TATE DUALITY IN POSITIVE DIMENSION
OVER FUNCTION FIELDS

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Abstract

We extend the classical duality results of Poitou and Tate for finite discrete Galois modules over local and global fields (local duality, nine-term exact sequence, etc.) to all affine commutative group schemes of finite type, building on the recent work of Česnavičius [Čes2] extending these results to all finite commutative group schemes. We concentrate mainly on the more difficult function field setting, giving some remarks about the number field case along the way.
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Chapter 1

Introduction

1.1 Motivation

Tate’s theorems on the Galois cohomology of finite discrete Galois modules over local and global fields are the cornerstone for the arithmetic theory of abelian varieties, Galois deformation theory, and beyond. In the function field case, these results traditionally required the hypothesis that the order of the Galois module is not divisible by the characteristic.

The restriction on the order was recently removed in work of Česnavičius ([Čes1], [Čes2]), also permitting general (i.e., not necessarily étale) finite commutative group schemes over local and global function fields. This requires using fppf cohomology rather than étale cohomology (the two coincide for coefficients in an étale finite commutative group scheme, or equivalently a finite discrete Galois module; more generally, they agree for coefficients in a smooth commutative group scheme, by [BrIII, Thm. 11.7]). It also requires using cohomology over the spectrum of an adele ring in the global case in place of various restricted direct products and direct sums as in the traditional formulation.

Our aim in the present work is to build on those results in the finite case to establish analogous duality theorems for the fppf cohomology of commutative affine group schemes $G$ of finite type with positive dimension over local and global fields $k$. This is not so difficult in characteristic 0 (as we discuss briefly in Appendix F), so our main focus in this work is on the function field case, where the imperfectness of such fields leads to substantial difficulties. Even the case $G = \mathbb{G}_a$ presents serious challenges for duality theorems when $\text{char}(k) > 0$ because the $\mathbb{G}_m$-dual functor $\widehat{G} := \mathcal{H}om(G, \mathbb{G}_m)$ over the category of $k$-schemes is non-representable (see Proposition 2.2.7).

The primary motivation for the present work, especially for proving a version of the Poitou-Tate 9-term global exact sequence, is to understand Tamagawa numbers of general smooth (not necessarily commutative!) connected affine groups over global function fields, going beyond the connected reductive case due to Sansuc [San]. (Sansuc’s work was conditional on Weil’s Tamagawa Number Conjecture that was fully proved later, with the
function field case of Weil's Conjecture settled only very recently by Gaitsgory and Lurie [GL]. We shall soon review Sansuc's result and indicate why, due to the structure of general smooth connected affine groups, any attempt to generalize beyond reductive groups is intimately tied up with our global duality theorems in positive dimension.

In principle all of our work on local and global duality could have been done in the late 1960’s were there some compelling motivation to carry it out at that time. However, it is only with the recent solution of the Weil’s Tamagawa Number Conjecture and the development of the structure theory of pseudo-reductive groups [CGP] and related finiteness applications [Con] (including that certain terms in Sansuc’s formula are always finite in the function field case) that it finally became reasonable to try to go beyond Sansuc’s work and tackle the most general case over function fields.

Inspired by earlier work of T. Ono and Voskresenskii for tori, in [San] Sansuc proved the “Tamagawa number formula”

$$\tau(G) = \frac{\# \text{Pic}(G)}{\# \text{III}^1(G)}, \quad (1.1.1)$$

where the terms are defined as follows:

- The Tamagawa number $\tau(G)$ of $G$ is the volume of a certain adelic coset space under a canonical Tamagawa measure. Its finiteness in the reductive case is due to global class field theory and classical results concerning reduction theory for arithmetic groups (see [Oes] Ch.I, §5] for a general definition and references on finiteness results in the reductive case). The finiteness of $\tau(G)$ in general over number fields is an elementary reduction to the reductive case, whereas over global function fields one has to bring in additional techniques (see [Con], Thm.1.3.6).

- The Tate–Shafarevich set $\text{III}^1(G)$ is the set of (isomorphism classes of) $G$-torsors over $k$ with a $k_v$-point for all places $v$ of $k$. Its finiteness in the context of abelian varieties is a major open problem in the arithmetic theory of abelian varieties. For smooth connected affine groups $G$ over global fields, the finiteness of $\text{III}^1(G)$ is due to Borel and Serre over number fields (where passage to the reductive case is easy), Harder and Oesterlé in the reductive and solvable cases respectively over global function fields, and Conrad in the general case over global function fields; see [Con], §1.3] and references therein for more details and references.

- The Picard group $\text{Pic}(G)$ consists of isomorphism classes of line bundles on $G$. Its finiteness for connected reductive $G$ is part of Sansuc’s work. This finiteness is of a much more elementary nature than the finiteness of the other two terms.

The settled simply connected case of (1.1.1) involves very deep arithmetic input on which we have nothing to add and about which we will say nothing here except to note that all terms appearing in (1.1.1) are equal to 1 in such cases.
Sansuc’s work was carried out over number fields, but his methods work essentially without change in the reductive case over global function fields. For general smooth connected affine groups $G$ over a global field $k$, to express how far $G$ is from the reductive case we define the $k$-unipotent radical $\mathcal{R}_{u,k}(G)$ to be the maximal unipotent normal smooth connected $k$-subgroup of $G$ and consider the exact sequence

$$1 \rightarrow \mathcal{R}_{u,k}(G) \rightarrow G \rightarrow R \rightarrow 1.$$ (1.1.2)

In general $\mathcal{R}_{u,k}(G)_{\overline{k}} \subset \mathcal{R}_{u}(G_{\overline{k}})$, and equality holds for perfect $k$ (i.e., number fields) by Galois descent. The perfectness of number fields also ensures that $\mathcal{R}_{u,k}(G)$ is split unipotent for such $k$ (i.e., it has a composition series by smooth connected $k$-subgroups such that the successive quotients are $\mathbf{G}_a$). This allows one to easily deduce Sansuc’s formula (1.1.1) for all connected linear algebraic groups over number fields from the known reductive case.

The gulf between the general case and the reductive case over global function fields $k$ is vast, for two reasons. The first problem is that the smooth connected affine quotient $R$ in (1.1.2) is not generally reductive because $k$ is imperfect. Rather, such $R$ is only pseudo-reductive, which means (by definition) that $\mathcal{R}_{u,k}(R) = 1$. Over any imperfect field, there exist (many) pseudo-reductive groups that are not reductive. (An interesting basic example is $R_{k'/k}(\text{SL}_p)/R_{k'/k}(\mu_p)$ for a nontrivial purely inseparable finite extension $k'/k$ in characteristic $p > 0$, where $R_{k'/k}$ denotes Weil restriction of scalars; there are many other examples with more intricate structure.) The second problem is that $\mathcal{R}_{u,k}(G)$ is generally not $k$-split, again because $k$ is imperfect; this creates tremendous cohomological difficulties. So even though there are (somewhat complicated) formulas for the behavior of $\tau(G)$ relative to short exact sequences in $G$ (proved by Oesterlé in [Oes, Ch.III, §5] conditional on some finiteness conjectures that were settled in [Con]), there does not appear to even be a simple devissage to reduce the study of $\tau(G)$ to the pseudo-reductive case.

In fact, Sansuc’s formula fails for most (all if $\text{char}(k) > 2$) nontrivial forms of $\mathbf{G}_a$ over global function fields. (This follows from [Ros1, Prop.9.8], in conjunction with Theorem 1.1.5 below.) To obtain a plausible candidate for a suitable formula in general, we consider the subgroup $\text{Ext}^1(G, \mathbf{G}_m) \subset \text{Pic}(G)$ consisting of “primitive” line bundles. That is, if we let $m, p_i : G \times G \rightarrow G$ denote the multiplication and projection maps, then

$$\text{Ext}^1(G, \mathbf{G}_m) := \{ \mathcal{L} \in \text{Pic}(G) \mid m^* \mathcal{L} \simeq p_1^* \mathcal{L} \otimes p_2^* \mathcal{L} \}.$$ 

The reason for the notation is that this group turns out to be naturally isomorphic to the Yoneda Ext-group of central extensions of $G$ by $\mathbf{G}_m$. (This actually holds in the more general context of arbitrary smooth connected $k$-group schemes [Ros1, Prop. 5.1].) Likewise, if $G$ is commutative then $\text{Ext}^1(G, \mathbf{G}_m)$ is canonically isomorphic to the derived-functor group $\text{Ext}^1_{\text{fppf}}(G, \mathbf{G}_m)$ for the fppf topology [Ros1, Prop. 5.4], so there is no ambiguity in the notation.

The following facts give an initial glimpse into the favorable behavior of $\text{Ext}^1(G, \mathbf{G}_m)$:
(i) For smooth connected affine $G$ over general fields $k$, the inclusion $\text{Ext}^1(G, G_m) \subset \text{Pic}(G)$ is an equality if $k$ is perfect [Ros1, Prop. 2.7] or if $G_{k_s}$ is rational [Ros1, Prop. 2.2]. Reductive groups (over any field) become rational over a separable closure (for example, by the open cell decomposition associated to a split maximal torus and a choice of set of positive roots). This implies that in the context of Sansuc’s work with connected reductive groups $G$ over global fields $k$ (as well as for general connected linear algebraic groups over number fields, since such fields are perfect) it is harmless to replace $\text{Pic}(G)$ with $\text{Ext}^1(G, G_m)$.

(ii) The group $\text{Ext}^1(G, G_m)$ is finite for every smooth connected affine group $G$ over a global field $k$ [Ros1, Th. 5.5]. This arithmetic result is truly global in the sense that over every local function field it is infinite for most nontrivial fppf-forms of $G_a$. More precisely, over local function fields of characteristic $p$ it is infinite for all nontrivial forms of $G_a$ when $p > 2$ and for all nontrivial forms of $G_a$ other than those given by affine plane conics when $p = 2$.

Here is a formula for Tamagawa numbers beyond the reductive case, proved in a forthcoming work on Tamagawa numbers (for the current version, see [Ros2]):

**Theorem 1.1.1.** Let $G$ be a smooth connected affine group over a global function field $k$. Assume that $G$ is either commutative or pseudo-reductive, and that $\text{char}(k) > 3$. Then

$$\tau(G) = \frac{\# \text{Ext}^1(G, G_m)}{\# \text{III}^1(G)}.$$  

**Remark 1.1.2.** The restriction on the characteristic in Theorem 1.1.1 is due to the existence of certain exceptional pseudo-reductive groups in characteristics 2 and 3 introduced in [CGP, §7.2]. For such groups $G$, by [CGP] Prop 7.3.3 there is a canonical surjection $f : G \to \overline{G}$ with $\overline{G}$ a simply connected semisimple group such that $\ker f$ is non-smooth with positive dimension but (i) the natural maps $G(k) \to \overline{G}(k)$ and $G(A_k) \to \overline{G}(A_k)$ are bijective, with the latter a topological isomorphism; and (ii) the natural maps

$$H^1(k, G) \to H^1(k, \overline{G}), \quad H^1(k_v, G) \to H^1(k_v, \overline{G})$$

are bijective, so also $\text{III}^1(k, G) \sim \text{III}^1(k, \overline{G})$.

Thus, the desired formula for $G$ reduces to the settled case of $\overline{G}$ if $\text{Ext}^1(G, G_m) = 1$ and the homeomorphism $G(A_k) \to \overline{G}(A_k)$ identifies the Tamagawa measures. The triviality of $\text{Ext}^1(G, G_m)$ is proved similarly to the case of $\overline{G}$. If $G$ has a split maximal $k$-torus then one can explicitly describe the effect of $f$ between open cells and use this description to identify the Tamagawa measures, but it is not yet clear how to compare the measures without a split maximal $k$-torus.

At any rate, the restriction on $\text{char}(k)$ in Theorem 1.1.1 has nothing to do with the results of the present work, and it would be removed by establishing the above identification of Tamagawa measures (which we hope to do in the future). One can also obtain a
formula beyond the commutative and pseudo-reductive cases by modifying the formula in Theorem 1.1.1 by a suitable power of $p := \text{char}(k)$. Describing this modification, however, is somewhat complicated; it will be discussed in forthcoming work.

One sense in which Theorem 1.1.1 is a more natural formulation than (1.1.1) is that $\text{Pic}(G)$ only involves the scheme structure on $G$, whereas $\text{Ext}^1(G, \mathbb{G}_m)$ also keeps track of the group structure. Perhaps more significantly, Theorem 1.1.1 is the analogue for linear algebraic groups of Bloch’s volume-theoretic formulation of the Birch and Swinnerton-Dyer Conjecture for abelian varieties as discussed in [Bl].

To explain the connection in the special case of finite Mordell–Weil group (Bloch handles any Mordell–Weil rank), let $B$ be a $g$-dimensional abelian variety with finite Mordell–Weil group over a global function field $k$ with characteristic $p > 0$ and let $B$ be its Néron model over the smooth connected proper curve $X$ over $\mathbb{F}_p$ with function field $k$. Let $A$ denote the adele ring of $k$. By the theory of abelian varieties over finite fields, the local $L$-factor for the abelian variety at a place $v$ of $k$ satisfies

$$L_v(B, 1) = \frac{q_v^g}{\#B_v^0(F_v)}$$

where $F_v$ is the residue field of at $v$, $q_v := \#F_v$, and (for globalization purposes) $L_v$ is expressed as a function of the parameter $s \in \mathbb{C}$ rather than the more traditional $q_v^{-s}$.

The integral structure at $v$ arising from the Néron model defines a canonical Haar measure on $B(k_v)$ for each $v$, and a calculation using smoothness of the Néron model to compute the volume of the preimage in $B(k_v) = B(O_v)$ of $0 \in B(F_v)$ shows that relative to this measure we have

$$\text{vol}(B(k_v)) = \frac{\#B(F_v) - \#B^0(F_v) \cdot \#\Phi_v(F_v)}{q_v^g} = L_v(B, 1)^{-1} \cdot \#\Phi_v(F_v)$$

where $\Phi_v := B/F_v$ is the finite étale component group of the $v$-fiber (and the second equality uses Lang’s Theorem to ensure that $B(F_v) \rightarrow \Phi(F_v)$ is surjective). The quantity $c_v := \#\Phi(F_v)$ is usually called the Tamagawa factor at $v$; for all but finitely many $v$ we have $c_v = 1$. A reasonable definition of $\tau(B)$ as a “volume” for $B(A)/B(k)$ is therefore

$$\tau(B) := \frac{L(B, 1)^{-1} \prod_v c_v}{\#B(k)}$$

if $L(B, 1) \neq 0$. (There is a procedure that puts a canonically normalized Haar measure on $B(A)$ relative to which one can instead define $\tau(B)$ as the actual volume of $B(A)/B(k)$, but we do not discuss that here except in Remark [1.1.3 below].)

The group $\text{Ext}^1(B, \mathbb{G}_m)$ consists of the translation-invariant line bundles on $B$. The Weil–Barsotti formula identifies this with $B'(k)$ where $B'$ is the dual abelian variety. Note
that \( B'(k) \) is finite, since \( B(k) \) is finite. The analogue of Theorem 1.1.1 in this situation is therefore
\[
\frac{L(B,1)\prod_v c_v}{\#B(k)} = \frac{\#B'(k)}{\#\mathcal{I}^1(B)}
\]
(assuming finiteness of \( \mathcal{I}^1(B) \)). This is precisely the Birch and Swinnerton-Dyer Conjecture for \( B \), including the non-vanishing of \( L(B,1) \), since \( B(k) \) is finite.

**Remark 1.1.3.** In [Bl], \( \tau(B) \) is defined to be an actual volume (suitably normalized). Under that definition and assuming finiteness of \( \mathcal{I}^1(B) \), by [Bl, Thm. 1.17] the equality \( \tau(B) = \#B'(k)/\#\mathcal{I}^1(B) \) is equivalent to the Birch and Swinnerton-Dyer Conjecture for such \( B \).

**Remark 1.1.4.** In the general (possibly higher rank) case of the Birch and Swinnerton-Dyer Conjecture, the analogue of \( \text{Ext}^1(G, G_m) \) is the torsion subgroup \( B'(k)_{\text{tor}} = \text{Ext}^1(B, G_m)_{\text{tor}} = \text{Pic}(B)_{\text{tor}} \), the last equality coming from the fact that \( \text{Pic}(B)/\text{Pic}^0(B) \) is torsion-free. It is therefore natural to wonder whether we have the equality \( \text{Ext}^1(G, G_m) = \text{Pic}(G)_{\text{tor}} \) for smooth connected commutative affine group schemes \( G \) over the global function field \( k \). This is false. Indeed, for an affine group scheme \( G \) of finite type over any field, the group \( \text{Pic}(G) \) is torsion-free [Ros1, Prop. 2.8], but it is not true in general even for commutative smooth connected affine \( G \) (over global function fields) that \( \text{Ext}^1(G, G_m) = \text{Pic}(G) \) [Ros1, Prop. 9.8].

How do our main results on duality for commutative affine group schemes of finite type play a role in the proof Theorem 1.1.1 (including cases with non-commutative \( G \))? In fact, both the commutative and pseudo-reductive cases turn out to be intimately tied up with problems for commutative group schemes that our duality theorems are well-equipped to handle. As a first illustration, we use these duality theorems and intermediate results occurring in their proofs to establish the general commutative case of Theorem 1.1.1 by transforming a result of Oesterlé into the desired formula. In what follows, all cohomology is understood to be for the fppf topology (which agrees with étale cohomology for smooth coefficients, by [BrIII, Thm. 11.7]).

**Theorem 1.1.5.** Let \( k \) be a global function field with adele ring \( A \) and let \( G \) a smooth connected commutative affine \( k \)-group. Then
\[
\tau(G) = \frac{\#\mathcal{I}^1(G) \cdot \#\mathcal{I}^2(G)}{\#\mathcal{I}^1(G)} = \frac{\#\text{Ext}^1(G, G_m)}{\#\mathcal{I}^1(G)},
\]
where \( \mathcal{I}^i(G) := \text{coker}(H^i(k, G) \to H^i(A, G)) \) and \( \mathcal{I}^2(G) := \ker(H^2(k, G) \to \mathcal{I}^2(A, G)) \).

Before we discuss the proof of Theorem 1.1.5, we address the compatibility of different-looking definitions of \( \mathcal{I}^1 \) and \( \mathcal{I}^2 \) in the smooth connected commutative affine case. Our definition of \( \mathcal{I}^i(G) \) agrees with the definition \( \text{coker}(H^i(k, G) \to \bigoplus_v H^i(k_v, G)) \) given in
for $i > 0$ because $\bigoplus_v H^i(k_v, G) = H^i(A, G)$ via the natural map for any $i > 0$ due to $G$ being smooth and connected (as we show in Proposition 6.1.1). For related reasons, our definition of $\prod^2(G)$ coincides with the one in [Oes] and likewise $\prod^1(G)$ defined in the traditional manner via $G$-torsors over $k$ coincides with $\ker(H^1(k, G) \to H^1(A, G))$. The formulation involving cohomology over the adele ring is the appropriate version for our work with $G$ that may not be smooth or connected (as is essential for the success of dévissage arguments even to prove results of interest primarily in the smooth connected case). Since $G$ is smooth, connected, and commutative, the finiteness of $\mathbb{Q}_1(G)$ is [Oes, Ch.IV, Prop.2.6(b)] and the finiteness of $\prod^2(G)$ is [Oes, Ch.IV, Thm.2.7(a)].

Proof. Define $\hat{G}$ to be the (possibly non-representable) Hom-functor $\mathcal{H}om(G, G_m)$ on the category of $k$-schemes. In Theorem 1.2.8 we state a 9-term exact sequence involving fppf cohomology of $G$ and of $\hat{G}$ over both $k$ and $A$; this involves some profinite completions and some $\mathbb{Q}/\mathbb{Z}$-duals (with $A^*$ denoting $\hom(A, \mathbb{Q}/\mathbb{Z})$ for an abelian group $A$). By [Oes, Ch.IV, Thm.3.2], we have

$$\tau(G) = \frac{\# \mathcal{C}^1(G) \cdot \# \prod^2(G)}{\# \prod^1(G) \cdot \# \ker(\lambda_G)}$$

where $\lambda_G : \mathcal{C}^2(G) \to \hat{G}(k)^*$ is the map obtained from the last three nonzero terms in the complex of Theorem 1.2.8. The exactness in Theorem 1.2.8 yields that $\lambda_G$ is an isomorphism! This proves the first equality.

For the second equality, we need to show that $\# \text{Ext}^1(G, G_m) = \# \mathcal{C}^1(G) \cdot \# \prod^2(G)$. As we have already remarked, $\text{Ext}^1(G, G_m)$ agrees with the derived-functor $\text{Ext}$. By Corollary 2.2.9, which concerns the derived-functor $\text{Ext}$, we have $\text{Ext}^1(G, G_m) \simeq H^1(k, \hat{G})$. Hence, $H^1(k, \hat{G})$ is finite and it suffices to show $\# H^1(k, \hat{G}) = \# \mathcal{C}^1(G) \cdot \# \prod^2(G)$. But Theorem 1.2.8 provides an exact sequence

$$0 \to \mathcal{C}^1(G) \to H^1(k, \hat{G})^* \to \prod^2(G) \to 0,$$

so the $\mathbb{Q}/\mathbb{Z}$-dual $H^1(k, \hat{G})^*$ of the finite group $H^1(k, \hat{G})$ has size $\# \mathcal{C}^1(G) \cdot \# \prod^2(G)$. Since any finite abelian group has the same size as its $\mathbb{Q}/\mathbb{Z}$-dual, we are done! 

Coming back to the non-commutative case of Theorem 1.1.1, consider the unipotent case. By the nilpotence of unipotent $k$-groups, they are built up from commutative unipotent smooth connected $k$-groups via successive central extensions. Thus, one can hope that the preceding result in the commutative case, combined with Oesterlé’s results on the behavior of Tamagawa numbers in short exact sequences, can lead to progress in understanding the Tamagawa numbers of unipotent and more general solvable linear algebraic groups.

Let’s now consider the pseudo-reductive case, for which we need something more than the mere definition of pseudo-reductivity. The substance behind this is given in [CGP].
where a structure theory for pseudo-reductive groups over general (imperfect) fields is established in the spirit of the Borel–Tits theory for connected reductive groups but requiring entirely different techniques of proof. (Further refinements given in [CP] address some loose ends in [CGP] over fields $F$ of characteristic 2 satisfying $[F : F^2] > 2$, but this is not needed for work over local and global function fields of characteristic 2.)

Away from some exceptional situations in characteristics 2 and 3 that can nonetheless be completely described, the main structure theorem in [CGP] describes all non-commutative pseudo-reductive $k$-groups in terms of a central quotient construction involving two classes of groups for which Theorem 1.1.1 has now been established: commutative pseudo-reductive $k$-groups (which admit no convenient description at all, so Theorem 1.1.1 for such $C$ via Theorem 1.1.5 relies crucially on the generality of our present work with arbitrary commutative affine groups of finite type) and Weil restrictions to $k$ of simply connected groups $G'$ over finite (typically inseparable) extensions $k'$ of $k$ (for which Theorem 1.1.1 reduces to the now-settled conjecture of Weil for the simply connected $k'$-group $G'$).

1.2 Main results

Let us now record our main results over local and global fields, giving references for the finite case already treated by Poitou, Tate, and Česnavičius. To state the local results, we need some notation. Given any locally compact Hausdorff topological abelian group $A$, we let $A^D := \text{Hom}_{cts}(A, \mathbb{R}/\mathbb{Z})$ denote the Pontryagin dual group of $A$, and we denote by $A_{\text{pro}}$ the profinite completion of $A$. Next, we need to discuss some basic topological generalities. Consider the class $\mathcal{C}$ of locally compact Hausdorff abelian groups $A$ that are either (i) profinite, (ii) discrete torsion, or (iii) of finite exponent. Many of the groups that we will be concerned with in the local setting lie in this class (but there are important exceptions, such as $k^\times$ for a non-archimedean local field $k$). The class $\mathcal{C}$ has several nice properties. For example, such $A$ satisfy

$$A^D = \text{Hom}_{cts}(A, (\mathbb{Q}/\mathbb{Z})_{\text{disc}})$$

where $(\mathbb{Q}/\mathbb{Z})_{\text{disc}}$ denotes $\mathbb{Q}/\mathbb{Z}$ with its discrete topology. Another nice property of $\mathcal{C}$ is that it is stable under $A \rightsquigarrow A^D$: if $A$ is profinite then $A^D$ is discrete torsion, and vice-versa, and the dual of a group of finite exponent is still of finite exponent.

Given two locally compact Hausdorff abelian groups $A, B$ and a continuous bilinear pairing $A \times B \to \mathbb{R}/\mathbb{Z}$, the induced map $A \to B^D$ is a topological isomorphism if and only if the map $B \to A^D$ is, by Pontryagin double duality. If both of these equivalent conditions hold, then we say that the pairing is perfect.

Let $k$ be a local field. Given an fpqc abelian sheaf $\mathcal{F}$ on $\text{Spec}(k)$, we define $\widehat{\mathcal{F}} = \mathcal{H}om(\mathcal{F}, \mathbb{G}_m)$ and we give $\mathbb{H}^2(k, \mathcal{F})$ the discrete topology (which trivially makes it into a locally compact Hausdorff abelian group). Given a commutative affine $k$-group scheme $G$ of finite type, we also give $\widehat{G}(k)$ the discrete topology. On the other hand, we give $G(k)$ its...
natural topology arising from that on \( k \). That is, we choose a closed immersion \( G \hookrightarrow \mathbb{A}^n_k \) into an affine space over \( k \), give \( \mathbb{A}^n_k(k) = k^n \) its usual topology, and then endow \( G(k) \) with the subspace topology. This is independent of the choice of embedding and makes \( G(k) \) into a locally compact Hausdorff topological group functorially in \( G \). If \( G \) is finite, this is simply the discrete topology on \( G(k) \).

Letting \( \hat{A}_{\text{pro}} \) denote the profinite completion of an abelian group \( A \), the first main local result is the following.

**Theorem 1.2.1.** Let \( k \) be a local function field, and \( G \) an affine commutative \( k \)-group scheme of finite type. The cohomology group \( H^2(k, G) \) is a torsion group, \( \hat{G}(k) \) is finitely generated, and cup product \( H^2(k, G) \times \hat{G}(k) \to H^2(k, G_m) \overset{\text{inv}}{\sim} \mathbb{Q}/\mathbb{Z} \) induces a functorial continuous perfect pairing of locally compact Hausdorff abelian groups \( H^2(k, G) \times \hat{G}(k)_{\text{pro}} \to \mathbb{Q}/\mathbb{Z} \), where \( H^2(k, G) \) is discrete.

If \( G \) is finite then \( \hat{G}(k) = \hat{G}(k)_{\text{pro}} \), so this case is \([Mi, \text{Ch.III, Thm.6.10}]\). By “functoriality” in Theorem 1.2.1 we mean that if we have a homomorphism \( f : G \to H \) of \( k \)-group schemes as in Theorem 1.2.1 then the diagram

\[
\begin{array}{ccc}
H^2(k, G) & \times & \hat{G}(k)_{\text{pro}} \\
\downarrow \text{H}^2(f) \downarrow & & \downarrow \text{H}^2(f) \\
H^2(k, H) & \times & \hat{H}(k)_{\text{pro}} \\
\end{array}
\]

commutes. This functoriality is clear by the functoriality of cup product.

Our second local result is the analogue of the preceding one with the roles of \( G \) and \( \hat{G} \) reversed. Unlike in Tate local duality for finite commutative group schemes, there is no analogous symmetry between \( G \) and \( \hat{G} \) in the generality of arbitrary affine commutative group schemes of finite type. In fact, the sheaf \( \hat{G} \) may not be representable (as we have already noted for \( G = \mathbb{G}_a \) due to Proposition 2.2.7); even if representable, it may not be finite type (as for \( G = \mathbb{G}_m \), whose fppf dual sheaf is \( \mathbb{Z} \)). Theorem 1.2.1 above and Theorem 1.2.2 below therefore really are distinct results.

**Theorem 1.2.2.** Let \( k \) be a local function field, and \( G \) an affine commutative \( k \)-group scheme of finite type. The cohomology group \( H^2(k, \hat{G}) \) is torsion, and the cup product \( H^2(k, \hat{G}) \times G(k) \to H^2(k, G_m) \overset{\text{inv}}{\sim} \mathbb{Q}/\mathbb{Z} \) induces a functorial continuous perfect pairing of locally compact Hausdorff groups \( H^2(k, \hat{G}) \times G(k)_{\text{pro}} \to \mathbb{Q}/\mathbb{Z} \) with \( H^2(k, \hat{G}) \) is discrete.

When \( G \) is finite, this is the same as Theorem 1.2.1 and therefore is once again \([Mi, \text{Ch.III, Thm.6.10}]\). Finally, we come to the duality between the cohomology groups \( H^1(k, G) \) and \( H^1(k, \hat{G}) \). This is subtle in positive characteristic because it involves endowing \( H^1(k, G) \) and \( H^1(k, \hat{G}) \)
with suitable topologies that are not quite as obvious as the ones for $H^0$ and $H^2$. (In characteristic 0, both groups are finite, and we simply take them to be discrete.) In Česnavičius adapts an idea of Moret-Bailly to define a locally compact topology on $H^1(k,G)$ for $G$ a $k$-group scheme locally of finite type. The topology is defined as follows: a subset $U \subset H^1(k,G)$ is open if for every locally finite type $k$-scheme $X$ and every $G$-torsor $\mathcal{X} \rightarrow X$, the subset
$$\{ x \in X(k) \mid \mathcal{X}_x \in U \} \subset X(k)$$
is open. (In effect, this amounts to declaring $X(k) \rightarrow (BG)(k)$ to be continuous for all $k$-morphisms $X \rightarrow BG$.) In Česnavičius, various desirable properties are proved (e.g., given a map $G \rightarrow H$ between two such group schemes, the induced map $H^1(k,G) \rightarrow H^1(k,H)$ is continuous and likewise for connecting maps in the commutative case from degree $i$ to degree $i + 1$ for $i = 0, 1$ in a long exact sequence associated to a short exact sequence of such group schemes); precise references for proof of these results in Česnavičius are given near the start of §4.2.

This procedure defines a topology on $H^1(k,\hat{G})$ when $G$ is an almost torus (in the sense of Definition 2.1.4), by the representability of $\hat{G}$ in Proposition 2.2.7 for such $G$. A generalization to define a locally compact Hausdorff topology on $H^1(k,\hat{G})$ for general (affine commutative finite type) $G$ will be given in §4.6.

Remark 1.2.3. Let us warn the reader that the topology on $H^1(k,\hat{G})$ is not in general $\delta$-functorial, in the sense that for a short exact sequence of affine commutative $k$-group schemes of finite type
$$1 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 1, \quad (1.2.1)$$
the connecting map $H^1(k,\hat{G}') \rightarrow H^2(k,\hat{G}'')$ is not necessarily continuous, where (as usual) $H^2(k,\hat{G}'')$ has the discrete topology. To give an example, consider the exact sequence
$$1 \rightarrow \alpha_p \rightarrow G_a \rightarrow G_a \rightarrow 1$$
where the last map on the right is the Frobenius $k$-isogeny. Since $H^1(k,\hat{G}_a) = 0$ (Proposition 2.4.2), the map $H^1(k,\hat{\alpha}_p) \rightarrow H^2(k,\hat{G}_a)$ is injective. To say that it is continuous, therefore, would be equivalent to saying that $H^1(k,\alpha_p)$ is discrete. But $\hat{\alpha}_p \simeq \alpha_p$, and we will see in (4.2.1) near the end of §4.2 that $H^1(k,\alpha_p) \simeq k/k^p$ as topological groups. This quotient is not discrete since $k^p \subset k$ is not open, so the connecting map is not continuous. This bad behavior cannot happen if all groups in (1.2.1) are almost tori, since then their fppf dual sheaves are represented by locally finite type $k$-group schemes (Lemma 2.2.7) and so the connecting map is continuous by Česnavičius, Prop. 4.2.

With these preliminaries out of the way, the final local duality result is:

Theorem 1.2.4. Let $k$ be a local function field, and $G$ an affine commutative $k$-group scheme of finite type. The fppf cohomology groups $H^1(k,G)$ and $H^1(k,\hat{G})$ are of finite
exponent, and cup product $H^1(k, G) \times H^1(k, \hat{G}) \to H^2(k, G_m)$ \text{inv} \sim \mathbb{Q}/\mathbb{Z}$ is a functorial continuous perfect pairing of locally compact Hausdorff abelian groups.

When $G$ is finite, this is once again [Mi, Ch. III, Thm. 6.10]. We remark that the topology we use on the cohomology groups agrees with Milne’s Čech topology, as will be discussed in §4.2.

All three of the preceding theorems are proved in Chapter 4. They will be deduced from the already-known finite cases.

Part of classical local duality describes integral (or in Galois-theoretic contexts: unramified) cohomology classes on each side as the exact annihilators of those on the other side. More precisely, let $k$ be a local function field with ring of integers $\mathcal{O}$, and let $\mathcal{G}$ be a finite flat commutative $\mathcal{O}$-group scheme with generic fiber $G$. Then the maps $H^i(\mathcal{O}, \mathcal{G}) \to H^i(k, G)$ are injective, and similarly for $\mathcal{H}$ by Cartier duality, and via the perfect cup product pairing $H^i(k, G) \times H^{2-i}(k, \hat{G}) \to \mathbb{Q}/\mathbb{Z}$, the exact annihilator of $H^{2-i}(\mathcal{O}, \mathcal{G})$ is $H^i(\mathcal{O}, \mathcal{G})$ (these groups clearly annihilate each other because $H^2(\mathcal{O}, G_m) = 0$): for $i = 1$ this is [Mi, Ch. III, Cor. 7.2], and for $i = 2$ it follows from Lemma 3.3.12 and the fact that $\mathcal{G}(\mathcal{O}) = \hat{G}(k)$ due to $\mathcal{G}$ being finite. The case $i = 0$ follows from the case $i = 2$ by Cartier duality.

Unfortunately, due to the lack of a good structure theory for flat affine commutative $\mathcal{O}$-group schemes of finite type, we do not obtain such a satisfactory result in general when $\dim G > 0$. However, we do obtain useful replacements that are not only interesting in their own right but also play an essential role in our work in §6.1 that relates cohomology over adele rings to cohomology over local fields. Part of our interest in these theorems lies in the injectivity statements below for the pullback maps $H^i(\mathcal{O}_v, \cdot) \to H^i(k_v, \cdot)$ for all but finitely many $v$, as may be proved by a direct dévissage for number fields but appears to be much more subtle in the function field setting.

To state these local duality results when $\dim G > 0$, let us first note that any affine commutative group scheme $G$ of finite type over a global field $k$ spreads out to an $\mathcal{O}_S$-model $\mathcal{G}$ (i.e., an affine commutative flat $\mathcal{O}_S$-group scheme of finite type) for some non-empty finite set $S$ of places of $k$ containing the archimedean places.

**Theorem 1.2.5.** Let $k$ be a global function field, and $G$ an affine commutative $k$-group scheme of finite type, $\mathcal{G}$ an $\mathcal{O}_S$-model of $G$. Then for all but finitely many places $v$ of $k$, we have $H^2(\mathcal{O}_v, \mathcal{G}) = 0$ and the map $\mathcal{G}(\mathcal{O}_v) \to \hat{G}(k_v)$ is an isomorphism.

In particular, for such $v$ the maps $H^2(\mathcal{O}_v, \mathcal{G}) \to H^2(k_v, G)$ and $H^0(\mathcal{O}_v, \mathcal{G}) \to H^0(k_v, \hat{G})$ are injective, and $H^2(\mathcal{O}_v, \mathcal{G})$ is the exact annihilator of $H^0(\mathcal{O}_v, \mathcal{G})$.

**Theorem 1.2.6.** Let $k$ be a global function field, and $G$ an affine commutative $k$-group scheme of finite type, $\mathcal{G}$ an $\mathcal{O}_S$-model of $G$. Then for all but finitely many places $v$ of $k$, the maps $H^0(\mathcal{O}_v, \mathcal{G}) \to H^0(k_v, G)$ and $H^2(\mathcal{O}_v, \mathcal{G}) \to H^2(k_v, \hat{G})$ are injective and $H^2(\mathcal{O}_v, \mathcal{G})$ is the exact annihilator of $H^0(\mathcal{O}_v, \mathcal{G})$.

**Theorem 1.2.7.** Let $k$ be a global function field, and $G$ an affine commutative $k$-group scheme of finite type, $\mathcal{G}$ an $\mathcal{O}_S$-model of $G$. Then for all but finitely many places $v$ of $k$, the
maps \( H^1(O_v, \mathcal{F}) \rightarrow H^1(k_v, G) \) and \( H^1(O_v, \mathcal{F}) \rightarrow H^1(k_v, \hat{G}) \) are injective and \( H^1(O_v, \mathcal{F}) \) and \( H^1(O_v, \mathcal{F}) \) are orthogonal complements under the local duality (i.e., cup product) pairing.

Note that these results are independent of the chosen \( O_S \)-model \( \mathcal{F} \), since any two such models become isomorphic over \( O_{S'} \) for some \( S' \supset S \). Theorems 1.2.5–1.2.7 are all proved in Chapter 5.

