1 Introduction

Let $G$ be a connected linear algebraic group over a global function field $k$. Recall that $\tau(G)$ denotes the Tamagawa number of $G$ (cf. [Oes] for the precise definition); $\III(G) := \ker(H^1(k, G) \to \prod_v H^1(k_v, G))$ the Tate-Shafarevich set of $G$ (where the product is over all places $v$ of $k$); and $\text{Ext}^1(G, \mathbb{G}_m) := \{L \in \text{Pic}(G) \mid m^*L \simeq p_1^*L \otimes p_2^*L\}$, where $m, p_i : G \times G \to G$ are the multiplication and projection maps ($i = 1, 2$). Recall Conjecture 1.1.1 in [Ros2].

**Conjecture 1.1.** Let $k$ be a global function field, $G$ a connected linear algebraic group over $k$. Then

$$\tau(G) = \frac{\# \text{Ext}^1(G, \mathbb{G}_m)}{\# \III(G)}$$

The goal of this document is to prove that Conjecture 1.1 holds for pseudo-reductive groups.

**Theorem 1.2.** Let $k$ be a global function field, $G$ a pseudo-reductive $k$-group. Then

$$\tau(G) = \frac{\# \text{Ext}^1(G, \mathbb{G}_m)}{\# \III(G)}$$

The proof proceeds via dévissage, taking as input three crucial ingredients:

(i) the classification of pseudo-reductive groups by Conrad, Gabber, and Prasad [CGP]

(ii) the proof by Gaitsgory and Lurie of Weil’s conjecture that $\tau(G) = 1$ for any simply-connected group over a global function field [GL]

(iii) Conjecture 1.1 holds for all connected commutative linear algebraic groups over global function fields [Ros2, Th. 1.1.3]. This is a consequence of the extension of Tate duality to positive-dimensional affine groups proved in [Ros2].
2 Standard pseudo-reductive groups

In this section we will prove Theorem 1.2 for standard pseudo-reductive groups. These exhaust all pseudo-reductive groups in characteristics greater than 3.

So let $G$ be a standard pseudo-reductive $k$-group. Then we have

$$G = (C \ltimes R_{k'/k}(G'))/\mathcal{E}$$

(2.1)

where the objects on the right are defined as follows. First, $k'$ is a finite reduced $k$-algebra (i.e., a product of finitely many finite extension fields of $k$), and $G'$ is a $k'$-group scheme with simply-connected (absolutely simple) fibers. We have $\mathcal{E} = R_{k'/k}(T')$ for some maximal $k'$-torus $T' \subset G'$. Let $Z_{G'} \subset G'$ denote the center. Then the conjugation action of $C$ on $R_{k'/k}(G')$ factors through $R_{k'/k}(T'/Z_{G'})$. We suppose that we have a factorization

$$\mathcal{E} \xrightarrow{\phi} C \rightarrow R_{k'/k}(T'/Z_{G'})$$

with $C$ commutative pseudo-reductive, and $C$ then acts on $R_{k'/k}(G')$ via this factorization. Then the quotient on the right side of (2.1) is obtained via the anti-diagonal embedding $C \rightarrow C \ltimes R_{k'/k}(G')$, where $j : \mathcal{E} \rightarrow R_{k'/k}(G')$ is the inclusion. It is easy to see that this anti-diagonal map is an isomorphism onto a central subgroup scheme, hence the quotient makes sense. Let us note that we may take $T'$ to be any maximal $k'$-torus in $G'$, cf. the third paragraph of §4.1 in [CGP]. By [H], Lemmas 5.5.3 and 5.5.4, and the remark after Lemma 5.5.4 (on p. 408), therefore, we may choose $T'$ so that $\Pi^2(k', T') = 0$. It follows that we also have

$$\Pi^2(k, R_{k'/k}(T')) = 0$$

(2.2)

Now there is an embedding $\mathcal{E} \hookrightarrow B$, where $B = R_{k''/k}(T'')$ for some finite reduced $k$-algebra $k''$ and some split $k''$-torus $T''$. Indeed, we may choose a finite étale $k'$-algebra $k''$ such that $T'_{k''}^{'}$ is split. Then we have the canonical inclusion $\mathcal{E} = R_{k'/k}(T') \hookrightarrow R_{k'/k} R_{k''/k}(T'_{k''}) = R_{k''/k}(T'_{k''})$. Consider the following pushout diagram:

$$\begin{array}{cccc}
1 & \rightarrow & \mathcal{E} & \rightarrow & C \ltimes R_{k'/k}(G') & \rightarrow & G & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & B & \rightarrow & II & \rightarrow & G & \rightarrow & 1
\end{array}$$

(2.3)

We need the following lemma, which is a slight variant of [San, Cor. 10.5].

