Recall that given a graph \( G \), its complement graph \( \overline{G} \) is the graph whose vertices are the same as those of \( G \), but whose edges are precisely the complement of the edges of \( G \)—in other words, if \((u, v)\) is an edge of \( G \), then \((u, v)\) is not an edge of \( \overline{G} \), and vice versa.

(a) Let \( C_n \) be a cycle graph of length \( n \), and let \( P_n \) be a path graph on \( n \) vertices (i.e. \( C_n \) minus an edge). Draw the complement graphs \( \overline{C_4}, \overline{C_5}, \overline{C_6}, \overline{P_3}, \overline{P_4}, \overline{P_5} \).

(b) Verify that \( \overline{C_5} \) is isomorphic to \( C_5 \) and that \( \overline{P_4} \) is isomorphic to \( P_4 \). Graphs with this property (that \( G \cong \overline{G} \)) are called self-complementary.

(c) Using the isomorphism between \( C_5 \) and \( \overline{C_5} \), reconstruct the example I gave in class of five length-2 strings that are not confusable from \( C_5 \).

(d) Generalize the construction above to prove that if \( G \) is a self-complementary graph with \( n \) vertices, then \( \Theta(G) \geq \sqrt{n} \).

(e)* Try to find more self-complementary graphs!

2 In this problem, we’ll explore a very important quantity associated to a graph, called its fractional chromatic number.

(a) Let \( \mathcal{I}(G) \) denote the collection of all non-empty independent sets in a graph \( G \). Write down all elements of \( \mathcal{I}(C_5) \) (make sure you write down 10 independent sets!).

(b) A fractional coloring of \( G \) is an assignment of a real number \( a_I \geq 0 \) for each \( I \in \mathcal{I}(G) \) such that for every vertex \( v \) of \( G \),

\[
\sum_{I \in \mathcal{I}(G), v \in I} a_I \geq 1
\]

Recall that a proper coloring of \( G \) is just a choice of some independent sets in \( G \) that cover all the vertices; these chosen independent sets are called color classes. Check that if we declare

\[
a_I = \begin{cases} 
1 & \text{if } I \text{ is a color class} \\
0 & \text{otherwise}
\end{cases}
\]

then this yields a fractional coloring. Thus, fractional coloring is a more general notion than proper coloring.

(c) The fractional chromatic number of \( G \) is defined as

\[
\chi_f(G) = \min \left\{ \sum_{I \in \mathcal{I}(G)} a_I \mid \{a_I\}_{I \in \mathcal{I}(G)} \text{ is a fractional coloring of } G \right\}
\]

Using (2b), check that \( \chi_f(G) \leq \chi(G) \).

(d) Find a fractional coloring \( \{a_I\}_{I \in \mathcal{I}(C_5)} \) of \( C_5 \) such that \( \sum_{I \in \mathcal{I}(C_5)} a_I = \frac{5}{2} \), and thus conclude that \( \chi_f(C_5) \leq \frac{5}{2} \). Tomorrow, we will see that the fractional chromatic number is an upper bound for the Shannon capacity (i.e. \( \Theta(G) \leq \chi_f(G) \) for all graphs \( G \)), so this implies that \( \Theta(C_5) \leq \frac{5}{2} \), which is better than our current upper bound of \( \chi(C_5) = 3 \).

(e)* Try to prove that \( \chi_f(C_5) = \frac{5}{2} \).
1. A sequence of real numbers $a_1, a_2, \ldots$ is called superadditive if for every $m, n$, we have that

$$a_{m+n} \geq a_m + a_n$$

A very useful theorem from analysis is the following, called Fekete’s Lemma.

**Lemma.** If $a_1, a_2, \ldots$ is a superadditive sequence, then $\lim_{n \to \infty} \frac{a_n}{n}$ exists.

Check that if $G$ is a graph and if we define

$$a_n = \log \omega(G^n)$$

then $a_1, a_2, \ldots$ is a superadditive sequence. Then apply Fekete’s Lemma to conclude that the Shannon capacity is well defined.

2. We know that for any graph, $\omega(G) \leq \chi(G)$. We also know that for the five-cycle $C_5$, we have that $\omega(C_5) = \chi(C_5) - 1$. A natural question is how large the gap between $\omega(G)$ and $\chi(G)$ can be. In this problem, we’ll explore the Mycielski construction, which shows that this gap can be arbitrarily large.

