Market equilibria for homothetic, quasi-concave utilities and economies of scale in production

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Abstract

Eisenberg and Gale (1959) gave a convex program for computing market equilibrium for Fisher's model for linear utility functions, and Eisenberg (1961) generalized this to concave homogeneous functions of degree one. We further generalize to:

1. Homothetic, quasi-concave utilities. This also helps extend Eisenberg's result to concave homogeneous functions of arbitrary degree.

2. We introduce the notion of a trading cone which enables us to compute market equilibrium in the presence of economies of scale in production provided differential pricing is allowed. Applications to network pricing are provided.

1 Introduction

In a classic work Eisenberg and Gale [9] give a convex optimization program whose solution yields equilibrium allocations for the linear case of Fisher's market equilibrium problem [4], and Eisenberg [10] extended this approach to derive a convex program for concave homogeneous functions of degree one. Their program consists of maximizing a joint utility function of all buyers (a concave, logarithmic function) over a convex region defined via linear constraints. Their formulation has a number of attractive properties: Their joint utility function is the unique one satisfying the property that the joint utility of buyers remains unchanged if the money of one of the buyers, say b, is split among several new buyers with the same utility function as b (this follows from Nash bargaining problem [22]). The dual of their program yields equilibrium prices. The utility derived by a buyer is the same in all equilibria (contrast this with the very diverse payoffs received in various Nash equilibria of a game). For the linear case of Fisher's model, uniqueness and rationality of equilibrium prices follow easily from this formulation. Furthermore, equilibrium allocations can be shown to satisfy proportional fairness [20]. This formulation also gives the only known combinatorial characterization of the equilibria [16].

Our first result is to extend this approach further to derive a convex program for continuous, monotone, homothetic, quasi-concave utility functions. Using our technique, one can also extend Eisenberg's result to concave homogeneous functions of arbitrary degree. Our model also includes producers. At the heart of our proof is the following: we give a monotone transformation that yields a log-concave function that is "equivalent" to such a utility function. Our proof of this fact relies on a theorem of Friedman [11]. Furthermore, using [11] one can show that homotheticity is necessary for our result. Our convex program also inherits some of the fundamental properties of the Eisenberg-Gale's convex program, such as uniqueness of utilities, proportional fairness [20], and the combinatorial characterization [16].

The study of market equilibria occupies a central place in mathematical economics. This study was formally started by Walras [27] over a hundred years ago, and its climax came with the celebrated Arrow-Debreu Theorem, establishing existence of market equilibria under very general conditions. Despite this progress, the question of efficient computability of equilibria via polynomial time algorithms was not properly addressed until recently. The paper of Deng, Papadimitriou and Safra [5] brought this question to the forefront in the theoretical computer science community and has led to a surge of activity on this issue [6, 18, 8, 14, 15, 26, 7, 16, 29, 17].

Arrow and Debreu introduced production in their exchange model, a generalization of Fisher's model which does not demarcate between buyers and sellers, and showed the existence of equilibrium when producers satisfy decreasing economies of scale, i.e. production becomes less and less efficient with the quantity produced [2]. [23] reports that V. M. Polterovich enhanced Fisher's model with linear utilities to include producers, and extended the Eisenberg-Gale approach to derive a convex program for this setting. As reported in [23], Polterovich assumed only one producer who can

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not consume raw materials to produce a finish good.

Over the years, several attempts have been made on establishing existence of equilibria in the presence of economies of scale in production, but these attempts have had only limited success, and typically involve weakening the notion of equilibrium [24]. Indeed, this remains an important issue in mathematical economics. Using price differentiation, we can incorporate economies of scale in production in the following sense: production becomes more and more efficient as a function of the number of consumers of this good (rather than the amount of the good produced). We show existence of equilibrium and present a polynomial time algorithm for computing it. Such economies of scale are natural for instance in software, media and entertainment industries. Applications to network pricing are provided.

2 The model

Consider a market with sets N of buyers, G of goods, and M of producers, with $|G| = n$. Each buyer $i \in N$ has a specified initial endowment of money $e_i > 0$ and a concave, scalable utility function $u_i : \mathbf{R}^n_+ \to \mathbf{R}_+$ for the goods. ($\mathbf{R}^n$ denotes the n-dimensional Euclidean space; $\mathbf{R}^n_+$ denotes the subset of $\mathbf{R}^n$ where each coordinate is non-negative; $\mathbf{R}$ and $\mathbf{R}_+$ denote the set of reals and the set of non-negative reals, respectively; the j-th coordinate of a point in $\mathbf{R}^n$ corresponds to the j-th good in n.)

