

1 Introduction

Given positive integers t_1, \dots, t_ℓ , let $r(t_1, \dots, t_\ell)$ denote the *Ramsey number* of t_1, \dots, t_ℓ , namely the least integer N such that every ℓ -coloring of the edges of K_N contains a copy of K_{t_i} in color i for some $i \in [\ell]$. In case $t_1 = \dots = t_\ell = t$, then we write $r(t; \ell) := r(t, \dots, t)$ to denote the ℓ -color *Ramsey number* of t .

Obtaining good upper and lower bounds on $r(t; \ell)$ is a central question in Ramsey theory, and despite more than 80 years of effort, still not much is known. For upper bounds, the best result is still essentially that given by the Erdős–Székere argument from 1935.

Theorem 1.1. $r(t; \ell) \leq \ell^{t\ell}$ for any integers t, ℓ .

Proof. Fix an ℓ -coloring of $E(K_N)$. For any vertex v_1 , at least N/ℓ of the edges incident with v_1 have the same color, say c_1 . Discard all other vertices. For any remaining vertex $v_2 \neq v_1$, at least $(N/\ell)/\ell$ of the remaining incident edges to v_2 have the same color, say c_2 . Continue in the same way, generating a sequence of vertices v_1, \dots, v_k and colors c_1, \dots, c_k . Note that since we cut down by a factor of at most $1/\ell$ the number of vertices at each step, we must have $k \geq \log_\ell N$. Moreover, by the pigeonhole principle, at least k/ℓ of the colors c_1, \dots, c_k are the same color. But the vertices corresponding to these k/ℓ identical colors form a monochromatic clique of size $t \geq k/\ell \geq (\log_\ell N)/\ell$. Rearranging shows that $N \leq \ell^{t\ell}$. \square

In fact, by tracking the recursion in this argument more carefully, one finds that

$$r(t_1, \dots, t_\ell) \leq \binom{t_1 + \dots + t_\ell - \ell}{t_1 - 1, \dots, t_\ell - 1},$$

and this multinomial coefficient is always smaller (by a lower-order term) than $\ell^{t\ell}$. Slight further improvements to this bound follow from Conlon’s approach to diagonal Ramsey numbers, but $\ell^{t\ell}$ is still roughly the best upper bound we have.

For lower bounds, the simplest thing to do is to consider a uniformly random ℓ -coloring of K_N . A simple union bound then shows that we may take $N \approx \ell^{t/2}$ and have no monochromatic clique of order t , implying that $r(t; \ell) \gtrsim \ell^{t/2}$. While this is essentially the best bound we know for 2 colors, there is a simple trick, usually attributed to Lefmann, which does better for more colors.

Lemma 1.2 (Product coloring). *For any integers t, ℓ_1, ℓ_2 ,*

$$r(t; \ell_1 + \ell_2) - 1 \geq (r(t; \ell_1) - 1)(r(t; \ell_2) - 1).$$

Proof. Fix colorings on $N_1 := r(t; \ell_1) - 1$ and $N_2 := r(t; \ell_2) - 1$ vertices with no monochromatic K_t . We form a coloring on $N_1 N_2$ vertices by blowing up each of the N_1 vertices in the first coloring to N_2 new vertices, and putting a copy of the second coloring in each blowup part. Equivalently, we make our vertex set be $[N_1] \times [N_2]$, and color edges lexicographically according to the two colorings on $[N_1], [N_2]$. Then it is immediate that this coloring has no monochromatic K_t , implying that $r(t; \ell_1 + \ell_2) - 1 \geq N_1 N_2$. \square

Plugging in the bound $r(t; 2) > 2^{t/2}$, we find that $r(t; \ell) \geq 2^{t\ell/4}$, since we can take an $(\ell/2)$ -fold product coloring. Putting this together, we find that

$$2^{t\ell/4} \leq r(t; \ell) \leq \ell^{t\ell} = 2^{t\ell \log \ell}.$$

There are two interesting parameter regimes, and the above indicates what the major problems in each are. First, let us suppose that t is fixed and $\ell \rightarrow \infty$. Then we find that $r(t; \ell)$ is between $2^{\Omega(\ell)}$ and $2^{O(\ell \log \ell)}$, i.e. between exponential and super-exponential in ℓ . It is a major open problem to determine whether the truth is exponential or super-exponential. Even in the simplest case, of $t = 3$, our knowledge is very limited, and Erdős offered \$100 to determine whether $r(3; \ell)$ is exponential or super-exponential. The current best bounds are

$$1073^{\ell/6} \approx 3.199^\ell \leq r(3; \ell) \leq \left(\frac{e - e^{-1} + 3}{2} \right) \ell! \approx 2.675\ell!.$$

