, we had to essentially reprove from nothing that the limit of a sum is the sum of the limits, or the analogous fact for products. These proofs are not hard, but having to do them in every single argument can easily make things, well, exhausting.

Using the method of exhaustion, Eudoxus was able to rigorously prove various formulas for the areas and volumes of curved shapes. For instance, he is likely the first person to prove that the ratio of volumes of spheres is proportional to the cube of the ratio of their diameters (by inscribing the spheres with polyhedra). He was also the first person to prove that the volume of a cone is $1/3$ the volume of the cylinder enclosing it, by inscribing the

cone with pyramids and the cylinder with prisms. Then, he needs the analogous fact for pyramids and prisms, which in Euclid is *again* proven with the method of exhaustion (by filling a pyramid with smaller pyramids similar to it, together with prisms).

Eudoxus's achievements with the method of exhaustion are remarkable, since they were the first (mostly) rigorous analyses that dealt with things like the area of a circle. However, they were also a tad disappointing in that none of these results was really that surprising. All Greek mathematicians knew that the area of a polygon scales like the square of its side length, so it can't have been very surprising that the same holds for circles; moreover, the fact that the argument is "fill a circle with polygons and then apply that well-known fact" is a bit underwhelming. Basically, all these results were great achievements in terms of foundations (i.e. rigorous proofs), but none of the results were all that surprising, and the proof technique was not that insightful; it was always "apply the known result from rectilinear shapes and take a limit". It took another generation until Archimedes would use these ideas to really blow everyone away.

4 After Eudoxus

The hero of this section is Archimedes, easily the most accomplished mathematician of the ancient world, and arguably of all of history. He was born about 50 years after Eudoxus's death (around 287 BC), in Syracuse, a Greek city on Sicily. We know relatively little about his life, but we are lucky in that a fairly large number of his mathematical writings have survived. Though he did much, much more, I'd like to sketch here a few of his accomplishments using the method of exhaustion. As mentioned above, unlike Eudoxus, Archimedes did not simply rigorously prove results that everyone expected; instead, he made a point of finding seemingly impossible problems and solving them.

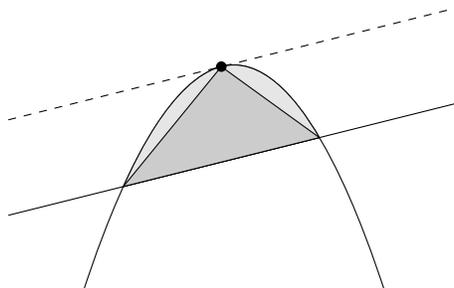
First, he did the thing that everyone since at least Hippocrates had been trying, by finding a rectilinear figure with the same area as a circle. Specifically, he proved that the area of a circle is the same as the area of a right triangle with one leg the radius and the other leg the circumference of the circle. Of course, this is in some sense cheating the problem that everyone had been trying to solve, since it doesn't tell us how to compute the circumference. In modern terms, Archimedes proved that the same constant of proportionality, π , appears both as the ratio between the area of a circle and its radius squared, and between the circumference and diameter of a circle, but did not express π in terms of other known constants. Today we know that π is transcendental, and therefore this task (often called "squaring the circle") is impossible. However, in the same text, Archimedes accurately approximated π , proving that

$$3\frac{10}{71} < \pi < 3\frac{1}{7}.$$

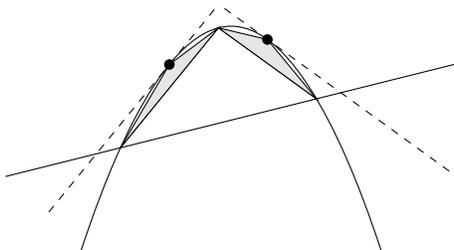
He did this by inscribing and circumscribing 96-gons in a circle, and then approximating their areas.

Next, Archimedes determined the area between a parabola and a line cutting it; in effect, this amounts to him integrating a quadratic function, though of course the argument is

geometric rather than analytic. Specifically, he shows that the area of the segment is $4/3$ the area of a triangle contained in it, whose base is the cutting line segment and whose third point is the point on the parabola where the tangent is parallel to the cutting line.