A utility function $u : \mathbf{R}^n_+ \to \mathbf{R}_+$ is said to be *concave* if for any $x, y \in \mathbf{R}^n_+$ and any $0 \leq \alpha \leq 1$, we have $u(\alpha x + (1-\alpha)y) \geq \alpha u(x) + (1-\alpha)u(y)$. It is *quasi-concave* if for any $x \in \mathbf{R}^n_+$ and $\alpha \in \mathbf{R}_+$, the set $\{x \in \mathbf{R}^n_+ : u(x) \geq \alpha\}$ is convex. For example, the function $e^x - 1$ is quasi-concave but not concave.

A utility fnction is *homothetic* if for any $x, y \in \mathbf{R}^n_+$ and any $\alpha > 0$, $u(x) \geq u(y)$ iff $u(\alpha x) \geq u(\alpha y)$. It is *monotone* if for any $x, y \in \mathbf{R}^n_+$ $x \geq y$ implies that $u(x) \geq u(y)$. It is *homogeneous* of degree d if for any $x \in \mathbf{R}^n_+$ and any $\alpha > 0$, $f(\alpha x) = \alpha^d f(x)$. We assume that $u(0) = 0$. The function $\log 1 + x$ is homothetic but not homogeneous.

Each producer $k \in M$ has the ability to produce certain goods and in doing so he is allowed to consume other goods. A *production point* for a producer is a point in $\mathbf{R}^n$. A production point can have positive and as well as negative coordinates, with positive coordinates representing output of the corresponding goods and negative coordinates representing consumption. The set of production points P_k of each producer k is given and forms a closed, bounded, convex set. We assume that there is a production point for each k at which the net amount (over all producers) of each good produced is strictly positive. The Fisher setting is a special case of the above when there is only one producer and his set of production points is a singleton set consisting of a point in $\mathbf{R}^n_+$ with each coordinate strictly positive.

An equilibrium is defined as a non-negative price vector $\pi \in \mathbf{R}^n_+$ at which there exist a bundle of goods $x_i \in \mathbf{R}^n_+$ for each buyer i, and production point $y_k \in P_k$ for each producer k such that the following conditions hold:

1. The vector x_i optimizes the utility of buyer i given her endowment e_i and the prices π, that is, x_i maximizes u_i over all $x \in \mathbf{R}^n_+$ such that $\pi^T x \leq e_i$.
2. The vector y_k maximizes the profit $\pi^T y$ over all $y \in P_k$.
3. For each good j, the total amount produced by the producers equals the total amount consumed by the buyers, that is, $\sum_{i \in N} x_{ij} = \sum_{k \in M} y_{kj}$.
4. The sum of the profits of all producers equals the sum of the money possessed by all buyers, that is, $\sum_{k \in M} \pi^T y_k = \sum_{i \in N} e_i$.

Equilibrium prices are also known as market clearing prices. We give a convex optimization problem whose optimal solution gives market clearing prices; the proof of this fact follows from the method of variational calculus. We assume that each convex set corresponding to producers is either an explicitly given polyhedron or a convex set with the corresponding strong separation oracle.

We assume that the utility function of a buyer is given via an oracle. That is, given $x \in \mathbf{R}^n_+$ and $\alpha \in \mathbf{R}_+$, the oracle tells us whether $\alpha \leq f(x)$ or not.

3 Obtaining a concave function from a quasi-concave homothetic function

Given a function $u : \mathbf{R}^n_+ \to \mathbf{R}_+$, a transformation yielding function $f : \mathbf{R}^n_+ \to \mathbf{R}_+$ is said to be a *monotone transformation* if for any $x, y \in \mathbf{R}^n_+$, if $u(x) > u(y)$ ($u(x) = u(y)$) then $f(x) > f(y)$ ($f(x) = f(y)$). It is easy to see that monotone transformations preserve monotonicity, quasi-concavity and homotheticity.

In this section, we prove the following central theorem.

THEOREM 3.1. *Let $u : \mathbf{R}^n_+ \to \mathbf{R}_+$ be a continuous, monotone, quasi-concave, homothetic function. Then there is a monotone transformation yielding a function $f : \mathbf{R}^n_+ \to \mathbf{R}_+$ that is homogeneous of degree one and is log-concave. Given an oracle for u and a point x such that $u(x) \neq 0$, the transformation can be approximated to any degree in polynomial time.*

We may assume w.l.o.g. that u is not identically zero. Suppose $u(y) = c \neq 0$. Then we can scale u by $1/c$ to ensure that u attains the value of 1 at some point. We assume this w.l.o.g. Let us define function f as follows. For $x \in \mathbf{R}_+^n$, if $u(x) = 0$ then $f(x) = 0$. Otherwise, $f(x) = \alpha$, where α is such that $u(x/\alpha) = 1$. We first prove that this transformation is well-defined, i.e., that such an α exists and is unique.