The question of whether $r(t; \ell)$ is exponential or super-exponential in ℓ is closely related to a number of other questions in graph theory and other fields of math. One of my favorite interpretations has to do with the Shannon capacity of graphs, which is a very mysterious quantity: it was a major achievement when Lovász determined the Shannon capacity of C_5 , and the Shannon capacity of any longer odd cycle is still unknown. There is a sequence of inequalities

$$\alpha(G) \leq \Theta(G) \leq \vartheta(G) \leq \chi_f(\overline{G}) \leq \chi(\overline{G}),$$

where α is the independence number, Θ is the Shannon capacity, ϑ is the Lovász theta function, χ_f is the fractional chromatic number, χ is the chromatic number, and \overline{G} is the complement graph of G . For almost every pair of quantities in this chain of inequalities, it is known that the larger cannot be bounded as a function of the smaller. The unique exception is the pair α and Θ , where we can't decide whether the Shannon capacity is bounded as a function of the independence number. However, it turns out that for any $\alpha \geq 2$,

$$\lim_{\ell \rightarrow \infty} r(\alpha + 1; \ell)^{1/\ell} = \max_{G: \alpha(G) = \alpha} \Theta(G).$$

Thus, determining whether $r(t; \ell)$ is exponential or super-exponential is equivalent to determining whether Θ can be bounded as a function of α .

The other parameter regime is where ℓ , the number of colors, is fixed, but $t \rightarrow \infty$. Here even for $\ell = 2$ our knowledge is very limited, stuck at the simple bounds $2^{t/2} \leq r(t; 2) \leq 2^{2t}$, other than lower-order improvements. Similarly, by the above, we know that for each fixed ℓ , the value of $r(t; \ell)$ is exponential in t as $t \rightarrow \infty$, so the main question in this regime is to determine the correct exponential constant. The goal of this talk is to tell you about some improvements, due to Conlon and Ferber [2], to the exponential constant of $r(t; \ell)$ for fixed $\ell \geq 3$ and $t \rightarrow \infty$.

Conlon and Ferber's construction mixes algebraic and probabilistic approaches, a technique that seems to be increasingly important in extremal combinatorics. Before describing their construction, I'll begin by discussing some earlier work which is either thematically or technically related, as a way of building up some of the main ideas.

2 Random subgraphs, random blowups, and random homomorphisms

Let's suppose we wish to prove a lower bound on the two-color Ramsey number $r(s, t)$. If we can find a graph G that has no clique of order s and no independent set of order t , then we've found such a lower bound: $r(s, t)$ is greater than the number of vertices of G . But since finding such graphs is hard, it would be nice to be able to lower-bound $r(s, t)$ by finding a graph G with some weaker property.

It turns out that this is possible. Suppose we now have a graph G with no K_s , but let's not assume that it has no independent sets of order t . Instead, let's suppose that G has "few" independent sets of order t . Concretely, assume that G has at most M^t independent sets of order t , for some parameter M (note that it is natural to parametrize things in this way, since there are exponentially many t -sets of vertices in G). It turns out that as long as M is not too big, we can use this G to get a good lower bound on $r(s, t)$, by *random sampling*.