Now we turn to our main results in the global setting. First, we introduce some notation. Let \( k \) be a global function field. We denote by \( \mathbf{A}_k \) (or \( \mathbf{A} \) when there is no confusion as to which field we are working over) the ring of adeles for \( k \). For any abelian fppf sheaf \( \mathcal{F} \) on the category of all \( k \)-schemes, we define

\[
\mathbb{H}^i(k, \mathcal{F}) := \ker(H^i(k, \mathcal{F}) \rightarrow H^i(\mathbf{A}, \mathcal{F}))
\]

(cohomology for the small fppf sites on \( k \) and \( \mathbf{A} \) respectively). When there is no confusion, we will often simply write \( \mathbb{H}^i(\mathcal{F}) \). This agrees with the more classical definition

\[
\mathbb{H}^i(k, \mathcal{F}) := \ker\left(H^i(k, \mathcal{F}) \rightarrow \prod_v H^i(k_v, \mathcal{F})\right)
\]

(where the product is over all places \( v \) of \( k \) if \( i \leq 2 \) and \( \mathcal{F} = G \) or \( \hat{G} \) for some affine commutative \( k \)-group scheme \( G \) of finite type, by Propositions 6.1.1 and 6.1.2.

For an abelian group \( A \), we denote by \( A^* \) the group \( \text{Hom}(A, \mathbb{Q}/\mathbb{Z}) \). The functor \( A \mapsto A^* \) is exact precisely because \( \mathbb{Q}/\mathbb{Z} \) is an injective abelian group. We aim to prove the following result, which extends the classical Poitou-Tate exact sequence for finite commutative group schemes to all affine commutative group schemes of finite type:

**Theorem 1.2.8.** Let \( k \) be a global function field, and \( G \) an affine commutative \( k \)-group scheme of finite type. The following sequence (with maps to be defined below) is exact and functorial in \( G \):

\[
\begin{array}{cccccc}
0 & \to & G(k)_{\text{pro}} & \to & G(\mathbf{A})_{\text{pro}} & \to & H^2(k, \hat{G})^* & \to & H^1(k, G) \\
& & \downarrow & & \downarrow & & \downarrow & \\
0 & \leftarrow & \hat{G}(k)^* & \leftarrow & H^2(\mathbf{A}, G) & \leftarrow & H^2(k, G) & \leftarrow & H^1(\mathbf{A}, G)
\end{array}
\]

Let us now describe the maps in this sequence. We endow \( G(k) \) with the discrete topology and \( G(\mathbf{A}) \) with the natural topology arising from that on \( \mathbf{A} \) (via a closed embedding from \( G \) into some affine space). This latter topology coincides with the restricted product topology defined as follows. We spread out \( G \) to an \( O_S \)-model \( \mathcal{F} \) (where \( S \) is a non-empty finite set of places of \( k \)), and declare a fundamental system of neighborhoods of the identity to be the sets of the form \( \prod_{v \in S'} U_v \times \prod_{v \notin S'} \mathcal{O}_v \), where \( S' \supset S \) is a finite set of places and
$U_v \subset G(k_v)$ is a neighborhood of $0 \in G(k_v)$. This topology is independent of the chosen model, since any two become isomorphic over some $O_{S'}$. The natural map $G(k) \to G(A)$ induced by the diagonal inclusion $k \hookrightarrow A$ is trivially continuous, and therefore uniquely extends to a continuous map $G(k)_{\text{pro}} \to G(A)_{\text{pro}}$. The maps $H^1(k, G) \to H^1(A, G)$ and $H^2(k, G) \to H^2(A, G)$ are also induced by the diagonal inclusion $k \hookrightarrow A$.

The map $H^1(A, G) \to H^1(k, \hat{G})^*$ is defined by the pairing $H^1(A, G) \times H^1(k, \hat{G}) \to \mathbb{Q}/\mathbb{Z}$ which for every place $v$ of $k$ cups the corresponding elements of $H^1(k_v, G)$ and $H^1(k_v, \hat{G})$ to obtain an element of $H^2(k_v, G_m)$, takes the invariant, and then sums the result over all places $v$. We need to check that this sum contains only finitely many nonzero terms. In order to do this, we spread out $G$ to an $O_{S'}$-model $\mathcal{G}$ for some non-empty finite set $S$ of places of $k$. Since $A = \lim_{\rightarrow S'} \prod_{v \in S'} k_v \times \prod_{v \in S'} O_v$, where the limit is over all finite sets $S'$ of places of $k$ that contain $S$, the direct limit formalism in Appendix [D] shows that $H^1(A, G) = \lim_{\rightarrow S'} \prod_{v \in S'} H^1(k_v, G) \times \prod_{v \in S'} O_v, \mathcal{G}$. We similarly have $H^1(k, G) = \lim_{\rightarrow S'} H^1(O_{S'}, \mathcal{G})$. (In the future, we leave such direct limit reasoning to the reader.)

We conclude that any element of $H^1(A, G)$ lands inside the image of $H^1(O_v, \mathcal{G}) \to H^1(k_v, G)$ for all but finitely many places $v$ of $k$, and any element of $H^1(k, \hat{G})$ comes from $H^1(O_{S'}, \mathcal{G})$ for sufficiently large $S' \supset S$ and so restricts into the image of $H^1(O_v, \mathcal{G}) \to H^1(k_v, \hat{G})$ for all but finitely many $v$. For any $\alpha \in H^1(A, G)$ and $\beta \in H^1(k, \hat{G})$, therefore, $\langle \alpha_v, \beta_v \rangle \in \text{im}(H^2(O_v, G_m) \to H^2(k_v, G_m))$ for all but finitely many $v$. But $H^2(O_v, G_m) = \text{Br}(O_v) = 0$, so the local pairing is 0 for all but finitely many places, as desired. The map $H^2(A, G) \to \hat{G}(k)^*$ is similarly defined by cupping everywhere locally and then adding the invariants. The corresponding sum has at most finitely many nonzero terms, by similar reasoning.

We similarly obtain a map $G(A) \to H^2(k, \hat{G})^*$. To extend it to a map from $G(A)_{\text{pro}}$, endow $H^2(k, \hat{G})$ with the discrete topology. Since $H^2(k, \hat{G})$ is torsion (Lemma [4.4.1]), $H^2(k, \hat{G})^*$ is the Pontryagin dual of the discrete group $\hat{H}^2(k, \hat{G})$, and so it is naturally a profinite group. We claim that the map $G(A) \to H^2(k, \hat{G})^*$ is continuous, hence it uniquely extends to a continuous map $G(A)_{\text{pro}} \to H^2(k, \hat{G})^*$.

To see such continuity, we need to show that for any finite subset $T \subset H^2(k, \hat{G})$ and any $\epsilon > 0$, there exists a neighborhood $U \subset G(A)$ of the identity such that $|\langle u, \alpha \rangle| < \epsilon$ for all $u \in U, \alpha \in T$. Choose some $O_{S'}$-model $\mathcal{G}$ of $G$, and enlarge $S$ if necessary so that each $t \in T$ extends to a class in $H^2(O_S, \mathcal{G})$. By continuity of the local duality pairing (Theorem [1.22]), we may choose for each $v \in S$ a neighborhood $U_v \subset G(k_v)$ of the identity such that $\langle U_v, T \rangle \subset (-\epsilon/\#S, \epsilon/\#S)$. Now we may take $U := \bigcap_{v \in S} U_v \times \prod_{v \notin S} \mathcal{G}(O_v)$. Indeed, given $u \in U$ and $t \in T$, we have $|\langle u_v, t_v \rangle| < \epsilon/\#S$ for $v \in S$ by our choice of $U$, while for $v \notin S$, $\langle u_v, t_v \rangle$ factors through $H^2(O_v, G_m) = 0$, hence vanishes. Thus, $|\langle u, t \rangle| = |\sum_v \langle u_v, t_v \rangle| < \epsilon$, as desired.

The maps $H^2(k, \hat{G})^* \to H^1(k, G)$ and $H^1(k, \hat{G})^* \to H^2(k, G)$ are the hardest ones to describe in Theorem [1.28]. They rest on the following result that for finite $G$ is an analogue of a classical result of Tate for finite Galois modules over global fields:
Theorem 1.2.9. Let $k$ be a global function field, $G$ an affine commutative $k$-group scheme of finite type. Then the groups $\text{III}^i(k, G)$ and $\text{III}^i(k, \hat{G})$ are finite for $i = 1, 2$. Further, for $i = 1, 2$, we have perfect pairings, functorial in $G$,

$$\text{III}^i(G) \times \text{III}^{3-i}(\hat{G}) \rightarrow \mathbb{Q}/\mathbb{Z}$$

Theorem 1.2.9 furnishes an isomorphism

$$\text{III}^2(k, \hat{G})^* \sim \text{III}^1(k, G)$$

and the map $H^2(k, \hat{G})^* \rightarrow H^1(k, G)$ in Theorem 1.2.8 is defined to be the composition

$$H^2(k, \hat{G})^* \rightarrow \text{III}^2(k, \hat{G})^* \sim \text{III}^1(k, G) \rightarrow H^1(k, G).$$

Similarly, the map $H^1(k, \hat{G})^* \rightarrow H^2(k, G)$ is defined to be the composition $H^1(k, \hat{G})^* \rightarrow \text{III}^1(k, \hat{G})^* \sim \text{III}^2(k, G) \rightarrow H^2(k, G)$.

Some easy parts of the proof of Theorem 1.2.8 can be settled immediately. First we check that the sequence is a complex. That it is a complex at $H^1(k, G)$ and $H^2(k, G)$ is immediate from the definitions. That the sequence is a complex at $G(A), H^1(A, G)$, and $H^2(A, G)$ follows from the fact that the sum of the local invariants of a class in $H^2(k, G_m) = Br(k)$ is 0. It is a complex at $H^1(k, \hat{G})^*$ due to the fact that the pairing between $H^1(A, G)$ and $\text{III}^1(k, \hat{G})$ is trivial (since by definition any element of $\text{III}^1(k, \hat{G})$ has trivial image in $H^1(k_v, \hat{G})$ for each $v$). The sequence is a complex at $H^2(k, \hat{G})$ by similar reasoning.

Let us also note that exactness at $H^1(k, G)$ and $H^2(k, G)$ is immediate from the definitions. Proving that the rest of the sequence is exact will be the main work of Chapter 6 building on results in the finite case from Česnokov [6, §1.2].

In the finite case, there is a natural symmetry between $G$ and $\hat{G}$ due to Cartier duality and double duality. For positive-dimensional groups, there is no symmetry in the hypotheses between $G$ and $\hat{G}$, since the latter sheaf need not be finite type or even representable for general affine commutative $G$ of finite type over $k$ (cf. Proposition 2.2.7). Nevertheless, we still have double duality for such $G$ (Proposition 2.3.1), and the statements of the local results (taken together) are unchanged if we switch the roles of $G$ and $\hat{G}$; the same goes for Theorem 1.2.9. There should therefore be an analogue of Theorem 1.2.8 with the roles of $G$ and $\hat{G}$ reversed. This is indeed the case:

Theorem 1.2.10. Let $G$ be an affine commutative group scheme of finite type over a global function field $k$. Then the sequence

$$0 \rightarrow \hat{G}(k)_{\text{pro}} \rightarrow \hat{G}(A)_{\text{pro}} \rightarrow H^2(k, G)^* \rightarrow H^1(k, \hat{G}) \rightarrow 0$$

$$(G(k)_{\text{pro}})^D \rightarrow \bigoplus_v H^2(k_v, \hat{G}) \rightarrow H^2(k, \hat{G}) \rightarrow H^1(k, G)^* \rightarrow \text{III}^1(A, \hat{G})$$

is exact and functorial in $G$. 

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The maps are entirely analogous to those in Theorem 1.2.8 and as usual, all cohomology groups \( H^i(k, \cdot) \) are equipped with the discrete topology. The topology on \( \hat{G}(A) \) is defined to be the restricted product topology coming from some \( \mathcal{O}_S \)-model \( G \) (\( S \) a non-empty finite set of places, as usual) and the discrete topology on each group \( \hat{G}(k_v) \) of local characters. (The choice of the discrete topology on such character groups is appropriate, since even the character group of any affine group scheme of finite type over an algebraically closed field is a finitely generated abelian group.) By Theorem 1.2.5, this identifies \( \hat{G}(A) \) with \( \prod_v \hat{G}(k_v) \) endowed with the product topology. We will deduce Theorem 1.2.10 as the Pontryagin dual of Theorem 1.2.8 by applying local duality.

Remark 1.2.11. One might compare Theorems 1.2.8 and 1.2.10 and wonder whether one might not replace \((G(k)_{pro})^\cdot \) with \( G(k)^* \), or \( \oplus_v H^2(k_v, \hat{G}) \) with \( H^2(A, \hat{G}) \), or both at the same time. In fact, none of these combinations work.

To see that Theorem 1.2.10 becomes false if we replace the term \((G(k)_{pro})^\cdot \) with \( G(k)^* \), consider the case \( G = G_m \). Then the group \( G_m(k)^* = (k^\times)^* \) is not torsion (since \( k^\times \) has a quotient isomorphic to a countable direct sum of copies of \( \mathbb{Z} \)), whereas we claim that the group \( H^2(A, \hat{G}(m)) = H^2(A, \mathbb{Z}) \) (and therefore also the subgroup \( \oplus_v H^2(k_v, \hat{G}) \)) is torsion, and so there cannot be a surjection from it onto \( G_m(k)^* \). Indeed, we have an exact sequence of sheaves

\[
0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0,
\]

and \( H^i(A, \mathbb{Q}) = \lim_{\rightarrow} \prod_{v \in s^0} H^i(k_v, \mathbb{Q}) \times H^i(\prod_{v \notin s^0} \mathcal{O}_v, \mathbb{Q}) \), so by Lemmas 6.1.13 and 6.1.14 we see that \( H^i(A, \mathbb{Q}) = 0 \) for \( i > 0 \). Hence, \( H^2(A, \mathbb{Z}) \simeq H^1(A, \mathbb{Q}/\mathbb{Z}) = \lim_{\rightarrow} H^1(A, \mathbb{Q}/n\mathbb{Z}) \), and this last group is clearly torsion. Similarly, we cannot replace the group \( \mathop{\oplus}_v H^2(k_v, \hat{G}) \) with \( H^2(A, \hat{G}) \). Indeed, if we could make such a replacement, then by comparing the two sequences, one with \( \oplus_v H^2(k_v, \hat{G}) \) and the other with \( H^2(A, \hat{G}) \), we would deduce that the inclusion \( \oplus_v H^2(k_v, \hat{G}) \to H^2(A, \hat{G}) \) is an isomorphism. This is false in general, as may already be seen in the case \( G = G_m \). Indeed, we need to show that the map \( \oplus_v H^2(k_v, \mathbb{Z}) \to H^2(A, \mathbb{Z}) \) is not an isomorphism. Using the fact that \( H^2(A, \mathbb{Z}) = \lim_{\rightarrow} (\prod_{v \in s} \mathbb{H}^2(k_v, \mathbb{Z}) \times \mathbb{H}^2(\prod_{v \notin s} \mathcal{O}_v, \mathbb{Z})) \), in conjunction with Lemmas 6.1.13 and 6.1.14 and the exact sequence

\[
0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0,
\]

we deduce that this is equivalent to showing that the map \( \oplus_v \mathbb{H}^1(k_v, \mathbb{Q}/\mathbb{Z}) \to \mathbb{H}^1(A, \mathbb{Q}/\mathbb{Z}) \) is not an isomorphism. Let \( \hat{\mathcal{O}} := \prod_v \mathcal{O}_v \). Then it suffices to show that the image of the composition

\[
\mathbb{H}^1(\hat{\mathcal{O}}, \mathbb{Q}/\mathbb{Z}) \to \mathbb{H}^1(A, \mathbb{Q}/\mathbb{Z}) \to \prod_v \mathbb{H}^1(k_v, \mathbb{Q}/\mathbb{Z})
\]

does not lie in \( \oplus_v \mathbb{H}^1(k_v, \mathbb{Q}/\mathbb{Z}) \). Since the map \( \mathbb{H}^1(\mathcal{O}_v, \mathbb{Q}/\mathbb{Z}) \to \mathbb{H}^1(k_v, \mathbb{Q}/\mathbb{Z}) \) is injective (as may be easily seen directly, or alternatively, by writing \( \mathbb{Q}/\mathbb{Z} = \lim_{\rightarrow} \mathbb{Z}/n\mathbb{Z} \) and using
the fact that a (necessarily finite) \( \mathbb{Z}/n\mathbb{Z} \)-torsor over \( \mathcal{O}_v \) has an \( \mathcal{O}_v \)-point if it has a \( k_v \)-point, by the valuative criterion for properness), it suffices to show that the image of the map \( H^1(\hat{\mathcal{O}}, \mathbb{Q}/\mathbb{Z}) \to \prod_v H^1(\mathcal{O}_v, \mathbb{Q}/\mathbb{Z}) \) doesn’t live inside \( \oplus_v H^1(\mathcal{O}_v, \mathbb{Q}/\mathbb{Z}) \). The inclusion \( \frac{1}{n}\mathbb{Z}/\mathbb{Z} \hookrightarrow \mathbb{Q}/\mathbb{Z} \) induces an inclusion \( H^1(\mathcal{O}_v, \frac{1}{n}\mathbb{Z}/\mathbb{Z}) \hookrightarrow H^1(\mathcal{O}_v, \mathbb{Q}/\mathbb{Z}) \), as may be seen by considering the exact sequence

\[
0 \longrightarrow \frac{1}{n}\mathbb{Z}/\mathbb{Z} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0
\]

It therefore suffices to show that the image of the map \( H^1(\hat{\mathcal{O}}, \mathbb{Z}/n\mathbb{Z}) \to \prod_v H^1(\mathcal{O}_v, \mathbb{Z}/n\mathbb{Z}) \) isn’t contained in \( \oplus_v H^1(\mathcal{O}_v, \mathbb{Z}/n\mathbb{Z}) \). In fact, this map is surjective by Proposition 6.1.3 applied to the Cartier dual \( \mu_n \). It therefore suffices to show that \( H^1(\mathcal{O}_v, \mathbb{Z}/n\mathbb{Z}) \neq 0 \). Since \( \mathcal{O}_v \) is Henselian, \( H^1(\mathcal{O}_v, \mathbb{Z}/n\mathbb{Z}) = H^1(\kappa_v, \mathbb{Z}/n\mathbb{Z}) \), where \( \kappa_v \) is the residue field of \( \mathcal{O}_v \). Since \( \kappa_v \) is finite, \( H^1(\kappa_v, \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z} \neq 0 \) for \( n > 1 \), as desired.

This suggests that, in the spirit of preserving the symmetry between \( G \) and \( \hat{G} \), one should really replace the terms \( \hat{G}(k)^* \) and \( H^2(A, G) \) in Theorem 1.2.8 with \( \hat{G}(k, \mu_0)^D \) and \( \bigoplus H^2(k, G) \). This makes no difference, in the first case because \( G(k) \) is a finitely generated abelian group, and in the second because of Proposition 6.1.1.

**Remark 1.2.12.** Let us be very precise about which aspects of Tate local and global duality for finite commutative group schemes we are taking as input from prior work of others. First, the local results in Theorems 1.2.1–1.2.4 and 1.2.7 for finite commutative group schemes we are taking as input from prior work of others. References for these results in the finite case were given along with their statements above. As for the global results, consider the three sequences obtained by deleting the maps from \( H^i(k, \hat{G})^* \), \( i = 1, 2 \):

\[
0 \longrightarrow G(k) \longrightarrow G(A) \longrightarrow H^2(k, \hat{G})^*
\]

\[
H^1(k, G) \longrightarrow H^1(A, G) \longrightarrow H^1(k, \hat{G})^*
\]

\[
H^2(k, G) \longrightarrow H^2(A, G) \longrightarrow G(k)^* \longrightarrow 0
\]

The exactness of these sequences for finite commutative \( G \) holds by Česněık [Čes2, §1.2].

We do not assume Theorem 1.2.9 for finite group schemes, even though this case has in fact been fully proved in prior work of others: we will prove this case directly. The reason that we do not treat the exactness of the full 9-term exact sequence as known at the outset for finite \( G \) is that Česnavičius defines the maps from \( H^i(k, \hat{G})^* \) in a very different manner from the way we do, and then deduces Theorem 1.2.9 from the exactness of the resulting sequence. Our approach proceeds in the opposite order. To avoid verifying the compatibility between our sequence and Česnavičius’, and between our pairings in Theorem 1.2.9 and his, we have chosen to simply prove Theorem 1.2.9 directly, even in the finite case.

Throughout this work, whenever we refer to “local duality” or “Poitou-Tate” for finite commutative group schemes, we will mean the results that have just been summarized above.
1.3 Overview of methods

To orient the reader, we now provide a brief overview of the manuscript, and indicate how the various parts fit together to establish the main results.

In Chapter 2, we establish some crucial results about affine commutative group schemes of finite type over general fields, including their structure theory (§2.1), their dual sheaves (§2.2), and the cohomology of $\hat{\mathbb{G}}_a$ (§2.4–§2.6).

Chapter 3 deals with several vital preliminaries on the cohomology of local and global fields. The main highlights are cohomological vanishing results for $G$ and $\hat{G}$ (Propositions 3.1.1 and 3.4.1), the explicit computation of $H^2(k, \hat{\mathbb{G}}_a)$ for local and global function fields (Propositions 3.2.1 and 3.2.2), and the equivalence between Čech and derived functor cohomology of $G$ and $\hat{G}$ in low degrees (Proposition 3.5.1). This last equivalence plays a crucial role in proving the continuity of the local duality pairings, as well as in defining the $\mathcal{X}$-pairings appearing in Theorem 1.2.9.

In Chapter 4, we prove the main local duality results (Theorems 1.2.1, 1.2.2, and 1.2.4). In Chapter 5, we establish the main results on local integral cohomology (Theorems 1.2.5, 1.2.6, and 1.2.7). Finally, in Chapter 6, we establish the global Tate duality results (Theorems 1.2.8–1.2.10), essentially analyzing the sequence bit by bit. An interesting feature of the proof of the 9-term exact sequence for positive-dimensional groups is that if we grant that the entire sequence is exact for finite commutative group schemes (cf. Remark 1.2.12) then the different parts of the sequence may be shown to be exact essentially independently, in stark contrast to Tate’s original proof of his duality results.

It is difficult to briefly explain the main ideas of the proofs of the preceding theorems except to say that the arguments typically proceed by a long dévissage from the cases of finite commutative group schemes, $\mathbb{G}_m$, and $\mathbb{G}_a$. This dévissage typically (though not always) proceeds as follows. One uses Lemma 2.1.7 to reduce to the case when $G$ is split unipotent or an almost torus (see Definition 2.1.4). The split unipotent case reduces to that of $\mathbb{G}_a$, which is often very difficult due to the non-representability of $\hat{\mathbb{G}}_a$. The case of almost tori is related to the cases of finite commutative group schemes and the case of $\mathbb{G}_m$ by means of Lemma 2.1.3(iv).

The case of finite $G$ in the preceding theorems will generally amount to known results of others, but for Theorem 1.2.9 our proof also treats the case of finite $G$ from scratch to avoid any need to address compatibility with the $\mathcal{X}$-pairings in [Čes2] that rest on $S$-integral excision constructions that do not seem to be useful in our work that incorporates positive-dimensional $G$. The main results for $\mathbb{G}_m$ typically amount to the main results of local and global class field theory. Finally, the main results for $\mathbb{G}_a$ rest on §3.2.

Let us also briefly explain how we analyze the group $H^2(k, \hat{\mathbb{G}}_a)$ when $p = \text{char}(k) > 0$, since working with the non-representable sheaf $\hat{\mathbb{G}}_a$ is one of the major features of Tate duality in positive dimension that does not appear for finite commutative group schemes, and (unlike $\mathbb{G}_m$) results for $\mathbb{G}_a$ don’t simply emerge from class field theory. Proposition
2.4.6 relates the $p$-torsion group $H^2(k, \hat{G}_a)$ to $p$-torsion in the Brauer group of $G_{a,k}$. In §2.5, we adapt an idea of Kato to relate such $p$-torsion Brauer classes to differential forms on $G_a$ (Lemma 2.5.2). Finally, we utilize this relationship in §2.6 to explicitly compute the group $H^2(k, \hat{G}_a)$ for all fields $k$ of characteristic $p > 0$ such that $[k : k^p] = p$ (Proposition 2.6.1), a class of fields that includes local and global function fields.

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1.5 Notation and terminology

Throughout this work, $k$ denotes a field and $k_s, \overline{k}$ denote separable and algebraic closures of $k$, respectively. Also, $p$ denotes a prime, equal to $\text{char}(k)$ when $\text{char}(k) > 0$. All cohomology is fppf unless stated otherwise.

We will frequently use without comment the fact that $H^i_{\text{ét}}(X, G) = H^i_{\text{fppf}}(X, G)$ for smooth $X$-group schemes $G$ when $i = 0, 1$, and for all $i$ if we also assume that $G$ is commutative. For $i = 0$, this is obvious. For $i = 1$, it follows from the fact that any fppf $G$-torsor over $X$ has a section étale-locally. The agreement for all $i$ in the commutative case follows from [GrIII, Thm. 11.7].

For an abelian sheaf $\mathcal{F}$ for the fppf topology on the category of schemes over a base scheme, the functor $\mathcal{H}om(\mathcal{F}, \mathbb{G}_m)$ is denoted $\hat{\mathcal{F}}$.

An isogeny $f : G \to H$ between finite type group schemes over a field is a finite flat surjective homomorphism. A split unipotent group over a field $k$ is a $k$-group admitting a composition series of (smooth connected) $k$-subgroups such that the successive quotients are each $k$-isomorphic to $\mathbb{G}_a$.

When we are working with the set of places of a global field, we say “almost every” to mean “all but finitely many”.

If $G$ is a scheme of finite type over a global field $k$ and $S$ is a non-empty finite set of places of $k$ containing the archimedean places, then an $\mathcal{O}_S$-model of $G$ is a flat separated finite type $\mathcal{O}_S$-group scheme $\mathcal{G}$ equipped with a $k$-group scheme isomorphism $\mathcal{G}_k \simeq G$. It is well-known that for sufficiently large $S$ (depending on $G$) such an $\mathcal{O}_S$-model exists, that any two become isomorphic over $\mathcal{O}_{S'}$ for some $S' \supset S$, and that if $G$ is affine then $\mathcal{G}_{\mathcal{O}_{S'}}$ is affine for sufficiently large $S' \supset S$. By flatness and separatedness, if $G$ is commutative then $\mathcal{G}$ is commutative. We will nearly always work with affine $G$, though occasionally we will permit more general $G$ when a proof does not require affineness. Consequently, we adopt the convention that if $G$ is affine, it is always understood that we only use $\mathcal{G}$ that are affine.
Chapter 2

General fields

In this chapter we prove various results about affine commutative group schemes of finite type over general fields, before turning to the special case of local and global function fields in later chapters. We begin by discussing the structure of such groups (§2.1), and then turn to a discussion of their fpf $\mathbb{G}_m$-dual sheaves (§§2.2-2.3), the main result being the exactness of the $\mathbb{G}_m$-dual functor (Proposition 2.2.3). We then turn to an in-depth discussion of the cohomology of $\hat{\mathbb{G}}_a$, especially $H^2(k, \hat{\mathbb{G}}_a)$, and its relation to primitive elements of the Brauer group of $\mathbb{G}_a$ (§§2.4-2.6). This discussion lies at the core of our extension of classical Tate duality beyond the case of tori and finite group schemes (and extensions thereof). The central result here is that $\dim_k H^2(k, \hat{\mathbb{G}}_a) = 1$ for any field $k$ of characteristic $p$ with $[k : k^p] = p$ (Cor. 2.6.2); this will play a crucial role in subsequent chapters.

2.1 Structure of affine commutative group schemes of finite type

In this section we prove some results that we shall require about the structure of affine commutative group schemes of finite type over a general field.

Lemma 2.1.1. Let $k$ be a field, $G$ an affine commutative $k$-group scheme of finite type. Then there is a finite $k$-subgroup scheme $E \subset G$ such that $G/E$ is smooth and connected.

Proof. By [SGA3, VII, Prop. 8.3], there is an infinitesimal subgroup scheme $I \subset G$ such that $G/I$ is smooth. Replacing $G$ with $G/I$, therefore, we may assume that $G$ is smooth. We next reduce to the case when $k$ is perfect (so $\overline{k}/k$ is Galois). Without loss of generality we may assume $\text{char}(k) = p > 0$. Let $F : G \to G^{(p)}$ be the relative Frobenius homomorphism; this is an isogeny since $G$ is smooth. If $E' \subset G^{(p)}$ is a finite $k$-subgroup scheme such that the smooth $k$-group $G^{(p)}/E'$ is connected then $E := F^{-1}(E')$ is finite and clearly the smooth
$k$-group $G/E$ is connected (as $G/E \simeq G^{(p)}/E'$). Hence, we may replace $G$ with $G^{(p)}$, or more generally with $G^{(p^n)}$ for any desired $n \geq 0$.

If the problem can be solved over the perfect closure of $k$ then by standard “spreading out” considerations it is solved over some purely inseparable finite extension of $k$. Any such extension is contained in $k^{p^n}$ for some large $n$, and the field extension $k \hookrightarrow k^{p^n}$ is identified with the $p^n$-power map $k \to k$. Hence, passing to $G^{(p^n)}$ would then do the job. Thus, now we may and do assume $k$ is perfect (of any characteristic).

We claim that each component of $G$ contains a torsion point of $G(\kbar)$. Assuming this, we can choose one such point in each component of $G$, and the subgroup of $G(\kbar)$ generated by their Galois-orbits is $\Gal(\kbar/k)$-stable and hence descends to a finite $k$-subgroup $E \subset G$ with the desired property.

To prove the claim, we may assume that $k = \kbar$. Let $N$ be the exponent of $G/G^0$. First suppose that char($k$) = 0. Then multiplication by $N$ sends $G$ into $G^0$. Further, given an arbitrary component $X$ of $G$, $[N] : X \to G^0$ is surjective, as can be seen by fixing $e \in X(k)$, noting that any element of $X(k)$ may be uniquely written in the form $e + g$ with $g \in G^0(k)$, and then using the fact that $[N] : G^0 \to G^0$ is surjective (since $G^0$ has a composition series with successive quotients $G_a$ or $G_m$, on each of which $[N]$ is surjective). If char($k$) = $p > 0$, then we use the same argument with $G^0$ replaced by its maximal torus $T$, and with $N$ replaced by $Np^n$, where $n$ is chosen so large that $[p^n]G^0 = T$.

Remark 2.1.2. Actually, by only slightly modifying the proof of Lemma 2.1.1 one can remove the affineness hypothesis.

Next we will introduce the notion of an almost torus.

**Lemma 2.1.3.** Let $k$ be a field, $G$ a commutative affine $k$-group scheme of finite type. The following are equivalent:

(i) $(G_{\kbar})^0_{\text{red}}$ is a torus.

(ii) There is a $k$-torus $T \subset G$ such that $G/T$ is finite.

(iii) There exist a $k$-torus $T$, a finite $k$-group scheme $A$, and an isogeny $A \times T \to G$.

(iv) There exist a positive integer $n$, a finite $k$-group scheme $A$, finite separable extensions $k_1, k_2/k$, split $k_i$-tori $T_i$, and an isogeny $A \times \text{R}_{k_1/k}(T_1) \to G^n \times \text{R}_{k_2/k}(T_2)$.

**Proof.** (i) $\implies$ (ii): Let $l \neq \text{char}(k)$ be a fixed prime, and let $G[l^{\infty}] := \cup_{n=1}^{\infty} G[l^n](\kbar)$. Let $T' \subset G_{\kbar}$ be the identity component of the Zariski closure of $G[l^{\infty}]$. Since $(G_{\kbar})^0_{\text{red}}$ is a torus, and the $l$-power torsion is dense in any $\kbar$-torus, we see that $T' = (G_{\kbar})^0_{\text{red}}$. We claim that $T'$ descends to a $k$-torus in $G$. Since $G_{\kbar}/T'$ is finite, this will show what we want.

First, $G[l^n](k) = G[l^n](\kbar)$, since $G[l^n]$ is étale, so, since the Zariski closure of a set of rational points commutes with field extension, $T'$ descends to a $k_s$-torus $T'' \subset G_{k_s}$. Now $G[l^{\infty}]$ is clearly preserved by $\text{Aut}(\kbar/k) = \text{Gal}(k_s/k)$, so $T'$ is also preserved and hence
descends to a \( k \)-torus \( T \), as claimed.

(ii) \( \Rightarrow \) (iii): Let \( A \subset G \) be a finite \( k \)-subgroup scheme such that \( G/A \) is smooth and connected (Lemma 2.1.1). Then \( T \times A \to G \) is the desired isogeny, since the cokernel of this map is smooth, connected, and finite, hence trivial.

(iii) \( \Rightarrow \) (iv): We may assume that \( G = T \) is a torus. Then (iii) is essentially [Ono, Thm.1.5.1]; there the theorem is stated over number fields, but it works over any field. The idea of the proof is to use the equivalence between tori and Galois lattices, combined with Artin’s theorem on induced representations.

(iv) \( \Rightarrow \) (i): Since \( k_i/k \) are separable, each \( \text{R}_{k_i/k}(T_i) \) is a torus. So \( ((G_{k_i})^{0}_{\text{red}})^n \times T'_{2} \) is the isogenous quotient of a torus, for some torus \( T'_{2} \). It follows that \( (G_{k_i})^{0}_{\text{red}} \) is itself a torus. \( \square \)

**Definition 2.1.4.** A \( k \)-group scheme \( G \) satisfying the equivalent conditions of Lemma 2.1.3 is called an **almost torus**.

Condition (iv) in Lemma 2.1.3 will be the most useful for us, since it reduces many questions about tori to the case of separable Weil restrictions of split tori. Note in particular that finite group schemes are almost tori. Let us also note the following easy fact.

**Lemma 2.1.5.** Suppose that we have a short exact sequence of affine commutative \( k \)-group schemes of finite type:

\[
1 \to G' \to G \to G'' \to 1
\]

Then \( G \) is an almost torus if and only if both \( G' \) and \( G'' \) are almost tori.

**Proof.** This follows easily from the well-known analogous statement for tori. \( \square \)

Now we study unipotent groups. The key result is the following lemma.

**Lemma 2.1.6.** Let \( k \) be a field, \( U \) a smooth connected commutative unipotent \( k \)-group. Then there is an infinitesimal \( k \)-subgroup scheme \( A \subset U \) such that \( U/A \) is split unipotent. The same result also holds with \( A \) étale.

**Proof.** When \( \text{char}(k) = 0 \), this is trivial, since every smooth connected unipotent \( k \)-group is split. So assume that \( \text{char}(k) = p > 0 \). First we treat the infinitesimal case. \( U \) splits over the perfect closure of \( k \), hence over \( k^{1/p^n} \) for some \( n > 0 \). Hence \( U^{(p^n)} \) is split. Since \( U \) is smooth, the \( n \)-fold relative Frobenius map \( U \to U^{(p^n)} \) is an isogeny. Therefore, letting \( I \) denote its infinitesimal kernel, we have \( U/I \simeq U^{(p^n)} \), hence \( U/I \) is split.

Next we treat the étale case. When \( U \) is \( p \)-torsion, this is part of [CGP, Lemma B.1.10]. Otherwise, we proceed by induction: suppose that we have a short exact sequence

\[
1 \to U' \to U \to U'' \to 1
\]

such that \( U', U'' \) are nontrivial smooth connected commutative unipotent \( k \)-groups for which the lemma holds. (We may take \( U' = [p]U, U'' = U/U' \).) That is, there exist finite étale
k-subgroup schemes $E' \subset U'$, $E'' \subset U''$ such that $U'/E'$ and $U''/E''$ are split. Replacing $U'$ with $U'/E'$ and $U$ with $U/E'$, we may assume that $U'$ is split. So we only need to find a finite étale $k$-subgroup scheme $E \subset U$ such that $E \to E''$, as $U/E$ is then split. To do this, we may assume that $k = k_s$, as we may then replace $E$ with the subgroup generated by its (finitely many) Galois translates in order to ensure that it is defined over $k$.

So suppose that $k = k_s$. Then $E''$ is constant, hence we merely need to show that for every $e'' \in E''(k)$, there is $e \in U(k)$ such that $e \mapsto e''$ since $U$ is torsion (if we choose one such $e$ for each $e'' \in E''(k)$, then the finite subgroup generated by the various $e$ surjects onto $E''$). But the existence of such $e$ is clear: the map $U \to U''$ is smooth (having smooth kernel $U'$), so the fiber above each $e'' \in E''(k)$ contains a $k$-point, since $k = k_s$.

The following lemma is the main result that we will use on the structure of affine commutative group schemes.

**Lemma 2.1.7.** Let $k$ be a field, $G$ an affine commutative $k$-group scheme of finite type. Then there is a short exact sequence of $k$-group schemes

$$1 \to H \to G \to U \to 1$$

with $H$ an almost torus and $U$ a split unipotent $k$-group.

**Proof.** By Lemma 2.1.1, there is a finite $k$-subgroup scheme $A \subset G$ such that $G/A$ is smooth and connected. Since an extension of an almost torus by an almost torus is an almost torus (Lemma 2.1.5), and $A$ is an almost torus, we may therefore assume that $G$ is smooth and connected. Letting $T \subset G$ be the maximal torus, $G/T$ is smooth connected unipotent. We may therefore assume that $G = U$ is smooth connected unipotent. But then by Lemma 2.1.6, there is a finite $k$-subgroup scheme $A \subset U$ such that $U/A$ is split unipotent, so we are done.