LEMMA 3.1. *If $u(x) \neq 0$ then there exists a unique $\alpha \in \mathbf{R}_+$ such that $u(x/\alpha) = 1$.*

Proof. By the assumption made above, $u(y) = 1$ for some $y \in \mathbf{R}_+^n$. Since $u(0) = 0$ and u is continuous and monotone, there exists $\beta \in \mathbf{R}_+$ such that $u(\beta y) < u(x)$. Now by homotheticity of u,

$$u(x/\beta) > u(y) = 1.$$

Finally, the continuity of u implies the existence of α.

Next we prove uniqueness of α. Since $u(0) = 0$ and u is continuous and homothetic, we get that if $u(x) = u(cx)$ for $x, c \neq 0$ then $u(dx) = 0$ for all $d \in \mathbf{R}_+$. Hence, non-uniqueness of α will contradict the assumption that $u(x) \neq 0$.

The definition of f and the monotonicity of u clearly imply that the transformation given above is monotone, i.e., $u(x) > u(y)$ ($u(x) = u(y)$) implies that $f(x) > f(y)$ ($f(x) = f(y)$).

LEMMA 3.2. *f is a homogeneous function of degree 1.*

Proof. Let $f(x) = \alpha \neq 0$. Then $u(x/\alpha) = 1$. Therefore $u(cx/c\alpha) = 1$. Hence $f(cx) = c\alpha$. The lemma follows.

We finally prove Theorem 3.1.

Proof. [Proof of Theorem 3.1] We apply the monotone transformation given above to obtain f from u and we need to show that f is log-concave. Clearly, f inherits monotonicity and quasi-concavity from u. Moreover, since f is a homogeneous function, $\log(f)$ is concave along any ray passing through the origin. We now apply Friedman's theorem [11] which states that a monotone, homothetic, quasi-concave function that is concave along any ray passing through the origin is concave. Hence we get that $\log(f)$ is concave.

Remark : Friedman gives an example showing that homotheticity is essential for his result. This example also shows that homotheticity is essential for Theorem 3.1 to hold.

Remark : Observe that the proof of Theorem 3.1 can be used to show that if f is a concave homogeneous function of degree d, then $f^{1/d}$ is a log-concave homogeneous function of degree 1. This helps extend Eisenberg's [10] result to concave homogeneous functions of arbitrary degree.

4 The convex program yielding market equilibrium

We will assume that the monotone transformation of Theorem 3.1 has already been applied to the given continuous, quasi-concave, monotone, homothetic utility function to yield an equivalent utility function that is homogeneous of degree one and is log-concave.

LEMMA 4.1. *Let $u(x)$ from $\Omega \subset R^n \to R$ be a homogeneous continuous function of degree d in C^1, that is, $u(\alpha x) = \alpha^d \cdot u(x)$ then*

$$\nabla u(x)^T x = \sum_j u_j(x) x_j = d \cdot u(x),$$

where $\nabla u(x)$ is the gradient vector function of $u(x)$, and $u_j(x)$ is the partial derivative function of $u(x)$ with respect to x_j.

Proof. For any given $x \in \Omega$, Consider $u((1+\epsilon)x))$ We have

$$\begin{aligned}(1+\epsilon)^d u(x) &= u((1+\epsilon)x)) \\ &= u(x) + \nabla u(x)^T((1+\epsilon)x - x) + o(\epsilon) \\ &= u(x) + \epsilon \nabla u(x)^T x + o(\epsilon)\end{aligned}$$

Thus,

$$(1+\epsilon)^d u(x) - u(x) = \epsilon \nabla u(x)^T x + o(\epsilon)$$

or

$$\frac{(1+\epsilon)^d - 1}{\epsilon} \cdot u(x) = \nabla u(x)^T x + \frac{o(\epsilon)}{\epsilon}.$$

Let $\epsilon \to 0$, we have the desired result.

Remark : In Lemma 4.1 and its proof we assumed that $u(x)$ is in C^1, i.e., $u(x)$ is differentiable. This is not necessary. One can use subdiffernties instead. For example, suppose

$$u(x) = \min\{u_1(x), ..., u_n(x)\}$$

where each $u_i(x) \in C^1$ with homogeneous degree d. Thus, $u(x)$ is not necessarily in C^1. At any point x, let

$$u(x) = u_1(x) = ... = u_m(x), \quad 1 \leq m \leq n.$$

Then, the subdifferentials of $u(x)$ is a convex combination of $\nabla u_1(x)$,..., $\nabla u_m(x)$, that is,

$$\nabla u(x) = \sum_{i=1}^{m} \alpha_i \nabla u_i(x)$$

where

$$\sum_{i=1}^{m} \alpha_i = 1,\ \alpha_i \geq 0,\ i = 1, ..., m.$$

Furthermore,

$$\begin{aligned}\nabla u(x)^T x &= \left(\sum_{i=1}^{m} \alpha_i \nabla u_i(x)\right)^T x \\ &= \sum_{i=1}^{m} \alpha_i (\nabla u_i(x)^T x) \\ &= \sum_{i=1}^{m} \alpha_i (d \cdot u_i(x)) \\ &= \sum_{i=1}^{m} \alpha_i (d \cdot u(x)) \\ &= d \cdot u(x)\end{aligned}$$