**Lemma 2.1.** Let $k$ be a global field. Suppose that we have an exact sequence of connected linear algebraic $k$-groups

$$1 \rightarrow G' \xrightarrow{i} G \xrightarrow{\pi} G'' \rightarrow 1$$
where $G'$ is such that $G'_{k_v}$ is rational; $H^1(k_v, G') = 1$ for every place $v$ of $k$; $H^1(k, G') = 1$; and $\text{Ext}^1(G', G_m) = 1$. Then

$$\frac{\tau(G') \tau(G'')}{\tau(G)} = \frac{\# \text{Ext}^1(G'', G_m)}{\# \text{Ext}^1(G, G_m)}.$$

**Proof.** By [Oes, Ch. III, Th. 5.3],

$$\tau(G)|G''(A)/(\pi(G(A))G''(k))| = \tau(G')|\tau(G'')|\ker(\text{III}^1(k, i))||\text{coker}(\hat{i})|^{-1}$$

Here, $A$ denotes the adeles of $k$, $\text{III}^1(k, i)$ is the map on $\text{III}$'s induced by $i$, and $\hat{i} : \hat{G}(k) \to \hat{G'}(k)$ is the map on the groups of $k$-characters induced by $i$. The lemma will then follow if we prove that (i) $|G''(A)/(\pi(G(A))G''(k))| = 1$; (ii) $|\ker(\text{III}^1(k, i))| = 1$; and (iii) $|\text{coker}(\hat{i})| = \# \text{Ext}^1(G, G'', G_m)/\# \text{Ext}^1(G, G_m)$.

(i) By [Oes, Ch. I, §3.6], $\pi(G(A)) \subset G''(A)$ is open, hence $G''(O_v) \subset \pi(G(O_v))$ for almost every place $v$ of $k$. Thus, in order to prove (i), it's enough to show that $\pi(G(k_v)) = G''(k_v)$ for every $v$. But this follows from our assumption that $H^1(k_v, G') = 1$.

(ii) follows immediately from our assumption that $H^1(k, G') = 1$.

(iii) By [Ros1, Lem. 6.3], we have an exact sequence

$$\hat{G}(k) \xrightarrow{\hat{i}} \hat{G'}(k) \to \text{Ext}^1(G'', G_m) \to \text{Ext}^1(G, G_m) \to \text{Ext}^1(G', G_m)$$

The formula (iii) then follows from our assumption that $\text{Ext}^1(G', G_m) = 0$. □

Thanks to Lemma 2.1, the bottom sequence in diagram (2.3) yields

$$\frac{\tau(G)}{\tau(H)} = \frac{\# \text{Ext}^1(G, G_m)}{\# \text{Ext}^1(H, G_m)} \quad (2.4)$$

Now let $A$ be the (smooth connected commutative) cokernel of the obvious inclusion $R_{k'/k}(G') \to H$. Applying Lemma 2.1 to the exact sequence

$$1 \to R_{k'/k}(G') \to H \to A \to 1 \quad (2.5)$$

yields

$$\frac{\tau(H)}{\tau(A)} = \frac{\# \text{Ext}^1(H, G_m)}{\# \text{Ext}^1(A, G_m)} \quad (2.6)$$

Multiplying equations (2.4) and (2.6), we obtain

$$\frac{\tau(G)}{\tau(A)} = \frac{\# \text{Ext}^1(G, G_m)}{\# \text{Ext}^1(A, G_m)} \quad (2.7)$$
Since $A$ is commutative, [Ros2, Th. 1.1.3] yields

$$\tau(A) = \frac{\# \text{Ext}^1(A, G_\text{m})}{\# \Pi^1(A)}$$

Substituting this into (2.7) yields

$$\tau(G) = \frac{\# \text{Ext}^1(G, G_\text{m})}{\Pi^1(A)} \quad (2.8)$$

Now, the bottom sequence in diagram (2.3) yields a bijection $\Pi^1(H) \cong \Pi^1(G)$. Indeed, the surjectivity follows from the fact that $\Pi^2(B) = 0$ (by class field theory), and the injectivity may be deduced by twisting the sequence and using the fact that in any such twisted sequence, the kernel is still $B$, because $B \subset H$ is central. On the other hand, I claim that the map $\Pi^1(H) \to \Pi^1(A)$ coming from (2.5) is a bijection. Assuming this, we see that $\#\Pi^1(A) = \#\Pi^1(H) = \#\Pi^1(G)$. Combining this with equation (2.8) proves Theorem 1.2.