By analogy with the situation of almost tori, we will on occasion have use for the notion of almost-unipotence (though this will come up far less often than almost tori):

**Lemma 2.1.8.** Let $G$ be an affine commutative $k$-group scheme of finite type. The following are equivalent:

(i) $(G_{k/\text{red}})^0$ is unipotent,

(ii) $G$ contains no nontrivial $k$-torus,

(iii) There is an exact sequence

$$1 \to A \to G \to U \to 1$$

with $A$ a finite commutative $k$-group scheme and $U$ split unipotent over $k$. 26
Proof. (i) \(\Rightarrow\) (ii): Obvious.

(ii) \(\Rightarrow\) (iii): By Lemma 2.1.7 there is such a sequence with \(A\) an almost torus. Since \(G\), hence \(A\), contains no nontrivial tori, it follows from Lemma 2.1.3(ii) that \(A\) is finite.

(iii) \(\Rightarrow\) (i): We may assume that \(k = \bar{k}\). If \(G_{\text{red}}^0\) is not unipotent then it contains a nontrivial torus \(T\). The map \(T \to U\) is trivial, so \(T \subset A\), an absurdity.

Definition 2.1.9. A \(k\)-group scheme \(G\) satisfying the equivalent conditions of Lemma 2.1.8 is said to be almost-unipotent.

Almost-unipotent groups have the expected permanence properties:

Lemma 2.1.10. A closed \(k\)-subgroup scheme of an almost-unipotent \(k\)-group scheme is almost-unipotent, as is the quotient of an almost-unipotent \(k\)-group scheme by a closed \(k\)-subgroup scheme. Any commutative extension of almost-unipotent \(k\)-group schemes is almost-unipotent.

Proof. For subgroups, this is probably most easily seen by appealing to Lemma 2.1.8(ii). For quotients and extensions, the assertion follows from the corresponding fact for smooth connected unipotent groups, by applying Lemma 2.1.8(i).

A very useful property of almost-unipotence is that it is preserved by Weil restriction:

Lemma 2.1.11. Let \(k'/k\) be a finite extension of fields, and \(G'\) an almost-unipotent \(k'\)-group scheme. Then \(R_{k'/k}(G')\) is an almost-unipotent \(k\)-group scheme.

Proof. Since Weil restriction is transitive, we may assume that \(k'/k\) is either separable or purely inseparable. Further, almost-unipotence may be checked after replacing \(k\) by an algebraic extension. If \(k'/k\) is separable (so \(k'_s\) is identified with a separable closure \(k_s\) of \(k\)) then \(R_{k'/k}(G')_{k_s}\) becomes a product of Galois-twisted copies of \(G'_{k'_s}\), hence almost-unipotent.

Now suppose that \(k'/k\) is purely inseparable. By [Oes, App. 3, A.3.6], there is an exact sequence of \(k'\)-group schemes

\[
1 \rightarrow U \rightarrow R_{k'/k}(G')_{k'} \rightarrow G' \rightarrow 1
\]

with \(U\) split unipotent. By Lemma 2.1.10 \(R_{k'/k}(G')_{k'}\) is therefore almost-unipotent.

2.2 Dual sheaves

Let us begin by proving a basic result which we will need later. Let \(f : X' \to X\) be a finite flat map between Noetherian schemes, \(\mathcal{F}'\) an abelian fppf sheaf on \(X'_s\), and let \(\mathcal{F} := f_*\mathcal{F}'\). Then we have a natural map of abelian fppf sheaves \(N_{X'/X} : f_*(\mathcal{F}') \to \widehat{\mathcal{F}}\) defined as follows. Given an \(X\)-scheme \(Y\) and a character \(\chi \in f_*(\mathcal{F}')(Y) = \widehat{\mathcal{F}}(Y \times_X X')\), the character \(N_{X'/X}(\chi) \in \mathcal{F}(Y)\) is defined functorially on \(Y\)-schemes \(Z\) as the composition

\[
N_{X'/X}(\chi)(z) = \text{res}(\chi(f(z))\cdot \text{res}(z))
\]
\( F(Z) = \mathcal{F}'(Z \times_X X') \xrightarrow{\chi_{Z \times_X X'}} \Gamma(Z \times_X X', \mathcal{O}_{Z \times_X X'}^\times) \xrightarrow{N_{X'/X}} \Gamma(Z, \mathcal{O}_Z^\times), \) where \( N_{X'/X} \) is the norm map, defined locally on \( \text{Spec}(A) \subset Z \) and at \( x' \in X' \) as the map \( A \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X',x'} \rightarrow A \) \( (x = f(x')) \), given by sending \( \alpha \) in the finite free \( A \)-module \( A \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X',x'} \) to the determinant of the \( A \)-linear map given by multiplication by \( \alpha \).

The following lemma is a (generalized) sheafified version of (part of) [Oes, Ch.II, Thm.2.4].

**Lemma 2.2.1.** Let \( f : X' \rightarrow X \) be a finite étale map between Noetherian schemes, \( \mathcal{F}' \) an fppf abelian sheaf on \( X' \), and \( \mathcal{F} := f^*(\mathcal{F}') \). Then \( N_{X'/X} : f^*(\hat{\mathcal{F}}') \rightarrow \hat{\mathcal{F}} \) is an isomorphism of fppf sheaves.

**Proof.** The assertion is fppf local, so we may assume that \( X \) is the spectrum of a strictly Henselian local ring. But then \( X' \) is a finite disjoint union of copies of \( X \), so since the norm map and \( \hat{(\cdot)} \) functor send such a union to a product, we may assume that \( X' = X \), in which case the assertion is trivial. \( \square \)

**Remark 2.2.2.** The étale hypothesis in Lemma [2.2.1] is absolutely crucial: unlike [Oes, Ch.II, Thm.2.4], the lemma fails for a finite inseparable extension of fields \( k'/k \), even for \( \mathcal{F}' = \mathbb{G}_m \). Indeed, if \( k'/k \) is a finite nontrivial purely inseparable extension, then there is no isomorphism between \( R_{k'/k}(\mathbb{G}_m) \) and \( R_{k'/k}(\mathbb{G}_m) \simeq R_{k'/k}(\mathbb{Z}) \), let alone via \( N_{k'/k} \), since the latter sheaf is a smooth (even étale) group scheme whereas the former isn’t even representable. To prove such non-representability, by [Oes, App.3, A.3.6] there is a short exact sequence

\[ 1 \rightarrow U \rightarrow R_{k'/k}(\mathbb{G}_m)_{k'} \xrightarrow{\pi} \mathbb{G}_{m,k'} \rightarrow 1 \]

with \( U \) split unipotent and non-trivial. The map \( \pi \) is given functorially on \( k' \)-algebras \( R' \) by the map \( (R' \otimes_k k')^\times \rightarrow R'^\times \) induced by the map \( R' \otimes_k k' \rightarrow R \) defined by \( r \otimes \lambda \mapsto r\lambda \).

In particular, \( R_{k'/k}(\mathbb{G}_m) \) is not an almost-torus. That \( R_{k'/k}(\mathbb{G}_m) \) is not representable therefore follows from Proposition [2.2.7] below.

Now we turn to the main goal of this section, which is to prove the following crucial result.

**Proposition 2.2.3.** Let \( k \) be a field, and suppose that we have a short exact sequence of affine commutative \( k \)-group schemes of finite type:

\[ 1 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 1. \]

The corresponding pullback sequence of (fppf) dual sheaves

\[ 1 \rightarrow \hat{G}'' \rightarrow \hat{G} \rightarrow \hat{G}' \rightarrow 1 \]

is exact when \( \text{char}(k) = p > 0 \), and likewise for the small étale topology over \( k \) when \( \text{char}(k) = 0 \).
The left-exactness of the functor $\hat{\cdot}$ is clear. The real content of Proposition 2.2.3 is therefore the exactness on the right. Before turning to the proof, we give several applications.

**Corollary 2.2.4.** Let $k = \bar{k}$ be an algebraically closed field, and suppose that we have a short exact sequence of affine commutative $k$-group schemes of finite type:

$$1 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 1$$

Then the corresponding pullback sequence

$$1 \rightarrow \hat{G}''(k) \rightarrow \hat{G}(k) \rightarrow \hat{G}'(k) \rightarrow 1$$

is also exact.

**Proof.** When $\text{char}(k) > 0$, this follows from Proposition 2.2.3 due to the fact that the functor $\text{H}^0(k, \cdot)$ is exact on the category of abelian sheaves for the fppf and étale topologies (since any fppf cover of $\text{Spec}(k)$ may be refined by the trivial cover $\text{Spec}(k) \rightarrow \text{Spec}(k)$, by the Nullstellensatz). When $\text{char}(k) = 0$, the corollary follows from the étale analogue of Proposition 2.2.3, cf. Remark 2.2.20.

**Corollary 2.2.5.** Let $k$ be a field, and suppose that we have an inclusion $H \hookrightarrow G$ of affine commutative $k$-group schemes of finite type. Then the restriction map $\hat{G}(k) \rightarrow \hat{H}(k)$ has finite cokernel.

**Proof.** Since this cokernel is finitely-generated, it suffices to show that it is torsion. That is, given $\chi \in \hat{H}(k)$, we want to show that $\chi^n$ extends to a character of $\hat{G}(k)$ for some positive integer $n$. By Corollary 2.2.4, there exists $\psi \in \hat{G}(k)$ such that $\psi|_H = \chi$. If $\text{char}(k) = p > 0$ then $\psi^p \in \hat{G}(k_s)$ for some positive integer $n$, so by replacing $\chi$ with $\chi^p$ we may assume that $\psi \in \hat{G}(k_s)$; this latter condition on $\psi$ automatically holds when $\text{char}(k) = 0$. Thus, working with $k$ of any characteristic, $\psi \in \hat{G}(k')$ for some finite Galois extension $k'/k$ (of degree $d$, say). Hence, $\prod_{\sigma \in \text{Gal}(k'/k)} \psi^\sigma \in \hat{G}(k)$. But $\prod_{\sigma \in \text{Gal}(k'/k)} \psi^\sigma|_H = \chi^d$, so we are done.

**Remark 2.2.6.** Oesterlé proved that if we have an inclusion $H \hookrightarrow G$ with $G$ smooth, connected, and affine (and $H$ a closed $k$-subgroup scheme), then the restriction map $\hat{G}(k) \rightarrow \hat{H}(k)$ has finite cokernel [Oes A.1.4]. That is, he avoids commutativity hypotheses, but adds in smoothness and connectedness. The smoothness may be dispensed with: [SGA3 VII A, Prop. 8.3] furnishes an infinitesimal subgroup scheme $I \subset G$ such that $G/I$ is smooth, and we then have an inclusion $H/H \cap I \hookrightarrow G/I$. If $\chi \in \hat{H}(k)$, then $\chi^n|_{H\cap I} = 1$ for some positive integer $n$, hence some power of $\chi^n$ extends to a character of $G/I$ (by the smooth connected case), hence to one of $G$, so again the cokernel is torsion and finitely-generated, hence finite.
But in the absence of commutativity assumptions, the connectedness of $G$ is absolutely crucial (even when $k = \mathbb{F}_q$), as the following example due to Brian Conrad illustrates. Let $G = \mathbb{G}_m \rtimes \mathbb{Z}/2\mathbb{Z}$ with the nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ acting on $\mathbb{G}_m$ by inversion, and let $H$ be the normal $\mathbb{G}_m$ inside of $G$. Then we claim that for any $\chi \in \hat{G}(k)$, we have $\chi|_H = 1$. Indeed, $\chi$ is invariant under conjugation, hence due to the $\mathbb{Z}/2\mathbb{Z}$-action, $\chi|_{\mathbb{G}_m}$ must be invariant under inversion, hence trivial.

Let us also use Proposition 2.2.3 to prove the following result, which we will use in chapter 4. (We could make do without it, but it makes things more convenient, and is interesting in its own right.)

**Proposition 2.2.7.** Let $k$ be a field, $G$ an affine commutative $k$-group scheme of finite type. Then the fppf sheaf $\hat{G}$ is representable if and only if $G$ is an almost torus, in which case it is locally of finite type.

The method of proof adapts with almost no changes to show that beyond the case of almost tori, representability always fails even in the wider generality of algebraic spaces.

**Proof.** First suppose that $G$ is an almost torus. Then by Lemma 2.1.3(iii), there is an isogeny $T \times A \to G$. Let $B$ denote its (finite) kernel. Then we have an exact sequence

$$1 \to B \to T \times A \to G \to 1$$

hence an exact sequence

$$1 \to \hat{G} \to \hat{T} \times \hat{A} \to \hat{B}$$

(which by Proposition 2.2.3 extends to a short exact sequence, but we will not need this). As is well-known, the dual sheaves of finite commutative group schemes and of tori are representable, hence $\hat{G}$ is the kernel of a homomorphism of representable $k$-group schemes, hence is representable.

Now suppose that $G$ is not an almost torus, and we will show that $\hat{G}$ is not representable. By Lemma 2.1.7, there is an almost torus $H \subset G$ such that the quotient $U := G/H$ is split unipotent (and non-trivial!). We then have an exact sequence

$$1 \to \hat{U} \to \hat{G} \to \hat{H}$$

so since $\hat{H}$ is representable, if $\hat{G}$ is representable, then so is $\hat{U}$, as it is the kernel of a morphism of $k$-group schemes. We may therefore assume that $G = U$ is nontrivial split unipotent.

So suppose that $\hat{U}$ is representable. We have $\hat{U}(\lim_{\to} R_i) = \lim_{\to} \hat{U}(R_i)$ for any directed system of $k$-algebras $R_i$, so by EGA IV, Prop.8.14.2, $\hat{U}$ is locally of finite type. But $\hat{U}$ has no nontrivial field-valued points, so it must be infinitesimal.

If $\text{char}(k) = 0$, then this means that $\hat{U} = 0$. But this is false: there is a surjection $U \to \mathbb{G}_a$, hence an inclusion $\hat{\mathbb{G}}_a \to \hat{U}$, and $\hat{\mathbb{G}}_a \neq 0$. Indeed, over any ring $R$, $\hat{\mathbb{G}}_a(R)$
includes all of the global units \( \exp(aX) = \sum_{n=0}^{\infty} (aX)^n/n! \in R[X] = H^0(G_{a,R} \cdot \text{G}_m) \) with \( a \in R \) nilpotent. (Actually, these are all of the elements of \( \hat{G}_a(R) \), but we will not need this.) So \( \hat{G}_a(R) \neq 0 \) for any non-reduced \( k \)-algebra \( R \).

Next suppose that \( \text{char}(k) > 0 \). Then the infinitesimal \( \hat{u} \) is finite. But there is an inclusion \( G_a \hookrightarrow U \), hence an inclusion \( \alpha_{p^n} \hookrightarrow U \) for each positive integer \( n \). This is impossible since \( \hat{u} \) is finite.

To prove Proposition 2.2.3, we will make use of certain Ext sheaves. In order to do this, we will first make an observation. Given an fppf abelian sheaf \( F \) on the category of \( k \)-schemes and a short exact sequence of fppf abelian sheaves

\[
0 \to G' \to G \to G'' \to 0,
\]

we obtain the expected long exact sequence of Ext groups

\[
1 \to \operatorname{Hom}(G'', F) \to \operatorname{Hom}(G, F) \to \operatorname{Hom}(G', F) \to \operatorname{Ext}^1(G'', F) \to \operatorname{Ext}^1(G, F) \to \operatorname{Ext}^1(G', F) \to \ldots
\]

and likewise (or by sheafifying) a long exact sequence of sheaves

\[
1 \to \mathcal{H}om(G'', F) \to \mathcal{H}om(G, F) \to \mathcal{H}om(G', F) \to \mathcal{E}xt^1(G'', F) \to \mathcal{E}xt^1(G, F) \to \mathcal{E}xt^1(G', F) \to \ldots
\]

The point is that although Ext and \( \mathcal{E}xt \) are only derived functors in the second argument (due to the lack of enough projectives), we can argue as follows in any abelian category with enough injectives: for an injective resolution \( F \to I^\bullet \) we get the left-exact sequence of complexes

\[
0 \to \operatorname{Hom}(G'', I^\bullet) \to \operatorname{Hom}(G, I^\bullet) \to \operatorname{Hom}(G', I^\bullet) \to 0
\]

where exactness on the right holds because the \( I^n \)'s are injectives. The cohomology sequence associated to this short exact sequence of complexes is then the desired sequence (2.2.1), and similarity in the sheafified setting.

Returning to the proof of Proposition 2.2.3, first we address the easy case \( \text{char}(k) = 0 \). In this case every finite type \( k \)-group scheme is smooth, and it is elementary to check by Galois descent from \( \overline{k} \) that a smooth commutative affine \( k \)-group is uniquely and functorially a direct product of a unipotent \( k \)-group and one of multiplicative type. (Here we handle disconnectedness by using that multiplication by a nonzero integer is surjective on any unipotent \( k \)-group or \( k \)-torus.) Over any étale \( k \)-algebra (a finite product of finite separable extension fields) there is no nontrivial homomorphism from \( G_a \) to \( \text{GL}_1 \), so the functor \( \hat{G} \) on étale \( k \)-algebras is unaffected by relacing \( G \) with its multiplicative-type factor. We may therefore assume \( G' \), \( G \), and \( G'' \) are of multiplicative type. Under the duality with discrete Galois modules finitely generated over \( \mathbb{Z} \), \( \hat{G} \) is represented by the commutative étale \( k \)-group corresponding to the discrete Galois module \( \operatorname{Hom}(G, \text{GL}_1) \), so the desired exactness
result is for short-exactness of a 3-term complex of commutative étale \(k\)-groups. This is sufficient to check on \(k\)-points, which in turn is elementary.

Now we may and do assume \(\text{char}(k) = p > 0\). By \([2.2.2]\), it is enough to prove the following proposition that is of interest in its own right.

**Proposition 2.2.8.** Let \(k\) be a field of characteristic \(p > 0\), \(G\) an affine commutative \(k\)-group scheme of finite type. Then \(\mathcal{E}xt_k^1(G, \mathbb{G}_m) = 0\), where \(\mathcal{E}xt\) denotes the fppf Ext sheaf.

**Corollary 2.2.9.** Let \(k\) be a field, \(G\) an affine commutative \(k\)-group scheme of finite type. We have a canonical isomorphism \(H^1(k, \hat{G}) \to \text{Ext}^1(G, \mathbb{G}_m)\).

**Proof.** We have a Leray spectral sequence (using the étale topology when \(\text{char}(k) = 0\))

\[
E_2^{i,j} = H^i(k, \mathcal{E}xt^j(G, \mathbb{G}_m)) \Rightarrow \text{Ext}^{i+j}(G, \mathbb{G}_m)
\]

which yields an edge map \(E_2^{1,0} = H^1(k, \hat{G}) \to \text{Ext}^1(G, \mathbb{G}_m)\). To show that this is an isomorphism, it suffices to show that \(E_2^{0,1} = H^0(k, \mathcal{E}xt^1(G, \mathbb{G}_m))\) vanishes. If \(\text{char}(k) = p > 0\), this follows immediately from Proposition \([2.2.8]\). If \(\text{char}(k) = 0\) then we have to show that \(\mathcal{E}xt^1(G, \mathbb{G}_m)\) computed for the small étale site on \(\text{Spec}(k)\) vanishes. This vanishing is the self-contained Lemma \([2.2.21]\).

**Remark 2.2.10.** The map constructed in the proof of Corollary \([2.2.9]\) is rather abstract. Here is a more concrete description. By Proposition \([2.2.8]\) any element of \(\text{Ext}^1(G, \mathbb{G}_m)\) is an fppf form of the trivial extension \(\mathbb{G}_m \times G\). We therefore have \(H^1(k, A) \cong \text{Ext}^1(G, \mathbb{G}_m)\), where \(A\) is the automorphism sheaf of the trivial extension; that is, \(A\) is the sheaf whose sections over a \(k\)-algebra \(R\) are the automorphisms of \(\mathbb{G}_m \times G\) as an extension: the group of isomorphisms \(\phi : \mathbb{G}_m \times G \to \mathbb{G}_m \times G\) of \(R\)-group schemes such that the following diagram commutes:

\[
\begin{array}{c}
1 \longrightarrow \mathbb{G}_m \longrightarrow \mathbb{G}_m \times G \longrightarrow G \longrightarrow 1 \\
\downarrow \phi \\
1 \longrightarrow \mathbb{G}_m \longrightarrow \mathbb{G}_m \times G \longrightarrow G \longrightarrow 1
\end{array}
\]

One easily sees that we have a canonical identification \(A \cong \hat{G}\), hence we get an isomorphism \(H^1(k, \hat{G}) \cong \text{Ext}^1(G, \mathbb{G}_m)\). This is the isomorphism in Corollary \([2.2.9]\) We will not make use of this interpretation.

To prove Proposition \([2.2.8]\) we may replace \(k\) with a finite extension, and after making such an extension \(G\) is filtered by finite group schemes, \(\mathbb{G}_m\)'s, and \(\mathbb{G}_a\)'s. By the \(\delta\)-functoriality of \(\mathcal{E}xt\) in the first variable, therefore we are reduced to these three cases. The case in which \(G\) is finite or \(\mathbb{G}_m\) is handled by \([\text{SGA}7\text{, VIII, Prop.3.3.1}]\), so we are reduced to the case \(G = \mathbb{G}_a\). Since it is no more difficult, we will prove the following more general result.
Proposition 2.2.11. For any $\mathbb{F}_p$-scheme $S$, the fppf sheaf $\mathcal{E}xt^1_S(\mathbb{G}_a, \mathbb{G}_m)$ vanishes.

Remark 2.2.12. Proposition [2.2.11] is false in characteristic 0. Before constructing a counterexample due to Gabber, we will prove the following claim: given a $\mathbb{Q}$-scheme $X$ and a commutative extension $E$ of $\mathbb{G}_a$ by $\mathbb{G}_m$ over $X$, if $E$ splits fppf-locally over $X$, then it splits étale-locally over $X$. (Actually, our argument will show that it splits Nisnevich-locally, but we will not need this.) Indeed, $\mathcal{E}xt(\mathbb{G}_a, \mathbb{G}_m)$ commutes with direct limits, so we may assume that $X = \text{Spec}(R)$, with $R$ Henselian local. Any fppf cover of $X$ may be refined by one that is finite locally free with constant rank $n > 0$ because [EGA, IV, Cor. 17.16.2] any fppf cover of $\text{Spec}(R)$ may be refined by an affine quasi-finite cover, and (since $R$ is Henselian) any such cover may be refined by one of the form $\text{Spec}(S) \to \text{Spec}(R)$ with $S$ a finite local $R$-algebra due to [EGA, IV, Thm. 18.5.11]. We may therefore assume that $\pi : X' \to X$ is finite locally free of degree $n$, and that $E$ splits over $X'$. Let $R'$ be such that $X' = \text{Spec}(R')$, so $R'$ is a finite free $R$-module of constant rank $n$.

Now for some generalities. Given fppf abelian sheaves $\mathcal{F}$ on $X$ and $\mathcal{F}'$ on $X'$, we have a natural map
\[ \mathcal{E}xt^1_X(\mathcal{F}, \pi_*\mathcal{F}') \overset{\phi}{\to} \mathcal{E}xt^1_{X'}(\pi^*\mathcal{F}, \mathcal{F}') \] (2.2.3)
defined as follows: given an extension $\mathcal{E} \in \mathcal{E}xt^1_X(\mathcal{F}, \pi_*\mathcal{F}')$, we may pull it back to $X'$ to obtain the extension $\pi^*\mathcal{E} \in \mathcal{E}xt^1_{X'}(\pi^*\mathcal{F}, \pi^*\pi_*\mathcal{F}')$. Then we push this out along the adjunction map $\pi^*\pi_*\mathcal{F}' \to \mathcal{F}'$ to obtain an element of $\mathcal{E}xt^1_{X'}(\pi^*\mathcal{F}, \mathcal{F}')$. We claim that the map $\phi$ in (2.2.3) is an inclusion. This is easy. A section from our new extension $\phi(\mathcal{E})$ to $\mathcal{F}'$ is the same thing as a commutative diagram
\[
\begin{array}{ccc}
\pi^*\pi_*\mathcal{F}' & \longrightarrow & \pi^*\mathcal{E} \\
\downarrow & & \\
\mathcal{F}' & \leftarrow & \\
\end{array}
\]
which, by adjointness, is the same thing as a section $\mathcal{E} \to \pi_*\mathcal{F}'$. That is, $\mathcal{E}$ splits if and only if $\phi(\mathcal{E})$ does.

Suppose that $\mathcal{F}' = \pi^*\mathcal{G}$ for some sheaf $\mathcal{G}$ on $X$. Then the adjunction map $\mathcal{G} \to \pi_*\pi^*\mathcal{G} = \pi_*\mathcal{F}'$ induces a map
\[ \mathcal{E}xt^1_X(\mathcal{F}, \mathcal{G}) \overset{\psi}{\to} \mathcal{E}xt^1_X(\mathcal{F}, \pi_*\mathcal{F}') \] (2.2.4)
It is easy to check that the composition $\phi \circ \psi : \mathcal{E}xt^1_X(\mathcal{F}, \mathcal{G}) \to \mathcal{E}xt^1_{X'}(\pi^*\mathcal{F}, \pi^*\mathcal{G})$ is simply the pullback map, sending the extension $\mathcal{E}$ to $\pi^*\mathcal{E}$. This uses the well-known fact that the composition $\pi^*\mathcal{G} \to \pi^*(\pi_*\pi^*\mathcal{G}) = (\pi^*\pi_*)\pi^*\mathcal{G} \to \pi^*\mathcal{G}$ is the identity, where the first map is induced by the adjunction map $\text{id} \to \pi_*\pi^*$, and the second by the adjunction map $\pi^*\pi_* \to \text{id}$.
We apply this with $F = G_a$, $\mathcal{G} = G_m$. Note that $\pi^*G_a = G_a$, $\pi^*G_m = G_m$. Then we see, using the injectivity of $\phi$ mentioned above, that if an extension $E$ of $G_a$ by $G_m$ over $X$ splits over $X'$, then it is killed by the map

$$\psi : \text{Ext}^1_X(G_a, G_m) \to \text{Ext}^1_X(G_a, \pi_*G_m)$$

But we also have a norm map $\text{Nm} : \pi_*G_m \to G_m$, defined functorially on $R$-algebras $S$ by the norm map $(S \otimes_R R')^\times \to S^\times$, which is defined by sending $s' \in S \otimes_R R'$ to the determinant of multiplication by $s'$ on the finite free $R$-algebra $S \otimes_R R'$. This induces a norm map

$$\text{Nm} : \text{Ext}^1_X(G_a, \pi_*G_m) \to \text{Ext}^1_X(G_a, G_m)$$

such that the composition

$$\text{Ext}^1_X(G_a, G_m) \xrightarrow{\psi} \text{Ext}^1_X(G_a, \pi_*G_m) \xrightarrow{\text{Nm}} \text{Ext}^1_X(G_a, G_m)$$

is induced by the composition $G_m \to \pi_*\pi^*G_m = \pi_*G_m \xrightarrow{\text{Nm}} G_m$, which is the $n$th-power endomorphism. Thus, we have shown that if an element of $\text{Ext}^1_X(G_a, G_m)$ dies when pulled back to $X'$, then it is $n$-torsion. But multiplication by $n$ on $\text{Ext}^1_X(G_a, G_m)$ is induced by multiplication by $n$ on $(G_a)_X$, which is an automorphism because $X$ is a $\mathbb{Q}$-scheme. We deduce that any element of $\text{Ext}^1_X(G_a, G_m)$ which is killed fppf locally is already trivial.

It therefore suffices to construct a $\mathbb{Q}$-scheme $X$ and an element of $\text{Ext}^1_X(G_a, G_m)$ that does not die étale locally. Here is Gabber’s example. Let $X$ be two copies of the affine line glued along the non-reduced subscheme $Z := \text{Spec}(k[\varepsilon]/(\varepsilon^2))$ (a thickened copy of the origin). Then we take the trivial extension $G_a \times G_m$ of $G_a$ by $G_m$ on each copy of the affine line, and then glue these along the closed subscheme $Z$ by a nontrivial automorphism of the trivial extension. Note that this gluing process makes sense by [Sch] Thm. 3.4. Which nontrivial automorphism? Well, an automorphism of the trivial extension (as an extension) is given by a homomorphism $G_a \to G_m$. Over a ring of characteristic $0$, these are all given by $T \mapsto \exp(rT)$ for some $r \in R$ nilpotent (Lemma 2.3.4). Take the automorphism of the trivial extension on $Z$ given the homomorphism $T \mapsto \exp(\varepsilon T)$.

We claim that this extension $E$ of $G_a$ by $G_m$ over $X$ does not split over any étale neighborhood of the common point of the two copies of the affine line. Any étale $X$-scheme is reduced since $X$ is reduced. A section $s : G_a \to E$ over an étale $X$-scheme $U$ must be unique (since any two differ by a homomorphism from $G_a$ to $G_m$, and there are no nontrivial such over a reduced scheme). Its restriction to the preimage of the first copy $Y_1$ of the affine line must therefore be the obvious section of the trivial extension $E|_{Y_1}$, hence the same holds for its restriction to $U|_Z$. The same holds for its restriction to the preimage of the second affine line, hence also to $U|_Z$. But the difference between these two restrictions to $U|_Z$ must be the pullback of the automorphism $T \mapsto \exp(\varepsilon T)$ described earlier, so $\varepsilon$ pulls back to $0$ on $U|_Z$. If the étale map $U \to X$ has non-empty fiber over the common origin point then $U|_Z \to Z$ is an étale cover and so by étale descent it is impossible that
the nonzero $\varepsilon$ on $Z$ pulls back to $0$ on $U|Z$. Hence, $E$ cannot split over any étale cover of $X$, so it cannot split fppf-locally over $X$ either.

The rest of this section will be devoted to the proof of Proposition 2.2.11, which is equivalent to the statement that for any $F_p$-algebra $R$, any extension

$$1 \rightarrow G_m \rightarrow E \rightarrow G_a \rightarrow 1$$

of $R$-group schemes with $E$ commutative splits fppf locally on $\text{Spec}(R)$. (Any extension with $E$ merely a sheaf is actually representable, due to the effectivity of fpqc descent for relatively affine schemes.) First note that by repeatedly adjoining $p$th roots and taking a direct limit, we may assume that the Frobenius map $F : R \rightarrow R$ is surjective. Indeed, if the sequence splits over the direct limit, then the splitting descends to some fppf cover of our original ring. It therefore suffices to prove the following more precise result.

**Proposition 2.2.13.** Let $R$ be a ring of characteristic $p$ such that the Frobenius map $F : R \rightarrow R$ is surjective. Then $\text{Ext}^1_R(G_a, G_m) = 0$.

As before, let $E$ denote our extension. Assume temporarily that we know that the proposition holds when $E$ admits a scheme-theoretic (but not a priori group-theoretic) section $G_a \rightarrow E$. We first note that $E$ may be thought of as an fppf $G_m$-torsor $\mathcal{L} \in \text{Pic}(G_a, R)$, and to say that there is a scheme-theoretic section as above is the same as saying that $\mathcal{L}$ is trivial. Note that $\mathcal{L}$ is $p$-torsion, since $\text{Ext}^1_R(G_a, G_m)$ is $p$-torsion (because $G_a$ is, and $\text{Ext}^1_R(G, G_m) \rightarrow \text{Pic}(G)$ is a homomorphism for any $R$-group scheme $G$).

Consider the relative Frobenius $R$-isogeny $F_{G_a/R} : G_a \rightarrow G_a$. This is given on coordinate rings by the $R$-algebra map $x \mapsto x^p$. We claim that $F^*_{G_a/R}(\mathcal{L})$ is trivial. In order to prove this, we will need the following lemma.

**Lemma 2.2.14.** Let $X$ be an affine scheme. The natural map $H^i(X, G_m) \rightarrow H^i(X_{\text{red}}, G_m)$ is an isomorphism for all $i > 0$.

Actually, we will only need the injectivity assertion, and only in the case $i = 1$, which is easy by lifting over the affine $X$ a generating global section of $\mathcal{L}|_{X_{\text{red}}}$ for $\mathcal{L} \in \text{Pic}(X)$. But it isn’t very hard to prove the stronger result above, so we will do so.

**Proof.** In this proof all cohomology is étale unless stated otherwise. If we write $X = \text{Spec}(A)$, then $A$ is the direct limit of its Noetherian subrings, and $H^i(\cdot, G_m)$ and $(\cdot)_{\text{red}}$ commute with direct limits of rings, so we may assume that $X$ is Noetherian. We therefore have a filtration $0 = \mathcal{I}_0 \subset \cdots \subset \mathcal{I}_n = \mathcal{N}$ of the sheaf $\mathcal{N} \subset O_X$ of nilpotents by quasi-coherent sheaves of ideals such that $\mathcal{I}_{m+1} \subset \mathcal{I}_m$, so it suffices to prove the following assertion: given an affine scheme $X$ and a quasi-coherent ideal sheaf $\mathcal{I} \subset O_X$ such that $\mathcal{I}^2 = 0$, the map $H^i(X, G_m) \rightarrow H^i(V(\mathcal{I}), G_m)$ is an isomorphism, where $V(\mathcal{I})$ is the closed subscheme defined by $\mathcal{I}$. We have an exact sequence of étale sheaves

$$1 \rightarrow 1 + \mathcal{I} \rightarrow G_m \rightarrow i_* G_m \rightarrow 1$$

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where $i : V(\mathcal{I}) \to X$ is the canonical map. Here we abuse notation and let $\mathcal{I}$ denote the sheaf which associates to an étale $X$-scheme $Y$ the groups $H^0(Y, \mathcal{I} \mathcal{O}_Y)$. Since $i$ is a closed immersion, $\mathcal{I}_s$ is an exact functor, hence $H^1(X, i_s^* \mathcal{G}_m) = H^1(V(\mathcal{I}), \mathcal{G}_m)$. Since $\mathcal{I}^2 = 0$, we have $\mathcal{I} \simeq 1 + \mathcal{I}$ via the map $i \mapsto 1 + i$, so in order to prove the lemma we must show that $H^1_{\text{ét}}(X, \mathcal{I}) = 0$ for $i > 0$. Since the canonical map $\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{O}_Y \to \mathcal{I} \mathcal{O}_Y$ is an isomorphism for any flat $X$-scheme $Y$, $\mathcal{I}$ “is” a quasi-coherent sheaf, so $H^1_{\text{ét}}(X, \mathcal{I}) = H^1_{\text{Zar}}(X, \mathcal{I}) = 0$ for $i > 0$, where the last equality holds because $X$ is affine.

The above lemma shows that in order to prove our claim that $F_{\mathcal{G}_a/R}(\mathcal{L})$ is trivial, we may replace $R$ with its maximal reduced quotient $R_{\text{red}}$, so we may assume that the Frobenius map $F : R \to R$ is actually an isomorphism. Now the absolute Frobenius morphism $F_{\mathcal{G}_a/R}$ factors as a composition $(F_R \otimes 1_{\mathcal{G}_a, k}) \circ F_{\mathcal{G}_a/R}$, and the first map is an isomorphism, since $F_R$ is. But $F_{\mathcal{G}_a/R}$ induces multiplication by $p$ on $\text{Pic}(\mathcal{G}_a, R)$ and hence kills the $p$-torsion class $\mathcal{L}$. We therefore have that $(F_R \otimes 1_{\mathcal{G}_a, k})^* F_{\mathcal{G}_a/R}(\mathcal{L})$ vanishes in $\text{Pic}((\mathcal{G}_a)_R)$, so $F_{\mathcal{G}_a/R}(\mathcal{L})$ is trivial, as desired.

Now suppose that we knew that any commutative $R$-group extension of $\mathcal{G}_a$ by $\mathcal{G}_m$ with a scheme-theoretic section splits. Then $F_{\mathcal{G}_a/R}(E)$ splits, by what we just proved. The exact sequence of $R$-group schemes

$$1 \to \alpha_p \to \mathcal{G}_a \xrightarrow{F_{\mathcal{G}_a/R}} \mathcal{G}_a \to 1$$

yields an exact sequence

$$\mathcal{G}_a(R) \to \alpha_p(R) \to \text{Ext}^1_R(\mathcal{G}_a, \mathcal{G}_m) \xrightarrow{F_{\mathcal{G}_a/R}} \text{Ext}^1_R(\mathcal{G}_a, \mathcal{G}_m),$$

so to show that $E$ splits, it suffices to prove the following lemma.

**Lemma 2.2.15.** Let $R$ be an $\mathbb{F}_p$-algebra. The restriction map $\mathcal{G}_a(R) \to \alpha_p^\circ(R)$ is surjective for all $n \geq 1$. In particular, $\alpha_p^{n+1}(R) \to \alpha_p^n(R)$ is surjective for all $n \geq 1$.

**Proof.** Given $g \in R[X]$, define $\exp_p(g) := \sum_{i=0}^{p-1} g^i/i!$. We first note that the elements of $\alpha_p^n(R)$ are precisely the polynomials $\prod_{i=0}^{p-1} \exp_p(a_iX^p) \in (R[X]/(X^p))^\times$ such that $a_i^p = 0$ for all $i$. By this we mean that the above polynomials give global units on $\alpha_p^n = \text{Spec}(R[X]/(X^p))$, hence $R$-scheme maps $\alpha_p^n \to \mathcal{G}_m$, and these maps are precisely the $R$-homomorphisms from $\alpha_p^n$ to $\mathcal{G}_m$. This description of $\alpha_p^n$ is relatively well-known: one way to see it is to note that (easily) all of the above polynomials do in fact give characters of $\alpha_p^n$, and since the group scheme representing the above functor has an obvious $n$-step filtration by $\alpha_p^i$’s (by using the $p$-nilpotent elements $a_i \in R$), it must be the full dual.