We will also incorporate producers into the model and using ideas from [23], give the convex program yielding market equilibrium prices. This will set the stage for introducing the notion of trading cone in the next section. Let m index producer k's production inequalities. We will show that the optimal solution to the following convex program yields equilibrium allocations and productions. Here the first set of inequalities provide production constraints for producer k, and the second set ensure that the consumption of each good does not exceed its production.

$$\begin{aligned}(4.1)\quad \text{maximize} \quad & \sum_i e_i \log(u_i(x_i)) \\ \text{subject to} \quad & \\ & \forall k : \forall m : \ \sum_j a_{jk}^m y_{jk} \leq b_k^m \\ & \forall j : \ \sum_i x_{ij} \leq \sum_k y_{jk} \\ & \forall i, j : \ x_{ij} \geq 0 \\ & \forall i : \ u_i \geq 0\end{aligned}$$

Under the assumption that the production sets are closed and bounded, and the assumption that there is a production point for each producer so that the net amount of each good produced (over all producers) is positive, and the (obvious) assumption that none of the utility functions is the identically-zero function, the maximum is well-defined. Let $\bar{x}_{ij}$, $\bar{u}_i$, $\bar{y}_{jk}$ denote an optimal solution to this convex program. Note that $\bar{u}_i > 0$ for all i. Consider the Lagrangian relaxation for the convex program 4.1 by introducing dual variables β_k^m for the first set, and p_j for the second set. At optimality, p_j's will turn out to be the equilibrium prices. Clearly at optimality each buyer buys her optimal, i.e., utility maximizing, bundle. We will additionally show that each producer is at a production point that maximizes his profit. This is accomplished by showing that the optimal solution to convex program 4.1 provides optimal solutions to the following LP which corresponds to producer k.

$$\begin{aligned}(4.2)\quad \text{maximize} \quad & \sum_j p_j y_{jk} \\ \text{subject to} \quad & \\ & \forall m : \ \sum_j a_{jk}^m y_{jk} \leq b_k^m \\ & \forall j : \ y_{jk} \text{ is unconstrained}\end{aligned}$$

Theorem 4.1. *An optimal solution to the convex program 4.1 optimizes for each buyer i her utility and for each producer k LP 4.2, i.e., buyers are buying optimal bundles and producers maximizing profits. Moreover, the money of each buyer is fully spent and the total money earned by producers is precisely equal to the total money initially possessed by buyers.*

We will first use Lemma 4.1 to show that buyers are buying optimal bundles. Then, the dual variables introduced in obtaining the Lagrangian relaxation of 4.1 will be used in constructing the duals of LP's 4.2. The idea of the rest of the proof is to derive conditions on the primal and dual variables from the optimality of 4.1 which yields feasibility of dual solutions constructed to LP's 4.2. Since LP's 4.2 satisfy complementary slackness conditions w.r.t. these feasible duals, the primal solutions constructed are optimal.

Proof. Let us start by taking the Lagrangian relaxation of program 4.1:

$$\begin{aligned}f &= \min_{\alpha_i^l \geq 0, \beta_k^m \geq 0, p_j \geq 0} \ \max_{x_{ij} \geq 0, y_{jk}, u_i} \sum_i e_i \log u_i \\ & - \sum_{kl} (b_k^m - \sum_j a_{jk}^l y_{jk}) \beta_k^m - \sum_j (\sum_k y_{jk} - \sum_i x_{ij}) p_j.\end{aligned}$$

First, the feasible set of the optimization problem is compact, the maximal solution exists and the maximum value is finite. Moreover, the optimization problem is convex. Setting the partial differential of f w.r.t. x_{ij} to be zero, we get ([21], page 105) that there exist p_j such that the following conditions are necessary for optimality:

$$\begin{aligned} e_i \frac{u_{ij}(x)}{u_i(x)} &\leq p_j, \ \forall i, j \\ e_i \frac{u_{ij}(x) x_{ij}}{u_i(x)} &= p_j x_{ij}, \end{aligned}$$

where p is the n-dimensional optimal dual price (Lagrangean) vector. The second equality constraint is called the complementarity condition.