Lemma 2.1 (Random sampling). *Let G be a K_s -free graph on N vertices, and suppose that G has at most M^t independent sets of order t . Then*

$$r(s, t) \geq \frac{N}{4M}.$$

Proof. We will randomly sample a subgraph H of G , by keeping each vertex of G independently with probability p , to be chosen later. Since G is K_s -free, its subgraph H is K_s -free as well. Additionally, each independent set of order t in G will survive in H with probability p^t . So the expected number of independent sets of order t in H is at most $p^t M^t = (pM)^t$. By choosing $p = 1/(2M)$, this number is less than $1/2$, so the probability that H has no independent set of order t is at least $1/2$. Additionally, with high probability, H has at least $pN/2$ vertices, by the Chernoff bound¹. So we find that with positive probability, H is a graph on at least $N/(4M)$ vertices with no K_s or \overline{K}_t , proving that $r(s, t) \geq N/(4M)$, as claimed. \square

Though the idea in Lemma 2.1 is simple and was known for a long time, it gained recent prominence due to work of Mubayi and Verstraëte [4]. Recall that Erdős and Szekeres proved that $r(s, t) \leq t^{s-1}$ for fixed $s \geq 3$ and $t \rightarrow \infty$.

Corollary 2.2 (Mubayi–Verstraëte). *Fix $s \geq 3$. Suppose that for every t , there exists a K_s -free graph on $N = t^{2s-3+o(1)}$ vertices with at most M^t independent sets of order t , where $M = t^{s-2+o(1)}$. Then*

$$r(s, t) \geq \frac{N}{4M} = t^{s-1-o(1)},$$

which is asymptotically tight as $t \rightarrow \infty$.

¹Strictly speaking, we'd have to assume that e.g. $N > 10M$ for this step to work. But given that the lemma statement is uninteresting if M and N have the same order, let's not worry about this technicality.

Of course, as I've stated it, this does not seem like a particularly impressive result, since the assumptions are weird and unnatural. However, the reason this result is so important is due to a lemma of Alon and Rödl [1], as well as another observation of Mubayi and Verstraëte. Without getting into it, it turns out that an *optimally pseudorandom K_s -free graph* satisfies the assumptions of Corollary 2.2. Such graphs are known to exist for $s = 3$, and it is a major open problem to prove that they exist for all $s \geq 4$. However, with this observation, one sees that the Mubayi–Verstraëte result says that finding such graphs for $s \geq 4$ would also determine the asymptotic order of $r(s, t)$ as $t \rightarrow \infty$.

In order to extend these ideas further, it will be convenient to take a different perspective on Lemma 2.1. Specifically, rather than keeping each vertex of G with probability p , we will pick a random function from a set of pN vertices to $V(G)$, and “pull back” the graph structure. Of course, if $p \ll 1$, then this random function will have no collisions with high probability, and so we will exactly get the random induced subgraph we got before, except that we'll have exactly pN vertices (rather than a binomial distribution on the number of vertices), but this difference is immaterial. The reason for taking this change of perspective is that it is much more amenable to using more than two colors: we can just pick a more random functions and overlay them, as we'll soon see.

Concretely, suppose that G is a K_s -free graph on N vertices with at most M^t independent sets of order at most² t . Let $n = pN$ for some parameter p , and pick a uniformly random function $f : [n] \rightarrow V(G)$. Define a graph H on vertex set $[n]$ by setting $\{u, v\} \in E(H)$ if $\{f(u), f(v)\} \in E(G)$; note that in particular we only connect u and v if $f(u) \neq f(v)$, which implies that H is also K_s -free. Then for any given set $T \subset [n]$ of order $|T| = t$, the probability that its image under f lies in a given subset of $V(G)$ is at most $(t/N)^t$. Thus, the probability that T is independent in H is at most $(tM/N)^t$. As there are $\binom{n}{t}$ choices for this T , we see by the union bound that

$$\Pr(H \text{ has an independent set of order } t) \leq \binom{n}{t} \left(\frac{tM}{N}\right)^t \leq \left(\frac{epN}{t} \frac{tM}{N}\right)^t = (epM)^t,$$

and we can recover the result of Lemma 2.1 by setting $p = 1/(2eM)$.