At any rate, the above description makes the first assertion of the lemma clear. For given $a_i$ as above, the element $\prod_{i=0}^{p-1} \exp_p(a_iX^p) \in R[X]^\times$ is a lift of the above character, and it is easy to see that it is a character of $\mathcal{G}_a$. The second assertion may be seen similarly, or simply by choosing a lift of any element of $\alpha_p^n(R)$ to $\mathcal{G}_a(R)$, then restricting it to $\alpha_p^{n+1}$.  

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We have reduced to the study of extensions of commutative $R$-group schemes

$$1 \rightarrow \mathbb{G}_m \rightarrow j \rightarrow E \xrightarrow{\pi} \mathbb{G}_a \rightarrow 1 \quad (2.2.5)$$

for which there is a scheme-theoretic section $s : \mathbb{G}_a \rightarrow E$. We seek an $R$-group section when $F : R \rightarrow R$ is surjective. By translating $s$ by $-s(0) \in E(R)$, we may assume that $s(0) = 0_E$. Then by means of $s$, we may identify $E$ with $\mathbb{G}_m \times \mathbb{G}_a$ as $R$-schemes with $j$ the inclusion $t \mapsto (t,0)$, $s(x) = (1,x)$, and $(1,0)$ the identity of $E$. The group law on $E$ is $(t_1, x_1) \cdot (t_2, x_2) = (t_1 t_2 h(x_1, x_2), x_1 + x_2)$ for some $h : \mathbb{G}_a \times \mathbb{G}_a \rightarrow \mathbb{G}_m$. That is, $h(X,Y) \in R[X,Y]^\times$. Note that $h(X,0) = h(0,Y) = 1$, since $(1,0)$ is the identity.

A group-theoretic section $\mathbb{G}_a \rightarrow E$ is the same as an $f(X) \in R[X]^\times$ such that

$$f(X + Y) = f(X)f(Y)h(X,Y) \quad (2.2.6)$$

Indeed, there is a bijection between the set of such $f$ and the set of such sections via assigning to $f$ the map $x \mapsto (x, f(x))$. We need to show that there exists $f$ satisfying (2.2.6).

Let us first note that $h \equiv 1 \pmod{I}$, where $I \subset R$ is a finitely-generated nilpotent ideal; i.e., $I^e = 0$ for some $e > 0$. Indeed, this follows from the fact that $h(X,0) = h(0,Y) = 1$ if the non-constant coefficients of $h$ are nilpotent. In order to prove such nilpotence, it is enough to show that $h = 1$ in every residue field of $R$, and this in turn follows from the fact that $h \in R[X]^\times$, and that $L[X]^\times = L^\times$ for a field $L$. We need a lemma:

**Lemma 2.2.16.** Let $R$ be an $\mathbb{F}_p$-algebra such that the Frobenius map $F : R \rightarrow R$ is surjective. Then $\text{Ext}^1_R(\alpha_{p^n}, \mathbb{G}_m) = 0$.

**Proof.** We have a Leray spectral sequence

$$E_2^{i,j} = H^i(R, \mathcal{E}xt^j_R(\alpha_{p^n}, \mathbb{G}_m)) \Rightarrow \text{Ext}_R^{i+j}(\alpha_{p^n}, \mathbb{G}_m)$$

This yields a map $H^1(R, \alpha_{p^n}) = E_2^{1,0} \rightarrow \text{Ext}_R^1(\alpha_{p^n}, \mathbb{G}_m)$, and we claim that this map is an isomorphism. In order to prove this, it suffices to show that $E_2^{0,1} = H^0(R, \mathcal{E}xt^1(\alpha_{p^n}, \mathbb{G}_m))$ vanishes. This follows from [SGA7, VIII, Prop. 3.3.1], which implies that $\mathcal{E}xt^1(\alpha_{p^n}, \mathbb{G}_m) = 0$.

So we need to show that $H^1(R, \alpha_{p^n}) = 0$. But $\alpha_{p^n}$, and hence $\alpha_{p^n}$, admits a filtration by $\alpha_p$'s, so it is enough to show that $H^1(R, \alpha_p) = 0$. This in turn follows from the vanishing of $H^1_{\text{fppf}}(R, \mathbb{G}_a) = H^1_{\text{Zar}}(R, \mathbb{G}_a)$ and the exact sequence

$$1 \rightarrow \alpha_p \rightarrow \mathbb{G}_a \xrightarrow{F} \mathbb{G}_a \rightarrow 1$$

due to our assumption that the Frobenius map on $R$ is surjective. 

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We will leverage the fact that $\operatorname{Ext}^1_{R}(\alpha_{p^n}, G_m) = 0$ in order to construct $f \in R[X]^\times$ satisfying (2.2.6). The pullback of the extension $E$ over $\alpha_{p^n} \subset G_a$ splits. Further, if we let $s_n$ denote a chosen section of $E|_{\alpha_{p^n}}$, then $s_{n+1}|_{\alpha_{p^n}} = s_n + X$ for some $X \in \alpha_{p^n}(R)$. Choose a lift $\chi_n \in \widehat{\alpha_{p^{n+1}}}(R)$ of $\chi_n$, which exists by Lemma 2.2.15. Then replace $s_{n+1}$ with $s_{n+1} - \chi_n$. Carrying out this process inductively, we may assume that the sections $s_n$ are compatible; i.e., $s_{n+1}|_{\alpha_{p^n}} = s_n$. In terms of $f$ and $h$, this says that we have for each $n$ an element $f_n \in (R[X]/(X^{p^n}))^\times$ such that

$$f_n(X + Y) \equiv f_n(X)f_n(Y)h(X,Y) \mod (X^{p^n}, Y^{p^n})$$

and such that $f_{n+1}(X) \equiv f_n(X) \mod X^{p^n}$. The $f_n$ therefore “glue” to give an element $f \in R[X]^\times$ such that (2.2.6) holds; i.e.,

$$f(X + Y) = f(X)f(Y)h(X,Y)$$

(2.2.7)

Now given a ring $A$, we define a formal character over $A$ to be an element $\chi \in A[X]^\times$ such that $\chi(X + Y) = \chi(X)\chi(Y)$. Note that the (non-empty!) set of $f \in R[X]^\times$ satisfying (2.2.7) is the torsor for the set of formal characters over $R$. More precisely, the solutions $g \in R[X]^\times$ to (2.2.7) are precisely those power series of the form $g = f\chi$, with $\chi$ a formal character over $R$. We will therefore try to modify $f$ by formal characters to make it a polynomial (rather than just a power series). In order to do this, we need to understand what formal characters look like.

Given an $F_{p^r}$-algebra $A$ and $g \in A[X]$, recall that $\exp_p(g) := \sum_{i=0}^{p-1} g^i/i!$. We have the following description of the group of formal characters.

**Lemma 2.2.17.** Let $A$ be an $F_{p^r}$-algebra. The formal characters over $A$ are precisely the power series $\prod_{n=0}^{\infty} \exp_p(a_n X^{p^n}) \in A[X]$, where the $a_n \in A$ satisfy $a_n^p = 0$.

**Proof.** An element $g \in A[X]$ is a formal character if and only if it is a bona fide character mod $X^{p^n}$ for each $m$; i.e., if and only if $g \mod X^{p^n}$ is a bona fide character of $\alpha_{p^n}$. The lemma therefore is immediate from our earlier description (in the proof of Lemma 2.2.15) of $\widehat{\alpha_{p^n}}(R)$ as the polynomials mod $X^{p^n}$ of the form $\prod_{i=0}^{m-1} \exp_p(a_i X^{p^i})$ with $a_i^p = 0$.

Returning to the proof of Proposition 2.2.13, we recall that we want to multiply $f$ by a formal character to make it into a polynomial. We will accomplish this in several steps. Write

$$f(X) = 1 + \sum_{n>0} b_n X^n, \quad h(X,Y) = 1 + \sum_{0<i,j<N} a_{ij} X^i Y^j$$

for some $N$; note that $h$ is in this form since $h(X,0) = h(0,Y) = 1$ and that $f$ has constant term 1 by inspection upon setting $X = Y = 0$ on both sides of (2.2.7), since $f \in R[X]^\times$. Further, $a_{mm} \in I$, due to the fact that $h \equiv 1 \pmod I$. Comparing the coefficients of $X^m Y^s$ in the equation $f(X + Y) = f(X)f(Y)h(X,Y)$, we obtain

$$(m + s) b_{m+s} = b_m b_s + \sum_{0<i,j<N} a_{ij} b_{m-i} b_{s-j}$$

(2.2.8)
Step 1: \( b'_p^n = 0 \) for \( n \gg 0 \).

We note that in order to prove this assertion, we may multiply \( f \) by a formal character, due to the fact that if \( \chi \) is a formal character, then \( (f \chi)^p = f^p \). Let \( I = (c_1, \ldots, c_g) \) (recall that \( I \) is finitely generated), and choose \( d_1, \ldots, d_g \in R \) such that \( d_i^p = c_i \). Let \( J = (d_1, \ldots, d_g) \subset R \) be the ideal they generate.

By looking at \((2.2.7) \mod I \), and using the fact that \( h \equiv 1 \mod I \), we see that \( f \mod I \) is a formal character, so \( f^p \mod \overline{\chi} \) is equal to 1 (since \( G_a \) is \( p \)-torsion). Hence, \( b_n^p \in I \) for all \( n > 0 \), so \( b_{pm}^p \in I \) for all \( m \geq 0 \). Therefore, for each \( m \geq 0 \) we have \((\alpha_m - b_{pm})^p = 0 \) for some \( \alpha_m \in J \). This conclusion remains true after multiplying \( f \) by a formal character, since such a modification of \( f \) does not affect our current setup.

Let \( \beta_0 := \alpha_0 - b_1 \). Multiplying \( f \) by the formal character \( \exp_p(\beta_0 X) \), we may assume that \( b_1 \in J \). Now let \( \beta_1 = \alpha_1 - b_p \) for this new \( f \), and multiply \( f \) by \( \exp_p(\beta_1 X^p) \) to ensure that \( b_p \in J \). Continuing in this fashion, (more precisely, multiplying \( f \) by the formal character \( \prod \exp_p(\beta_i X^i) \)), we see that we may (and we will) assume that each \( b_{pm} \in J \) for all \( m \geq 0 \). We will need the following lemma

**Lemma 2.2.18.** Fix a positive integer \( M \). Then for all \( n \) sufficiently large (depending on \( M \)), \( b_{rp^n - i} = 0 \) for \( 0 < r < p \), \( 0 < i < M \).

**Proof.** The key observation is the following: Let \( s(n) \) denote the sum of the terms in the base \( p \) expansion of \( n \), so if \( n = c_0 + c_1 p + \cdots + c_r p^r \), \( 0 \leq c_j < p \), then \( s(n) := \sum_{i=0}^r c_i \). For any integer \( l \geq 0 \), we claim \( b_n \in J^l \) provided \( s(n) \) is sufficiently large (a priori depending on \( l \), though not really since \( I \), hence \( J \), is nilpotent). Then since \( J^k = 0 \) for some \( h \), taking \( l \geq h \) shows that \( b_n = 0 \) provided \( s(n) \) is sufficiently large. Since for each positive integer \( i \) and each \( 0 < r < p \), we have \( s(rp^n - i) \to \infty \) as \( n \to \infty \), this will prove the lemma.

We prove the above observation by induction on \( l \), the case \( l = 0 \) being trivial. So suppose that the assertion is true for \( l \), and we’ll prove it for \( l + 1 \). Suppose that \( s(n) \) is large and let \( n = c_0 + c_1 p + \cdots + c_r p^r \) be the base \( p \) expansion of \( n \), with \( c_l \neq 0 \). Then \( s(n) = s(n - p^l) + 1 \), and taking \( m = n - p^l \) and \( s = p^l \) in \((2.2.8) \) yields

\[
\binom{n}{p^l} b_n = b_{n - p^l} b_{p^l} + \sum_{0 < i, j < N} a_{ij} b_{n - p^l - i} b_{p^l - j}
\]

Now \( s(n - p^l) = s(n) - 1 \) is large, so by induction, \( b_{n - p^l} \in J^l \). Further, \( b_{p^l} \in J \) by design, so the first term on the right lies in \( J^{l+1} \) if \( s(n) \) is sufficiently large. Further, if \( s(n) \) (and hence \( s(n - p^l) \)) is large, then so are \( s(n - p^l - i) \) for \( 0 < i < N \), so \( b_{n - p^l - i} \in J^l \) for each such \( i \) by induction. Finally, \( a_{ij} \in I \subset J \), so each term in the sum on the right lies in \( J^{l+1} \) if \( s(n) \) is large.

Now we recall a fact from elementary number theory that we will use repeatedly. Given integers \( 0 \leq a_i, b_i < p \), \( 0 \leq i \leq m \), we have the congruence

\[
\begin{pmatrix} a_0 + a_1 p + a_2 p^2 + \cdots + a_m p^m \\ b_0 + b_1 p + b_2 p^2 + \cdots + b_m p^m \end{pmatrix} = \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & \cdots & a_m \\ b_1 & b_2 & \cdots & b_m \end{pmatrix}
\]

(2.2.9)
By \((2.2.9)\), \((n) \equiv c_t \pmod{p}\), hence is nonzero in \(F_p\), so \(b_n \in J^{t+1}\). This proves the lemma.

Returning to the proof of Step 1, we apply \((2.2.8)\) with \(m = p^n, s = (p-1)p^n\) to obtain

\[
\begin{pmatrix} p^{n+1} \\ p^n \end{pmatrix} b_{p^{n+1}} = b_{p^n} b_{(p-1)p^n} + \sum_{0 < i, j < N} a_{ij} b_{p^n-i} b_{(p-1)p^n-j}.
\]

Once again using \((2.2.9)\), we now see that \((p^{n+1})_p = 0\) in \(F_p\). Further, if we take \(M = N\) in Lemma 2.2.18 then for \(n\) sufficiently large the sum on the right side is 0. Therefore,

\[
0 = b_{p^n} b_{(p-1)p^n}
\]

for all sufficiently large \(n\). Now we will show by induction on 0 < \(r \leq p\) that, provided \(n\) is sufficiently large, \(b_{p^n} b_{(p-r)p^n} = 0\) for 0 < \(r \leq p\); the base case \(r = 1\) has just been handled. Taking \(r = p\) will complete the proof of Step 1. So suppose that the case of some positive \(r < p\) is settled, Taking \(m = p^n, s = (p-r-1)p^n\) in \((2.2.8)\), we see that

\[
\begin{pmatrix} (p-r)p^n \\ p^n \end{pmatrix} b_{(p-r)p^n} = b_{p^n} b_{(p-r-1)p^n} + \sum_{0 < i, j < N} a_{ij} b_{p^n-i} b_{(p-r-1)p^n-j}.
\]

Again using \((2.2.9)\), we see that \((p-r)p^n) = p-r \neq 0\) in \(F_p\). Further, taking \(M = N\) once again in Lemma 2.2.18 we see that for \(n \gg 0\), the sum on the right side of \((2.2.10)\) vanishes. Therefore, since \(b_{p^n} b_{(p-r)p^n} = 0\) by our inductive hypothesis on \(r\), we deduce \(b_{p^n} b_{(p-r-1)p^n} = 0\), which proves the desired result for \(r+1\), hence completes the induction and the proof of Step 1.

Step 2: By replacing \(f\) with \(f\chi\) for some formal character \(\chi\), we may arrange that \(b_{p^n} = 0\) for \(n \gg 0\).

Indeed, using Step 1, suppose that \(b_{p^n} = 0\) for \(n \geq N\). This remains true upon replacing \(f\) with \(f\chi\) for some formal character \(\chi\), since \(\chi^p = 1\). Let \(\chi_N = \exp_p(-b_{p^n} X^{p^n})\). This is a formal character, and the coefficient of \(X^{p^n}\) in \(f\chi_N\) is 0. Denote the new coefficient of \(X^{p^n}\) in \(f\chi_N\) by \(b_{p^n}^{N+1}\). Then multiplying by \(\chi_{N+1} = \exp_p(-b_{p^n}^{N+1} X^{p^{n+1}}\), we get that the coefficients of \(X^{p^n}\) and \(X^{p^{n+1}}\) in \(f\chi_{N+1}\) are both 0. Continuing in this way, we see that we may take the \(\chi\) in the statement of Step 2 to be \(\prod_{n=N}^\infty \chi_n\).

Since we are free to multiply \(f\) by a formal character, from now on we may and do assume that \(b_{p^n} = 0\) for \(n \gg 0\).

Step 3: \(f \in R[X]\); i.e., \(f\) is a polynomial.

We will prove by induction on \(r\) that \(f\) is a polynomial mod \(I^r\) for each positive integer \(r\). That is, \(b_n \in I^r\) for \(n \gg r\). Taking \(r = e\) (where \(I^e = 0\)) will prove that \(f\) is a polynomial.
polynomial. For the base case $r = 1$, we have arranged that $f$ mod $I$ is a formal character, and this formal character satisfies $b_{p^r} := b_{p^r} \mod I = 0$ for $n \gg 0$. It follows from the description of formal characters in Lemma 2.2.17 that $f$ mod $I$ is a polynomial, as desired.

Now suppose that $f$, $I$ is a polynomial mod $I^r$, and we will show that the same holds mod $I^{r+1}$; that is, $b_n \in I^{r+1}$ for $n \gg r$. Write $n$ in base $p$: $n = c_0 + c_1 p + \cdots + c_r p^r$ with $0 \leq c_i < p$ and $c_r \neq 0$. Now apply (2.2.8) with $m = p^r$, $s = n - p^r$ to conclude that

$$(n/p^r) b_n = b_{p^r} b_{n-p^r} + \sum_{0 \leq i+j < N} a_{ij} b_{p^r-i} b_{n-p^r-j}$$

Since $b_t \in I^r$ for $l \gg 0$, the sum on the right lies in $I^{r+1}$ (since $a_{ij} \in I$). We also have $b_{p^r} = 0$ if $n \gg 0$ (since then $t \gg 0$, and we have arranged that $b_{p^m} = 0$ for all sufficiently large $m$). Finally, by (2.2.9), $(n/p^r) = c_r \neq 0$ in $F_p$. We therefore deduce that $b_n \in I^{r+1}$, as desired. This completes the induction and the proof of Step 3.

The proof of Proposition 2.2.11 is now almost complete. We have found $f \in R[X] \cap R[X]^\times$ satisfying (2.2.7). We need to show that $f \in R[X]^\times$. But $h^i = 1$ when $p^r \geq e$, since $h \equiv 1$ (mod $I$), and $I^e = 0$, so $f^{p^e}$ is a formal character. Therefore, $f^{p^{r+1}} = 1$, so $f \in R[X]^\times$. The proof of Proposition 2.2.13 is now complete!

We conclude this section with some remarks on the case char($k$) = 0. For such $k$ it is easier to show that any commutative extension of $G_a$ by $G_m$ admitting a scheme-theoretic section splits (on the nose, not just fppf locally). Unfortunately, such an extension need not admit a section fppf locally; see example 2.2.12.

Remark 2.2.19. To obtain our results for such $k$ we shall (as we have already indicated) replace the fppf site with the small étale site throughout. But over perfect fields this change of sites has no effect on the cohomology of interest. To be precise, consider an fppf abelian sheaf $\mathcal{F}$ on the category of all $k$-schemes and assume naturally $H^i(L, \mathcal{F}) = \lim_{\rightarrow} H^i(L, \mathcal{F})$, where the limit is over all finite extensions $L/k$; all sheaves of interest in this work (e.g., $\hat{G}$ for a commutative affine $k$-group scheme of finite type) will have this property. Then in
particular, \( \lim_{\rightarrow} H^i(\mathcal{F}) = H^i(\mathcal{F}) = 0 \) for \( i > 0 \), since the functor \( H^0(\mathcal{F}) \) is exact on the category of fppf abelian sheaves (because any fppf cover of Spec(\( \overline{k} \)) may be refined by the trivial cover Spec(\( \overline{k} \)) \( \rightarrow \) Spec(\( \overline{k} \)), due to the Nullstellensatz).

Now for any finite extension \( L/k \), we have the Čech-to-derived functor spectral sequences for fppf and étale cohomologies

\[
H^i(L/k, \mathcal{H}^j(\mathcal{F})) \implies H^{i+j}(k, \mathcal{F})
\]

Taking the direct limit over all \( L \), using the above observation, and the description of étale cohomology over fields in terms of Galois cocycles, we deduce that we have natural isomorphisms

\[
H^i_\text{ét}(k, \mathcal{F}) \simeq H^i_{\text{fppf}}(k, \mathcal{F})
\]

for all \( i \) for such \( \mathcal{F} \). We are therefore free to replace fppf with étale cohomology for such sheaves when \( k \) is perfect.

For our applications, when \( \text{char}(k) = 0 \) it suffices to prove Proposition 2.2.3 only for the small étale rather than the fppf site. Just as for the fppf topology, it suffices to show:

**Lemma 2.2.21.** On the small étale site of Spec(\( k \)) for a field \( k \) with characteristic 0, the sheaf \( \mathcal{E}xt^1(G, \mathbb{G}_m) \) for the étale topology vanishes for every affine commutative \( k \)-group scheme \( G \) of finite type.

**Proof.** As in our study of positive characteristic, we are reduced to the cases when \( G \) is finite, \( \mathbb{G}_m \), or \( \mathbb{G}_a \). When \( G \) is finite, since \( \text{char}(k) = 0 \) we may even assume that \( G \) is constant. Then the proof that the extension splits is easy (and actually, the proof that the fppf Ext sheaf vanishes given in [SGA7, VIII, Prop. 3.3.1] shows that the extension splits after replacing Spec(\( L \)) with the fppf cover given by some finite \( L \)-group scheme, which is necessarily étale, since \( \text{char}(L) = 0 \)).

When \( G = \mathbb{G}_m \), for any extension

\[
1 \rightarrow \mathbb{G}_m \rightarrow T \rightarrow \mathbb{G}_m \rightarrow 1
\]

over a field, \( T \) is étale-locally \( \mathbb{G}_m^2 \). Thus, one easily sees that the extension splits étale-locally on the base. Finally, as is well-known, any extension of \( \mathbb{G}_a \) by \( \mathbb{G}_m \) over a field splits. (Proof sketch: Such an extension is trivial as a \( \mathbb{G}_m \)-torsor over \( \mathbb{G}_a \), since Pic(\( \mathbb{G}_a \)) = 0, so it admits a scheme-theoretic section. But all of the units on \( \mathbb{G}_a \times \mathbb{G}_a \) are constant, so \( h \) as in the proof of Proposition 2.2.11 must equal 1.)

### 2.3 Double duality

In accordance with the philosophy that our results should be symmetric in \( G \) and \( \hat{G} \), we will now prove the following result.
Proposition 2.3.1. Let $k$ be a field, $G$ an affine commutative $k$-group scheme of finite type. The canonical map $G \to G^\wedge$ is an isomorphism of fppf sheaves.

We will never use this, so the reader who does not care may safely skip this section. We already know Proposition 2.3.1 for finite commutative group schemes, and it is trivial for $G = \mathbb{G}_m$, since clearly $\hat{\mathbb{Z}} = \mathbb{G}_m$. The hardest case is $G = \mathbb{G}_a$:

Lemma 2.3.2. Proposition 2.3.1 holds for $G = \mathbb{G}_a$.

We first need to describe all characters of $\mathbb{G}_a$ over a ring. Lemma 2.2.17 basically did this over $\mathbb{F}_p$-algebras:

Lemma 2.3.3. Let $A$ be an $\mathbb{F}_p$-algebra. The elements of $\hat{\mathbb{G}}_a(A)$, thought of as global units on $\mathbb{G}_{a,A}$, are precisely the polynomials of the form $\prod_{n=0}^N \exp_p(a_nX^{p^n}) \in A[X]$ for some positive integer $N$, where the $a_n \in A$ satisfy $a_n^p = 0$.

Proof. This follows immediately from Lemma 2.2.17 since characters are nothing more than formal characters that are actually polynomials (and products as in the statement of the present lemma have degree at most $(p - 1)(1 + p + \cdots + p^n) = p^{n+1} - 1 < p^{n+1}$, so a formal character $\prod_{n \geq 0} \exp_p(b_nX^{p^n})$ with $b_n \in \alpha_p(A)$ can only be a polynomial when $b_n = 0$ for all large $n$).

We will also need to describe the characters of $\mathbb{G}_a$ over $\mathbb{Q}$-algebras. For this, we use the usual exponential function $\exp(T) := \sum_{n=0}^\infty T^n/n!$. As in characteristic $p$, a formal character is an element $f \in R[[X]]^\times$ such that $f(X + Y) = f(X)f(Y)$.

Lemma 2.3.4. Let $A$ be a $\mathbb{Q}$-algebra. The elements of $\hat{\mathbb{G}}_a(A)$, viewed as global units on $\mathbb{G}_{a,A}$, are precisely the polynomials of the form $\exp(aX)$ for nilpotent $a \in A$. (Nilpotence of $a$ ensures that $\exp(aX)$ has only finitely many nonzero terms.) The formal characters over $A$ are the power series of the form $\exp(aX)$ for some $a \in A$.

Proof. Since characters are just formal characters that are polynomials, the first assertion is a consequence of the second. All power series of the given form are clearly formal characters, so we only need to check that there are now others. Let $f \in A[[X]]^\times$ be a formal character, which is to say

$$f(X + Y) = f(X)f(Y)$$

Substituting $X = Y = 0$ implies $f(0) = 1$, so $f(X) = 1 + \sum_{n \geq 1} c_nX^n$ for some $c_n \in A$. Comparing coefficients of $X^{m+1}$ on both sides of (2.3.1) implies $(m + 1)c_{m+1} = c_1c_m$; i.e., $c_{m+1} = c_1c_m/(m + 1)$. An easy induction now shows that $c_m = c_1^m/m!$ for all $m \geq 1$.

Proof of Lemma 2.3.2. We first treat the case $\text{char}(k) = 0$. Choose a homomorphic natural transformation $\phi : \hat{\mathbb{G}}_{a,R} \to \mathbb{G}_m$ (i.e., an element of $\hat{\mathbb{G}}_a^\wedge(R)$) for a $\mathbb{Q}$-algebra $R$; in particular, $\phi(1) = 1$. We need to find $r \in R$ such that $\phi(\exp(aX)) = \sum_{n \geq 0} a^nr^n/n!$
for any $R$-algebra $A$ and any nilpotent $a \in A$. Let $A_n := R[T]/(T^n)$, and let $f_n(T) := \phi(\exp(TX)) \in (R[T]/(T^n))^\times$. (Essentially, $\exp(TX)$ is the universal character obtained from nilpotents of order $\leq n$.) For any $a \in A$ satisfying $a^n = 0$ there is a unique $R$-algebra map $R[T]/(T^n) \to A$ satisfying $T \mapsto a$, so naturality of $\phi$ as a map of functors implies $\phi(\exp(aX)) = f_n(a)$ for any such $a \in A$. Further use of naturality of $\phi$ implies the “coherence condition” $f_{n+1} \mod T^n = f_n$, so the $f_n$’s arise are reductions of a single $f \in 1 + TR[T]$. (We have $f(0) = 1$ because $\phi(1) = 1$.) In particular, $\phi(\exp(aX)) = f(a)$ for any $R$-algebra $A$ and nilpotent $a \in A$.

Now $f(a + b) = \phi(\exp((a + b)X)) = \phi(\exp(aX) \exp(bX)) = \phi(\exp(aX)) \phi(\exp(bX)) = f(a)f(b)$ for any $R$-algebra $A$ and nilpotent $a, b \in A$. It follows that $f(S + T) = f(S)f(T)$ in $A[S, T]$. This says that $f$ is a formal character. It follows from Lemma 2.3.3 that $f(T) = \exp(rT)$ for some $r \in R$, so $\phi(\exp(aX)) = f(a) = \exp(ra)$ for any nilpotent $a$ belonging to any $R$-algebra $A$, which is what we wanted to show.

The treatment of $\text{char}(k) = p > 0$ is similar, but more complicated. Once again, choose $\phi \in G^{\alpha}(R)$. We need to find $r \in R$ such that $\phi(\prod_{n=0}^N \exp_p(a_n X^{p^n})) = \prod_{n=0}^N \exp_p(a_n r^{p^n})$ for all $R$-algebras $A$ and $a_n \in \alpha_p(A)$. Define $A_N := R[T_0, T_1, \ldots, T_N]/(T_0^p, \ldots, T_N^p)$ and $f_N(T_0, \ldots, T_N) := \phi(\prod_{n=0}^N \exp_p(T_n X^{p^n})) \in A_N^{\times}$ for $N \geq 1$. Note that $\prod_{n=0}^N \exp_p(T_n X^{p^n})$ is the “universal character of degree $< p^{N+1}p^p$”. For any $R$-algebra $A$ and $a_0, \ldots, a_N \in \alpha_p(A)$ we have $\phi(\prod_{n=0}^N \exp_p(a_n X^{p^n})) = f_N(a_0, \ldots, a_N)$. To proceed, it is convenient to introduce the completion $\hat{B}$ of $R[T_0, T_1, \ldots]/(T_0^p, T_1^p, \ldots)$ for the topology defined by the decreasing sequence of ideals $J_N = (T_{N+1}, T_{N+2}, \ldots)$. Elements of $\hat{B}$ have a unique expansion (that can be manipulated $R$-linearly) as formal series

$$f(T_0, T_1, \ldots) = \sum_I r_I T^I$$

where each multi-index $I = (i_0, i_1, \ldots)$ satisfies $0 \leq i_j \leq p - 1$ for all $j$, $i_j = 0$ for all but finitely many $j$ (so only finitely many such $I$ exist with $i_j$’s vanishing for $j$ outside a fixed finite set), and $T^I := \prod_{j \geq 0} T_j^{i_j}$. Since $f_{N+1} \mod T_{N+1} = f_N$ by naturality of $\phi$, the $f_N$’s are reductions of a single $f \in B$. (We will only ever actually need to consider $f$ after specializing all $T_m$ to be 0 for sufficiently large $m$, but to streamline notation it is convenient to express the subsequent considerations in terms of the single $f$.)

Since $\phi(1) = 1$, we have $f(0, 0, \ldots) = 1$. Thus, $f = 1 + \sum_{I \neq 0} r_I T^I$. It is clear that $\phi(\prod_{n=0}^N \exp_p(a_n X^{p^n})) = f(a_0, a_1, \ldots, a_N, 0, 0, \ldots)$ for any $a_0, a_1, \ldots, a_N \in \alpha_p(A)$ for any $R$-algebra $A$. By multiplicity, $\phi$ is determined by where it sends characters of the form $\exp_p(aX^{p^n})$ for $R$-algebras $A$ and $a \in \alpha_p(A)$. For $A = R[T_n]/(T_n^p)$ and $a = T_n \mod T_n^p \in A$ we have $\phi(\exp_p(T_n X^{p^n})) \in (R[T_n]/(T_n^p))^\times$ as a 1-unit, so $\phi$ is determined by the $r_I$’s for $I = (i_0, i_2, \ldots)$ such that $i_j = 0$ for all but one $j$ (and the remaining $i_n$ belonging to $\{1, \ldots, p - 1\}$).

Let $C(n, m) := r_{I(n, m)}$, where $I(n, m)$ has $n$th component equal to $m \in \{1, \ldots, p - 1\}$ and all other components equal to 0; i.e., it is the $T_n^m$-coefficient of $f$. We will now prove:
(i) \( C(n, m) = C(n, m - 1)C(n, 1)/m \) if \( 1 \leq m \leq p - 1, n \geq 0 \),

(ii) \( C(n + 1, 1) = (p - 1)C(n, 1)C(n, p - 1) \) for all \( n \geq 0 \).

This will imply what we want. Indeed, these facts imply that the \( C(n, m) \)'s are determined by \( C(0, 1) \), so \( \phi \) is determined once we specify the coefficient \( r \) of \( T_0 \) in \( f_0(T_0) \in R[T_0]/(T_0^p) \).

But the image \( \phi_r \) of \( r \) under the natural map \( G_a \to G_A^{\wedge} \) sends \( \exp(T_0X) \) to \( \exp(T_0r) \); i.e., its “\( f_0 \)” has linear coefficient \( r \). Since any \( \phi \) is determined by this coefficient, it follows that \( \phi = \phi_r \), which is what we wanted. It therefore only remains to prove (i) and (ii) above.

Let \( A = R[S,T]/(S^p,T^p) \). Viewing \( S \) and \( T \) as elements of \( A \), we have

\[
\exp_p(SX^n) \exp_p(TX^n) = \exp_p((S + T)X^n) \exp_p \left( \sum_{i=1}^{p-1} \frac{S^iT^{p-i}}{i!(p-i)!} X^{p+1} \right)
\]

in \( A[X]^\times \); this equality is easily checked directly, but the simplest way to see it is to note that both sides are characters over \( A \) such that the coefficients of \( X^n \) agree for all \( n \) (vanishing except possibly for \( m = n, n + 1 \)). To exploit this identity, it is convenient to introduce some notation as follows. For \( a, b \in A = R[S,T]/(S^p,T^p) \) we define \( f(a_n) \) to be \( f(0,0, \ldots, a,0, \ldots) \in A^\times \) (all entries vanishing away from the \( n \)th, which is \( a \)) and define \( f(a_n, b_{n+1}) \) to be the evaluation of \( f \) on the vector whose \( n \)th entry is \( a \), whose \( (n + 1) \)th entry is \( b \), and whose other entries vanish. In terms of this notation, we have

\[
f(S_n)f(T_n) = \phi(\exp_p(SX^n))\phi(\exp_p(TX^n))
\]

\[
= \phi \left( \exp_p(SX^n) \exp_p(TX^n) \right)
\]

\[
= \phi \left( \exp_p((S + T)X^n) \exp_p \left( \sum_{i=1}^{p-1} \frac{S^iT^{p-i}}{i!(p-i)!}X^{p+1} \right) \right)
\]

\[
= f \left( (S + T)_n; \left( \sum_{i=1}^{p-1} \frac{S^iT^{p-i}}{i!(p-i)!} \right)_{n+1} \right)
\]

(the final equality by the design of \( f \)). For \( 1 \leq m \leq p - 1 \), comparing coefficients \( S^{m-1}T \) in the first and last expressions for this string of equalities yields (i) and comparing coefficients of \( ST^{p-1} \) yields (ii).

Lemma 2.3.5. Suppose that we have a short exact sequence

\[
1 \to G' \to G \to G'' \to 1
\]

of commutative \( k \)-group schemes such that \( \epsilon \cdot x_1^1(G'', G_m) = 0 \). If Proposition 2.3.1 holds for \( G', G'' \), then it also holds for \( G \).
Proof. Our assumption on $\text{Ext}^1_k(G'', G_m)$ implies that the dual sequence

$$1 \to \hat{G}'' \to \hat{G}'' \to \hat{G}' \to 1$$

is exact. Dualizing once more yields a left-exact sequence

$$1 \to G'' \to G'' \to G'',$$

so we obtain a commutative diagram (of sheaves) with exact rows

$$
\begin{array}{ccccccccc}
1 & \to & G' & \to & G & \to & G'' & \to & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & G'' & \to & G'' & \to & G'' & \to & 1
\end{array}
$$

where the left and right vertical arrows are isomorphisms by assumption. A simple diagram chase now shows that the middle vertical arrow is an isomorphism.

We may now easily complete the proof of Proposition 2.3.1. By Lemma 2.3.5 and the fact that $\text{Ext}^1(E, G_m) = 0$ when $E$ is a finite $k$-group scheme [SGA7, VIII, Prop. 3.3.1], we may replace $G$ with $G^0$ and thereby assume that $G$ is connected. The assertion is fpqc local, so we may also replace $k$ by a finite extension and so assume that $G_{\text{red}} \subset G$ is a smooth $k$-subgroup scheme. Then $G/G_{\text{red}}$ is finite, so applying Lemma 2.3.5 again, we may assume that $G$ is smooth and connected. Replacing $k$ by a further finite extension, we may assume that $G$ is the product of a split torus and a split unipotent group. As we have already observed, Proposition 2.3.1 is trivial for $G_m$, so we are left with the case when $G = U$ is split unipotent. When $k$ is a number field, we have $U = G_a^n$ for some $n$, so we are done by Lemma 2.3.2. When $k$ is a function field, Lemma 2.3.5 and Proposition 2.2.8 reduce us to the case $U = G_a$, which is once again handled by Lemma 2.3.2.

2.4 Cohomology of $\hat{G}_a$

The crucial cases for the proofs of our results in a certain sense boil down to the groups $G_a$ and $G_m$, which are in a vague sense the fundamental building blocks for arbitrary affine commutative group schemes of finite type over fields of positive characteristic. The main cohomological results in the case $G = G_m$ essentially come down to the major statements of class field theory. There is no analogous theory, however, in the case of $G_a$, and it is therefore necessary for us to undertake a separate (but not entirely unrelated) study of the cohomology of its (not even representable) dual sheaf $\hat{G}_a$. That is the object of this section.