To prove p is a market clearing price, we sum the complementarity condition equations over j for agent i, and have

$$\begin{aligned} \sum_j p_j x_{ij} &= \sum_j e_i \frac{u_{ij}(x) x_{ij}}{u_i(x)} \\ &= e_i \frac{\sum_j u_{ij}(x) x_{ij}}{u_i(x)} \\ &= e_i \frac{u_i(x)}{u_i(x)} \\ &= e_i \end{aligned}$$

which implies that under prices p each buyer spends her money completely.

Next setting the partial differential of f w.r.t. y_{jk} to zero, we get that there exist non-negative β_k^m such that

$$\sum_m a_{jk}^m \beta_k^m = p_j \tag{4.3}$$

$$\beta_k^m \sum_j a_{jk}^m \bar{y}_{jk} = \beta_k^m b_k^m \tag{4.4}$$

$$p_j \sum_i \bar{x}_{ij} = p_j \sum_k \bar{y}_{jk} \tag{4.5}$$

We next obtain the dual of LP 4.2.

$$\begin{aligned} \text{(4.6)} \quad \text{minimize} \quad & \sum_m b_k^m \rho_k^m \\ \text{subject to} \quad & \forall j : \ \sum_m a_{jk}^m \rho_k^m = p_j \\ & \forall m : \ \rho_k^m \geq 0 \end{aligned}$$

Note that from Equation 4.3, $\rho_k^m = \beta_k^m$ forms a feasible dual solution for this LP. Also, $\bar{y}_{jk}$ is feasible for LP 4.2. It is easy to verify using Equation 4.4 that the complementary slackness conditions are satisfied. Hence $\bar{y}$ is an optimal solution for the primal LP 4.2.

Thus both consumers and producers are making optimal choices with respect to prices p_j. Now we have to show that the market clears. Note that whenever p_j is positive, then by Equation 4.5 the corresponding inequality for the good j is tight in convex program 4.1. In other words whenever there is a surplus of good j its price is zero. If there is surplus of some good that has zero price we just give this surplus to some buyer. Now we need to check the conservation of money. Again, using the fact that each buyer spends her endowment and Equation 4.5, we get:

$$\begin{aligned} \sum_i e_i &= \sum_i \sum_j p_j \bar{x}_{ij} = \sum_j p_j \sum_i \bar{x}_{ij} \\ &= \sum_j p_j \sum_k \bar{y}_{jk} = \sum_k \sum_j p_j \bar{y}_{jk}. \end{aligned}$$

Hence we have conservation of money.

Note that once we have an optimal solution to the convex program 4.1, the optimality conditions yield a linear program to find the dual variables p_j, α_i^l, and β_k^m. One can solve this linear program within some bounded precision. Alternatively, one can use primal-dual path following interior point methods to solve the convex program. As a side result one can derive the value of the dual variables. The case with separation oracle can be solved as in [19]. When we solve a convex program using the ellipsoid algorithm, the algorithm considers only polynomially many separating hyperplanes (because it is a polynomial time algorithm). This polynomial number of separating hyperplanes forms a proof that the run of the algorithm has found an optimal solution. In essence, if one writes a linear program consisting of half-spaces used by a run of the ellipsoid algorithm then this linear program will have the same solution. Again, one needs to consider the dual variables corresponding to these hyperplanes only; this is a polynomial sized program. Once we know a primal solution, optimality conditions give a small sized linear program. Alternatively, for certain utility functions polynomial-time interior-point algorithms can be used to further reduce the computational complexity of the problem, see [29].

Theorem 4.2. *There is a polynomial, in the input size and* $\log(1/\epsilon)$, *time algorithm to find a feasible primal and a dual for convex program 4.1 (and also for convex program 5.7 defined later) such that all the complementarity slacks are less that* ϵ.

5 The trading cone and economies of scale

The second set of inequalities in convex program 4.1 say that the quantity of good consumed is at most the quantity of good produced. This is not true for goods which can be simultaneously consumed, e.g., a software package which can be used by many users. Even in production, many inputs can be used simultaneously, e.g., an authors' efforts are used simultaneously whereas the physical book itself can't be. The situation is much more complicated with the service sector. To encompass such situations, we define a new notion of a *trading cone* which defines the feasibility of consumption with production.

We replace the second set of inequalities in convex program 4.1 as follows. These inequalities, which are indexed by h, will create price differentiation for buyers as well as producers. In return, they will enable us to impose desirable economic properties, such as introducing economies of scale for consumption.

$$
\begin{aligned}
(5.7) \quad & \text{maximize} \quad && \sum_i e_i \log(u_i(x_i)) \\
& \text{subject to} && \forall k : \forall m : \ \sum_j a_{jk}^m y_{jk} \leq b_k^m \\
& && \forall h : \ \sum_{i,j} d_{ij}^h x_{ij} \leq \sum_{j,k} g_{jk}^h y_{jk} \\
& && \forall i, j : \ x_{ij} \geq 0 \\
& && \forall j, k : \ y_{jk} \text{ is unconstrained}
\end{aligned}
$$