So it only remains to show that the map $\Pi^1(H) \to \Pi^1(A)$ is a bijection. First we check injectivity. Let us call a pseudo-reductive $k$-group quasi-trivial if it is of the form $R_{k''/k}(G'')$ for some finite reduced $k$-algebra $k''$ and some $k''$-group $G''$ with simply-connected fibers. Since such group has trivial $H^1$, in order to prove the desired injectivity it suffices to show that any étale twist of the sequence (2.5) still has quasi-trivial kernel. For this, we want to check that quasi-triviality may be checked over $k_s$. This follows from Galois descent, combined with the uniqueness of standard presentations compatible with a given torus, [CGP, Prop. 5.2.2].

Now we have to prove surjectivity. Let $F$ be defined by the following pushout diagram:

$$
\begin{array}{ccc}
\mathcal{C} & \longrightarrow & C \times_{k'} R_{k'/k}(T') \\
\downarrow & & \downarrow \\
B & \longrightarrow & F
\end{array}
$$

Then we have a commutative diagram of exact sequences

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & R_{k'/k}(T') & \longrightarrow & F & \longrightarrow & A & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & R_{k'/k}(G') & \longrightarrow & H & \longrightarrow & A & \longrightarrow & 1
\end{array}
$$

Since $\Pi^2(R_{k'/k}(T')) = 0$ by (2.2), any element of $\Pi^1(A)$ lifts to $H^1(k, F)$, hence to $H^1(k, H)$. Since $H^1(k_v, R_{k'/k}(G')) = 1$ for all places $v$ of $k$, it follows that this lift must actually lie in $\Pi^1(H)$, hence the map $\Pi^1(H) \to \Pi^1(A)$ is surjective, as desired.
3 Beyond the standard case

Now we will sketch how to prove Theorem 1.2 for general (not necessarily standard) pseudo-reductive groups. Such non-standard groups only occur in characteristic 2 and 3. So suppose that $G$ is pseudo-reductive. $G = G_1 \times G_2$, where $G_1$ is totally non-reduced (This can be nontrivial only in characteristic 2.) and $G_2$ is reduced. Since $(G_2)_k$ is rational (as we will see below), Ext is multiplicative on the factors, so we may treat these two cases separately. Since it is more analogous to the standard case, let us first treat the case in which $G$ is reduced. It is then necessarily either commutative or generalized standard. The commutative case follows from Poitou-Tate in positive dimension, so we may assume that $G$ is generalized standard. That is, we have

$$G = (C \ltimes R_{k'/k}(G'))/\mathscr{C}$$

where $k'$ is a finite reduced $k$-algebra; $\mathscr{C} = R_{k'/k}(Z_{G'})$; $T' \subset G'$ is a maximal $k'$-torus; there is a factorization

$$\mathscr{C} \twoheadrightarrow C \rightarrow R_{k'/k}(T'/Z_{G'})$$

through which $C$ acts on $G'$ via conjugation; and each factor of $G'$ is either absolutely simple and simply connected, or basic exotic pseudo-reductive.

We note once again that we may choose $T' \subset G'$ to be an arbitrary maximal $k'$-torus, by [CGP, Prop. 10.2.2(3)]. We choose $T'$ as follows. On the simply connected fibers, we simply choose $T'$ to be a maximal torus with trivial $X_2$. On the basic exotic fibers, we consider the natural map $G' \rightarrow \overline{G}$ to a simply connected $k'$-group. Choose a maximal $k'$-torus $\overline{T}' \subset \overline{G}$ such that $\overline{III}^2(\overline{T}') = 0$. We use [CGP, Cor. 7.3.4] to produce a (unique) maximal $k'$-torus $T' \subset G'$ such that $T' \rightarrow \overline{T}$ is an isogeny, and we take this to be our torus.