We first note that when $k$ is perfect, the cohomology of $\hat{G}_a$ is very simple:

**Proposition 2.4.1.** Let $k$ be a perfect field. Then $H^i(k, \hat{G}_a) = 0$ for all $i$. 

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Proof. By Remark 2.2.20, $H^i(k, \hat{G}_a) = H^i_{\acute{e}t}(k, \hat{G}_a)$. But the sheaf $\hat{G}_a$ on the small étale site of a field vanishes since $G_a$ has no nontrivial characters over a field.

Over general (possibly imperfect) fields, the above result is false, but we can still say something:

**Proposition 2.4.2.** Let $k$ be a field. Then $H^i(k, \hat{G}_a) = 0$ for $i = 0, 1$.

**Proof.** For $i = 0$ this result is clear, since $G_a$ has no nontrivial characters over a field. For $i = 1$, the lemma follows from Corollary 2.2.9 (actually, we only need something weaker, that $H^1(k, \hat{G}_a)$ injects into $\text{Ext}^1_{\text{ét}}(G_a, G_m)$), since any extension of $G_a$ by $G_m$ over a field splits (as we reviewed at the end of the proof of Lemma 2.2.21).

Let us actually give a somewhat more general result.

**Lemma 2.4.3.** Let $X$ be a reduced scheme such that $\text{Pic}(G_a, X) = 0$. Then $H^1(X, \hat{G}_a) = 0$.

**Proof.** The Leray spectral sequence $E_2^{i,j} = H^i(X, \mathcal{E}xt^j(G_a, G_m)) \implies \text{Ext}_{\text{ét}}^{i+j}(G_a, G_m)$ yields an injection $E_2^{1,0} = H^1(X, \hat{G}_a) \hookrightarrow \text{Ext}_{\text{ét}}^1(G_a, G_m)$, so it suffices to show that this latter group vanishes. Given a commutative extension $E$ of $G_a$ by $G_m$ over $X$, $E$ in particular is a $G_m$-torsor over $G_a, X$, so since $\text{Pic}(G_a, X) = 0$ by assumption, the map $E \to G_a, X$ has a scheme-theoretic section. Then we obtain an $h$ as in the proof of Proposition 2.2.11, which necessarily equals 1 because $X$ is reduced, hence the extension splits.

**Lemma 2.4.4.** Let $k$ be a field, $U$ a smooth connected commutative unipotent $k$-group.

(i) If $i > 1$, then $H^i(k, U) = 0$.

(ii) If $U$ is split, then $H^i(k, U) = 0$ for all positive $i$.

(iii) If $U$ is split, then $H^1(k, \hat{U}) = 0$.

As usual, when $\text{char}(k) = 0$ it is understood that we work with the small étale site over $k$.

**Proof.** Assertion (ii) follows from the well-known case $U = G_a$ via filtering $U$ by $G_a$’s over $k$. To prove (i), we first claim that there is a $k$-subgroup inclusion $U \hookrightarrow U'$ for some split unipotent $k$-group $U'$. Indeed, let $k'/k$ be a finite extension over which $U$ splits. Then the canonical inclusion $U \hookrightarrow R_{k'/k}(U_{k'})$ does the job. (The Weil restriction is split since $R_{k'/k}(G_a) \simeq G_a^{[k'/k]}$.) The quotient $U'' := U'/U$ is then also necessarily split, so the exact sequence

$$1 \rightarrow U \rightarrow U' \rightarrow U'' \rightarrow 1$$

reduces the vanishing of $H^i(k, U)$ for $i > 1$ to the vanishing in positive degrees in the split case as in the settled assertion (ii).

Finally, (iii) follows immediately from Propositions 2.4.2 and 2.2.3 by filtering $U$ by $G_a$. 

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As we shall see, $H^2(k, \hat{G}_a)$ is nontrivial for imperfect fields $k$. We will spend the rest of this section studying this cohomology group. Before stating the next result, we make a definition. If $G$ is a commutative group scheme, then an element $\alpha \in \text{Br}(G)$ is called \textit{primitive} if $m^*\alpha = p^*_1\alpha + p^*_2\alpha$, where $m, p_i : G \times G \to G$ are the multiplication and projection maps, respectively. We denote the subgroup of primitive Brauer elements by $\text{Br}(G)_{\text{prim}}$.

Note that $G_a$ has a natural $k$-linear action, given on $R$-valued points (for $R$ a $k$-algebra) by $\lambda \cdot r = \lambda r$ for $\lambda, r \in G_a(R) = R$. This yields a “multiplicative action” of $k$ on any functor evaluated at $G_a$. In certain circumstances, this action is also additive. That is, if the functor is valued in the category of abelian groups, then we obtain a $k$-vector space structure on the associated functor evaluated at $G_a$. This is the case, for example, for the groups $H^i(k, \hat{G}_a)$ and $\text{Ext}^i(G_a, G_m)$, by general nonsense, since $m_{\lambda+\mu} = m_\lambda + m_\mu$, where $m_\lambda : G_a \to G_a$ is multiplication by $\lambda \in k$. We do not obtain a $k$-linear action on $\text{Br}(G_a)$.

We do, however, obtain one on $\text{Br}(G_a)_{\text{prim}}$, since for $X \in \text{Br}(G_a)_{\text{prim}}$, pulling back the equality $m^*X = p^*_1X + p^*_2X$ along the map $G^2_a \to G^2_a$ given by $(r_1, r_2) \mapsto (\lambda_1 r_1, \lambda_2 r_2)$ yields $(\lambda_1 + \lambda_2)^*X = \lambda_1^*X + \lambda_2^*X$.

\textbf{Remark 2.4.5.} In fact, if $k$ is imperfect, then there is no $k$-linear action on $\text{Br}(G_a, k)$. (If $k$ is perfect, then $\text{Br}(G_a, k) = \text{Br}(k)$, via pullback along the structure map. This can be deduced from the case $k = \bar{k}$ by using a Hochschild-Serre spectral sequence.) Indeed, by Lemma 2.6.3 $\text{Br}(G_a, k) \neq 0$, while Lemma 3.3.3 shows that $\text{Br}(G_a, k)$ is $p$-divisible. It follows that $\text{Br}(G_a, k)$ is not $p$-torsion, hence admits no $k$-vector space structure.

\textbf{Proposition 2.4.6.} Let $k$ be a field. We have functorial isomorphisms of $k$-vector spaces $H^2(k, \hat{G}_a) \simeq \text{Ext}_k^2(G_a, G_m) \simeq \text{Br}(G_a)_{\text{prim}}$. If $k$ is perfect, then all three groups vanish.

By “functorial” in Proposition 2.4.6 we mean that the isomorphisms are functorial in $k$-homomorphisms $\phi : G_a \to G_a$. For imperfect fields $k$ we will see later that these three common groups can be rather interesting (and are always nontrivial).

\textbf{Proof.} We generally suppress $k$-subscripts during this proof. First assume $k$ is perfect. The argument that we will give below shows that the map $\text{Ext}^2(G_a, G_m) \to \text{Br}(G_a)_{\text{prim}}$ is injective, so it suffices to show that the outer two groups vanish. The group $H^2(k, \hat{G}_a)$ vanishes by Proposition 2.4.4. To show that $\text{Br}(G_a)_{\text{prim}} = 0$, we note that the map $\text{Br}(k) \to \text{Br}(G_a)_{\text{prim}}$ is an isomorphism, as may be deduced from the case $k = \bar{k}$ from a Hochschild-Serre spectral sequence. Restricting the equality $m^*\alpha = p^*_1\alpha + p^*_2\alpha$ to the point $(0, 0)$, we find that $\text{Br}(G_a)_{\text{prim}}$ contains no nontrivial constant Brauer classes, hence $\text{Br}(G_a)_{\text{prim}} = 0$.

Now we may and do assume that $\text{char}(k) = p > 0$. (This will be relevant only in one later step, where we appeal to Proposition 2.2.11) Let us first describe a natural map $\text{Ext}^2(G_a, G_m) \to \text{Br}(G_a)_{\text{prim}}$ (where the Ext-term is defined for the fppf topology). Given an fppf abelian sheaf $\mathcal{F}$ on the category of finite type $k$-schemes, Yoneda’s Lemma provides a canonical map $\text{Hom}(G_a, \mathcal{F}) \to \mathcal{F}(G_a)$. Further, the image of this map lives inside $\ker(m^* - p^*_1 - p^*_2 : \mathcal{F}(G_a) \to \mathcal{F}(G_a \times G_a))$. 

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It follows that we have a map of derived functors $\text{Ext}^i(G_\mathbb{A}, \mathcal{F}) \to H^i(G_\mathbb{A}, \mathcal{F})_{\text{prim}}$, and specializing this to $i = 2$, $\mathcal{F} = G_m$ yields the map of interest. Next we show that this map is an isomorphism.

We will use some spectral sequences for the fppf topology due to Breen that we now describe (for more details, see [Br1, §1]). Associated to any commutative $k$-group scheme $G$, there is a complex $A(G) = A(G)_\bullet$ of fppf abelian sheaves concentrated in nonnegative degrees such that each term of $A(G)$ is a product of sheaves of the form $\mathbb{Z}[G^n]$ (the sheaf freely generated by $G^n$). Further, $A(G)_0 = Z[G]$, the canonical map $G \to A(G)_0$ induces an isomorphism $G \simeq H_0(A(G))$, and we have $H_1(A(G)) = 0$, $H_2(A(G)) = G/2G$. We also have that $A(G)_1 = \mathbb{Z}[G^2]$, and the differential $A(G)_1 = \mathbb{Z}[G^2] \to \mathbb{Z}[G] = A(G)_0$ is the map induced by $m - p_1 - p_2 : G \times G \to G$, where, as before, $m, p_i : G \times G \to G$ are the multiplication and projection maps, respectively.

Breen obtains two spectral sequences

$$F_1^{i,j} = \text{Ext}^i(A(G)_1, H) \implies \text{Ext}^{i+j}(A(G), H)$$

$$'E_2^{i,j} = \text{Ext}^i(H_j(A(G)), H) \implies \text{Ext}^{i+j}(A(G), H)$$

We have a canonical isomorphism $\text{Ext}^i(\mathbb{Z}[G], \cdot) \simeq H^i(G, \cdot)$ of functors on fppf abelian sheaves, since both are the derived functors of $\Gamma(G, \cdot)$, thanks to Yoneda’s Lemma. Thus, the first sequence above becomes

$$F_1^{i,j} = H^j(X_i, H) \implies \text{Ext}^{i+j}(A(G), H)$$

where $X_i$ is some explicit disjoint union of products of copies of $G$, and $X_0 = G$. In fact, Breen shows that we may replace the above sequence with another one that is somewhat more convenient, involving “reduced” cohomology groups $\overline{H}^j(X_i, G_m)$ defined as follows. Let $Y_i$ be the analogue of $X_i$ for the 0 group; that is, $Y_i$ is a representing object for $\text{Hom}(A(0)_i, \cdot)$ (so $Y_i$ is a disjoint union of $\text{Spec}(k)$’s). Then via the trivial map $G \to 0$, we obtain maps $Y_i \to X_i$, and we define $\overline{H}^j(X_i, G_m) := \ker(H^j(X_i, G_m) \to H^j(Y_i, G_m))$ to be the kernel of the induced map on cohomology. Breen proved that the inclusions $\overline{H}^j(X_i, G_m) \to H^j(X_i, G_m)$ induce a map of spectral sequences with the same abutment. That is, we have a spectral sequence

$$E_1^{i,j} = \overline{H}^j(X_i, H) \implies \text{Ext}^{i+j}(A(G), H)$$

We shall apply this with $G = G_\mathbb{A}$, $H = G_m$. The two spectral sequences above give us a map

$$\text{Ext}^2(G_\mathbb{A}, G_m) \simeq \overline{H}^2(H_0(A(G_\mathbb{A})), G_m) = 'E_2^{2,0} \to \text{Ext}^2(A(G_\mathbb{A}), G_m) \to E_1^{0,2}$$

where

$$E_1^{0,2} := \overline{H}^2(G_\mathbb{A}, G_m) = \widetilde{\text{Br}}(G_\mathbb{A})$$

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for $\widetilde{\text{Br}}(G_a) := \ker(\text{Br}(G_a) \to \text{Br}(\mathbb{k}))$. (Note that $\text{Br}(G_a)_{\text{prim}} \subset \widetilde{\text{Br}}(G_a)$, as is easily seen by specializing to $0 \times 0$ the equality $m^*\alpha = p_1^*\alpha + p_2^*\alpha$ for $\alpha \in \text{Br}(G_a)_{\text{prim}}$.) Further, this is precisely the map constructed above, since the first isomorphism is obtained by the natural map $G_a \to A(G_a)$, and the last arrow above was induced by the Yoneda isomorphism $\text{Ext}^2(Z[G_a], G_m) \xrightarrow{\sim} H^2(G_a, G_m)$, so the composite map is that induced by the Yoneda map as defined previously. We want to show that this composite map is an isomorphism.

Let us first show that the map $'E_2^0 = \text{Ext}^2(G_a, G_m) \to \text{Ext}^2(A(G_a), G_m)$ is an isomorphism. For this, it suffices to show that $'E_2^{0,1} = 'E_2^{1,1} = 'E_2^{0,2} = 0$. That $'E_2^{0,1} = 'E_2^{1,1} = 0$ follows from the fact that $H_1(A(G_a)) = 0$. To show that $'E_2^{0,2} = \text{Hom}(H_2(A(G_a)), G_m) = 0$, we note that $H_2(A(G_a)) = G_a/2 \cdot G_a$, and there are no nontrivial homomorphisms from $G_a$ to $G_m$ over a field. So the map $\text{Ext}^2(G_a, G_m) \to \text{Ext}^2(A(G_a), G_m)$ is an isomorphism.

Next, we claim that $E_2^{0,2} = \text{Br}(G_a)_{\text{prim}}$. Indeed, since the map $A(G_a)_0 = Z[G_a] \to Z[G_a^2] = A(G_a)_1$ is the one induced by $m - p_1 - p_2$, we see that the edge map $E_1^{0,2} = \tilde{H}^2(G_a, G_m) = \tilde{H}^2(G_a^2, G_m) = E_1^{1,2}$ is the pullback map $m^* - p_1^* - p_2^*$. Hence, $E_2^{0,2} = \text{Br}(G_a)_{\text{prim}}$, and we claim that $E_2^{0,2} = E_2^{0,2} = 0$. To see this, it suffices to show that $E_2^{1,1} = E_1^{0,0} = 0$. First, $E_2^{1,1} = \tilde{H}^1(X_2, G_m) = \text{Pic}(X_2) = 0$ because $X_2$ is a disjoint union of products of $G_a$'s. Next we will show that $E_1^{1,0} = 0$. In fact, we can just as easily see that $E_1^{1,0} = 0$ for all $i$: we have $E_1^{i,0} = \tilde{H}^0(X_i, G_m) = 0$ because the only global units on $G_a^n$ are the constants.

Finally, we claim that the map $\text{Ext}^2(A(G_a), G_m) \to \text{Br}(G_a)_{\text{prim}} = E_2^{0,2}$ is an isomorphism. This will complete the proof that the Yoneda map $\text{Ext}^2(G_a, G_m) \to \text{Br}(G_a)_{\text{prim}}$ is an isomorphism for all $k$. In order to show this, it suffices to show that $E_1^{1,1} = E_1^{0,0} = 0$. We have already checked that $E_1^{1,0} = 0$ for all $i$ in the preceding paragraph, and we also have $E_1^{1,1} = H^1(X_1, G_m) = \text{Pic}(X_1) = 0$. This completes the proof that the Yoneda map $\text{Ext}^2(G_a, G_m) \simeq \text{Br}(G_a)_{\text{prim}}$ is an isomorphism.

Next we will construct a natural isomorphism $H^2(k, G_a) \simeq \text{Ext}^2(G_a, G_m)$. We have a Leray spectral sequence

$$E_2^{i,j} = H^i(k, \mathcal{E}xt^j(G_a, G_m)) \Rightarrow \text{Ext}^{i+j}(G_a, G_m).$$

This yields a map $E_2^{0,0} = H^0(k, G_a) \to \text{Ext}^2(G_a, G_m)$. To show that it is an isomorphism, it suffices to prove the vanishing of $E_2^{0,1}, E_2^{1,1}$, and $E_2^{0,2}$. That $E_2^{0,1} = H^0(k, \mathcal{E}xt^1(G_a, G_m))$ vanishes reduces to the elementary fact that any commutative $k$-group extension of $G_a$ by $G_m$ over an algebraically closed field splits (in fact, any such extension over any field splits). Alternatively, we could use a hammer and apply Proposition 2.2.11. That $E_2^{1,1} = H^1(k, \mathcal{E}xt^1(G_a, G_m))$ also follows from Proposition 2.2.11 since $\text{char}(k) > 0$.

Finally, we need to show that $E_2^{0,2} = H^0(k, \mathcal{E}xt^2(G_a, G_m))$ vanishes. Since any fppf cover of Spec($k$) may be refined by one of the form Spec($L$) for $L/k$ some finite extension (by the Nullstellensatz), this reduces to showing that $\lim \text{Ext}^2(G_a, G_m) = 0$. But thanks to the part of the proposition already proved, this direct limit is identified with $\lim \text{Br}(G_a, L)_{\text{prim}} = \text{Br}(G_a, \overline{k})_{\text{prim}}$, and it is well-known that $\text{Br}(G_a, \overline{k}) = 0$. 

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Finally, both of the isomorphisms defined above are clearly “functorial in $G_a$”. It follows that they are also $k$-linear. □

It will be useful for later purposes for us to have analogous results for more general rings. Note that the map $H^2(R, G_a) \to \text{Br}(G_{a,R})_{\text{prim}}$ defined above makes sense for any ring $R$.

**Proposition 2.4.7.** Let $R$ be a reduced $F_p$-algebra such that $\text{Pic}(R[X_1, \ldots, X_n]) = 0$ for every $n \geq 0$. Then we have functorial inclusions $H^2(R, \hat{G}_a) \to \text{Ext}^2_R(G_a, G_m) \to \text{Br}(G_{a,R})_{\text{prim}}$.

**Proof.** Consider the Leray spectral sequence

$$E_2^{i,j} = H^0(R, \mathcal{E}xt^1(G_a, G_m)) \Rightarrow \text{Ext}^{i+j}_R(G_a, G_m).$$

Since $E_2^{0,1} = H^0(R, \mathcal{E}xt^1(G_a, G_m)) = 0$ by Proposition 2.2.11, the map $H^2(R, \hat{G}_a) \to \text{Ext}^2(G_a, G_m)$ is injective. Thus, it suffices to show that the Yoneda map $\text{Ext}^2(G_a, G_m) \to H^2(G_{a,R}, G_m)$ is injective. For this, we will once again use the Breen spectral sequences.

The first spectral sequence is

$$E_2^{i,j} = \tilde{H}^i(R, A(A(G_a)), G_m) \Rightarrow \text{Ext}^{i+j}_R(A(G_a), G_m)$$

This yields a map $\text{Ext}^2(G_a, G_m) = E_2^{2,0} \to \text{Ext}^2(A(G_a), G_m)$, and to show that it is injective, it suffices to show that $E_2^{0,1} = \text{Hom}(H_1(A(G_a)), G_m)$ vanishes, and this holds because $H_1(A(G_a)) = 0$.

The second spectral sequence is

$$E_1^{i,j} = \tilde{H}^j(X_1, G_m) \Rightarrow \text{Ext}^{i+j}(A(G_a), G_m)$$

and this yields a map $\text{Ext}^2(A(G_a), G_m) \to \tilde{H}^2(G_{a,R}, G_m)$, which actually lands inside $\text{Br}(G_{a,R})_{\text{prim}}$. To show that this map is injective, and therefore complete the proof of the lemma, it is enough to show that $E_1^{2,0} = E_1^{1,1} = 0$. First, $E_1^{2,0} = \tilde{H}^0(X_2, G_m) = 0$ because the only global units on $G_{a,R}$ are the elements of $R^\times$, since $R$ is reduced. Next, $E_1^{1,1} = \tilde{H}^1(X_1, G_m)$, and this group vanishes by assumption. □

**Proposition 2.4.8.** Let $R$ be a Henselian DVR of characteristic $p$. Then the already-defined maps $H^2(R, \hat{G}_a) \to \text{Ext}^2_R(G_a, G_m) \to \text{Br}(G_{a,R})_{\text{prim}}$ are isomorphisms.

**Proof.** The proof is exactly the same as that of Proposition 2.4.6 except for the argument that $H^2(O_v, \mathcal{E}xt^2(G_a, G_m)) = 0$. For this, we use the fact that any fppf cover of $\text{Spec}(R)$ may be refined by one of the form $\text{Spec}(R_L)$, where $L/K$ is a finite extension field ($K := \text{Frac}(R)$), and $R_L$ is the ring of integers of $L$. Indeed, by [EGA IV, Cor. 17.16.2], any fppf cover of $\text{Spec}(R)$ may be refined by an affine quasi-finite cover, and since $R$ is Henselian,
any such cover may be refined by one of the form \( \text{Spec}(S) \to \text{Spec}(R) \) with \( S \) a finite local \( R \)-algebra (EGA IV, Thm. 18.5.11)), which is necessarily the ring of integers of its fraction field.

It suffices, therefore, to show that \( \text{Ext}^2_R(G_a, G_m) = 0 \), where \( \overline{R} \) is the ring of integers of \( \overline{K} \). Just as in the field case, for this it suffices to show that \( \text{Br}(G_{a,\overline{K}}) = 0 \), and for this, we note that the map \( \text{Br}(G_{a,\overline{R}_L}) \to \text{Br}(G_{a,L}) \) is an inclusion for all finite extensions \( L/K \), since \( G_{a,L} \) is regular. Hence, by the compatibility of Brauer groups with direct limits of rings, it suffices to show that \( \text{Br}(G_{a,\overline{K}}) = 0 \), and this is well-known.

It will be important later for us to know that the isomorphisms in Proposition 2.4.6 are compatible with evaluation at a point \( a \in G_a(k) \). More precisely, given such \( a \), we get a morphism of sheaves \( \hat{G}_a \to G_m \) on \( \text{Spec}(k) \) fppf given by evaluation at \( a \). This induces a map \( \text{ev}_a : H^2(k, \hat{G}_a) \to H^2(k, G_m) = \text{Br}(k) \). On the other hand, via evaluation (i.e., restriction) at \( a \), we have a map \( \text{Br}(G_a) \to \text{Br}(k) \) that we will also denote by \( \text{ev}_a \). Then we have the following lemma.

**Lemma 2.4.9.** The following diagram commutes:

\[
\begin{array}{ccc}
H^2(k, \hat{G}_a) & \longrightarrow & \text{Br}(G_a)_{\text{prim}} \\
\downarrow_{\text{ev}_a} & & \downarrow_{\text{ev}_a} \\
\text{Br}(k) & \longrightarrow & \text{Br}(k)
\end{array}
\]

where the horizontal map \( H^2(k, \hat{G}_a) \to \text{Br}(G_a)_{\text{prim}} \) is the isomorphism of Proposition 2.4.6.

**Proof.** First recall that the map \( H^2(k, \hat{G}_a) \to \text{Br}(G_a) \) was defined as the composition of two maps \( H^2(k, \hat{G}_a) \to \text{Ext}^2(G_a, G_m) \to H^2(G_a, G_m) \). We will define a map \( \text{ev}_a : \text{Ext}^2(G_a, G_m) \to H^2(k, G_m) \) and show that the following diagram commutes:

\[
\begin{array}{ccc}
H^2(k, \hat{G}_a) & \longrightarrow & \text{Ext}^2(G_a, G_m) \\
\downarrow_{\text{ev}_a} & & \downarrow_{\text{ev}_a} \\
H^2(k, G_m) & \longrightarrow & H^2(k, G_m)
\end{array}
\]

(2.4.1)

The map \( \text{ev}_a : \text{Ext}^2(G_a, G_m) \to H^2(k, G_m) \) is defined as follows. We have a natural transformation of functors \( \text{Hom}(G_a, \cdot) \to \Gamma(k, \cdot) \) from the category of sheaves on \( \text{Spec}(k) \) fppf to the category of abelian groups, defined by evaluation at \( a \in \mathbb{G}_a(k) \). That is, given an element of \( \text{Hom}(G_a, \mathcal{F}) \), we get a map \( G_a(k) \to \mathcal{F}(k) \) and we take the image of \( a \) under this map. This yields a corresponding map of derived functors \( \text{ev}_a : \text{Ext}^2(G_a, \mathcal{F}) \to H^2(k, \mathcal{F}) \), and we specialize this to the case \( \mathcal{F} = G_m \) in degree 2.
Now let's check the commutativity of the first square in \((2.4.1)\). Recall that the map $H^2(k, \hat{G}_a) \to \operatorname{Ext}^2(G_a, G_m)$ was defined as the map coming from the composite functor spectral sequence

$$E^{pq}_2 = H^p(k, \mathcal{E}xt^q(G_a, \mathcal{F})) \Longrightarrow \operatorname{Ext}^{p+q}(G_a, \mathcal{F})$$

where $\mathcal{F}$ is a fpf sheaf on the category of finite type $k$-schemes. (We apply this with $\mathcal{F} = G_m$.) But the composite functor spectral sequence is natural in the associated functors. That is, given two pairs of functors $F_1, G_1$ and $F_2, G_2$, together with natural transformations $F_1 \to F_2$, $G_1 \to G_2$, the induced maps of derived functors yield a natural transformation from the spectral sequence $R^p F_1 (R^q G_1) \Longrightarrow R^{p+q}(F_1 \circ G_1)$ to the sequence $R^p F_2 (R^q G_2) \Longrightarrow R^{p+q}(F_2 \circ G_2)$.

We now apply this with $F_1 = F_2 = \Gamma(k, \cdot)$ with the identity transformation $F_1 \to F_2$, and $G_1 = \mathcal{H}om(G_a, \cdot)$, $G_2 = \text{Identity}$, with the transformation $G_1 \to G_2$ being evaluation at $a$ (that is, given an fpf sheaf $\mathcal{F}$ on the category of finite type $k$-schemes, for each finite type $k$-scheme $U$ we have a map $\mathcal{H}om(G_a, \mathcal{F})(U) = \operatorname{Hom}((G_a)_U, \mathcal{F}|_U) \to \mathcal{F}(U)$ given by evaluating at the pullback of $a \in G_a(k)$ to a section $a_U \in G_a(U)$). This induces on the derived functor level the first two vertical maps $\text{ev}_a$ appearing in diagram \((2.4.1)\). The maps $H^p(k, \mathcal{F}) \to H^p(k, \mathcal{F})$ associated to the spectral sequence for $F_2, G_2$ are the identity map for all $p$; specializing the functoriality of the spectral sequence to this situation, for $\mathcal{F} = G_m$ and $p = 2$ we obtain the commutativity of the first square in \((2.4.1)\).

It now suffices to prove the commutativity of the second square. This is somewhat simpler. Recall that the map $\operatorname{Ext}^2(G_a, G_m) \to H^2(G_a, G_m)$ was defined as follows. We have a natural transformation of functors $\operatorname{Hom}(G_a, \cdot) \xrightarrow{\text{Yon}} \Gamma(G_a, \cdot)$ defined on an fpf abelian sheaf $\mathcal{F}$ on the category of finite type $k$-schemes by using Yoneda’s Lemma to assign to any $G_a \to \mathcal{F}$ the corresponding element of $\Gamma(G_a, \mathcal{F})$. Then we obtain an induced map on derived functors $\operatorname{Ext}^*(G_a, \mathcal{F}) \to H^*(G_a, \mathcal{F})$, and the map in \((2.4.1)\) is simply this map specialized to the case $\mathcal{F} = G_m$ and degree 2. So to check that the second square in \((2.4.1)\) commutes, we merely need to check that the associated square for the 0th derived functors commutes for any abelian fpf sheaf $\mathcal{F}$. That is, we need commutativity of

$$\begin{array}{ccc}
\text{Hom}(G_a, \mathcal{F}) & \xrightarrow{\text{Yon}} & \Gamma(G_a, \mathcal{F}) \\
\downarrow\text{ev}_a & & \downarrow\text{ev}_a \\
\Gamma(k, \mathcal{F}) & \longrightarrow & \Gamma(k, \mathcal{F})
\end{array}$$

and this is clear. \(\square\)

We will use the isomorphism $H^2(k, \hat{G}_a) \simeq \text{Br}(G_a)_{\text{prim}}$ to study $H^2(k, \hat{G}_a)$ if char($k$) = $p > 0$. (There is nothing to do when char($k$) = 0, since in that case these isomorphic groups vanish.) To this end, in the next section we turn to a method for studying $p$-torsion Brauer classes by relating them to differential forms.
2.5 Brauer groups and differential forms

Throughout this section, \( k \) denotes a field of characteristic \( p > 0 \). To understand \( H^2(k, \widehat{\mathbb{G}}_a) \), we need some way of understanding \( \text{Br}(\mathbb{G}_a)[p] \). In order to do this, we recall an observation of Kato that in geometrically favorable situations \( p \)-torsion Brauer elements can be related to differential forms by utilizing the (inverse) Cartier operator. Let us recall how this goes. Let \( X \) be an \( \mathbf{F}_p \)-scheme, and let \( \Omega^1_X \) be the sheaf of Kähler differential forms. Note that we are not forming the relative 1-forms over \( k \), but rather the sheaf of “absolute” 1-forms, or equivalently, the sheaf \( \Omega^1_{X/\mathbf{F}_p} \) of relative 1-forms over \( \mathbf{F}_p \). Let \( B^1_X \subset \Omega^1_X \) be the subsheaf of coboundaries; that is, \( B^1_X := \text{Im}(d : \mathcal{O}_X \to \Omega^1_X) \). There is a morphism \( C^{-1} : \Omega^1_X \to \Omega^1_X/B^1_X \) defined by

\[
C^{-1}(fdg) = f^p g^{p-1} dg
\]

(The reason for the “inverse” notation is that the Cartier operator is usually defined in a relative setting, and is essentially built as the inverse of the above operator. The “inverse” notation should not be taken to mean that the operator defined above is the inverse of some operator \( C \).) We remark that the above map is well-defined. The only nontrivial point is to check that \( C^{-1}(d(f + g)) = C^{-1}(df) + C^{-1}(dg) \). This is a consequence of the identity

\[
(f + g)^{p-1}d(f + g) - f^{p-1}df - g^{p-1}dg = d\left(\frac{(f + g)^p - f^p - g^p}{p}\right)
\]

Define the map \( \text{dlog} : \mathbf{G}_m/(\mathbf{G}_m)^p \to \Omega^1_X \) by \( f \mapsto df/f \), and let \( i : \Omega^1_X \to \Omega^1_X/B^1_X \) denote the projection. Then we have the following lemma, which is the key to relating Brauer elements to differential forms.

**Lemma 2.5.1.** Let \( X \) be a regular \( \mathbf{F}_p \)-scheme. The following sequence of étale sheaves on \( X \) is exact:

\[
0 \to \mathbf{G}_m/(\mathbf{G}_m)^p \xrightarrow{\text{dlog}} \Omega^1_X \xrightarrow{C^{-1}-i} \Omega^1_X/B^1_X \to 0.
\]

**Proof.** The first place regularity (actually, normality suffices) comes in in proving the injectivity of \( \text{dlog} \): by standard limit arguments we may assume that \( X \) is of finite type over \( \mathbf{F}_p \), and then normality reduces us to the case in which \( X = \text{Spec}(K) \), with \( K \) a field finitely generated over \( \mathbf{F}_p \). (The \( p \)-th root \( g \) at the generic points of a section \( f \) over \( U \) extends over all of \( U \) by normality, using the fact that \( f \), hence \( g \), has nonnegative order at all codimension-1 points.)

The exactness on the right is easy: to hit a class in \( \Omega^1_X/B^1_X \) (over an étale \( X \)-scheme \( U \)) represented by \( f dg \), over an étale cover of \( U \) we can find \( H \) such that \( H^p g^{p-1} - H = f \). Then \( (C^{-1} - i)(H dg) = f dg \). It is also easy to see that the sequence is a complex: we have \( (C^{-1} - i)(df/f) = (1/f^p)f^{p-1}df - df/f = 0 \).

It is harder to show that the sequence is exact at \( \Omega^1_X \), and this is where the regularity comes in. In the case that \( X = \text{Spec}(K) \) for a field \( K \), this is [GS, Thm. 9.2.2]. For the general case, the claimed exactness is a local assertion, so we may assume that \( R \) is a regular
local ring. Given $\omega \in \Omega^1_R$ such that $C^{-1}(\omega) = \iota(\omega)$, we need to check that $\omega = du/u$ for some $u \in R^\times$. By the already-known case in which $R$ is a field, we know that $\omega = df/f$ for some $f \in K := \text{Frac}(R)$. Since $R$ is regular local, it is a UFD, so we may write $f = u \prod \pi_i^{e_i}$ for some pairwise non-associate prime elements $\pi_i \in R$, some $e_i \in \mathbb{Z}$, and some $u \in R^\times$. We then have $df/f = du/u + \sum d\pi_i/e_i$. We may assume that $p \nmid e_i$ for each $i$, since the terms with $p \mid e_i$ disappear. We need to show that if $df/f$ extends to a differential form in $\Omega^1_R$, then the sum is empty, hence $\omega = du/u$. We may localize at one of the primes $(\pi_i)$ to reduce ourselves to the following assertion: if $R$ is an equicharacteristic discrete valuation ring with uniformizer $\pi$, then the element $\hat{d} := \lim dt/t \in K \otimes R \Omega^1_R = \Omega^1_K$ does not arise from an element of $\Omega^1_R$. It is difficult to describe $\Omega^1_R$ directly, so we will work with some completions instead.

We may certainly replace $R$ with its completion, and so we may assume that $R$ is a complete discrete valuation ring. Now $R$ is isomorphic to $\kappa[t]$, where $\kappa$ is the residue field of $R$. The module $\Omega^1_R$ of (absolute) differentials is rather huge, so we will work with a completed version of this that is more amenable to calculations. Any continuous (absolute) derivation $D : R \to M$ to an $m_R$-adically separated and complete $R$-module $M$ clearly arises from a unique compatible system of derivations $D_n : R/(t^{n+1}) \to M/t^n M$. Each $D_n$ uniquely factors through the universal (absolute) derivation $d_n$ from $R/(t^{n+1}) = \kappa[t]/(t^{n+1})$ to

$$\Omega^1_{R/(t^{n+1})} = (R/(t^{n+1}, (n+1)t^n)) dt \otimes \Omega^1_{\kappa[t]/(t^{n+1})}.$$  

Passing to the inverse limit, we see that the $m_R$-adic completion of $\Omega^1_R$ is $Rdt \oplus \Omega^1_{\kappa[t]}$ into which $\hat{d} := \lim n d_n$ is given by $\hat{d}g = \hat{g}(t) dt \oplus \sum t^i \alpha_i$ for $\hat{g} = \sum t^i \alpha_i \in \kappa[t]$. In particular, even though $\Omega^1_R$ is generally not finitely generated as an $R$-module (since it is purely algebraic, not defined with a continuity condition on the universal derivation), its $m_R$-adic completion is separated and complete by inspection with $\hat{d}$ the universal continuous derivation from $R$ into $m_R$-adically separated and complete $R$-modules.

It suffices to check that the element $\hat{dt}/t \in K \otimes R \Omega^1_R$ doesn’t arise from an element of $\Omega^1_R$. Using the above explicit description of $\Omega^1_R$ and $\hat{d}$ it is obvious that $\Omega^1_R$ is torsion-free, so the map $\Omega^1_R \to \Omega^1_R \otimes_R K$ is injective. Hence, it suffices to show that $\hat{dt}$ is not divisible by $t$ in $\Omega^1_R$, which in turn is obvious from the explicit description of $(\Omega^1_R, \hat{d})$.

Now we come to the key result relating Brauer classes to differential forms.

**Lemma 2.5.2.** Let $A$ be a regular ring (i.e., Spec$(A)$ is regular) of characteristic $p > 0$ such that Pic$(A) = 0$. Then there is an isomorphism, functorial in $A$:

$$\text{Br}(A)[p] \cong \frac{\Omega^1_A}{dA + (C^{-1} - i)\Omega^1_A}.$$  

Here, $d : A \to \Omega^1_A$ is the canonical derivation.
In this section we will compute arising from its functoriality applied to the \( q \).

This correspondence is functorial we will apply to the ring of characteristic \( p > 0 \) such that \( [k : k^p] = p \). (Recall from Proposition \( 2.4.6 \) that the isomorphism \( \text{Br}(k, \hat{G}_a) \simeq \text{Br}(G_{a, \text{prim}}) \) is \( k \)-linear.) Note that \( \text{Br}(G_{a, \text{prim}}) \subset \text{Br}(G_a)[p] \). This will allow us to exploit the relationship between Brauer elements and differential forms in Lemma \( 2.5.2 \), which we will apply to the ring \( A = k[X] \). Often without comment we identify an element of \( \Omega^1_A / (B^1_A + (C^{-1} - i) \Omega^1_A) \) with the corresponding class in \( \text{Br}(A)[p] \), so it is important that this correspondence is functorial in \( A \). For example, this ensures that the quotient map \( q : \Omega^1_{k[X]} \to \text{Br}(G_a)[p] \) from Lemma \( 2.5.2 \) is \( k \)-linear with \( k \) acting on \( \Omega^1_{k[X]} = \Omega^1_{G_a} \) through functoriality applied to the \( k \)-action on \( G_a \) (not to be confused with the \( k \)-action on \( \Omega^1_{k[G_a]} \) arising from its \( k[G_a] \)-module structure!).