Let the dual variable for the new set of inequalities be δ_h; those for the rest of the inequalities are as before. Setting the partial differential of f w.r.t. x_{ij}, y_{jk} respectively to be zero, we get:

$$\frac{e_i u_{ij}(x_i)}{u_i} = \sum_h \delta_h d_{ij}^h$$

$$\sum_m a_{jk}^m \beta_k^m = \sum_h \delta_h g_{jk}^h$$

Let p_{ij} denote the price of good j for buyer i and q_{jk} denote the price of good j for producer k. Then, by letting $p_{ij} = \sum_h \delta_h d_{ij}^h$ and $q_{jk} = \sum_h \delta_h g_{jk}^h$, one can show conservation of money as in Theorem 4.1.

Let us see by an example how the notion of trading cone can be used to model economies of scale, which had been a hard modeling question in the theory of equilibrium. Let us consider the set cover problem. This problem consists of a set of buyers, U, and a set of sellers, each with a fixed endowment of a subset of U. A buyer i has a money e_i and her utility function is linear in the number of sets she buys containing her. The supply of each set, S, is d_S. Also, for each set S, we have a submodular function f_S; for $T \subset S$, $f_S(T)$ says how much S is needed to serve each user in T to the extent of one. Economies of scale are being modeled since f_S is submodular (more precisely f_S is a polymatroid function, because f_S is submodular, monotonic, non-negative and zero at the empty set).

Let us now consider various scenarios based on f_S's. Suppose each f_S is just the cardinality function, i.e., $f_S(T) = |T|$. This means each copy of S can serve at most one user to the extent of one. We call such a set S an exclusive good. As an example, a book can be shared by two people to the extent of half. The corresponding trading cone can be written as: $\forall S : \ \sum_{i \in S} x_{iS} \leq y_S$, where x_{iS} denotes the consumption of S by i, and y_S denotes the supply of S. Another case is when $f_S(T) = 1$ for every nonempty set T. We call such a good a nonexclusive good. An example is services of an author to write a book. Another major example is software. Once some software is developed, it can used by many users on a nonexclusive basis. The corresponding trading cone can be written as: $\forall S : \ \max_{i \in S} x_{iS} \leq y_S$, which further can be written as: $\forall S$ and $i \in S : \ x_{iS} \leq y_S$.

These two cases are the extremes for submodular functions. Typically, in real life, for any manufacturing activity we have both kinds of input, exclusive and nonexclusive, e.g., exclusive inputs are physical raw material and nonexclusive inputs are research and development. Other inputs, like labor, are typically neither exclusive nor nonexclusive, but still satisfy economies of scale.

Next let us deal with general submodular functions. We want to come up with a cone which defines the feasibility of the x and y variables. Consider a set S. Suppose it is supplied to the extent of y_S. Suppose x_{iS} is the demand for this set by user i. We want to describe whether the supply y_S of S can satisfy the demand of x_{iS}'s. In other words we want to figure out whether y_S can be decomposed into y_S^T's, where T's are subsets of S, and y_S^T denotes the extent to which S is available for T, i.e.,

$$\sum_{T \subseteq S} y_S^T \leq y_S$$

$$\forall i \in S : \quad x_{iS} \leq \sum_{T : i \in T} \frac{y_S^T}{f_S(T)} .$$

For convenience we define $z_S^T = y_S^T / f_S(T)$. Using this equality the above two conditions become:

$$(5.8) \quad \sum_{T \subseteq S} f_S(T) z_S^T \leq y_S$$

$$(5.9) \quad \forall i \in S : \quad x_{iS} \leq \sum_{T : i \in T} z_S^T$$

Lemma 5.1. *Suppose z satisfy inequalities 5.8 and 5.9. We may assume that whenever $z_S^{T_1} > 0$ and $z_S^{T_2} > 0$, either $T_1 \subseteq T_2$ or $T_2 \subseteq T_1$. W.l.o.g. we may in fact assume that the former is the case.*

Lemma 5.1 says that we may assume that the z_S's are zero, except for a telescoping sequence of subsets. Let us consider a permutation σ on users, so that without loss of generality we may assume that z_S is zero except for the sets $T_1 = \{\sigma_1\}, T_2 = \{\sigma_1, \sigma_2\}, \cdots T_s = \{\sigma_1 . \sigma_2, \cdots, \sigma_s\}$, where s is the cardinality of S.