Given a graph $G$, its Mycielski $M(G)$ is a new graph defined as follows. If the vertices of $G$ are $u_1, \ldots, u_n$, then the vertices of $M(G)$ are $u_1, \ldots, u_n, v_1, \ldots, v_n, x$. Among the $u_i$s, we keep connectivity exactly as it was in $G$, namely if $u_i, u_j$ are adjacent in $G$, then they’re also adjacent in $M(G)$. Between the $v_i$s we don’t connect anything, although we connect $v_i$ to $u_j$ if and only if $(u_i, u_j)$ is an edge of $G$; in other words, each $v_i$ is a “clone” of $u_i$, and is connected to the $u_j$s that $u_i$ is connected to. Finally, the extra vertex $x$ is connected to all the $v_i$s. Here is a maybe-helpful schematic:

![Mycielski Construction Diagram]

The bottom layer are the $u_i$s (namely just a copy of $G$), the middle layer are the $v_i$s (with dotted lines not representing edges, just demonstrating the relationships copied from the bottom layer), and the top layer is $x$.

(a) Check that if $P_2 = \bullet \cdots \bullet$, then $M(P_2) = C_5$.

(b) Prove that if $G$ is triangle-free (i.e. there is no set of three vertices all pairs of which are adjacent), then $M(G)$ is triangle-free as well.

(c) Prove that $\chi(M(G)) = \chi(G) + 1$.

(d) We define the Mycielski graphs as $M_1 = P_2$, $M_2 = C_5 = M(M_1)$, and more generally $M_n = M(M_{n-1})$. Observe that each $M_n$ is triangle-free, so $\omega(M_n) = 2$ for all $n$, and that $\chi(M_n) = n+1$. Therefore, these graphs have arbitrarily large gaps between $\omega$ and $\chi$. Thus, for these graphs, we get arbitrarily bad bounds on $\Theta$. 
3* If you did the homework problem on the fractional chromatic number yesterday, check that it satisfies the two properties in the key lemma we proved today, namely that for any graphs $G, H$, we have

- $\chi_f(G) \geq \omega(G)$
- $\chi_f(G \cdot H) \leq \chi_f(G) \cdot \chi_f(H)$

Conclude that $\Theta(G) \leq \chi_f(G)$, and thus conclude that $\Theta(C_5) \leq \frac{5}{2}$.

4* In this problem, we will prove Fekete’s Lemma (from Problem 1). Let $a_1, a_2, \ldots$ be any superadditive sequence of real numbers.

(a) Pick any $m \geq 1$ and any $n \geq m$. Divide $n$ by $m$ with remainder to write $n = qm + r$, where $q, r$ are integers and $0 \leq r \leq m - 1$. By applying the superadditivity property many times, prove that

$$a_n \geq qa_m + ar$$

(b) Divide by $n$ to get

$$\frac{a_n}{n} \geq \frac{q}{n}a_m + \frac{ar}{n} \geq \frac{q}{n}a_m + \frac{\min_{0 \leq t \leq m-1} a_t}{n}$$

Now, let $n \to \infty$ while keeping $m$ fixed. Prove that the right-hand side of the above inequality converges to $a_m/m$.

(c) Conclude that the sequence $a_1/1, a_2/2, \ldots, a_m/m, \ldots$ can’t oscillate between a high and a low value infinitely many times, and thus that $\lim_{m \to \infty} a_m/m$ exists.
This problem will be about the geometry of regular pentagons; tomorrow, we will see that the facts here will end up being very useful for us.

(a) Prove that if a regular pentagon has side length 1, then the length of any diagonal is the golden ratio,
\[ \phi = \frac{1 + \sqrt{5}}{2} \]

(b) Prove that if a regular pentagon has side length 1, then the distance from any vertex to the center is
\[ \sqrt{\frac{1}{2} + \frac{\sqrt{5}}{10}} = \frac{\sqrt{\phi}}{\sqrt{5}} \]

(c) Check that
\[ \phi = \phi \sqrt{5} - 2 \]

2* For any \( k \geq 2 \), a \( k \)-vector coloring of a graph \( G \) is an assignment of vectors \( u_i \in \mathbb{R}^n \), one for each vertex of \( G \), such that if \( i \) and \( j \) are adjacent vertices, then
\[ u_i \cdot u_j = -\frac{1}{k-1} \]

(a) Prove that if we place three points \( v_1, v_2, v_3 \) in \( \mathbb{R}^2 \) so that they form an equilateral triangle centered at the origin, then
\[ v_1 \cdot v_2 = v_2 \cdot v_3 = v_1 \cdot v_3 = -\frac{1}{2} \]

(b) Similarly, prove that if we place four points in \( \mathbb{R}^3 \) so that they form a regular tetrahedron centered at the origin, then the dot product of any pair of them will be equal to \(-1/3\).