**Proof.** All cohomology in this proof is étale. Let \( X = \text{Spec}(A) \). The exact sequence of étale sheaves on \( X \)

\[
1 \to \mathbb{G}_m \xrightarrow{p} \mathbb{G}_m \to \mathbb{G}_m / (\mathbb{G}_m)^p \to 0
\]

yields an exact sequence

\[
\frac{\text{Pic}(X)}{p \cdot \text{Pic}(X)} \to H^1(X, \mathbb{G}_m / (\mathbb{G}_m)^p) \to \text{Br}(X)[p] \to 0
\]

By assumption, \( \text{Pic}(X) = 0 \), so we have an isomorphism

\[
H^1(X, \mathbb{G}_m / (\mathbb{G}_m)^p) \simeq \text{Br}(X)[p]
\]

Lemma \( 2.5.1 \) yields an exact sequence

\[
0 \to \frac{H^0(X, \Omega^1_X / B^1_X)}{(C^{-1} - i)(H^0(X, \Omega^1_X))} \to H^1(X, \mathbb{G}_m / (\mathbb{G}_m)^p) \to H^1(X, \Omega^1_X)
\]

Since \( \Omega^1_X \) is a quasi-coherent sheaf, its étale and Zariski cohomology agree. Therefore, since \( X \) is affine, \( H^1(X, \Omega^1_X) = 0 \). So combining \( 2.5.1 \) and \( 2.5.2 \), we see that the proof of the lemma will be complete if we show that the natural maps \( H^0(X, \Omega^1_X) \to H^0(X, \Omega^1_X / B^1_X) \) and \( d : A = H^0(X, \mathcal{O}_X) \to H^0(X, B^1_X) \) are surjective. For the surjectivity of the first map, it is enough show that \( H^1(X, B^1_X) = 0 \). We have an exact sequence

\[
0 \to \mathcal{O}_X \xrightarrow{F} \mathcal{O}_X \xrightarrow{d} B^1_X \to 0
\]

where \( F \) is the map \( f \mapsto f^p \). That \( H^1(X, B^1_X) = 0 \) therefore follows from the fact that \( H^1(X, \mathcal{O}_X) = 0 \) for all \( i > 0 \), since \( X \) is affine. The surjectivity of the map \( d : H^0(X, \mathcal{O}_X) \to H^0(X, B^1_X) \) follows from the exact sequence \( 2.5.3 \) and the fact that \( H^1(X, \mathcal{O}_X) = 0 \).

### 2.6 Computation of \( H^2(k, \hat{G}_a) \) and \( \text{Br}(G_{a, \text{prim}}) \)

In this section we will compute \( H^2(k, \hat{G}_a) \simeq \text{Br}(G_{a, \text{prim}}) \) for a field \( k \) of characteristic \( p > 0 \) such that \( [k : k^p] = p \).
The map $\Omega^1_k \to \Omega^1_{k[X]}$ defined by $\omega \mapsto X\omega$ is $k$-linear for that same $k$-action on $\Omega^1_{k[X]}$ since for $\lambda \in k$, the action of $\lambda$ on $G_a$ is given by $X \mapsto \lambda X$, and $X(\lambda\omega) = (\lambda X)\omega$. The composition of this map with $q$ clearly lands inside $\text{Br}(G_a)_{\text{prim}}$ and thereby defines a natural map $\phi: \Omega^1_k \to \text{Br}(G_a)_{\text{prim}}$ that is $k$-linear. Under a further hypothesis on $k$ that is satisfied by local and global function fields, the map $\phi$ is an isomorphism:

**Proposition 2.6.1.** Let $k$ be a field of characteristic $p > 0$ such that $[k : k^p] = p$. The $k$-linear map $\phi$ is an isomorphism:

$$\Omega^1_k \overset{\phi}{\sim} \text{Br}(G_a)_{\text{prim}} \simeq H^2(k, \hat{G}_a) \simeq \text{Ext}^2_k(G_a, G_m).$$

The isomorphisms between the last three groups is Proposition 2.4.6. What is new is that $\phi$ is an isomorphism when $[k : k^p] = p$. Under the hypotheses of Proposition 2.6.1, if $t \in k - k^p$, then $\Omega^1_k = k dt$, so we immediately obtain the following corollary.

**Corollary 2.6.2.** Let $k$ be a field of characteristic $p > 0$ such that $[k : k^p] = p$. Then $H^2(k, \hat{G}_a)$ is a one-dimensional $k$-vector space.

We first check that $\phi$ is injective, for which we do not need any hypothesis on $[k : k^p]$:

**Lemma 2.6.3.** Let $k$ be a field of characteristic $p > 0$. The $k$-linear map $\phi: \Omega^1_k \to \text{Br}(G_a)_{\text{prim}}$ given by $\omega \mapsto X\omega$ is injective.

*Proof.* Let $\{t_i\}_{i \in I}$ be a $p$-basis for $k$, so it is a differential basis; i.e., $\{dt_i\}$ is a $k$-basis for $\Omega^1_k$ [Mat Thm. 26.5]. The $k$-linear independence is therefore equivalent to showing that the Brauer classes $X dt_i \in \text{Br}(G_a)$ are $k$-linearly independent. In other words, for any relation $\sum_{i=1}^N \lambda_i X dt_{i_n} = 0$ as Brauer classes with $\lambda_i \in k$ we claim that every $\lambda_i$ vanishes. It suffices to prove $\lambda_1$ vanishes, as then we can argue by induction on $N$, so by passing to the extension $k(t_{i_2}^{1/p}, \ldots, t_{i_n}^{1/p})$ in place of $k$ we see via contradiction that it suffices to show for $t \in k - k^p$ that the Brauer class $X dt \in \text{Br}(G_a)$ is nonzero. Note that $t$ is transcendental over $F_p$. We assume that the Brauer class of $X dt$ vanishes and seek a contradiction.

Since the formation of Brauer groups commute with direct limits of rings (as for étale cohomology of $G_m$ on quasi-compact quasi-separated schemes rather generally), and $k$ is the direct limit of its subfields finitely generated over $F_p(t)$, we may assume (arguing by contradiction) that $k$ is finitely generated over the subfield $F_p(t)$ (and so is finitely generated over $F_p$). We can write $k$ as the function field of an integral affine scheme $Z$ of finite type over $F_p$ equipped with a map to the affine $t$-line $\text{Spec}(F_p[t])$ inducing $k/F_p(t)$ on function fields. By shrinking we can arrange that $Z$ is $F_p$-smooth, so $dt$ being part of a $k$-basis of the stalk $\Omega^1_k$ of $\Omega^1_{Z/F_p}$ at its generic point implies that $Z$ is generically smooth over the $t$-line. Thus, by passing to the generic fiber over the $t$-line and shrinking, we obtain a smooth affine $F_p(t)$-scheme $Y$ with function field $k$. The vanishing of the class of $X dt$ in $\text{Br}(G_{a,k})$ allows us to shrink $Y$ around its generic point so that $X dt$ vanishes in $\text{Br}(G_{a,Y})$. 

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The $k$-smooth $Y$ has a $k'$-point for some finite separable extension $k'/\mathbb{F}_p(t)$. Since $k'/\mathbb{F}_p(t)$ is separable, $t$ is not a $p$th power in $k'$. Thus, by specializing at this $k'$-point and renaming $k'$ as $k$, we are reduced to the case when $k$ is a global function field. The completion $k_v$ for a place $v$ of $k$ is separable over $k$, and more specifically $t$ is not a $p$th power in $k_v$, so we can rename $k_v$ as $k$ to instead reduce to the case that $k$ is a local function field.

Now $\Omega^1_k = k d\pi$ for $\pi \in k$ a uniformizer, so $\Omega^1_k$ is 1-dimensional over $k$. Thus, our injectivity assertion is equivalent to showing in the present circumstances that $\phi$ is nonzero (and the element $t$ has already serviced its purpose; we will not use it any further).

Now it suffices to show that the element $X d\omega \in Br(\mathbb{G}_{a,k})$ is nonzero. For this, it suffices to show specializing $X$ to some value in $k$ produces a nonzero element of $Br(k)$. But since $\Omega^1_k = k d\omega$, Lemma 2.5.2 combined with the well-known fact that $Br(k)[p] = \mathbb{Z}/p\mathbb{Z} \neq 0$ implies that $\lambda d\pi$ is a nonzero Brauer class for some $\lambda \in k$. Setting $X = \lambda$ then does the job.

**Remark 2.6.4.** The map $\phi$, which makes sense for any field of characteristic $p > 0$ and is always injective by Lemma 2.6.3, is an isomorphism if and only if $[k : k^p] \leq p$. Indeed, if $k$ is perfect then $Br(\mathbb{G}_{a})_{\text{prim}} = 0$, so $\phi$ is clearly surjective in such cases. The rest of this section will be devoted to proving that it is surjective if $[k : k^p] = p$.

Now suppose that $[k : k^p] > p$; we will show that $\phi$ is not surjective. Let $t, w \in k$ be $p$-independent; i.e., $\{t^i w^j\}_{0 \leq i,j < p}$ is linearly independent over $k^p$. We claim that the Brauer elements $tXdw, tw^pX^p dw/w$ (which are primitive) are linearly independent over $k$. This amounts to showing that if

$$\lambda_0 tXdw + \lambda_1 t^p w^p X^p dw/w = 0$$

with $\lambda_0, \lambda_1 \in k$ then the $\lambda_j$’s vanish. (Here we have used that the action of $\lambda \in k$ on the Brauer class of $X^p dw/w$ is given by $(\lambda X)^p dw/w = \lambda^p X^p dw/w$.) It suffices to check such vanishing after extending scalars to $k(t^{1/p})$. But over this larger field we have $(C^{-1} - i)(\lambda_1 t^{1/p} X dw) = \lambda_1 t^{1/p} X^p dw - \lambda_1 t^{1/p} Xdw$, so as Brauer classes $\lambda_1 t^p w^p X^p dw = \lambda_1 t^{1/p} Xdw$. We therefore have

$$\lambda_0 tXdw + \lambda_1 t^{1/p} Xdw = 0$$

in $Br(\mathbb{G}_{a,k(t^{1/p})})$. By Lemma 2.6.3, we deduce that

$$(\lambda_0 t + \lambda_1 t^{1/p})dw = 0$$

as differential forms in $k_{k(t^{1/p})}$. But $w \notin k(t^{1/p}) = k^p(t) \subset k(t)$, so $dw \neq 0$ in $\Omega^1_{k(t^{1/p})}$ and hence $\lambda_0 t + \lambda_1 t^{1/p} = 0$ in $k(t^{1/p})$. Raising to the $p$th power and using that $t \in k - k^p$, we see that $\lambda_1 = 0$, so also $\lambda_0 = 0$.

In fact, the failure of surjectivity is extreme: the cokernel of the $k$-linear $\phi$ is infinite-dimensional over $k$ whenever $[k : k^p] > p$. To see this, it suffices to show in the above
notation that no non-trivial $k$-linear combination of the Brauer elements $tw^{p^n}X^{p^n}dw/w$ ($n \geq 1$) is of the form $X\omega$ for some $\omega \in \Omega^1_k$. Suppose that some $k$-linear combination of these differential forms, say $\sum_{n=1}^N \lambda_n t^{p^n} X^{p^n} dw/w$, does lie in the image of $\phi$. We want to check that each $\lambda_n$ vanishes. It suffices to check such vanishing in $k(t^{1/p^N})$.

For any extension field $F/k$ and $n \geq 1$ we have

$$(C^{-1} - i)(\mu w^{p^{n-1}}X^{p^n} dw/w) = \mu^p w^{p^n}X^{p^n} dw/w - \mu w^{p^{n-1}} X^{p^{n-1}} dw/w$$

in $\Omega^1_{F[X]}$. Hence, as Brauer classes for $G_{a,F}$ we have

$$\mu^p w^{p^n}X^{p^n} dw/w = \mu w^{p^{n-1}} X^{p^{n-1}} dw/w$$

for $n \geq 1$. Applying this repeatedly with $F = k(t^{1/p^N})$, we see for $1 \leq n \leq N$ that $\lambda_n t^{1/p^n} Xdw/w = \lambda_n t^{1/p^n} Xdw/w$ as Brauer classes over $k(t^{1/p^N})$. By Lemma 2.6.3 therefore, we have

$$\left(\sum_{n=1}^N \lambda_n t^{1/p^n}\right) dw/w = f^* \omega$$

in $\Omega^1_{k(t^{1/p^N})}$ for some $\omega \in \Omega^1_k$, where $f : \text{Spec}(k(t^{1/p^N})) \to \text{Spec}(k)$ is the obvious map. Since $\{t, w\}$ are $p$-independent in $k$, so part of a $p$-basis, the relationship between $p$-bases and differential bases gives that $dw \neq 0$ in $\Omega^1_{k(t^{1/p^N})}$ and $\omega = adw + bdt + \sum_{i=1}^M \alpha_i dz_i$ for some $a, b, \alpha_i \in k$ and $\{w, t, z_1, \ldots, z_M\}$ part of a $p$-basis for $k$. Clearly $\{w, t^{1/p^n}, z_1, \ldots, z_M\}$ is part of a $p$-basis of $k(t^{1/p^N})$, so the $\alpha_i$’s vanish and

$$\frac{1}{w} \sum_{n=1}^N \lambda_n t^{1/p^n} = a \in k.$$ 

Raising both sides to the $p^N$th power then shows that we must have all $\lambda_n$ vanish, since $\{1, t, t^p, \ldots, t^{p^{N-1}}\}$ are linearly independent over $k^{p^N}$.

The rest of this section will be devoted to showing that $\phi$ is surjective under the hypothesis that $[k : k^p] = p$. First, we need the following lemma.

**Lemma 2.6.5.** Let $F : G_a \to G_a$ be the relative Frobenius $k$-isogeny, given on coordinate rings by $X \mapsto X^p$. Then the pullback map $F^* : \text{Br}(G_a)_{\text{prim}} \to \text{Br}(G_a)_{\text{prim}}$ is surjective.

**Proof.** By Proposition 2.4.6 it suffices to prove this with $\text{Br}(G_a)_{\text{prim}}$ replaced by $H^2(k, \hat{G}_a)$. We have an exact sequence

$$1 \to \alpha_p \to G_a \xrightarrow{F} G_a \to 1 \quad (2.6.1)$$

hence an exact sequence

$$1 \to \hat{G}_a \xrightarrow{\hat{F}} \hat{G}_a \to \hat{\alpha}_p \to 1.$$
so it suffices to show that \( H^2(k, \alpha_p) = 0 \). This follows from the sequence [2.6.1], since \( \alpha_p \simeq \alpha_p \).

For the rest of this section, fix \( t \in k - k^p \), so \( \{t\} \) is a \( p \)-basis for \( k \). Every differential form on \( G_\alpha \) is of the form \( G(X)dX + F(X)dt \) for some \( G, F \in k[X] \). A Brauer class is called quasi-constant if it can be represented by such a differential form with \( G = 0 \). Since \( d(X^p) = 0 \), Lemma 2.6.5 implies that every primitive Brauer class is quasi-constant.

**Lemma 2.6.6.** Let \( k \) be a field of characteristic \( p \) such that \([k : k^p] = p\). Every quasi-constant Brauer class may be represented by a sum of such Brauer classes of the form \( \lambda X^n dt \), with \( \lambda \in k \) and either \( n = 0 \) or \( p \nmid n \).

**Proof.** It is enough to show that every Brauer class represented by some \( \lambda X^n dt \) with \( \lambda \in k \) and \( n \geq 1 \) may be represented by a 1-form \( \lambda' X^n dt \) with \( \lambda' \in k \). Writing \( \lambda = \sum_{i=0}^{p-1} a_i t^i \) for \( a_i \in k \), we may assume that the representative 1-form is \( a^p t^i X^n dt \) for some \( a \in k \) and \( 0 \leq i < p \). If \( i \neq p-1 \) then \( a^p t^i X^n dt = (a^p X^{p(i+1)}) \), so this represents the trivial Brauer class. If \( i = p-1 \) then \( a^p t^{p-1} X^n dt = aX^n dt + (C-1-i)(aX^n dt) \), so as Brauer elements, \( a^p t^{p-1} X^n dt = aX^n dt \).

Now we will show that imposing the extra assumption in Lemma 2.6.6 that the Brauer class is primitive allows us to make every term linear in \( X \). First, we need a lemma.

**Lemma 2.6.7.** Let \( L/k \) be a (not necessarily algebraic) separable extension of fields. Then the map \( H^2(k, \hat{G}_\alpha) \to H^2(L, \hat{G}_\alpha) \) is injective.

**Proof.** Since \( L/k \) is separable, \( L \) is the direct limit of its smooth \( k \)-subalgebras. Specializing to a separable point of a suitable such algebra, we see that it suffices to treat the case in which \( L/k \) is finite Galois, say with Galois group \( g \). Then we have a Hochschild-Serre spectral sequence

\[
E_2^{i,j} = H^i(g, H^j(L, \hat{G}_\alpha)) \Rightarrow H^{i+j}(k, \hat{G}_\alpha)
\]

In order to prove the lemma, therefore, it suffices to show that \( E_2^{1,0} = E_2^{1,1} = 0 \). We have \( E_2^{2,0} = H^2(g, H^0(L, \hat{G}_\alpha)) = 0 \) because \( G_\alpha \) has no nontrivial characters over a field, and we have \( E_2^{1,1} = H^1(g, H^1(L, \hat{G}_\alpha)) = 0 \) by Lemma 2.4.4(iii).

The following lemma completes the proof of Proposition 2.6.1.

**Lemma 2.6.8.** Let \( k \) be a field of characteristic \( p > 0 \). Every primitive Brauer class may be represented by a differential form \( cX dt \) for some \( c \in k \).

**Proof.** Let \( \alpha \) be a primitive Brauer class. By Lemma 2.6.5 this class is quasi-constant, so by Lemma 2.6.6 it is represented by a differential form \( F(X)dt \) for some \( F \in k[X] \) such that each monomial appearing in \( F \) is either constant or of degree prime to \( p \). For any \( a \in k \) we have \( a(X + Y)dt = aX dt + aY dt \) as Brauer classes, so the degree-1 term of \( F \)
contributes a primitive class. By subtracting this away, we may assume that the degree-1 part of $F$ vanishes. In this case we will show that $\alpha = 0$.

We have

$$\alpha = \sum_{0 \leq n \leq N \atop \text{if } n > 0} c_n X^n dt$$

Since $\alpha$ is primitive,

$$\sum_{0 \leq n \leq N \atop \text{if } n > 0} c_n (X + Y)^n \, dt = \sum_{0 \leq n \leq N \atop \text{if } n > 0} c_n X^n dt + \sum_{0 \leq n \leq N \atop \text{if } n > 0} c_n Y^n dt$$

as Brauer classes; i.e., as elements of $\Omega^1_{k[X]}/(P^1_{k[X]} + (C^{-1} - 1)\Omega^1_{k[X]})$. We now pull $\alpha$ back to $\text{Br}(k_s)$ (and, by abuse of notation, we will still refer to this pullback as $\alpha$). By Lemma 2.6.7 it will suffice to show that this pullback is 0. Since $\text{Br}(k_s) = 0$, the term for $n = 0$ represents a vanishing Brauer class. Hence, we can drop that term, so we have

$$\alpha = \sum_{1 < n \leq N \atop \text{if } n \neq p} c_n X^n dt$$

with the primitivity relation

$$\sum_{1 < n \leq N \atop \text{if } n \neq p} c_n (X + Y)^n \, dt = \sum_{1 < n \leq N \atop \text{if } n \neq p} c_n X^n dt + \sum_{1 < n \leq N \atop \text{if } n \neq p} c_n Y^n dt$$

as Brauer classes. We will show that $\alpha = 0$ via induction on $N$, the case $N = 1$ being trivial. Now assume $N > 1$

Since $p \nmid N$ and $N > 1$, the polynomial $G(Z) = (Z + 1)^N - Z^N - 1 \in \mathbb{F}_p[Z]$ has positive degree. Choose $r \in \mathbb{F}_p \subset k_s$ such that $G(r) = 1$. Then setting $Y = rX$ in (2.6.2) (more precisely, pulling back to the closed subscheme $(Y = rX) \subset G_a \times G_a$) expresses the Brauer class of the differential form $c_N X^N dt$ as a sum of terms of the form $aX^n dt$ with $a \in k$ and $1 < m < N$ not divisible by $p$. Therefore, $\alpha$ is also a sum of such terms. This has replaced the representative differential form for the primitive Brauer class $\alpha$ with one involving a sum up to at most $N - 1$, so $\alpha = 0$ by induction. \[\square\]
Chapter 3

Preliminaries on the Cohomology of Local and Global Fields

In this chapter we turn to arithmetic, proving various preparatory results on the cohomology of local and global fields. We begin by proving that the cohomology of affine commutative group schemes of finite type over such fields vanishes beyond degree 2 (Proposition 3.1.1), and we then prove that the cohomology of the \( \mathbb{G}_m \)-dual sheaves of such groups vanishes in degree 3 (Proposition 3.4.1). We then show that for an affine commutative group scheme \( G \) of finite type over a local or global function field, the Čech and derived-functor cohomology with coefficients in either \( G \) or \( \hat{G} \) agree up to degree 2 (Proposition 3.5.1), the degree-2 assertion being the only part of this proposition that does not follow from general principles. This agreement will be used in several places in the sequel, first to show the continuity of the local duality pairings in Chapter 4, and then most crucially in §6.9 to define the pairings between Tate-Shafarevich groups described in Theorem 1.2.9, which will be done by imitating Tate’s original definition of these pairings (for finite discrete Galois modules) in terms of Galois cocycles.

3.1 Cohomological vanishing for \( G \)

The main goal of this section is to prove the following crucial result, which we shall use repeatedly.

**Proposition 3.1.1.** Let \( k \) be a non-archimedean local field or a global field with no real places. Let \( G \) be an affine commutative \( k \)-group scheme of finite type. Then \( H^i(k,G) = 0 \) for \( i > 2 \).

Before giving the proof, we need a lemma.

**Lemma 3.1.2.** Let \( k \) be a field, \( I \) a commutative infinitesimal \( k \)-group scheme. Let \( f : \text{Spec}(k)_{\text{fppf}} \to \text{Spec}(k)_{\text{ét}} \) denote the natural morphism. Then \( R^i f_* I = 0 \) for \( i \neq 1 \).
Proof. Since $I$ is infinitesimal, it is clear that $f_i I = 0$. By \[Mc\text{ II, 3.2.5}], there is an exact sequence

$$1 \rightarrow I \rightarrow G \rightarrow H \rightarrow 1$$

with $G,H$ smooth connected affine commutative $k$-groups. (Explicitly, $G = R_{I/k}(GL_1)$ and $H := G/A$, where the quotient exists as an affine $k$-group of finite type over which $G$ is faithfully flat by \[SGA3\text{ V, 4.1}]. The $k$-group $G$ is smooth and connected by \[CGP\text{ A.5.11}], and $H$ inherits smoothness and connectedness from its fppf cover $G$.)

It therefore suffices to show that $R^i f_* G = 0$ for any smooth $k$-group scheme $G$ and all $i > 0$. This in turn follows from the fact that fppf and étale cohomology agree for smooth commutative group schemes.

Proof of Proposition 3.1.1. Step 1: $G$ is finite. We may write $G$ as the product of its $l$-primary parts for different primes $l$, hence we may assume that it is of prime power order. In the case that $p \nmid |G|$, $G$ is étale, and the lemma follows from results of Poitou-Tate, using the fact that étale and fppf cohomology agree for commutative smooth group schemes \[BrIII\text{ Thm. 11.7}] (cf. \[Mi\text{ Ch.I, Thm. 4.10(c)}] for the case in which $k$ is global, and \[Ser1\text{ Ch.I, §5.3, Prop.15}] for the case in which $k$ is local). Thus we may assume that $G$ is of $p$-power order.

Using the connected-étale sequence, it suffices to treat the cases in which $G$ is étale or infinitesimal. If $G$ is étale, then since any field of characteristic $p$ has $p$-cohomological dimension $\leq 1$ \[Ser1\text{ Ch.II, §2.2, Prop.3}], we are done. If $G$ is infinitesimal, then by Lemma 3.1.2 $R^i f_* G \neq 0$ for $i \neq 1$, where $f : \text{Spec}(k)_{\text{fppf}} \rightarrow \text{Spec}(k)_{\text{ét}}$ is the natural morphism of sites. Therefore $H^i_{\text{fppf}}(k,G) = H^{i-1}_{\text{ét}}(k,R^1 f_* G)$, and this latter group is 0 for $i > 2$, again because fields of characteristic $p$ have $p$-cohomological dimension $\leq 1$.

Step 2: $G$ is a torus. By Lemma 2.1.3 (iv), we may harmlessly modify $G$ in order to assume that there is an isogeny $R_{k'/k}(T') \times A \rightarrow G$, for some finite $k$-group scheme $A$, some finite separable extension $k'/k$, and some split $k'$-torus $T'$. Thus, by Step 1, we may assume that $G = R_{k'/k}(G_m)$. Since $R_{k'/k}(G_m)$ is smooth, we may take our cohomology to be étale. Since finite pushforward is exact on categories of étale sites, we have $H^i(k,R_{k'/k}(G_m)) \simeq H^i(k',G_m)$, so (renaming $k'$ as $k$) we are reduced to the case $G = G_m$. To avoid sending the reader looking for another reference, we will treat this by reduction to the finite case treated in Step 1 (which is opposite to the order in which this is usually done). Since higher Galois cohomology is torsion, it suffices to show that $H^i(k,G_m)[n] = 0$ for $i > 2$ and any positive integer $n$. We have the Kummer sequence (of fppf sheaves! The corresponding sequence of étale sheaves is not exact if char($k$) | $n$.)

$$1 \rightarrow \mu_n \rightarrow G_m \xrightarrow{n} G_m \rightarrow 1$$

so the desired vanishing of $H^i(k,G_m)[n]$ for $i > 2$ follows from Step 1.

Step 3: General $G$. By Step 1 and Lemma 2.1.1, we may assume that $G$ is smooth and connected. If $T \subset G$ is the maximal torus, then $G/T$ is unipotent. By Step 2, we may therefore assume that $G$ is unipotent, and then we are done by Lemma 2.4.4(i). \[\square\]
3.2 Concrete description of \( H^2(k, \hat{G}_a) \)

Let \( k \) be a local function field. Then the cup product pairing combined with the isomorphism \( H^2(k, \mathbb{G}_m) \cong \mathbb{Q}/\mathbb{Z} \) given by taking invariants yields a map \( k = \mathbb{G}_a(k) \to H^2(k, \hat{G}_a)^* \). Since \( H^2(k, \hat{G}_a)^* \) is profinite (with respect to the usual compact-open topology given to the Pontryagin dual, once \( H^2(k, \hat{G}_a) \) is endowed with the discrete topology), this in turn induces a map \( k_{\text{pro}} \to H^2(k, \hat{G}_a)^* \).

**Proposition 3.2.1.** Let \( k \) be a local function field. Then the map \( k_{\text{pro}} \to H^2(k, \hat{G}_a)^* \) constructed above is a topological isomorphism.

If \( k \) is a global function field, then we obtain a map \( A = \mathbb{G}_a(A) \to H^2(k, \hat{G}_a)^* \) in the usual manner: cup everywhere locally and add the invariants. This map factors through \( A/k \) because the sum of the invariants of a global Brauer class is 0.

**Proposition 3.2.2.** Let \( k \) be a global function field. Then the map \( A/k \to H^2(k, \hat{G}_a)^* \) constructed above is a topological isomorphism.

**Proof of Proposition 3.2.1.** Since the source is compact and the target Hausdorff, it suffices to show that the map is a continuous bijection. We could prove continuity directly, but we shall instead merely invoke the continuity assertion in Theorem 1.2.2, which will be proved in chapter 7 (and whose proof in no way depends upon this result). We therefore only have to show that the map is an algebraic isomorphism.

Since the image of the map is compact, hence closed, it suffices by Pontryagin duality to show that the dual map is an isomorphism. But both dual groups are one-dimensional \( k \)-vector spaces. Indeed, \( H^2(k, \hat{G}_a) \) is by Corollary 2.6.2. To see that \( (k_{\text{pro}})^D \) is one-dimensional, we first note that the map \( (k_{\text{pro}})^D \to k^D \) is an (algebraic) isomorphism. Indeed, it is clearly injective, and surjectivity follows from the fact that any continuous homomorphism \( k \to R/\mathbb{Z} \) has kernel that is closed of finite index, since \( k \) has finite exponent. It therefore suffices to show that \( k^D \) is one-dimensional, and this is [CF] Ch. XV, Lemma 2.2.1.

Since the dual map is \( k \)-linear, by the functoriality of cup product, we only have to show that it is nonzero. So we only need to show that there exist \( \lambda \in k \) and \( \alpha \in H^2(k, \mathbb{G}_m) \) such that \( \alpha(\lambda) \in H^2(k, \mathbb{G}_m) = \text{Br}(k) \) is nonzero. Let \( \pi \in \mathcal{O}_k \) be a uniformizer. We take \( \alpha = X d\pi \) (under the isomorphism \( \phi \) in Proposition 2.6.1), and then \( \alpha(\lambda) = \lambda d\pi \), by Lemma 2.4.9. Since every \( p \)-torsion element of \( \text{Br}(k) \) is represented by an element of this form for some \( \lambda \in k \) (by Lemma 2.5.2), and since \( \text{Br}(k)[p] \neq 0 \), some such Brauer class is nonzero, hence has nontrivial invariant.

**Proof of Proposition 3.2.2.** Again, the source is compact and the target Hausdorff, so it suffices to show that the map is a continuous bijection. In fact, continuity of the map \( G(A) \to H^2(k, \hat{G})^* \) was proved for any affine commutative \( k \)-group scheme of finite type in §1.2 in the discussion following Theorem 1.2.8.
To show that the map is an isomorphism we once again pass to the dual groups. The discrete group \((A/k)^D\) dual to the compact quotient \(A/k\) is the annihilator of \(k\) in \(A^D\) by Pontryagin duality and as a \(k\)-vector space is 1-dimensional by [Weil, Ch. IV, §2, Thm. 3]. Likewise, \(H^2(k, \hat{G}_a)\) is 1-dimensional for its natural \(k\)-vector space structure by Corollary 2.6.2. Therefore, since the map is \(k\)-linear, in order to show that it is an isomorphism we only have to show that it is nonzero. This follows from the fact that the local maps \(k_v \rightarrow H^2(k_v, \hat{G}_a)\) are nonzero, combined with the fact that the map \(H^2(k, \hat{G}_a) \rightarrow H^2(k_v, \hat{G}_a)\) is injective (Lemma 2.6.7).

3.3 Vanishing of \(H^3(k, \hat{G}_a)\)

The main result of this section is the following.

**Proposition 3.3.1.** Let \(k\) be a local or global function field. Then \(H^3(k, \hat{G}_a) = 0\).

We will also later require the analogous statement for the ring of integers of a local field.

**Proposition 3.3.2.** For any local function field \(k\) with valuation ring \(O\), \(H^3(O, \hat{G}_a) = 0\).

We will give the proof of Proposition 3.3.1, most of which goes through mutatis mutandis to prove Proposition 3.3.2, and at the end of the section we will indicate the changes that must be made in order to make the entire proof go through.

The proof will use Breen’s spectral sequences which were discussed in the proof of Proposition 2.4.6. First, we have a Leray spectral sequence

\[ E^{i,j}_2 = H^i(k, \mathcal{E}xt^j(G_a, G_m)) \implies \text{Ext}^{i+j}(G_a, G_m) \]

which yields a map \(H^3(k, \hat{G}_a) \rightarrow \text{Ext}^3(G_a, G_m)\). We claim that this map is injective. To prove this, it suffices to show that \(E^{1,1}_2 = E^{0,2}_2 = 0\).

First, \(E^{1,1}_2 = H^1(k, \mathcal{E}xt^1(G_a, G_m)) = 0\) by Proposition 2.2.11. Since \(E^{0,2}_2 = H^0(k, \mathcal{E}xt^2(G_a, G_m))\),

to show \(E^{0,2}_2 = 0\) it suffices to prove that for \(L\) varying through finite extensions of \(k\) inside \(\overline{k}\), we have \(\lim \mathcal{E}xt^2(G_a, G_m) = 0\). By Proposition 2.4.6, it suffices to show the vanishing of \(\lim \text{Br}((G_a)_L)\). The limit coincides with \(\text{Br}((G_a)_{\overline{k}})\), whose vanishing is well-known, so the vanishing of \(E^{0,2}_2\) is proved.

Now we use the Breen spectral sequences. Our notation is the same as in the proof of Proposition 2.4.6; there we also recall the various properties of the objects occurring in these spectral sequences. Recall that Breen constructs two spectral sequences

\[ 'E^{i,j}_2 = \text{Ext}^i(H_j(A(G_a)), G_m) \implies \text{Ext}^{i+j}(A(G_a), G_m) \]
which are functorial in $\mathbb{G}_a$. Since $H_0(A(\mathbb{G}_a)) = \mathbb{G}_a$, the first spectral sequence gives us a map $\text{Ext}^3(\mathbb{G}_a, \mathbb{G}_m) \rightarrow \text{Ext}^3(A(\mathbb{G}_a), \mathbb{G}_m)$, which we claim is injective. It suffices to show that $E_2^{1,1} = E_2^{0,2} = 0$. That $E_2^{1,1} = \text{Ext}^1(H_1(A(\mathbb{G}_a)), \mathbb{G}_m)$ vanishes is because $H_1(A(\mathbb{G}_a)) = 0$. That $E_2^{0,2} = \text{Hom}(H_2(A(\mathbb{G}_a)), \mathbb{G}_m) = 0$ holds is due to the fact that $H_2(A(\mathbb{G}_a)) = \mathbb{G}_a/2\mathbb{G}_a$, and there are no nontrivial homomorphisms over a field from $\mathbb{G}_a$ to $\mathbb{G}_m$. This proves the desired injectivity. Composing the two injective maps thus far defined, we obtain an inclusion $H^3(k, \hat{\mathbb{G}}_a) \hookrightarrow \text{Ext}^3(A(\mathbb{G}_a), \mathbb{G}_m)$.

Let us compute some terms in the second spectral sequence. First note that $E_1^{i,0} = E_1^{i,1} = 0$ for all $i$. Indeed, $E_1^{i,0} = \tilde{H}^0(X_i, \mathbb{G}_m)$, and the global units on $X_i$ are the ones that are constants on each component, hence the map $H^0(X_i, \mathbb{G}_m) \rightarrow H^0(Y_i, \mathbb{G}_m)$ is an isomorphism, where the $Y_i$ are the schemes appearing in the Breen spectral sequence applied with $\mathbb{G}_a$ replaced by the trivial group scheme. The group $E_1^{i,1} = \tilde{H}^1(X_i, \mathbb{G}_m) \subset \text{Pic}(X_i)$ vanishes because $X_i$ is a finite disjoint union of products of $\mathbb{G}_a$ (and hence has trivial Picard group).

The composite map

$$H^3(k, \hat{\mathbb{G}}_a) \hookrightarrow \text{Ext}^3(A(\mathbb{G}_a), \mathbb{G}_m) \rightarrow E_0^{0,3} \hookrightarrow E_1^{0,3} = \tilde{H}^3(X_0, \mathbb{G}_m) \subset H^3(X_0, \mathbb{G}_m)$$

is the zero map. Indeed, the image is contained in $\tilde{H}^3(X_0, \mathbb{G}_m)[p]$, which vanishes by Lemma \ref{3.3.3} below. Since $E_1^{3,0} = E_2^{2,1} = 0$ by the general vanishing proved above, it follows that we have an injective map $H^3(k, \hat{\mathbb{G}}_a) \hookrightarrow E_\infty^{1,2}$, and this map is functorial in $k$-homomorphisms $\mathbb{G}_a \rightarrow \mathbb{G}_a$.

**Lemma 3.3.3.** *Let $X$ be a regular affine $\mathbb{F}_p$-scheme. Then $H^i(X, \mathbb{G}_m)[p] = 0$ for $i \geq 3$, and $H^i(X, \mathbb{G}_m)/pH^i(X, \mathbb{G}_m) = 0$ for $i \geq 2$.*

Note that it is equivalent to use étale and fppf cohomology in the statement of this lemma; the proof will use the étale topology.

**Proof.** We have an exact sequence of étale sheaves on $X$

$$0 \rightarrow \mathbb{G}_m \xrightarrow{p} \mathbb{G}_m \rightarrow \mathbb{G}_m/(\mathbb{G}_m)^p \rightarrow 0,$$

hence an exact sequence

$$0 \rightarrow H^i(X, \mathbb{G}_m)/pH^i(X, \mathbb{G}_m) \rightarrow H^i(X, \mathbb{G}_m/(\mathbb{G}_m)^p) \rightarrow H^{i+1}(X, \mathbb{G}_m)[p] \rightarrow 0,$$

so it is equivalent to prove $H^i(X, \mathbb{G}_m/(\mathbb{G}_m)^p) = 0$ for $i \geq 2$. By Lemma \ref{2.5.1} (which requires regularity), there is an exact sequence of étale sheaves on $X$

$$0 \rightarrow \mathbb{G}_m/(\mathbb{G}_m)^p \rightarrow \Omega^1_X \rightarrow \Omega^1_X/B^1_X \rightarrow 0$$
Since $X$ is affine and $\Omega_X^1$ is quasi-coherent, we have $H^i(X, \Omega_X^1) = 0$ for $i \geq 1$. It remains to show that $H^i(X, \Omega_X^1/B_X^1) = 0$ for $i \geq 1$. Since the higher cohomology of $\Omega_X^1$ vanishes, it suffices to show that $H^i(X, B_X^1) = 0$ for $i \geq 2$.