We have to verify four things:

(i) $R_{k'/k}(G')_k$ is rational.
(ii) $H^1(k, R_{k'/k}(G')) = H^1(k, R_{k'/k}(G')) = 1$ for every place $v$ of $k$.
(iii) $\text{Ext}^1(R_{k'/k}(G_m)) = 1$.
(iv) $\tau(R_{k'/k}(G')) = 1$.

We may then find an injection $\mathscr{C} \hookrightarrow B := U \times R_{k'/k}(T'')$ for some split unipotent $U$, some finite extension $k''/k$, and some split $k''$-torus $T''$ (as we may do for any smooth connected commutative $k$-group). The proof then almost proceeds exactly as in the standard case. There is, however, one wrinkle. In the notation of the proof for standard groups, we need to show that the map $H \rightarrow A$ induces a bijection on $\overline{III}$. The proof of injectivity is essentially the same, but surjectivity requires a different argument, since we don’t know that our torus $T'$ has trivial $\overline{III}^2$ on the basic exotic fibers. We may fix this problem as follows. We have a map $G' \rightarrow \overline{G}$, where $\overline{G}$ agrees with $G'$ on the simply connected fibers, and it is the natural simply connected quotient of $G'$ (which appears in the definition of basic exotic groups) on the basic exotic fibers. Now let $F$ be defined by the following pushout
Then we have a commutative diagram of exact sequences

\[
\begin{array}{ccccccccc}
1 & \longrightarrow & R_{k'/k}(G') & \longrightarrow & H & \longrightarrow & A & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & R_{k'/k}(G) & \longrightarrow & F & \longrightarrow & A & \longrightarrow & 1 \\
\end{array}
\]

Now since the maximal torus in \(\overline{G}\) which is the image of our chosen maximal torus in \(G'\) has trivial \(\Omega_2\), we may imitate the earlier proof in the standard case to show that the map \(\Omega^1(F) \rightarrow \Omega^2(A)\) is surjective. It suffices, therefore, to show that the map \(H^1(k, H) \rightarrow H^1(k, F)\) is bijective, as then any class in \(\Omega^1(A)\) lifts to \(H^1(k, H)\), hence to \(\Omega^1(H)\), since \(H^1(k_v, R_{k'/k}(G')) = 1\) for all \(v\) (by [CGP, Prop. 7.3.3(i)]), since the same holds for \(R_{k'/k}(\overline{G'})\), because \(\overline{G'}\) is simply connected. In order to prove this bijection, it suffices to show that the Galois-equivariant map \(H(k_s) \rightarrow F(k_s)\) is bijective. This follows from a simple diagram chase in the following commutative diagram:

\[
\begin{array}{ccccccccc}
1 & \longrightarrow & R_{k'/k}(G')(k_s) & \longrightarrow & H(k_s) & \longrightarrow & A(k_s) & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & R_{k'/k}(\overline{G})(k_s) & \longrightarrow & F(k_s) & \longrightarrow & A(k_s) & \longrightarrow & 1 \\
\end{array}
\]

in which the first vertical arrow is an isomorphism by [CGP, Prop. 7.3.3(1)]. It therefore only remains to prove (i)-(iv) above. Note that each one may be proven fiber by fiber. (For the vanishing of \(\text{Ext}^1\), this uses the separable rationality (i), which implies that Pic of a product is the product of the Pic’s.) Since we already know them all on the simply connected fibers, it only remains to treat the basic exotic fibers.

For (i), we may use the open cell decomposition ([CGP], Remark 7.2.8), since any maximal torus is split over \(k_s\). Property (ii) follows from the analogous property of the simply connected quotient and [CGP, Prop. 7.3.3(i)]. For property (iii), we use the following lemma.