However, as indicated above, the power of this perspective is that it easily extends to more colors. Indeed, suppose that we instead pick independent uniformly random functions $f_1, \dots, f_r : [n] \rightarrow V(G)$. We color the edges of K_n in $r + 1$ colors, as follows. If there is some $i \in [r]$ such that $\{f_i(u), f_i(v)\} \in E(G)$, then we color $\{u, v\}$ by the minimum such i . If not, we color $\{u, v\}$ by color $r + 1$. Then each of the first r colors is K_s -free, by the above. Additionally, the probability that some fixed t -set T is monochromatic in the last color is at most $(tM/N)^{rt}$, since we have a probability $(tM/N)^t$ for each function f_i , and these probabilities are independent. Therefore, by the union bound, we find that the

²Note that we've slightly strengthened this assumption, bounding the number of independent sets of order at most t . As it turns out, this is usually OK: many techniques that bound the number of independent sets of order exactly t will also work here, including the Alon–Rödl technique mentioned above.

probability that the last color has a clique of order t is at most

$$\binom{n}{t} \left(\frac{tM}{N} \right)^{rt} \leq \left(\frac{pt^r M^r}{N^{r-1}} \right)^t. \quad (1)$$

We conclude the following generalization of Lemma 2.1.

Lemma 2.3 (Random homomorphisms). *Let G be a K_s -free graph on N vertices, and suppose that G has at most M^t independent sets of order at most t . Then*

$$r(\underbrace{s, \dots, s}_r, t) \geq \frac{N^r}{2t^r M^r}.$$

Proof. We set $p = N^{r-1}/(2t^r M^r)$, so that the quantity in equation (1) is less than 1. Then we see that the coloring described above has no K_s in the first r colors, and no K_t in the final color, and has $n = pN$ vertices. \square

Of course, even this isn't the most general form of this lemma that we could prove, since there's no real reason to have f_1, \dots, f_r all have the same codomain. Thus, as in [3], we can use this idea to obtain lower bounds on many off-diagonal multicolor Ramsey numbers.

The crucial thing to observe about Lemma 2.3 is that p is not a probability, and in particular, it does not need to be less than 1! If $p > 1$, then $n = pN$ will be larger than N , and the functions f_1, \dots, f_r will no longer be making random subgraphs of G . Instead, they will be forming random *blowups* of G , and thus the coloring we use in Lemma 2.3 is gotten by randomly overlaying r random blowups of G , and then coloring all uncolored edges with the final color. This idea of overlaying random blowups to obtain lower bounds on multicolor Ramsey numbers goes back to Alon and Rödl [1], though they didn't use the perspective of random homomorphisms. The observation that the Alon–Rödl approach and the Mubayi–Verstraëte approach are both instances of the same general technique is due to Xiaoyu He, and our paper [3] uses this observation to combine the Alon–Rödl and Mubayi–Verstraëte approaches and obtain unified bounds on multicolor Ramsey numbers, at least under the assumption that there exist optimal K_s -free pseudorandom graphs. In my opinion, the fact that random induced subgraphs and random blowups are “the same thing” is a very powerful observation, and it's the main message I'd like to get across today.

3 The Conlon–Ferber argument (and beyond)

The Conlon–Ferber construction can be broken up into two parts. The first is a simple but powerful generalization of Lemmas 2.1 and 2.3, which is that by adding one more random color, we are able to significantly reduce the size of monochromatic cliques that occur. Then the second step is finding an appropriate graph G to apply this strengthened lemma to. Let's discuss each step in turn.

For the first step, in Lemma 2.3, we gave all remaining edges the same color, and then used a simple union bound to estimate the probability of a monochromatic K_t . The Conlon–Ferber idea is to actually use two colors for these remaining edges, choosing randomly for each edge. Since we know that random colorings generally have small monochromatic cliques, it stands to reason that doing this will improve the lower bound on the Ramsey number. Of course, doing this is costly, in the sense that we have to add a new color, so we are obtaining a strengthened bound on a different Ramsey number. The precise statement, implicit in [2, 5], is as follows.

Lemma 3.1. *Let G be a K_s -free graph on N vertices, and suppose that G has at most M^t independent sets of order at most t . Then*