We have an exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{F} \mathcal{O}_X \xrightarrow{d} B_X^1 \to 0$$

where $F$ is the map $a \mapsto a^p$. Since $X$ is affine, so $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$, $H^i(X, B_X^1) = 0$ for $i > 0$, as desired. \[\square\]

Let us compute $E^{1,2}_\infty$. We first note that, since $E^{i,0}_1 = E^{i,1}_1 = 0$ for all $i$, $E^{1,2}_\infty = E^{1,2}_2$. Thus, $E^{1,2}_\infty$ is the middle homology of a complex

$$\tilde{H}^2(X_0, G_m) \xrightarrow{f_0} \tilde{H}^2(X_1, G_m) \xrightarrow{f_1} \tilde{H}^2(X_2, G_m)$$ \hspace{1cm} (3.3.1)

By the construction of the Breen complex $A(G_a)$ (see [Br2]), we have $X_0 = G_a$, $X_1 = G_a^2$, and $X_2 = G_a^2 \amalg G_a^3$. Further, the maps $f_i : \tilde{H}^2(X_i, G_m) \to \tilde{H}^2(X_{i+1}, G_m)$ are given as follows. We have $f_0 = m^* - p_1^* - p_2^*$, where $m, p_i : G_a^2 \to G_a$ are the multiplication and projection maps. The map $f_1 : \tilde{H}^2(G_a^2; G_m) \to \tilde{H}^2(G_a^2; G_m) \times \tilde{H}^2(G_a^3; G_m)$ is given on the first component by $\sigma^*$, where $\sigma : G_a^2 \to G_a^2$ is the map $(x, y) \mapsto (y, x)$. On the second component, $f_1$ is given by $(p_1, m \circ p_{12})^* + p_{23}^* - (m \circ p_{12}, p_3)^* - p_{12}^*$, where again $m : G_a^2 \to G_a$ is multiplication and $p_i : G_a^3 \to G_a$, $p_{ij} : G_a^3 \to G_a^2$ are the various projections.

We conclude that $E^{1,2}_2$ is the middle homology of the complex

$$\widetilde{Br}(G_a) \xrightarrow{f_0} \widetilde{Br}(G_a^2) \xrightarrow{f_1} \widetilde{Br}(G_a^2) \times \widetilde{Br}(G_a^3)$$

where $\widetilde{Br}(X) := \ker(\text{Br}(X) \xrightarrow{0^*} \text{Br}(k))$ for any $k$-group scheme $X$. We have therefore constructed a functorial inclusion

$$H^3(k, \widetilde{G}_a) \hookrightarrow (\ker(f_1)/\text{im}(f_0))[p].$$

We claim that any element of $(\ker(f_1)/\text{im}(f_0))[p]$ is represented by an element of $\ker(f_1)[p] \subset \text{Br}(G_a^2)[p]$. Given $\alpha \in (\ker(f_1)/\text{im}(f_0))[p]$, we have $\alpha \circ \beta = f_0(\beta)$ for some $\beta \in \text{Br}(G_a)$. By Lemma 3.3.3 we have $\beta = p\gamma$ for some $\gamma \in \text{Br}(G_a)$. Then $\alpha - f_0(\gamma) \in \text{Br}(G_a^2)[p]$, so $\alpha$ is indeed equivalent to an element of $\text{Br}(G_a^2)[p]$ modulo $\text{im}(f_0)$, and hence is represented by an element of $(\ker(f_1))[p]$.

**Lemma 3.3.4.** Let $k$ be a field of characteristic $p > 0$ such that $[k : k^p] = p$, and let $L/k$ be a (not necessarily algebraic) separable extension. Then the map $H^3(k, \widetilde{G}_a) \to H^3(L, \widetilde{G}_a)$ is injective.
Proof. Since $L/k$ is separable, $L$ is the direct limit of smooth $k$-algebras. By specializing to a separable point of a suitable such algebra, we are reduced to the case in which $L/k$ is finite Galois, say with Galois group $\mathfrak{g}$. We then have a Hochschild-Serre spectral sequence

$$E_{2}^{i,j} = H^{i}(\mathfrak{g}, H^{j}(L, \hat{G}_{a})) \implies H^{i+j}(k, \hat{G}_{a})$$

It thus suffices to show that $E_{2}^{3,0} = E_{2}^{2,1} = E_{2}^{1,2} = 0$. The group $E_{2}^{3,0} = H^{3}(\mathfrak{g}, H^{0}(L, \hat{G}_{a}))$ vanishes because $H^{0}(L, \hat{G}_{a}) = 0$, since $G_{a}$ has no nontrivial characters over a field. The group $E_{2}^{2,1} = H^{2}(\mathfrak{g}, H^{1}(L, \hat{G}_{a}))$ vanishes because $H^{1}(L, \hat{G}_{a}) = 0$ by Lemma 2.4.4 (iii). Finally, we need to show that $E_{2}^{1,2} = H^{1}(\mathfrak{g}, H^{2}(L, \hat{G}_{a}))$ vanishes. Since $L/k$ is finite separable, we have $[L : L^{p}] = p$, and the map $L \otimes_{k} \Omega_{L}^{1} \rightarrow \Omega_{L}^{1}$ is an isomorphism. By Proposition 2.6.1 we have an isomorphism $\Omega_{L}^{1} \simeq H^{2}(L, \hat{G}_{a})$ of $\mathfrak{g}$-modules. Using the fact that $\Omega_{L}^{1}$ is a one-dimensional $k$-vector space, therefore, we obtain a (non-canonical) isomorphism $H^{2}(L, \hat{G}_{a}) \simeq L$ of $\mathfrak{g}$-modules. Therefore, $H^{1}(\mathfrak{g}, H^{2}(L, \hat{G}_{a})) = H^{1}(\mathfrak{g}, L) = 0$, as desired. \qed

If $k$ is a local function field, then we have the natural map $k \rightarrow H^{2}(k, \hat{G}_{a})^{*}$ obtained from

$$G_{a}(k) \times H^{2}(k, \hat{G}_{a}) \hookrightarrow H^{2}(k, G_{m}) \simeq \mathbb{Q}/\mathbb{Z}$$

(which is continuous when the $H^{2}$-term is viewed discretely and even $\mathbb{Q}/\mathbb{Z}$ is viewed discretely). If $k$ is a global function field, we obtain a map $A \rightarrow H^{2}(k, \hat{G}_{a})^{*}$ from

$$A \times H^{2}(k, \hat{G}_{a}) \rightarrow H^{2}(A, G_{m}) \rightarrow \bigoplus H^{2}(k_{v}, G_{m}) \rightarrow \mathbb{Q}/\mathbb{Z}$$

(the second map resting on the vanishing of $H^{2}(\mathcal{O}_{v}, G_{m}) = Br(\mathcal{O}_{v})$); this pairing is continuous when the $H^{2}$-term is viewed discretely and even $\mathbb{Q}/\mathbb{Z}$ is viewed discretely. The induced map $A \rightarrow H^{2}(k, \hat{G}_{a})^{*}$ kills $k$ due to the fundamental exact sequence of class field theory. The purpose of this discussion is to prove the following lemma.

Lemma 3.3.5. Let $k$ be a local function field or a global function field, and $\phi : G_{a} \rightarrow G_{a}$ a $k$-isogeny. The induced map $H^{3}(\phi) : H^{3}(k, \hat{G}_{a}) \rightarrow H^{3}(k, \hat{G}_{a})$ is an isomorphism.

Proof. Let $B := \ker(\phi)$. By Proposition 2.2.3 we have an exact sequence

$$0 \rightarrow \hat{G}_{a} \xrightarrow{\phi} \hat{G}_{a} \xrightarrow{\gamma} \hat{B} \rightarrow 0$$

We have $H^{3}(k, \hat{B}) = 0$ by Proposition 3.1.1 so $H^{3}(\hat{\phi})$ is surjective. To show that it is injective, we need to show that the map $H^{3}(k, \hat{G}_{a}) \rightarrow H^{3}(k, \hat{B})$ is surjective. Since the cokernel is torsion and hence admits a non-trivial homomorphism to $\mathbb{Q}/\mathbb{Z}$ if it is nonzero, it suffices to prove that the $\mathbb{Q}/\mathbb{Z}$-dual map $H^{2}(k, \hat{B})^{*} \rightarrow H^{2}(k, \hat{G}_{a})^{*}$ is injective.
If \( k \) is a local function field, then in the commutative diagram

\[
\begin{array}{c}
B(k) \leftarrow k \\
\downarrow \quad \downarrow \quad \downarrow \\
H^2(k, \hat{B})^* \rightarrow H^2(k, \hat{G}_a)^*
\end{array}
\]

the first vertical arrow is an isomorphism by local duality for finite group schemes. To see that the second vertical arrow is an inclusion, by Proposition 3.2.1 we need to check that the map \( k \rightarrow k_{\pro} \) is injective. In order to do this, it suffices to show that the dual map \( k_{\pro} \rightarrow k^D \) is surjective. But this is clear, by using the fact that any \( \alpha \in k^D \) has closed finite kernel, since \( k \) is \( p \)-torsion. Now the commutativity of the diagram implies that the map \( H^2(k, \hat{B})^* \rightarrow H^2(k, \hat{G}_a)^* \) is injective.

Similarly, the assertion for global function fields \( k \) may be deduced from the following commutative diagram

\[
\begin{array}{c}
B(A)/B(k) \leftarrow A/k \\
\downarrow \quad \downarrow \quad \downarrow \\
H^2(k, \hat{B})^* \rightarrow H^2(k, \hat{G}_a)^*
\end{array}
\]

where the second vertical arrow is an isomorphism by Proposition 3.2.2, combined with the fact that any element of \( H^2(k, \hat{B})^* \) that pulls back to 0 on \( H^2(k, \hat{G}_a)^* \) comes from \( B(A)/B(k) \). Indeed, in order to show this, it suffices by Poitou-Tate for the finite group scheme \( B \) to show that any such element restricts to 0 in \( \Pi^2(\hat{B})^* \). This in turn follows if we show that any element of \( \Pi^2(\hat{B}) \) lifts to \( H^2(k, \hat{G}_a) \), and this follows from the fact that \( \Pi^2(\hat{G}_a) = 0 \), by Lemma 3.3.4 (applied with \( L = k_v \) for any place \( v \) of \( k \)). This gives the desired result in the global case.

We resume the study of \( (\ker f_1)[p] \subset \hat{\Br}(G_a^2) \) in order to analyze \( H^3(k, \hat{G}_a) \). Choose \( t \in k - k^p \), so \( k = k^p(t) \), and hence \( \Omega^1_k \) is a 1-dimensional \( k \)-vector space generated by \( dt \). Now we use the equivalence between \( p \)-torsion Brauer classes and differential forms from Lemma 2.5.2. This says that any element of \( \ker(f_1)[p] \) is represented by some absolute differential form on \( G_a^2 \), say \( G_1(X,Y)dX + G_2(X,Y)dY + H(X,Y)dt \) for some \( G_1, G_2, H \in k[X,Y] \). Pulling back under the relative Frobenius \( k \)-isogeny kills \( dX \) and \( dY \) (since \( d(X^p) = d(Y^p) = 0 \)), so by Lemma 3.3.5 any (necessarily \( p \)-torsion) element of \( \ker(f_1) \) representing a class coming from \( H^3(k, \hat{G}_a) \) may be represented by \( F(X,Y)dt \) for some \( F \in k[X,Y] \).

Suppose there is some monomial \( aX^{p^h}Y^{p^j} \) appearing in \( F \) (with \( a \in k \)). We may replace it with a monomial of the form \( bX^hY^j \) (with \( b \in k \)) as follows. Write \( a = \sum_{n=0}^{p-1} b_n t^n \) with \( b_n \in k \), so clearly \( aX^{p^h}Y^{p^j} = \sum_{n=0}^{p-1} t^n (b_n X^h Y^j)^p dt \). For \( 0 \leq n < p - 1 \), we have

\[
t^n (b_n X^h Y^j)^p dt = d((b_n X^h Y^j)^p (\frac{p^{n+1}}{n+1})),
\]

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which is 0 as a Brauer class. For \( n = p - 1 \), we have
\[
(b_n X^h Y^j)^p t^{p-1} dt = b_n X^h Y^j dt + (C^{-1} - i)(b_n X^h Y^j dt),
\]
so as Brauer classes we have \((b_n X^h Y^j)^p t^{p-1} dt = b_n X^h Y^j dt\). Applying this procedure repeatedly, we may arrange that no monomial of the form \(X^h Y^j\) appears in \(F\) unless \(h = j = 0\). We may also arrange that the constant term of \(F\) vanishes, since by assumption our Brauer class lies in \(\overline{Br}(\mathbb{G}_a^n)\). Now we need the following lemma.

**Lemma 3.3.6.** Let \(k\) be a field of characteristic \(p > 0\), and \(F \in k[X_1, \ldots, X_n]\) a nonzero element not containing any monomial of the form \(a X_1^{\pi_1} X_2^{\pi_2} \ldots X_n^{\pi_n} (a \in k^\times)\). For \(t \in k - k^p\), the class in \(Br(\mathbb{G}_a^n)[p]\) represented by \(F dt\) is nonzero.

**Proof.** The usual spreading out and specializing argument (as in the proof of Lemma 2.6.3) reduces us to the case in which \(k\) is a local function field. The key case is \(n = 1\), for which we will prove a stronger result: if \(E \in k[X]\) contains a monomial term with degree \(e\) not divisible by \(p\) then \(F dt\) represents a nonzero class in \(Br(\mathbb{G}_a)[p]\).

Note that any \(p\)-torsion Brauer class represented by \(a X_1^{\pi_1} (a \in k)\) is equivalent to one of the form \(b X_1^{\pi_1} (b \in k)\), by the same argument as given in the paragraph preceding the present Lemma. Applying this repeatedly, we may assume that \(p \nmid \deg(F)\). Choose \(a \in k^\times\) such that the class \(a dt \in Br(k)\) has invariant \(1/p\). Clearly \(F(X) - a\) has an irreducible factor \(\pi(X)\) of degree prime to \(p\). Let \(\beta\) be a root of \(\pi\) in a finite extension of \(k\), so specializing along \(X = \beta\) yields the class \(a dt \in Br(k(\beta))[p]\). If we let \(m = [k(\beta) : k]\), then \(inv_{k(\beta)}(a dt) = m/p \in \mathbb{Q}/\mathbb{Z}\) is nonzero since \(p \nmid m\). Since specializing along \(X = \beta\) yields a nonzero Brauer class, the class of \(F dt\) must be nonzero. This settles our stronger claim when \(n = 1\).

Now consider the general case. The hypothesis on \(F\) implies that its constant term vanishes, so \(F\) contains a monomial with positive total degree. Without loss of generality, \(X_1\) appears with exponent \(e\) prime to \(p\) in some monomial occurring in \(F\). Let \(G(X_2, X_3, \ldots, X_n)\) be the coefficient of \(X_1^e\) in \(F\). Specialize \((X_2, \ldots, X_n)\) to some point \((\beta_2, \ldots, \beta_n) \in k^{n-1}\) such that \(G(\beta_2, \ldots, \beta_n) \neq 0\). Then \(F(X_1, \beta_2, \beta_3, \ldots, \beta_n) dt \in Br(\mathbb{G}_a)[p]\) is a nonzero Brauer class, by the stronger result proved in the case \(n = 1\). It follows that the class of \(F dt\) is nonzero as well.

**Remark 3.3.7.** We proved the stronger result that the Brauer class of \(F dt\) is nonzero if \(F\) contains some term \(a X_1^{\pi_1} X_2^{\pi_2} \ldots X_n^{\pi_n}\) with some \(e_i\) not divisible by \(p\). We will not need this.

Consider the class \(F(X, Y) dt \in ker(f_1)[p]\). By the definition of \(f_1\), we have equalities of Brauer classes
\[
F(X, Y) dt = F(Y, X) dt
\]
\[
(F(X, Y + Z) + F(Y, Z) - F(X + Y, Z) - F(X, Y)) dt = 0.
\]
But one easily checks that since no monomials of the form \(a X_1^{\pi_1} X_2^{\pi_2} \ldots X_n^{\pi_n}\) appear in \(F\), the same holds for \(F(X, Y) - F(Y, X)\) and \(F(X, Y + Z) + F(Y, Z) - F(X + Y, Z) - F(X, Y)\).
It follows from Lemma 3.3.6 that \( F(X, Y) = F(Y, X) \) and
\[
F(X, Y + Z) + F(Y, Z) - F(X + Y, Z) - F(X, Y) = 0. \tag{3.3.2}
\]
Examples of such \( F \) include \( G(X + Y) - G(X) - G(Y) \) for any \( G \in k[Z] \) and the additive Witt polynomial \( F = S(X, Y) \pmod{p} \in F_p[X, Y] \), where
\[
S(X, Y) = \frac{(X + Y)^p - X^p - Y^p}{p} \in Z[X, Y].
\]
The following lemma says that these examples exhaust all possibilities.

**Lemma 3.3.8.** Let \( k \) be a field of characteristic \( p > 0 \), and consider \( F \in k[X, Y] \) that contains no monomial of the form \( aX^pY^p \) with \( a \in k^\times \) (so \( F(0, 0) = 0 \)). Assume \( F(X, Y) = F(Y, X) \) and that \( F \) satisfies (3.3.2). Then there exist \( G \in k[Z] \) and \( \alpha \in k \) such that
\[
F(X, Y) = G(X + Y) - G(X) - G(Y) + \alpha \cdot S(X, Y)
\]
where \( S = ((X + Y)^p - X^p - Y^p)/p \in Z[X, Y] \).

**Proof.** The identities (3.3.2) break up degree by degree, so we may assume that \( F \) is homogeneous of degree \( n \geq 0 \), say \( F(X, Y) = \sum_{i=0}^n \beta_{i,n-i}X^iY^{n-i} \) with \( \beta_{i,j} \in k \). If \( n = 0 \) then \( F = 0 \) by assumption, so assume that \( n > 0 \). The symmetry of \( F \) says \( \beta_{i,j} = \beta_{j,i} \) for any \( i, j \).

Comparing coefficients of \( X^n \) on both sides of the identity
\[
F(X, Y + Z) + F(Y, Z) - F(X + Y, Z) - F(X, Y) = 0 \tag{3.3.3}
\]
gives \( \beta_{n,0} = 0 \), so \( \beta_{0,n} = 0 \) by symmetry. Next, comparing coefficients of \( X^aY^bZ^c \) on both sides of (3.3.3) gives
\[
\left( \begin{array}{c} b + c \\ b \end{array} \right) \beta_{a,b+c} = \left( \begin{array}{c} a+b \\ a \end{array} \right) \beta_{a+b,c} \quad \text{if } a, c > 0. \tag{3.3.4}
\]

Suppose for now that \( n \neq p \) and consider \( 0 < i < n \). We shall prove that if \( \binom{n}{i} \neq 0 \) in \( F_p \), then \( \beta_{i,n-i} = 0 \). The vanishing of \( \binom{n}{i} \) in \( F_p \) says exactly that if we write \( n \) and \( i \) in base \( p \), say
\[
n = d_0 + d_1p + \ldots + d_rp^r, \quad i = e_0 + e_1p + \cdots + e_rp^r
\]
with \( 0 \leq d_l, e_l < p \), then \( e_l < d_l \) for some \( l \). Let \( j \) be the smallest index such that \( e_j > d_j \).

Write \( n-i = f_0 + f_1p + \cdots + f_rp^r \) with \( 0 \leq f_l < p \) (so \( f_l = d_l - e_l \) for \( l < j \), and \( e_j + f_j \geq p \); i.e., \( f_j \geq p - e_j \)). We want to apply (3.3.4) with \( a = i, b = (p - e_j)p^j, c = n - i - (p - e_j)p^j \). This only works if \( i + (p - e_j)p^j \neq n \); i.e., if \( n - i \neq (p - e_j)p^j \).
So suppose first that \( n - i \neq (p - e_j)p^j \). By construction, the \( j \)th digit in the base-\( p \) expansion of \( i + (p - e_j)p^j \) is \( 0 \leq e_j \). It follows that \( (i+(p-e_j)p^j) = 0 \) in \( F_p \). We therefore obtain
\[
\left( \frac{n-i}{(p-e_j)p^j} \right) \beta_{i,n-i} = 0
\]
Since the \( j \)th digit in the base-\( p \) expansion of \( n - i \) is \( f_j \geq (p - e_j), \left( \frac{n-i}{(p-e_j)p^j} \right) \neq 0 \) in \( F_p \). We conclude that \( \beta_{i,n-i} = 0 \), as desired.

Suppose instead that \( n - i = (p - e_j)p^j \) (still assuming \( n \neq p \) and \( \binom{n}{i} \neq 0 \) in \( F_p \)). By symmetry of \( F \) it suffices to show \( \beta_{n-i,i} = 0 \). Switching the roles of \( i \) and \( n-i \) in the above argument gives \( \beta_{n-i,i} = 0 \) unless \( i = e_jp^j \) and \( n - i = (p - e_j)p^j \). But \( 0 < i < n \) and \( n \neq p \), so \( j > 0 \) and hence \( p \mid i, n - i \), so \( \beta_{i,n-i} = 0 \) by assumption. This completes the proof that \( \beta_{i,n-i} = 0 \) if \( \binom{n}{i} = 0 \) in \( F_p \) with \( n \neq p \).

Now suppose that \( 0 < i < n \) and that \( \binom{n}{i} \neq 0 \) in \( F_p \) (so \( n \) is not a power of \( p \)). That is, if we again write \( n \) and \( i \) in base \( p \) as
\[
n = d_0 + d_1p + \cdots + d_rp^r, \ i = e_0 + e_1p + \cdots + e_r p^r
\]
then \( e_l \leq d_l \) for each \( l \). Let \( j \) be such that \( e_j \neq 0 \) (recall \( i \neq 0 \)). We may apply \( (3.3.4) \) with \( a = i - p^j, b = p^j, c = n - i \) provided that \( i \neq p^j \). Hence, when \( i \neq p^j \) we have
\[
\left( \frac{n-i+p^j}{p^j} \right) \beta_{i-p^j,n-i+p^j} = \binom{i}{p^j} \beta_{i,n-i}.
\]
Clearly \( \binom{i}{p^j} \neq 0 \) in \( F_p \) because the \( j \)th digit in the base-\( p \) expansion of \( i \) is nonzero. We therefore get that \( \beta_{i,n-i} \) is some "universal" \( k \)-multiple (i.e., independent of \( F \)) of \( \beta_{i-p^j,n-i+p^j} \). That is, the \( X^iY^{n-i} \)-coefficient of \( F \) is some "universal" \( k \)-multiple (i.e., independent of \( F \)) of the \( X^{i-p^j}Y^{n-i+p^j} \)-coefficient. Applying this inductively, we see that the \( X^iY^{n-i} \)-coefficient is some "universal" \( k \)-multiple of \( \beta_{p^j,n-p^j} \), where \( l \) is any index such that \( e_l \neq 0 \) (and for some such \( l \) we have \( e_l \leq d_l \) since \( i < n \), so \( d_l \neq 0 \) for some \( l \) for which \( e_l \neq 0 \)).

We claim that if \( l, l' \) are distinct indices such that \( d_l, d_{l'} \neq 0 \) then \( \beta_{p^j,n-p^j} \) is some "universal" \( k \)-multiple of \( \beta_{p^j,n-p^j} \). It will then follow from everything we have shown that the space of homogeneous degree-\( n \) solutions \( F \) to (3.3.2) is at most 1-dimensional. If \( n \) is not a power of \( p \), then it would follow that every such polynomial is a \( k \)-multiple of the nonzero polynomial \( (X + Y)^n - X^n - Y^n \). If \( n > p \) is a power of \( p \), then \( \binom{n}{i} = 0 \) for every \( 0 < i < n \), so we know that \( F = 0 \). Thus the lemma for \( n \neq p \) is proved once we show the preceding claim concerning \( l, l' \).

We have \( \beta_{p^j,n-p^j} = \beta_{n-p^j,p^j} \), and for every index \( l' \neq l \) such that \( d_{l'} \neq 0 \), the \( l' \)th digit in the base-\( p \) expansion of \( n - p^l \) is nonzero. By what we have already shown, \( \beta_{n-p^j,p^j} \) is some "universal" \( k \)-multiple of \( \beta_{p^j,n-p^j} \) for every such index \( l' \). This proves the claim, and the lemma for \( n \neq p \).
Finally, suppose that $n = p$. Consider $1 < i < p$. Apply (3.3.4) with $a = i - 1, b = 1, c = p - i$. This gives
\[
\begin{pmatrix}
    i \\
    1
\end{pmatrix}
\beta_{i,p-i} = \begin{pmatrix} p - i + 1 \\ 1 \end{pmatrix} \beta_{i-1,p-i+1}.
\]
Since $\binom{i}{1} \neq 0$, this tells us that $\beta_{i,p-i}$ is some $k$-multiple, independent of $F$, of $\beta_{i-1,p-i+1}$. Applying this repeatedly, we see that each $\beta_{i,p-i}$ with $0 < i < p$ is some $k$-multiple, independent of $F$, of $\beta_{1,p-1}$. This implies that the space of homogeneous degree-$p$ solutions to (3.3.2) is at most 1-dimensional. Since the polynomial $S(X,Y) \mod p$ satisfies (3.3.2), it follows that every such polynomial is a $k$-multiple of this one, completing the proof of the lemma.

For a $p$-torsion Brauer class
\[
G(X)dt \in \widetilde{\text{Br}}(G_a) = \ker(\text{Br}(G_a) \xrightarrow{0^*} \text{Br}(k))
\]
(so $G(0) = 0$), clearly $f_0(G(X)dt) = [G(X + Y) - G(X) - G(Y)]dt$. Conversely, for any $G \in k[X]$ with vanishing constant term, the Brauer class represented by $G(X)dt$ clearly lies in $\text{Br}(G_a)$. It therefore follows from Lemma 3.3.8 that for our $F \in k[X,Y]$ with vanishing constant term such that $F(X,Y)dt$ represents a chosen class in $(\ker(f_1)/\text{im}(f_0))[[p]]$, we may assume $F = \alpha \cdot S(X,Y)$ for some $0 \neq \alpha \in k$. We shall prove that any such Brauer class is killed by some $k$-isogeny $\phi : G_a \to G_a$, so Lemma 3.3.5 then implies that any Brauer class coming from $H^2(k,G_a)$ is 0, thereby completing the proof that $H^2(k,G_a) = 0$.

Write $\alpha = \sum_{i=0}^{p-1} a_i t^i$ for some $a_i \in k$. We will find $\gamma, \beta \in k$, not both 0, such that the isogeny $\phi$ given on $k$-algebras by $X \mapsto \gamma X + \beta X^p$ kills the class of $z := \alpha S(X,Y)dt \in (\ker(f_1)/\text{im}(f_0))[[p]]$. If we write $S(X,Y) = \sum c_{hj}X^hY^j$ with $c_{hj} \in \mathbb{F}_p$ (and $h + j = p$) then
\[
\hat{\phi}(z) = \sum_{hj} (\alpha \gamma^p X^h Y^j + \alpha \beta^p X^p Y^j)dt + F(X,Y)dt,
\]
where $F$ is a $k$-linear combination of terms $X^h Y^{h'}$ with $p \nmid h$ or $p \nmid h'$ (so $F(0,0) = 0$), and $h + h' \neq p$, and $F$ satisfies (3.3.2). It follows from Lemma 3.3.8 that $F(X,Y) = G(X + Y) - G(X) - G(Y)$ for some $G \in k[X]$ with $G(0) = 0$, so $\hat{\phi}(z) = \sum c_{hj} (\alpha \gamma^p X^h Y^j + \alpha \beta^p X^p Y^j)dt + F(X,Y)dt$. For $0 \leq n < p - 1$, we have $c_{hj} \gamma^p a_n X^h Y^j dt = d((c_{hj}a_n \beta X^h Y^j)^{p^{n+1}}) dt$, so this is 0 as a Brauer class. Therefore, $\hat{\phi}(z) = \sum c_{hj} (\alpha \gamma^p X^h Y^j + p^{n+1} a_{n+1} \beta^p X^p Y^j) dt$.

Since
\[
\begin{align*}
\sum c_{hj} (\alpha \gamma^p + p^{n+1} a_{n+1} \beta^p) X^h Y^j dt = \sum c_{hj} \alpha \gamma^p X^h Y^j dt + (C-1-b)(c_{hj}a_{n-1} \beta X^h Y^j dt),
\end{align*}
\]
we have $c_{hj} (\alpha \gamma^p + p^{n+1} a_{n+1} \beta^p) X^h Y^j dt = c_{hj}a_{n-1} \beta X^h Y^j dt$ as Brauer classes. Therefore, $\hat{\phi}(z) = \sum c_{hj} (\alpha \gamma^p + p^{n+1} a_{n+1} \beta^p) X^h Y^j dt = 0$. If we choose $\gamma, \beta$ not both 0 so that $\alpha \gamma^p + p^{n+1} a_{n+1} \beta^p = 0$ (clearly possible: if $a_{n-1} = 0$, take $\gamma = 0, \beta = 1$, otherwise choose $\gamma \neq 0$ arbitrarily and let $\beta = -\alpha \gamma^p / a_{n-1}$), then $\hat{\phi}(z) = 0$. This finally completes the proof of Proposition 3.3.1. \(\square\)
Now we will discuss the proof of Proposition 3.3.2, which, as we said, is almost entirely the same as that of Proposition 3.3.1. There are only a few modifications that must be made. The first is that the \( t \) in the proof of Proposition 3.3.1 that was allowed to be any element of \( k - k^p \) must now be chosen to be a uniformizer of \( \mathcal{O} \). Then it is still true that \( \mathcal{O} = \oplus_{i=0}^{t-1} \mathcal{O}^p \mathcal{O}^i \), and therefore that \( \Omega^1_{\mathcal{O}} = \mathcal{O} \). Other than this, there are four steps in the proof of Proposition 3.3.11 that require modification. The first is right at the beginning, when proving that there is an inclusion \( H^3(k, \hat{G}_a) \to \text{Ext}^2_k(\mathcal{G}_a, \mathcal{G}_m) \). Most of that argument still goes through, with the exception of the proof that \( H^0(k, \mathcal{E}xt^2(\mathcal{G}_a, \mathcal{G}_m)) = 0 \). We must therefore prove the following result.

**Lemma 3.3.9.** \( H^0(\mathcal{O}, \mathcal{E}xt^2(\mathcal{G}_a, \mathcal{G}_m)) = 0 \).

The second step is the proof of Lemma 3.3.6.

**Lemma 3.3.10.** Let \( F \in \mathcal{O}[X_1, \ldots, X_n] \) be a nonzero element not containing any monomial of the form \( aX_1^{p_1}X_2^{p_2} \ldots X_n^{p_n} \) (\( a \neq 0 \)). Then the class in \( \text{Br}(\mathcal{G}_a, \mathcal{O})[p] \) represented by \( Fdt \) is nonzero.

Indeed, this follows immediately from Lemma 3.3.6 by pulling back to the generic fiber.

The third step is the proof of Lemma 3.3.11.

**Lemma 3.3.11.** Let \( \phi : \mathcal{G}_a \to \mathcal{G}_a \) be an \( \mathcal{O} \)-isogeny. Then the induced map \( H^3(\hat{\mathcal{O}}) : H^3(\hat{\mathcal{O}}, \hat{\mathcal{G}}_a) \to H^3(\mathcal{O}, \hat{\mathcal{G}}_a) \) is an isomorphism.

The last step is at the very end of the proof in the construction of the isogeny \( \phi \) such that \( \hat{\phi}(z) = 0 \). We must choose \( \beta, \gamma \in \mathcal{O} \) and not just in \( k \). This is easy: if \( a_{\mu-1} \neq 0 \), then simply choose \( \gamma \) to be of such high valuation that \( \beta = -\alpha \gamma^p / a_{\mu-1} \in \mathcal{O} \).

Now we will prove the remaining lemmas above, and thereby complete the proof of Proposition 3.3.2.

**Proof of Lemma 3.3.9.** Any fppf cover of \( \text{Spec}(\mathcal{O}_K) \) may be refined by one of the form \( \text{Spec}(\mathcal{O}_{K'}) \) for some finite extension \( K'/K \), cf. the first paragraph of the proof of Proposition 2.4.8. It therefore suffices to show that for any local function field \( K \) and any \( \alpha \in \text{Ext}^2_{\mathcal{O}_K}(\mathcal{G}_a, \mathcal{G}_m) \), there is a finite extension \( K'/K \) such that \( 0 = \alpha_{\mathcal{O}_{K'}} \in \text{Ext}^2_{\mathcal{O}_{K'}}(\mathcal{G}_a, \mathcal{G}_m) \). By Lemma 2.4.7 it is enough to show that for any \( \alpha \in \text{Br}(\mathcal{G}_a, \mathcal{O}_K) \), there is a finite extension \( K'/K \) such that \( 0 = \alpha_{\mathcal{O}_{K'}} \in \text{Br}(\mathcal{G}_a, \mathcal{O}_{K'}) \). But \( \mathcal{G}_a, \mathcal{O}_{K'} \) is regular, so the map \( \mathcal{G}_a, \mathcal{O}_K \to \mathcal{G}_{a,K} \) is injective (indeed, the map \( \text{Br}(\mathcal{G}_n, \mathcal{O}) \to \text{Br}(K(X_1, \ldots, X_n)) \) is injective), hence it suffices to show that for any \( \alpha \in \text{Br}(\mathcal{G}_a, \mathcal{O}_K) \), there is a finite extension \( K'/K \) such that \( 0 = \alpha_{K'} \in \text{Br}(\mathcal{G}_a, \mathcal{O}_{K'}) \), and this in turn follows from the well-known fact that \( \text{Br}(\mathcal{G}_a, \mathcal{O}_K) = 0 \).

**Proof of Lemma 3.3.11.** Let \( A := \ker(\phi) \). Then we have an exact sequence

\[
1 \to A \to \mathcal{G}_a \xrightarrow{\phi} \mathcal{G}_a \to 1
\]

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and dualizing, I claim that we obtain an exact sequence of fppf sheaves

$$1 \rightarrow \widehat{G}_a \rightarrow \widehat{G}_a \rightarrow \widehat{A} \rightarrow 1$$

The only point that is not clear is exactness at \( \widehat{A} \), but this follows from the vanishing of \( \mathcal{E}xt^1_{\mathcal{O}}(G_a, G_m) \), which follows from Proposition 2.2.11. In order to prove the lemma, therefore, it is enough to show that \( H^i(\mathcal{O}, \widehat{A}) = 0 \) for \( i > 1 \). This follows from Lemma 3.3.12 below.

**Lemma 3.3.12.** Let \( \mathcal{O} \) be a Henselian local ring with finite residue field, and let \( A \) be a finite flat commutative \( \mathcal{O} \)-group scheme. Then \( H^i(\mathcal{O}, A) = 0 \) for \( i > 1 \).

**Proof.** We will essentially prove a slightly weaker version of this statement later, but let us directly prove this stronger version now. By Cartier duality, it suffices to show that for any finite flat \( \mathcal{O} \)-group scheme \( A \), we have \( H^i(\mathcal{O}, A) = 0 \) for \( i > 1 \). By [Ma II, 3.2.5] we have an exact sequence

$$1 \rightarrow A \rightarrow G \rightarrow H \rightarrow 1$$

with \( G, H \) smooth commutative affine \( \mathcal{O} \)-group schemes with connected fibers. (Indeed, generalizing the construction in the proof of Lemma 3.1.2 we may use \( G := R_{A/\mathcal{O}}(GL_1) \) and \( H := G/A \), where the quotient \( H \) exists as an affine \( \mathcal{O} \)-group of finite type over which \( G \) is faithfully flat by [SGA3 V, 4.1], so \( \mathcal{O} \)-smoothness of \( G \) implies \( \mathcal{O} \)-smoothness of \( H \) and the fibral connectedness for \( G \) and hence \( H \) follows from [CGP A.5.11]). It therefore suffices to show that \( H^i(\mathcal{O}, G) = 0 \) for all \( i > 0 \) whenever \( G \) is a smooth \( \mathcal{O} \)-group scheme with connected fibers. We proceed by induction on \( i \). Let us note that the smoothness of \( G \) allows us to take our cohomology to be étale rather than fppf.

Let \( \kappa \) denote the (finite)residue field of \( \mathcal{O} \). To show that \( H^1(\mathcal{O}, G) = 0 \), we need to check that for any étale \( G \)-torsor \( X \) over \( \mathcal{O} \), we have \( X(\mathcal{O}) \neq \emptyset \). First, since \( G \) has connected special fiber, \( X(\kappa) \neq \emptyset \) by Lang’s Theorem. Since \( G \) is smooth, so is \( X \), hence, since \( \mathcal{O} \) is Henselian, the map \( X(\mathcal{O}) \rightarrow X(\kappa) \) is surjective. In particular, \( X(\mathcal{O}) \neq \emptyset \), as desired.