Lemma 5.2. *Inequalities 5.8 and 5.9 imply the following:*

$$f_S(T_1)x_{\sigma_1 S} + (f_S(T_2) - f_S(T_1))\, x_{\sigma_2 S}$$
$$+ (f_S(T_3) - f_S(T_2))\, x_{\sigma_3 S} \cdots (f_S(T_s) - f_S(T_{s-1}))\, x_{\sigma_s S} \leq y_S .$$

Proof. Denote the empty set by T_0, so we have $f_S(T_0) = 0$. Now the left hand side of the above inequality becomes:

$$\sum_{i=1}^{s} (f_S(T_i) - f_S(T_{i-1}))\, x_{\sigma_i S}$$
$$\leq \sum_{i=1}^{s} (f_S(T_i) - f_S(T_{i-1})) \sum_{T_j : i \in T_j} z_S^{T_j}$$
$$= \sum_{j=1}^{s} \sum_{i=1}^{j} (f_S(T_i) - f_S(T_{i-1}))\, z_S^{T_j} .$$

The first inequality follows from the inequality 5.9 and the second equality follows by changing the order of summation.

$$\sum_{j=1}^{s} \sum_{i=1}^{j} (f_S(T_i) - f_S(T_{i-1}))\, z_S^{T_j} = \sum_{j=1}^{s} f_S(T_j) z_S^{T_j} \leq y_S .$$

Here the first equality follows from cancellation and the second follows from the inequality 5.8.

Theorem 5.1. *The trading cone corresponding to submodular function f_S consists of the following inequality for each permutation σ on the users in S:*

$$f_S(T_1)x_{\sigma_1 S} + (f_S(T_2) - f_S(T_1))\, x_{\sigma_2 S}$$
$$+ (f_S(T_3) - f_S(T_2))\, x_{\sigma_3 S} \cdots (f_S(T_s) - f_S(T_{s-1}))\, x_{\sigma_s S}$$
$$\leq y_S .$$

Proof. Let us prove the easier direction first. Suppose the inequality in this theorem is valid for every permutation then we need to show that we can find z_S^T's so that inequalities 5.8 and 5.9 hold. Choose the permutation σ which puts x_{iS}'s in descending order. Choose $z_S^{T_j} = x_{\sigma_j S} - x_{\sigma_{j+1} S}$ except for $z_S^{T_s}$ which is simply $x_{\sigma_s S}$. It is easy to verify inequalities 5.8 and 5.9.

For the harder direction, suppose we have x_{iS}'s and y_S so that z_S^T's exist that satisfy inequalities 5.8 and 5.9. Again consider a permutation σ which puts x_{iS}'s in descending order. It is not difficult to see that the inequality in the theorem is satisfied for this permutation. But what about the other permutations? Consider π as an arbitrary permutation. Using submodularity we will show that the left hand side of the inequality in the theorem for permutation π is at most the left hand side of the inequality for permutation σ. It is easy to verify that when π also puts x_{iS}'s in descending order then the left hand side of the inequality in theorem is the same for π and σ.

Let us assume that π does not put $x'_{iS}s$ in descending order. Suppose for some j, $x_{\pi_j S} < x_{\pi_{j+1} S}$. We create another permutation by interchanging the places of j and $j+1$ and keeping the rest the same. We claim that this procedure can not decrease the left hand side of the inequality in the theorem.

Suppose it does, then we have:

$$(f_S(\{\pi_1, \ldots, \pi_j\}) - f_S(\{\pi_1, \ldots, \pi_{j-1}\}))\, x_{\pi_j S}$$
$$+ (f_S(\{\pi_1, \ldots, \pi_{j+1}\}) - f_S(\{\pi_1, \ldots, \pi_j\}))\, x_{\pi_{j+1} S}$$
$$> (f_S(\{\pi_1, \ldots, \pi_{j-1}, \pi_{j+1}\}) - f_S(\{\pi_1, \ldots, \pi_{j-1}\}))\, x_{\pi_{j+1} S}$$
$$+ (f_S(\{\pi_1, \ldots, \pi_{j+1}\}) - f_S(\{\pi_1, \ldots, \pi_{j-1}, \pi_{j+1}\}))\, x_{\pi_j S} .$$

Rearranging terms we get:

$$(f_S(\{\pi_1, \ldots, \pi_j\}) + f_S(\{\pi_1, \ldots, \pi_{j-1}, \pi_{j+1}\})$$
$$- f_S(\{\pi_1, \ldots, \pi_{j-1}\}) - f_S(\{\pi_1, \ldots, \pi_{j+1}\})\, (x_{\pi_j S} - x_{\pi_{j+1} S})$$
$$> 0$$

This gives a contradiction because the first factor is nonnegative by submodularity of f_S and the second factor is negative because of the assumption. This completes the proof.

This theorem shows that an equilibrium exists. One can consider the trading cone consisting of all the s-factorial inequalities for every set S. The number of such inequalities can't be explicitly written. The proof of the above theorem also gives us a separation oracle. Given $x'_{iS}s$ and y_S, verify the inequality corresponding to that permutation which put x_{iS}'s in descending order.