$$r(\underbrace{s, \dots, s}_r, t, t) \geq \frac{2^{t/2} N^r}{4t^r M^r}.$$

Proof. As indicated above, we pick a parameter p , set $n = pN$, and choose r random functions $f_1, \dots, f_r : [n] \rightarrow V(G)$. We color $E(K_n)$ by assigning the first r colors as before, with $\{u, v\}$ getting color i only if $\{f_i(u), f_i(v)\} \in E(G)$. For the uncolored edges, we assign one of the colors $r + 1, r + 2$ uniformly at random, independently for each uncolored edge. Then as above, we know that the first r colors are K_s -free. For the final two colors, let's estimate the probability that a t -set $T \subset [n]$ is monochromatic. For T to be monochromatic, it must first not contain any edges of the first r colors, which we know happens with probability at most $(tM/N)^{rt}$. Then, there is a probability $2^{1-\binom{t}{2}}$ that all the pairs of T get assigned the same color among $\{r + 1, r + 2\}$. Putting this all together with the union bound, we see that the probability that K_n has a monochromatic K_t in one of the last two colors is at most

$$\binom{n}{t} 2^{1-\binom{t}{2}} \left(\frac{tM}{N}\right)^{rt} \leq \left(pN \cdot 2^{1-\frac{t}{2}} \cdot \frac{t^r M^r}{N^r}\right)^t = \left(p \frac{2t^r M^r}{2^{t/2} N^{r-1}}\right)^t.$$

So if we take $p = 2^{t/2} N^{r-1} / (4t^r M^r)$, this probability will be less than 1, and we'll obtain a coloring with no K_s in the first r colors and no K_t in the final two colors. This gives that

$$r(\underbrace{s, \dots, s}_r, t, t) \geq n = pN = \frac{2^{t/2} N^r}{4t^r M^r}. \quad \square$$

Now, for the second step of the Conlon–Ferber argument, we need to find a graph which has no small cliques and few small independent sets, for an appropriate choice of “small” and “few”. They used the following graph.

Definition 3.2. Let t be even and let $V \subset \mathbb{F}_2^t$ be the subspace of all vectors of even Hamming weight. We define a graph G_0 with vertex set V by letting $\{u, v\} \in E(G_0)$ if $u \cdot v = 1$, where $u \cdot v = \sum_{i=1}^t u_i v_i$ denotes the scalar product over \mathbb{F}_2 . Note that G_0 has $N = |V| = 2^{t-1}$ vertices.

The two properties we need of G_0 are the content of the next two lemmas.

Lemma 3.3. G_0 is K_t -free.

Proof. This is essentially the Oddtown theorem. We claim that every clique of even order in G_0 must consist of linearly independent vectors. Indeed, suppose that m is even and that v_1, \dots, v_m form a clique in G_0 . Suppose we had a linear relation $\sum \alpha_i v_i = 0$. Note that by the definition of V , we have $v_i \cdot v_i = 0$ for all i , while $v_i \cdot v_j = 1$ for $i \neq j$, since these vectors form a clique in G_0 . So taking the scalar product of the linear relation with each v_j in turn, we find that

$$0 = v_j \cdot \left(\sum_{i=1}^m \alpha_i v_i \right) = \sum_{\substack{1 \leq i \leq m \\ i \neq j}} \alpha_i = \left(\sum_{i=1}^m \alpha_i \right) - \alpha_j.$$

Adding these equations over all $j \in [m]$ shows that

$$\sum_{j=1}^m \alpha_j = m \sum_{i=1}^m \alpha_i \quad \implies \quad (m-1) \sum_{i=1}^m \alpha_i = 0.$$

Since m is even, $m-1$ is invertible over \mathbb{F}_2 , so we conclude that $\sum \alpha_i = 0$. But $\alpha_j = \sum \alpha_i$ for all j , so we conclude that all the α_j are 0, implying that v_1, \dots, v_m are linearly independent.

Since we assumed that t is even, this implies that any K_t in G must consist of linearly independent vectors. But $\dim V = t-1$, so this is impossible. \square

Lemma 3.4. G_0 has at most $2^{\frac{5t^2}{8} + o(t^2)}$ independent sets of order at most t .

Proof. For an independent set T in G_0 , let its *rank* be the number of linearly independent vectors in it, or equivalently the dimension of its span. For $0 \leq r \leq m \leq t$, we will upper-bound the number of independent sets of order m and rank r , and then add this up over all r to obtain an upper bound on the total number of independent sets of order at most t .