(c)* More generally, prove that \( n+1 \) points in \( \mathbb{R}^n \) that form a regular “simplex” centered at the origin have the property that the dot product of any pair equals \(-1/n\).

(d) Prove that if \( k \) is an integer and we have a proper \( k \)-coloring of a graph \( G \), then we can also get a \( k \)-vertex coloring of \( G \).

Hint: Place each color class at a vertex of a regular simplex in \( \mathbb{R}^{k-1} \).

(e) The vertex chromatic number of \( G \) is defined as
\[ \chi_v(G) = \min\{k : \text{there exists a } k \text{-vector coloring of } G\} \]

Note that we allow \( k \) to be any real number, not just an integer. Using the previous part, prove that \( \chi_v(G) \leq \chi(G) \).

(f)* Try to prove that \( \chi_v(G) = \theta(G) \).
1 To solve this problem, you’ll need some familiarity with modular arithmetic and a little bit of number theory.

If \( p \) is a prime that is 1 mod 4, then we define the Paley graph of order \( p \), denoted \( P_p \), as follows. The vertices are just the integers mod \( p \), namely 0, 1, \ldots, \( p - 1 \). Two distinct vertices are adjacent in this graph if their difference is a perfect square mod \( p \).

(a) Prove that this is a well-defined graph, namely that if \( x \) is adjacent to \( y \), then also that \( y \) is adjacent to \( x \) (notice that this is not immediate from the definition). To prove this, you may assume a standard fact from number theory, namely that if \( p \equiv 1 \mod 4 \), then \(-1\) is a perfect square mod \( p \).

(b) Check that the Paley graph of order 5 is just \( C_5 \).

(c) Prove that all Paley graphs are self-complementary (and thus solve a homework problem from the first assignment).

\textit{Hint:} To construct an isomorphism between \( P_p \) and \( \overline{P}_p \), fix a non-perfect-square \( z \mod p \), and map every vertex \( x \) to \( xz \).

(d) Conclude that \( \Theta(P_p) \geq \sqrt{p} \), using a problem on the first homework assignment.

(e) Try to prove that \( \vartheta(P_p) \leq \sqrt{p} \), which implies that \( \Theta(P_p) = \sqrt{p} \).

It is worth remarking that the clique number and chromatic number of \( P_p \) are unknown in general; it is believed that \( \omega(P_p) \approx \log p \) and \( \chi(P_p) \approx p/\log p \) for most values of \( p \), although it seems that proving anything like this will involve a major breakthrough in number theory (e.g. proving the Riemann Hypothesis). Thus, Paley graphs are in sharp contrast to the graphs we’ve been studying for much of this course (e.g. odd cycles), for which we know the clique and chromatic numbers, but have to work really hard to determine the Shannon capacity.

2 We’ve almost finished determining the Shannon capacity of \( C_5 \). However, the Shannon capacity of any larger odd cycle is still unknown (and new bounds, even if they are a very small improvement, would likely be considered a major breakthrough). In this problem, we’ll look a bit at the best known results for these larger odd cycles.

(a) It turns out that if \( k \) is odd, then

\[ \vartheta(C_k) = \frac{1 + \cos\left(\frac{\pi}{k}\right)}{\cos\left(\frac{\pi}{k}\right)} \]

and this is our best-known upper bound on \( \Theta(C_k) \). For the case \( k = 7 \), try to prove this, or at least prove that

\[ \vartheta(C_7) \leq 1 + \cos\left(\frac{\pi}{7}\right) \]

I believe that you can do this by exhibiting a suitable 4-dimensional orthonormal representation of \( C_7 \), but I’m not sure; I can’t picture \( \mathbb{R}^4 \) well enough.

(b) As for lower bounds, the best known result is the following, due to Bohman and Holzman:

\[ \Theta(C_{2n+3}) \geq \left(2^{2^n} + 1\right)^{1/2^n} > 2 \]

Note that though this beats the clique-number lower bound of 2, it does so \textit{very} slightly. They proved this result by exhibiting a clique of size \( 2^{2^n} + 1 \) in the OR power \( C_{2n+3}^2 \). Try to do this for the 7-cycle, namely the case \( n = 2 \): try to find a clique of size 17 in \( C_7^4 \).

(c) Can you improve on either of these bounds?