Economies of scale via submodular functions are just an example of the power we get by introducing trading cones. It is worthwhile studying this notion further.

5.1 A natural application to network pricing

Using the framework proposed by [1] for utilizing Network Coding, we present a natural application of the notion of trading cone to network pricing.

Suppose we are given a directed network with capacities on edges and a special node s, the sender, that is running a broadcasting session. There is a set of receivers, R, who want to receive this broadcast. The sender s is running the broadcasting session in an asynchronous fashion. The sender has M packets to broadcast and keeps sending out random linear combinations of these M packets. Random linear combinations are linearly independent with high probability so each receiver needs to accumulate only a little more than M packets to retrieve the information. This scheme allows the receivers to accumulate packets at different rate.

Using the framework of Network Coding [3], each receiver can accumulate the packets at a rate which equals the bandwidth between the sender and the receiver (see [28]). Hence an edge can simultaneously augment the flow to more than one receiver. This is precisely the notion of economies of scale which the trading cone can deal with. Let us define a market in this setting. Each receiver is a buyer and each edge is a seller of bandwidth – the most it can sell to a buyer is its capacity. Assume that receiver i has money m_i. ([1] use this notion to control congestion in a scenario where multiple broadcasting sessions are being run on the same network. This is along the lines of [20].) Each receiver wants to maximize the rate at which packets are accumulated and we assume that a buyer's utility is proportional to this rate. Using network coding, this rate is the maximum flow from the sender to the receiver. The convex program whose optimal solution gives market equilibrium is as follows:

$$\begin{aligned}
(5.10)\quad & \text{maximize} && \sum_i m_i \log(r_i) \\
& \text{subject to} && \forall i: \sum_{p \in P_i} x_p \geq r_i \\
& && \forall e: \ y_e \leq Cap(e) \\
& && \forall e, i: \sum_{p \in P_i : e \in p} x_p \leq y_e \\
& && \forall i: \ r_i \geq 0 \\
& && \forall p: \ x_p \geq 0
\end{aligned}$$

Here variable r_i denotes the rate of consumption by the i-th receiver, P_i the set of all paths from the sender to receiver, $Cap(e)$ the capacity of e, and y_e the amount of capacity sold by edge e. The first set of constraints chooses paths for routing r_i amount of flow from the sender to the i-th receiver. The second set of constraints implies that an edge can not sell more capacity than available. The third set of constraints represents the trading cone between what the receivers bought and what the edges sold. Note that a unit capacity bought on some edge e can potentially be used simultaneously by all the receivers but can't be used to more than a unit extent by any one receiver.

In the Lagrangian relaxation, the dual variable corresponding to the first set of constraints for the i-the receiver denotes the amount of money paid by the receivers for one unit of flow. The dual variables corresponding to the second set of constraints for the edge e denote the amount of money received by edge e for one unit of capacity. The dual variables for the third set of constraints denote how the money received from the receivers is distributed among various edges. Note that an edge can receive money from more than one receiver and that too at an unequal rate.

A more enlightening example is when there are more than one broadcasting sessions happening on the same network. Let the senders of these sessions be denoted by s_j's. Let R_j denote the set of receivers interested in getting data from s_j. We superscript the variables corresponding to the j-th broadcast by j. The Eisenberg-Gale convex program becomes:

$$\begin{aligned}
(5.11)\quad & \text{maximize} && \sum_{j, i \in R_j} m_i^j \log(r_i^j) \\
& \text{subject to} && \forall j, i \in R_j: \sum_{p \in P_i^j} x_p^j \geq r_i^j \\
& && \forall e: \ y_e \leq Cap(e) \\
& && \forall e: \sum_j y_e^j \leq y_e \\
& && \forall e, j, i \in R_j: \sum_{p \in P_i^j : e \in p} x_p^j \leq y_e^j \\
& && \forall j, i \in R_j: \ r_i^j \geq 0 \\
& && \forall p: \ x_p^j \geq 0
\end{aligned}$$

The variables have the same meaning as in the previous scenario except they are in the context of j. The first set of constraints represent the paths bought by the i-th receiver in j-th sessions. The second constraints represent the capacity sold by the edge e. The third and the fourth sets of constraints represent the capacity bought by different sessions. Note that a unit of capacity sold on one edge can help only one session. So there is no economies of scale. This means that the capacity on an edge e is sold to different sessions at the same price. Within a session this capacity can be used by many receivers simultaneously hence the price paid by the session to the edge e is the sum of prices paid by the receivers in the session. The situation can be thought of as if a session acts as an intermediary to buy capacity on the edges for its receivers. This example demonstrates that the trading cone can be quite sophisticated. Agarwal et. al. [1] is exploiting this interpretation of trading cone in controlling the congestion in case there are many broadcasting sessions running based on Network Coding.

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