So let $\{v_1, \dots, v_m\}$ be an independent set of rank r . We may assume without loss of generality that the first r vectors are linearly independent. We have 2^{t-1} choices for v_1 , since $\dim V = t-1$. Next, v_2 must be linearly independent of v_1 , so we have at most 2^{t-2} choices for it, and similarly we have at most 2^{t-i} choices for v_i . However, once $i > r$, v_i must be in the span of v_1, \dots, v_r , which is an r -dimensional space, so we have at most 2^r choices for these vectors. In all, we find that the number of independent sets of order m and rank r is at most

$$\left(\prod_{i=1}^r 2^{t-i} \right) 2^{r(m-r)} \leq 2^{2tr - \binom{r+1}{2} r^2} \leq 2^{2tr - \frac{3r^2}{2}}, \quad (2)$$

using that $m \leq t$. It is straightforward to see that this quantity is maximized when $r = 2t/3$, and in Conlon and Ferber's original version, they used this bound. However, one can improve on this by noting that the rank of any independent set is at most $t/2$. Indeed, the span of any independent set is a subspace $W \subset \mathbb{F}_2^t$ with the property that $w_1 \cdot w_2 = 0$ for all $w_1, w_2 \in W$. This implies that $W \subseteq W^\perp$. But we also have that $\dim W + \dim W^\perp = t$, which implies

that $\dim W \leq t/2$. Thus, the quantity in equation (2) is maximized when $r = t/2$, and we conclude that the total number of independent sets of order at most t is at most

$$\frac{t}{2} \left[2^{2tr - \frac{3r^2}{2}} \right]_{r=t/2} = 2^{\frac{5t^2}{8} + o(t^2)}. \quad \square$$

We now have all the pieces and can put them together.

Theorem 3.5 (Conlon–Ferber [2], W. [5]). *For any fixed $\ell \geq 2$,*

$$r(t; \ell) \geq \left(2^{\frac{3\ell - 1}{8}} \right)^{t - o(t)}.$$

Proof. Let t be a large even number. By Lemma 3.3, G_0 is K_t -free and has $N = 2^{t-1}$ vertices. By Lemma 3.4, G_0 has at most M^t independent sets of order at most t , where $M = 2^{\frac{5t}{8} + o(t)}$. So by applying Lemma 3.1 with $r = \ell - 2$, we find that

$$r(t; \ell) \geq \frac{2^{t/2} N^{\ell-2}}{4t^{\ell-2} M^{\ell-2}} = \left(\frac{2^{\frac{1}{2}} \cdot 2^{\ell-2}}{2^{\frac{5}{8}(\ell-2)}} \right)^{t - o(t)} = \left(2^{\frac{3}{8}(\ell-2) + \frac{1}{2}} \right)^{t - o(t)} = \left(2^{\frac{3\ell - 1}{8}} \right)^{t - o(t)}. \quad \square$$

Remark. Conlon and Ferber actually only obtained this bound for $\ell = 3$, but didn't use the random homomorphism trick. For more colors, they instead applied a different coloring to a variant of the graph G_0 . Specifically, they repeated the construction of G_0 , but over some other field \mathbb{F}_q . They then used $q - 1$ colors to color most of the edges, giving $\{u, v\}$ color i if $u \cdot v = i$ for $i \neq 0$. Finally, they colored all remaining edges with two more colors randomly, and took a random subgraph. While this works for all prime q , it ends up giving the best bounds for $q = 2$ and $q = 3$, and they obtained bounds for more colors by using Lefmann's product coloring trick. As it turns out, the random homomorphism framework obtains better bounds for this problem, and means that one doesn't have to use product colorings at all.

References

- [1] N. Alon and V. Rödl, Sharp bounds for some multicolor Ramsey numbers, *Combinatorica* **25** (2005), 125–141.
- [2] D. Conlon and A. Ferber, Lower bounds for multicolor Ramsey numbers, 2020. Preprint available at arXiv:2009.10458.
- [3] X. He and Y. Wigderson, Multicolor Ramsey numbers via pseudorandom graphs, *Electron. J. Combin.* **27** (2020), Article No. P1.32.
- [4] D. Mubayi and J. Verstraëte, A note on pseudorandom Ramsey graphs, 2019. Preprint available at arXiv:1909.01461.
- [5] Y. Wigderson, An improved lower bound on multicolor Ramsey numbers, 2020. Preprint available at arXiv:2009.12020.