**Lemma 3.1.** Let \(k\) be a field, \(G\) a perfect (i.e., \(G = \mathcal{O}G\)) connected linear algebraic \(k\)-group such that the geometric reductive quotient \((G_{\overline{k}})^{\text{red}}\) is simply connected. Then \(\text{Ext}^1(G, \mathbb{G}_m) = 1\).
Proof. Suppose that we have a (necessarily central) extension
\[ 1 \to G_m \to E \to G \to 1 \]
We will show that it admits a section, hence splits. Indeed, we will show that the map \( \mathcal{D}E \to G \) is an isomorphism. For this, we may extend scalars to \( k \). First, since \( G = \mathcal{D}G \), we have an exact sequence
\[ 1 \to \mu \to \mathcal{D}E \to G \to 1 \]
I claim that \( \mu \) is finite. Indeed, more generally, for any connected linear algebraic group \( H \) over \( k \), and any central \( G_m \subset H \), \( \mathcal{D}H \cap G_m \) is finite. To see this, let \( U \) be the unipotent radical of \( H \). Then \( \mathcal{D}H \cap G_m \subset G_m \to H/U \), hence we may assume that \( H \) is reductive. But then the claim follows from the well-known fact that for any reductive \( H \), \( \mathcal{D}H \cap Z \) is finite, where \( Z \subset H \) is the maximal central torus. This proves the finiteness of \( \mu \).
It only remains to show that \( \mu = 1 \). But since \( \mu \subset G_m \), we have \( \mu \hookrightarrow \mathcal{D}E/U \), where \( U \subset \mathcal{D}E \) is the unipotent radical. Hence we have an exact sequence
\[ 1 \to \mu \to \mathcal{D}E/U \to G/\pi(U) \to 1 \]
Since \( \mathcal{D}E/U \) is reductive, so is \( G/\pi(U) \), which is therefore the reductive quotient of \( G \), hence simply connected by assumption. It therefore has no nontrivial finite central covers, so \( \mu = 1 \), as claimed.
Finally, it remains to prove (iv). This is the hardest part. Since Tamagawa numbers are invariant under Weil restriction, what we want to show is that \( \tau(\mathcal{G}) = 1 \) when \( \mathcal{G} \) is a basic exotic pseudo-reductive group. We have the natural map \( \mathcal{G} \to \mathcal{G}' \), where \( \mathcal{G}' \) is simply connected, and this map induces a topological isomorphism \( \mathcal{G}(A) \to \mathcal{G}'(A) \) that restricts to an isomorphism on \( k \)-points, by [CGP, Prop. 7.3.3]. Since \( \tau(\mathcal{G}') = 1 \), therefore, the natural question is whether under this isomorphism, the Tamagawa measure on \( \mathcal{G}' \) pulls back to the Tamagawa measure on \( \mathcal{G} \). The affirmative answer is below.

**Proposition 3.2.** Let \( \mathcal{G} \) be a basic exotic pseudo-reductive group, \( \mathcal{G} \to \mathcal{G}' \) the natural surjection onto a simply connected group. Then via the isomorphism \( \mathcal{G}(A) \to \mathcal{G}'(A) \), Tamagawa measure pulls back to Tamagawa measure. In particular, \( \tau(\mathcal{G}) = 1 \).

**Proof.**

\[ \Box \]

### 4 Totally non-reduced pseudo-reductive groups

In order to prove Theorem 1.2, it only remains to treat the case of totally non-reduced pseudo-reductive groups (which only occur in characteristic 2). We will actually show that for such a group \( G \), we have \( \tau(G) = \# \mathrm{III}^1(G) = \# \mathrm{Ext}^1(G, G_m) = 1 \). In fact, all such
groups are basic non-reduced. First we check that simple such groups are rational over \( k \), hence Ext is multiplicative on the factors, so we may work factor by factor. The separable rationality holds by the way such groups are constructed. The vanishing of Ext\(^1\) follows from [CGP, Prop. 10.1.5]. The vanishing of \( \text{III} \)\(^1\) follows from [CGP, Prop. 9.9.4]. It only remains to prove that \( \tau(G) = 1 \). If \( K \) is the minimal field of definition of the geometric radical of \( G \), then we let \( G' := G'_{ss} \cong \text{Sp}_{2n} \). Then the natural maps \( G(k) \to G'(K) \) and \( G(A_k) \to G'(A_K) \) are isomorphisms, by [CGP, Prop. 9.9.4], so as with basic exotic groups, the natural approach is to check that via this isomorphism, Tamagawa measure pulls back to Tamagawa measure. This is indeed the case.

**Proposition 4.1.** Let \( G \) be a basic non-reduced pseudo-reductive group, \( K/k \) the minimal field of definition for its geometric radical, and \( G' := G'_{ss} \cong \text{Sp}_{2n} \). Then via the natural isomorphism \( G(A_k) \xrightarrow{\sim} G'(A_K) \) (which restricts to an isomorphism \( G(k) \xrightarrow{\sim} G'(K) \)), Tamagawa measure pulls back to Tamagawa measure. In particular, \( \tau(G) = 1 \).

**References**