To prove this, Archimedes repeats this operation in each of the small remaining segments (namely finds the point where the tangent is parallel and inscribe a triangle).

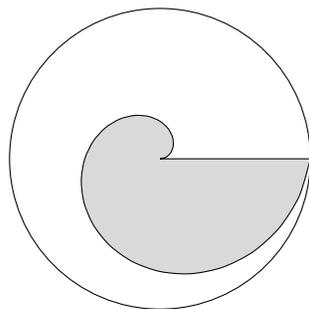


He then shows that each resulting triangle will have base $1/2$ the base of our original triangle, and height $1/4$ the height of our original triangle, using simple properties of parabolas. Thus, if we iterate this once, we will obtain two triangles, each with area $1/8$ of the area of our first triangle. Thus, the total area enclosed after this step is $(1 + 1/4)$ times the area of our first triangle. Now, we iterate this, obtaining at each step $1/4$ of the previous step, and doing this infinitely many times will exhaust the parabolic segment. Therefore, the ratio of the area of the segment to the triangle is

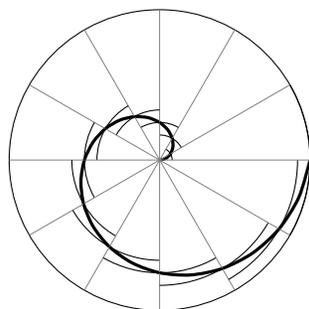
$$1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots = \frac{4}{3}.$$

To evaluate this geometric series, Archimedes *again* uses the method of exhaustion; the sum of every finite geometric series was known to Euclid, and Archimedes uses the method of exhaustion to show that the partial sums converge to $4/3$. In another text, on basic mechanics, he also determined the center of gravity of a parabolic segment, which amounts to the integration of a different quadratic function.

He studied the figure now known as the Archimedes spiral; in polar coordinates, this is the curve $r = a\theta + b$ for fixed a, b , though Archimedes thought of it as the path traced out by a point moving along a rotating segment at constant speed. His main accomplishment here was the determination of its area relative to the circle enclosing it. Though he did this for all b , it's cleanest when we set $b = 0$; then he found that the spiral contains $1/3$ the area of the enclosing circle.



His proof relies on the following extremely slick geometric argument. He drew a large number of equally spaced radii from the center of the circle. Wherever one of these radii intersects the spiral, he drew a circular arc to the previous and subsequent radius.



This yields two new shapes, one of which encloses the spiral and one of which is enclosed by it. Moreover, the difference between the areas of these two shapes is the sum of the differences in each sector, and by shifting sectors, we see that this sum is simply the area of one of the sectors. Thus, by making our number of radii go to infinity, the discrepancy between the two area approximations will go to 0, so we can use the method of exhaustion. Moreover, to compute each of these areas, we simply need to add up the area of a bunch of sectors. Since they all have the same angle, their area is proportional to their radius squared, so this essentially amounts to summing up the series $\sum_{i=1}^n i^2$. This Archimedes did, and that is where the $1/3$ comes from; in modern terminology, he in effect computed

$$\int_0^1 x^2 dx = \frac{1}{3}.$$

In another text, he compared a sphere to a cylinder that encloses it (namely one whose height is the diameter of the sphere and whose radius is that of the sphere). He managed to compare both their volumes and surface areas, showing that the sphere has $2/3$ the volume and surface area of the cylinder. He was particularly proud of these results because comparing a cylinder and inscribed sphere is a three-dimensional analogue to comparing a square and inscribed circle. In the two-dimensional problem, determining either the ratio of areas or the ratio of perimeters amounts to determining π , and was therefore essentially impossible; Archimedes nevertheless managed to solve the (a priori harder) three-dimensional problem. There is a story, which is likely false, that he was so proud of this result that he

had a drawing of a sphere and cylinder inscribed on his tomb. He also generalized these results, and was able to compute the volumes of spheres, ellipsoids, and paraboloids and hyperboloids of revolution cut by planes, essentially a higher-dimensional generalization of the quadratic integrals he previously computed.