Now suppose that \( i > 1 \) and that the claim holds for \( i - 1 \). Let \( \alpha \in H^i(\mathcal{O}, G) \). We need to show that \( \alpha = 0 \). Since \( \mathcal{O} \) is Henselian local, its finite étale covers are cofinal among all étale covers. There is therefore a finite étale cover \( \mathcal{O}' \) of \( \mathcal{O} \) such that \( \alpha \) pulled back to \( \mathcal{O}' \) is 0. Consider the composition \( H^i(\mathcal{O}, G) \rightarrow H^i(\mathcal{O}, R_{\mathcal{O}'/\mathcal{O}}(G_{\mathcal{O}'})) \rightarrow H^i(\mathcal{O}', G_{\mathcal{O}'}) \), where the first map is induced by the canonical inclusion \( G \hookrightarrow R_{\mathcal{O}'/\mathcal{O}}(G_{\mathcal{O}'}) \) and the second is the canonical map. This composition is none other than the pullback map. Indeed, we have such a map with \( G \) replaced by any étale sheaf on \( \mathcal{O} \), so this is the specialization to \( G \) of a map of \( \delta \)-functors, hence this agreement (for all sheaves) reduces to the \( H^0 \) case, which is clear. The map \( H^i(\mathcal{O}, R_{\mathcal{O}'/\mathcal{O}}(G_{\mathcal{O}'})) \rightarrow H^i(\mathcal{O}', G_{\mathcal{O}'}) \) is an isomorphism because \( \text{Spec}(\mathcal{O}') \rightarrow \text{Spec}(\mathcal{O}) \) is finite, so it follows that the map \( H^i(\mathcal{O}, G) \rightarrow H^i(\mathcal{O}, R_{\mathcal{O}'/\mathcal{O}}(G_{\mathcal{O}'})) \) kills \( \alpha \). Let \( H := R_{\mathcal{O}'/\mathcal{O}}(G_{\mathcal{O}'})/G \), a smooth affine \( \mathcal{O} \)-group scheme with connected fibers.
(as we may check by working over a finite étale cover of \(\mathcal{O}\) that splits \(\mathcal{O}'\)). Then we have an exact sequence
\[
1 \to G \to R_{\mathcal{O}'/\mathcal{O}}(G_{\mathcal{O}'}) \to H \to 1
\]
Since \(\alpha\) dies in \(H^i(\mathcal{O}, R_{\mathcal{O}'/\mathcal{O}}(G_{\mathcal{O}'}))\), and \(H^{i-1}(\mathcal{O}, H) = 0\) by induction, we deduce that \(\alpha = 0\).

3.4 Vanishing of \(H^3(k, \hat{G})\)

The goal of this section is to prove the following result, generalizing Proposition 3.3.1.

**Proposition 3.4.1.** Let \(k\) be a non-archimedean local or a global field with no real places, and let \(G\) be an affine commutative \(k\)-group scheme of finite type. Then \(H^3(k, \hat{G}) = 0\).

**Proof.** We will give the proof for function fields. The proof in the characteristic 0 case is the same, replacing the fppf with the étale site everywhere (see Remark 2.2.20). First, if we have a short exact sequence
\[
1 \to G' \to G \to G'' \to 1
\]
of group schemes as in the proposition, then (by Proposition 2.2.3) if the result holds for \(G'\) and \(G''\) it holds for \(G\). Thanks to Lemma 2.1.7, therefore, we may assume that \(G\) is either \(G_a\) or an almost torus. The \(G_a\) case is Proposition 3.3.1. Thanks to Lemma 2.1.3(ii), we may assume that \(G\) is either finite or a torus. When \(G\) is finite, then so is \(\hat{G}\), so we are done by Proposition 3.1.1. We may therefore assume that \(G = T\) is a torus.

Let \(k'/k\) be a finite separable extension such that \(T\) splits over \(k'\). Then we have the natural inclusion \(T \hookrightarrow R_{k'/k}(T')\), where \(T'\) is the split \(k'\)-torus \(T_{k'}\), so we have an exact sequence
\[
1 \to T \to R_{k'/k}(T') \to S \to 1
\]
for some \(k\)-torus \(S\). This yields an exact sequence
\[
H^3(k, R_{k'/k}(T')) \to H^3(k, \hat{T}) \to H^4(k, \hat{S})
\]
The functor \(\hat{S}\) is represented by an étale \(k\)-group scheme, so by [BrIII, Thm. 11.7] we have \(H^4(k, \hat{S}) = H^4_{\text{ét}}(k, \hat{S})\), and this vanishes because local and global function fields have strict cohomological dimension 3. We may therefore assume that \(T = R_{k'/k}(G_m)\) for some finite separable extension \(k'/k\).

By Lemma 2.2.1, we have \(R_{k'/k}(G_m) \simeq R_{k'/k}(\hat{G}_m) = R_{k'/k}(\mathbb{Z})\). We may therefore take our cohomology to be étale, and since finite pushforward is exact between categories of étale sheaves, we have \(H^3(k, R_{k'/k}(G_m)) \simeq H^3(k', \mathbb{Z})\). Replacing \(k\) with \(k'\), it therefore suffices to show that \(H^3(k, \mathbb{Z}) = 0\). Using the exact sequence
\[
0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0
\]
and the fact that the constant Galois module $\mathbb{Q}$ is uniquely divisible (hence its higher Galois cohomology vanishes), this is equivalent to the assertion that $H^2(k, \mathbb{Q}/\mathbb{Z}) = 0$. This is proved when $\text{char}(k) = 0$ in [Ser2 §6.5]. The proof goes through verbatim in characteristic $p$ except that one must show that the map $H^1(k, \mathbb{Q}_p/\mathbb{Z}_p) \to H^2(k, \mathbb{Z}/p\mathbb{Z})$ coming from the exact sequence

$$0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{Q}_p/\mathbb{Z}_p \to \mathbb{Q}_p/\mathbb{Z}_p \to 0$$

is surjective. But this is clear, since the latter group vanishes (as any field of characteristic $p$ has $p$-cohomological dimension at most 1, by [Ser1 Ch.II, §2.2, Prop.3]).

3.5 Relating Čech and sheaf cohomology of $G$

We shall in several places require an explicit description of certain cohomology classes in terms of Čech cocycles. We therefore need the following result.

**Proposition 3.5.1.** Let $k$ be a local or global field, $G$ an affine commutative $k$-group scheme of finite type. Let the fppf sheaf $\mathcal{F}$ denote either $G$ or $\hat{G}$. Then the canonical map $\tilde{H}^i(k, \mathcal{F}) \rightarrow H^i(k, F)$ is an isomorphism for $i \leq 2$.

Proposition 3.5.1 is true for $i = 0, 1$ by general principles. The map $\tilde{H}^2(k, \mathcal{F}) \rightarrow H^2(k, \mathcal{F})$ is also injective by general principles. The real content of Proposition 3.5.1 is therefore the surjectivity of this map; that is, every derived $H^2$-class comes from a Čech class. The rest of this section will be occupied with the proof of Proposition 3.5.1 for $\mathcal{F} = G$. We will give the proof for $\mathcal{F} = \hat{G}$ in §3.6.

First, let us note that the case $\mathcal{F} = G$ in Proposition 3.5.1 is trivial when $G$ is smooth, since then its fppf and étale cohomology agree, and Galois cohomology may always be computed via Galois cocycles (which are nothing other than étale Čech cocycles over fields). In particular, Proposition 3.5.1 is true when $\text{char}(k) = 0$, so we may and do from now on assume that $\text{char}(k) = p > 0$. Let us also note that, by the Nullstellensatz, and because the $\mathcal{F}$ in Proposition 3.5.1 commutes with direct limits of rings, the Čech cohomology of $\mathcal{F}$ may be computed by using the (not technically fppf) covering $\text{Spec}(k')/\text{Spec}(k)$.

We will now prove Proposition 3.5.1 in various steps of increasing generality.

**Step 1:** $G = R_{k'/k}(\mu_{p^n})$ for some finite separable extension $k'/k$. The key point is the following lemma.

**Lemma 3.5.2.** Let $k'/k$ be a finite separable extension, $\mu := R_{k'/k}(\mu_{p^n})$. The Weil-restricted Kummer sequence

$$1 \rightarrow \mu \rightarrow R_{k'/k}(\mathbb{G}_m) \xrightarrow{p^n} R_{k'/k}(\mathbb{G}_m) \rightarrow 1$$

is an exact sequence of fppf sheaves on $\text{Spec}(k)$. Further, it induces a long exact sequence in Čech cohomology

$$\cdots \rightarrow \tilde{H}^i(k, \mu) \rightarrow \tilde{H}^i(k, R_{k'/k}(\mathbb{G}_m)) \rightarrow \tilde{H}^i(k, \mu) \rightarrow \tilde{H}^{i+1}(k, R_{k'/k}(\mu_{p^n})) \rightarrow \cdots$$

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that is compatible with the derived functor cohomology long exact sequence.

Proof. The first assertion follows from the fact that separable Weil restriction is exact (as exactness may be checked after extending scalars to a normal closure of $k'/k$, and then the sequence just becomes a product of scalar-extended copies of the $p^n$-power Kummer sequence over $k'$ twisted by the Galois action).

In order to prove the second assertion of the Lemma, we need to show that the associated sequence of Čech complexes is short exact. The left-exactness is immediate, so the only thing that needs to be checked is that the $p^n$th power map from $R_{k'/k}(G_m)\otimes_{k'}k'\times$ to itself is surjective. For this, it suffices to show that the $p^n$th power map on $\bar{k}^\otimes_{k'} \otimes_k k'$ is surjective, and even the $p$th power map (by iterating $n$ times). Since the $p$th power map is surjective on $\bar{k}$, we only need to show that any tensor of the form $1^\otimes_{k'} \otimes \lambda$ for $\lambda \in k'$ is a $p$th power.

Let $t \in k$ be a $p$-basis; i.e., $k = \sum_{i=0}^{p-1} k^p t^i$. Then since $k'/k$ is finite separable, $t$ is also a $p$-basis for $k'$ (since $t$ being a $p$-basis is equivalent to $dt$ spanning $\Omega^1_{k'}$, and this is preserved by étale base change). Therefore, we may write $\lambda = \sum_{i=0}^{p-1} t^i a_i^p$ for some $a_i \in k'$. Thus $1 \otimes \ldots \otimes 1 \otimes \lambda = \sum_{i=0}^{p-1} 1 \otimes \ldots \otimes t^i \otimes a_i^p$. Again, the $t^i$ term (more precisely, $1 \otimes \ldots \otimes 1 \otimes t^i \otimes 1$) is a $p$th power because the $p$th power map is surjective on $\bar{k}$. Clearly $1 \otimes \ldots \otimes 1 \otimes a_i^p$ is a $p$th power. So every element of $\bar{k}^\otimes_{k'} \otimes_k k'$ is a $p$th power, as desired. The compatibility with the derived functor cohomology sequence follows from Proposition E.2.2.1.

We may now complete the proof of Step 1. By Lemma 3.5.2, we have a commutative diagram of exact sequences

$$
\begin{array}{cccc}
\hat{H}^2(k, R_{k'/k}(\mu_{p^n})) & \longrightarrow & \hat{H}^2(k, R_{k'/k}(G_m)) & \longrightarrow & \hat{H}^2(k, R_{k'/k}(G_m)) \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^2(k, R_{k'/k}(\mu_{p^n})) & \longrightarrow & H^2(k, R_{k'/k}(G_m))
\end{array}
$$

where the 0 in the bottom row comes from the fact that $H^1(k, R_{k'/k}(G_m)) = 0$ (since it equals $H^1(k', G_m)$, as the cohomology may be taken to be étale, and finite pushforward is exact between categories of étale sheaves), and the last two vertical maps are isomorphisms because $R_{k'/k}(G_m)$ is smooth. A simple diagram chase now yields the surjectivity of the first vertical map (which, recall, is all that we need, because injectivity is automatic).

Step 2: $G$ is multiplicative of $p$-power order; i.e., $G_T$ (or equivalently $G_{k'}$), is a product of various $\mu_{p^n}$'s. The key point in this case is the following lemma.

Lemma 3.5.3. Let $G$ be multiplicative of $p$-power order. There are integers $n_1, \ldots, n_j \geq 1$ and a finite separable extension $k'/k$ such that there is an isogeny $\prod_{i=1}^j R_{k'/k}(\mu_{p^{n_i}}) \rightarrow G$.

Proof. Let $k'/k$ be a finite Galois extension splitting $G$; i.e., $G_{k'}$ is a product of $\mu_{p^{n_i}}$'s. Let $\Gamma := \mathrm{Gal}(k'/k)$. Then using the anti-equivalence between finite multiplicative-type
group schemes and finite Galois modules, the lemma is equivalent to showing that any finite $\Gamma$-module of $p$-power order may be embedded into a product of various $(\mathbb{Z}/p^n\mathbb{Z})[\Gamma]$’s. Dualizing, this would be clear if $(\mathbb{Z}/p^n\mathbb{Z})[\Gamma]$ is a self-dual $\Gamma$-module (as the assertion is then that any $p$-power order $\Gamma$-module $A$ is a quotient of a product of various $(\mathbb{Z}/p^n\mathbb{Z})[\Gamma]$’s, which is clear simply by choosing generators for $A$).

It remains to check the asserted self-duality of $(\mathbb{Z}/p^n\mathbb{Z})[\Gamma]$. If $\phi_g \in ((\mathbb{Z}/p^n\mathbb{Z})[\Gamma])^\wedge$ is the element defined by $\phi_g(\sum_{\gamma \in \Gamma} N_{\gamma}\gamma) = N_g$ for any $g \in \Gamma$, then the map $\sum N_gg \mapsto \sum N_g\phi_g$ is the desired isomorphism between $(\mathbb{Z}/p^n\mathbb{Z})[\Gamma]$ and $((\mathbb{Z}/p^n\mathbb{Z})[\Gamma])^\wedge$. (This is $\Gamma$-equivariant because $\phi_{\gamma g} = \phi_g(\gamma^{-1}(\cdot)) =: \gamma.\phi_g$ from the definitions.)

We may now complete the proof of Step 2. Let $G$ be $p$-power order of multiplicative type, and choose an isogeny as in Lemma 3.5.3. We then have a commutative diagram of exact sequences

$$
\begin{array}{ccc}
\hat{H}^2(k, \prod_i R_{k'/k}(\mu_{p^{n_i}})) & \longrightarrow & \hat{H}^2(k, G) \\
\downarrow & & \downarrow \\
H^2(k, \prod_i R_{k'/k}(\mu_{p^{n_i}})) & \longrightarrow & H^2(k, G) \longrightarrow 0
\end{array}
$$

The 0 in the bottom row comes from the fact that $H^3(k, G') = 0$ for any affine commutative $k$-group scheme $G'$ of finite type (Proposition 3.1.1). The first vertical map is an isomorphism by Step 1. The above diagram immediately implies that the map $\hat{H}^2(k, G) \rightarrow H^2(k, G)$ is surjective, which completes the proof of Step 2.

Step 3: $G$ finite. We may write $G$ as the product of its $l$-primary parts for various primes $l$ and so assume that $G$ is $l$-primary. When $l \neq p$, then $G$ is étale, so Proposition 3.5.1 holds for $G$ by the settled smooth case. So we may assume that $G$ is of $p$-power order. The key to proving Proposition 3.5.1 in this case is the following lemma.

**Lemma 3.5.4.** Let $k$ be a field of characteristic $p > 0$. Let $A$ be a finite commutative $k$-group scheme of $p$-power order, and suppose that $A$ is either étale or local-local. Then $H^2(k, A) = 0$.

**Proof.** If $A$ is étale then we may conclude by using that any field of characteristic $p$ has $p$-cohomological dimension $\leq 1$ [Ser1, Ch. II, §2.2, Prop. 3]. Suppose instead that $A$ is local-local. It suffices to build an embedding $A \rightarrow U$ for some commutative split unipotent $k$-group $U$, as then $\overline{U} := U/A$ is necessarily also split, and the exact sequence

$$
1 \rightarrow A \rightarrow U \rightarrow \overline{U} \rightarrow 1
$$

combined with the fact that $H^i(k, V) = 0$ for all $i > 0$ when $V$ is split unipotent will complete the proof.

We may filter $A$ by group schemes whose relative Frobenius and Verschiebung morphisms vanish, so we can assume that the Frobenius and Verschiebung morphisms for $A$
vanish. Any such group schemes always equal $\alpha_p^n$ for some $n$, and $\alpha_p \hookrightarrow \mathbf{G}_a$ as the kernel of the Frobenius map over $k$. Alternatively, it suffices to use the elementary fact that $A_{k'} = \alpha_p^n$ for some finite extension $k'/k$ (as this holds for $A_{k}$), and then use the Weil restriction trick that $A$ is naturally a $k$-subgroup of $R_{k'/k}(A_{k'})$.

Now we return to the problem at hand. We want to prove Proposition 3.5.1 for finite $G$ of $p$-power order. First suppose that $G$ is infinitesimal (hence with $p$-power order). Then applying the connected-étale sequence to $\hat{G}$ and dualizing, we obtain an exact sequence

$$1 \to M \to G \to A \to 1$$

where $M$ is multiplicative of $p$-power order and $A$ is local-local (as $G$ is infinitesimal). We then have a commutative diagram

$$\begin{array}{c}
\tilde{H}^2(k, M) \to \tilde{H}^2(k, G) \\
\downarrow \quad \downarrow \\
H^2(k, M) \to H^2(k, G) \to 0
\end{array}$$

where the bottom row is exact by Lemma 3.5.4 and the first vertical map is an isomorphism by Step 2. The above diagram immediately proves the surjectivity of the map $\tilde{H}^2(k, G) \to H^2(k, G)$, hence proves Proposition 3.5.1 in this case.

Suppose that $G$ is an arbitrary finite group scheme of $p$-power order. By the connected-étale sequence, we have an exact sequence

$$1 \to I \to G \to E \to 1$$

where $I$ is infinitesimal and $E$ is étale. We then have a commutative diagram

$$\begin{array}{c}
\tilde{H}^2(k, I) \to \tilde{H}^2(k, G) \\
\downarrow \quad \downarrow \\
H^2(k, I) \to H^2(k, G) \to 0
\end{array}$$

in which the bottom row is exact by Lemma 3.5.4 and the first vertical map is an isomorphism by the already-treated infinitesimal case. The above diagram then completes the proof of Proposition 3.5.1 for $G$.

Step 4: $G$ is an almost torus; i.e., $(G_{\text{tor}})_{\text{red}}$ is a torus. By Lemma 2.1.3, there is an isogeny $\hat{T} \times A \to G$ for some torus $T$ and some finite $A$. We then have a commutative diagram

$$\begin{array}{c}
\tilde{H}^2(k, T \times A) \to \tilde{H}^2(k, G) \\
\downarrow \quad \downarrow \\
H^2(k, T \times A) \to H^2(k, G) \to 0
\end{array}$$

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in which the first vertical map is an isomorphism by the already-treated finite case (Step 3) and because tori are smooth. The bottom row is exact because $H := \ker(T \times A \to G)$ satisfies $H^2(k,H) = 0$ (by Proposition 3.1.1 in positive characteristic). The above diagram immediately proves Proposition 3.5.1 for $G$.

Step 5: General $G$. Now assume that $G$ is an arbitrary affine commutative $k$-group scheme of finite type. Then by Lemma 2.1.7, there is an exact sequence

$$1 \to H \to G \to U \to 1 \quad (3.5.1)$$

with $H$ an almost torus and $U$ smooth connected unipotent. The sequence (3.5.1) provides us with a commutative diagram

$$\begin{CD}
\tilde{H}^2(k,H) @>>> \tilde{H}^2(k,G) \\
@VVV @VVV \\
H^2(k,H) @>>> H^2(k,G) @>>> 0
\end{CD}$$

in which the first vertical map is an isomorphism by Step 4 and the bottom row is exact because $H^2(k,V) = 0$ for any smooth connected commutative unipotent $k$-group $V$ (Lemma 2.4.4 (i)). The above diagram shows that the map $\tilde{H}^2(k,G) \to H^2(k,G)$ is surjective, and completes the proof of Proposition 3.5.1. \hfill \Box

3.6 Relating Čech and sheaf cohomology of $\hat{G}$

Now we will prove Proposition 3.5.1 for $\mathcal{F} = \hat{G}$. That is, we will show that the map $\tilde{H}^2(k,\hat{G}) \to H^2(k,\hat{G})$, which is known to be injective, is surjective as well. We may once again assume that $\text{char}(k) = p > 0$, since otherwise we may take our cohomology to be étale, by Remark 2.2.20 and étale cohomology over fields may be computed via Galois cocycles.

We first note that the proposition is true for finite $G$, by the already-treated case when $\mathcal{F}$ is a group scheme, since $\hat{G}$ is a finite group scheme if $G$ is. Now we will once again prove the proposition in a sequence of increasingly general steps.

Step 1: $G$ is a smooth almost torus. There is an exact sequence

$$1 \to H \to G \to E \to 1$$

with $H$ an almost torus such that $p \nmid \#(H/H^0)$, and $E$ finite étale of $p$-power order: take $H$ to be the preimage of the prime-to-$p$ part in the natural map $G \to G/G^0$.

Now $\hat{H}$ is smooth (even étale), though usually not affine. Indeed, the exact sequence

$$1 \to H^0 \to H \to H/H^0 \to 1$$
yields by Proposition 2.2.3 an exact sequence
\[ 1 \rightarrow \hat{H}/H^0 \rightarrow \hat{H} \rightarrow \hat{H}^0 \rightarrow 1 \]
and \( \hat{H}^0 \) is étale because \( H^0 \) is a torus, while \( \hat{H}/H^0 \) is étale because \( \#(H/H^0) \in k^\times \). Proposition 3.5.1 therefore holds for \( H \) (by passing to étale cohomology, and then Galois cohomology). It also holds for the finite étale group \( E \). In order to prove it for \( G \), we will need the following lemma.

**Lemma 3.6.1.** With notation as above, there is a long exact Čech cohomology sequence
\[ \cdots \rightarrow \check{H}^i(k, \hat{E}) \rightarrow \check{H}^i(k, \hat{G}) \rightarrow \check{H}^i(k, \hat{H}) \rightarrow \check{H}^{i+1}(k, \hat{E}) \rightarrow \cdots \]
that is compatible with the derived functor cohomology sequence.

*Proof.* We want to show that the associated sequence of Čech complexes is short exact. The left-exactness is immediate, so what we need to show is that for every positive integer \( n \), the map \( \hat{G}(k^\otimes n) \rightarrow H(k^\otimes n) \) is surjective. For this, it suffices to show that \( H^1(k^\otimes n, \hat{E}) = 0 \).

Over \( k^\otimes n \), \( E \) is a constant group scheme (of \( p \)-power order). Hence, we need to show that \( H^1(k^\otimes n, \mu_{p^m}) = 1 \) for all \( m \geq 1 \), and this follows from the (elementary) triviality of \( \text{Pic}(k^\otimes n) \) and the fact that the \( p \)-th power map \( (k^\otimes n)^\times \rightarrow (k^\otimes n)^\times \) is surjective (since the \( p \)-th power map \( k \rightarrow k \) is surjective with \( p = \text{char}(k) > 0 \)). The compatibility with the derived functor sequence follows from Proposition E.2.1.

We need another lemma.

**Lemma 3.6.2.** The map \( \check{H}^3(k, \hat{E}) \rightarrow H^3(k, \hat{E}) \) is injective.

*Proof.* We have the Čech-to-derived functor spectral sequence
\[ E^{i,j}_2 = \check{H}^i(k, \mathcal{H}^j(\hat{E})) \Rightarrow H^{i+j}(k, \hat{E}) \]
so it suffices to show that \( E^{0,2}_2 = E^{1,1}_2 = 0 \). We have \( E^{0,2}_2 = \check{H}^0(k, \mathcal{H}^2(\hat{E})) \). To see that this vanishes, it suffices to show that \( \lim \check{H}^2(L, \hat{E}) = 0 \) where \( L/k \) varies through finite-degree subextensions of an algebraic closure \( \bar{k} \). But by general cohomological interaction with limits (and the compatibility of the functor \( \hat{E} \) with direct limits of \( k \)-algebras) this limit coincides with \( H^2(\bar{k}, \hat{E}) \). This latter cohomology vanishes since \( H^0(\bar{k}, \cdot) \) is an exact functor on the fppf abelian sheaves over \( \bar{k} \). To see that \( E^{1,1}_2 = \check{H}^1(k, \mathcal{H}^1(\hat{E})) \) vanishes, it likewise suffices to prove the vanishing of
\[ \lim \check{H}^1(L \otimes_k L, \hat{E}) = H^1(\bar{k} \otimes_k \bar{k}, \hat{E}). \]
We showed this latter vanishing in the course of the proof of Lemma 3.6.1. \( \square \)
Now we have a commutative diagram of exact sequences

\[
\begin{array}{cccc}
\hat{H}^2(k, \hat{E}) & \longrightarrow & \hat{H}^2(k, \hat{G}) & \longrightarrow & \hat{H}^2(k, \hat{H}) & \longrightarrow & \hat{H}^3(k, \hat{E}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^2(k, \hat{E}) & \longrightarrow & H^2(k, \hat{G}) & \longrightarrow & H^2(k, \hat{H}) & \longrightarrow & H^3(k, \hat{E})
\end{array}
\]

The first and third vertical arrows are isomorphisms because we already know Proposition 3.5.1 for $E$ and $H$. The last vertical arrow is an injection by Lemma 3.6.2. The top row comes from Lemma 3.6.1. A simple diagram chase now shows that the map $\hat{H}^2(k, \hat{G}) \rightarrow H^2(k, \hat{G})$ is surjective, hence an isomorphism. This completes the proof of Step 1.

Step 2: $G$ is an almost torus. We need the following lemma.

**Lemma 3.6.3.** Let $k$ be a field with characteristic $p > 0$, $I$ an infinitesimal commutative $k$-group scheme. Then $H^2(k, \hat{I}) = 0$.

**Proof.** The connected-étale sequence for $\hat{I}$ is

\[
1 \longrightarrow A \longrightarrow \hat{I} \longrightarrow E \longrightarrow 1
\]

with $A$ local-local and $E$ étale of $p$-power order. The lemma therefore follows from Lemma 3.5.4. \qed

By [SGA3, VII A, Prop. 8.3], there is an infinitesimal $k$-subgroup scheme $I \subset G$ such that $\hat{H} := G/I$ is a smooth almost torus. Then by Proposition 2.2.3, we have an exact sequence

\[
1 \longrightarrow \hat{H} \longrightarrow \hat{G} \longrightarrow \hat{I} \longrightarrow 1
\]

We therefore obtain a commutative diagram of exact sequences

\[
\begin{array}{cccc}
\hat{H}^2(k, \hat{H}) & \longrightarrow & \hat{H}^2(k, \hat{G}) & \longrightarrow & \hat{H}^2(k, \hat{H}) & \longrightarrow & \hat{H}^3(k, \hat{E}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^2(k, \hat{H}) & \longrightarrow & H^2(k, \hat{G}) & \longrightarrow & H^2(k, \hat{H}) & \longrightarrow & H^3(k, \hat{E})
\end{array}
\]

where the first vertical arrow is an isomorphism by Step 1, and the 0 comes from Lemma 3.6.3. The above diagram implies that the map $\hat{H}^2(k, \hat{G}) \rightarrow H^2(k, \hat{G})$ is an isomorphism.

Step 3: $G = U$ is split unipotent. We have the Čech-to-derived functor spectral sequence

\[
E^{i,j}_2 = H^i(k, \mathcal{H}^{j}(\hat{U})) \Rightarrow H^{i+j}(k, \hat{U})
\]

To show that $\hat{H}^2(k, \hat{U}) \hookrightarrow H^2(k, \hat{U})$ is surjective, it suffices to show that $E^{0,2}_2 = E^{1,1}_2 = 0.$
To prove that $E^{0,2}_2 = \tilde{H}^0(k, \mathcal{H}^2(\hat{U}))$ vanishes, it suffices to prove $\lim_{\rightarrow} \mathbb{H}^2(L, \hat{U}) = 0$, and this limit is identified with $\mathbb{H}^2(\overline{k}, \hat{U})$ which indeed vanishes, again because $\mathbb{H}^0(\overline{k}, \cdot)$ is an exact functor.

That $E^{1,1}_2 = \tilde{H}^1(k, \mathcal{H}^1(\hat{U}))$ vanishes reduces likewise to proving $\tilde{H}^1(\overline{k} \otimes_k \overline{k}, \hat{U}) = 0$. This latter vanishing follows from Corollary 2.2.19.

Step 4: General $G$. By Lemma 2.1.7, there is an exact sequence

$$1 \rightarrow H \rightarrow G \rightarrow U \rightarrow 1$$

with $H$ an almost torus and $U$ split unipotent. By Proposition 2.2.3, we have an exact sequence

$$1 \rightarrow \hat{U} \rightarrow \hat{G} \rightarrow \hat{H} \rightarrow 1 \quad (3.6.1)$$

We shall feed this into the following lemma.

**Lemma 3.6.4.** The sequence $(3.6.1)$ yields a long exact Čech cohomology sequence

$$\cdots \rightarrow \tilde{H}^i(k, \hat{U}) \rightarrow \tilde{H}^i(k, \hat{G}) \rightarrow \tilde{H}^i(k, \hat{H}) \rightarrow \tilde{H}^{i+1}(k, \hat{U}) \rightarrow \cdots$$

that is compatible with the derived functor long exact sequence.

**Proof.** We want to show that the sequence of Čech complexes associated to $(3.6.1)$ is short exact. This amounts to showing that for every $n > 0$, the map $\hat{G}(\overline{k} \otimes_k k^n) \rightarrow H(\overline{k} \otimes_k k^n)$ is surjective. In order to show this, it suffices to show that $\tilde{H}^1(\overline{k} \otimes_k k^n, \hat{U}) = 0$. This follows from Corollary 2.2.19. The compatibility with the derived functor sequence follows from Proposition E.2.1. □

**Lemma 3.6.5.** The map $\tilde{H}^3(k, \hat{U}) \rightarrow H^3(k, \hat{U})$ is injective.

**Proof.** We have the Čech-to-derived functor spectral sequence

$$E^{i,j}_2 = \tilde{H}^i(k, \mathcal{H}^j(\hat{U})) \implies H^{i+j}(k, \hat{U})$$

In order to show that the map $\tilde{H}^3(k, \hat{U}) = E^{0,3}_2 \rightarrow H^3(k, \hat{U})$ is injective, it suffices to show that $E^{0,2}_2 = E^{1,1}_2 = 0$.

That $E^{0,2}_2 = \tilde{H}^0(k, \mathcal{H}^2(\hat{U}))$ vanishes holds for a reason that we’ve seen several times before: It suffices to show that $\lim_{\rightarrow} \mathbb{H}^2(L, \hat{U}) = 0$ with $L$ varying through finite extensions of $k$ contained in $\overline{k}$. This limit equals $\mathbb{H}^2(\overline{k}, \hat{U})$, which vanishes because of the Nullstellensatz.

To prove that $E^{1,1}_2 = \tilde{H}^1(k, \mathcal{H}^1(\hat{U})) = 0$, it suffices to show that $H^1(\overline{k} \otimes_k \overline{k}, \hat{U}) = 0$. This in turn follows from Corollary 2.2.19. □
Lemmas 3.6.4 and 3.6.5 furnish us with a commutative diagram with exact rows

\[
\begin{array}{cccccc}
\tilde{H}^2(k, \hat{U}) & \longrightarrow & \tilde{H}^2(k, \hat{G}) & \longrightarrow & \tilde{H}^2(k, \hat{H}) & \longrightarrow & \tilde{H}^3(k, \hat{U}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^2(k, \hat{U}) & \longrightarrow & H^2(k, \hat{G}) & \longrightarrow & H^2(k, \hat{H}) & \longrightarrow & H^3(k, \hat{U})
\end{array}
\]

The first vertical arrow is an isomorphism by Step 3, and the third is by Step 2. A simple diagram chase shows that the map \( \tilde{H}^2(k, \hat{G}) \rightarrow H^2(k, \hat{G}) \) is surjective, hence an isomorphism. This completes the proof of Proposition 3.5.1. \( \square \)
Chapter 4

Local Fields

In this chapter we establish the local results stated in §1.2, particularly Theorems 1.2.1, 1.2.2, and 1.2.4. An important step in doing this is defining and studying the cohomology on the groups $H^1(k, G)$ and $\hat{H}^1(k, \hat{G})$, where $k$ is a local field, and $G$ is an affine commutative $k$-group scheme of finite type. This is done in §4.2. The proofs of Theorems 1.2.4 and 1.2.2 are intertwined, so we begin by only proving Theorem 1.2.4 for almost tori (§4.3), and then turn to proving Theorem 1.2.2 (§§4.4-4.5), before finally completing the proof of Theorem 1.2.4 (§4.6).

4.1 Duality between $H^2(k, G)$ and $\hat{G}(k)_{\pro}$

Let us first show that $H^2(k, G)$ is a torsion group.

**Lemma 4.1.1.** Let $k$ be a field, $G$ an affine commutative $k$-group scheme of finite type. Then $H^2(k, G)$ is a torsion group.

**Proof.** If $G$ is smooth then this is immediate from the fact that $H^2(k, G) = H^2_{\text{et}}(k, G)$ (Remark 2.2.20), and higher Galois cohomology is torsion. For general $G$, there exists by [SGA3 VII A, Prop. 8.3] an infinitesimal $k$-subgroup scheme $I \subset G$ such that $H := G/I$ is smooth. Since $H^2(k, I)$ is clearly torsion, and $H^2(k, H)$ is torsion because $H$ is smooth, $H^2(k, G)$ is torsion as well. \( \square \)

The continuity of the pairing in Theorem 1.2.1 is trivial, since by definition both groups $H^2(k, G)$ and $\hat{G}(k)$ are discrete. The abelian group $\hat{G}(k)$ is finitely generated, by reduction to the case of finite group schemes and tori, using Lemmas 2.1.7 and 2.1.3(ii), so $\text{Hom}_{\text{cts}}(\hat{G}(k)_{\text{pro}}, \mathbb{Q}/\mathbb{Z}) = \text{Hom}(\hat{G}(k), \mathbb{Q}/\mathbb{Z})$. Theorem 1.2.1 is therefore equivalent to the assertion that the map $H^2(k, G) \to \hat{G}(k)^*$ is an isomorphism. Indeed, it then follows that the map $\hat{G}(k)_{\text{pro}} \to H^2(k, G)^*$ is a continuous bijective homomorphism, and since the source is compact and the target is Hausdorff it must therefore be a homeomorphism as well. We
already know Theorem 1.2.1 if \( G \) is finite, thanks to Tate local duality for finite group schemes (as supplemented in [Čes2] for group schemes of \( p \)-power order in characteristic \( p > 0 \)). Before proceeding with the proof in general, we have to check it in the special case \( G = R_{k'/k}(G_m) \) with \( k'/k \) a finite separable extension.

We have a natural isomorphism \( H^2(k', G_m) \cong H^2(k, R_{k'/k}(G_m)) \), since we may take the cohomology to be étale, and finite pushforward is exact between categories of étale sheaves. In addition, by Lemma 2.2.1, the norm map \( N_{k'/k} : \hat{G}_m(k') \to R_{k'/k}(G_m)(k) \) is an isomorphism. By Proposition C.1.3, we have a commutative diagram

\[
\begin{array}{ccc}
H^2(k', G_m) & \times & \hat{G}_m(k') \\
\downarrow \cong & & \downarrow N_{k'/k} \\
H^2(k, R_{k'/k}(G_m)) & \times & R_{k'/k}(G_m)(k) \\
\end{array}
\]

Therefore, in order to prove Theorem 1.2.1 for \( G = R_{k'/k}(G_m) \) over \( k \), it suffices to prove it for \( G = G_m \) over \( k' \), for which it is trivial (using the fact that \( H^2(k', G_m)^* = \mathbb{Z} \), since \( H^2(k', G_m) \cong \mathbb{Q}/\mathbb{Z} \), since the pairing \( H^2(k', G_m) \times \hat{G}_m(k') \to H^2(k', G_m) \) is the one which sends a generator for \( \hat{G}_m(k') \), namely the identity character of \( G_m \), to the identity map \( H^2(k', G_m) \to H^2(k', G_m) \).

Now suppose that \( G \) is an almost torus. By Lemma 2.1.3(iv), after harmlessly modifying \( G \) we may assume that there is an isogeny \( B \times R_{k'/k}(T') \to G \) for some finite separable extension \( k'/k \), some split \( k' \)-torus \( T' \), and some finite commutative \( k \)-group scheme \( B \). For notational convenience, let us denote \( B \times R_{k'/k}(T') \) by \( X \), and let \( A := \ker(X \to G) \).

The exact sequence
\[
1 \to A \to X \to G \to 1
\]
yields a commutative diagram of exact sequences

\[
\begin{array}{cccc}
H^2(k, A) & \to & H^2(k, X) & \to & H^2(k, G) & \to & 0 \\
\downarrow \cong & & \downarrow \cong & & \downarrow & & \\
\hat{A}(k)^* & \to & \hat{X}(k)^* & \to & \hat{G}(k)^* & \to & 0
\end{array}
\]

The first two vertical arrows are isomorphisms because we already know Theorem 1.2.1 for \( X \) and the \( k \)-finite \( A \). The bottom row is exact because \((\cdot)^*\) is an exact functor on abelian groups, and the 0 in the top row is because \( H^3(k, A) = 0 \) by Proposition 3.1.1. The diagram shows that \( H^2(k, G) \to \hat{G}(k)^* \) is an isomorphism, hence proves Theorem 1.2.1 for \( G \).

Finally, suppose \( G \) is an arbitrary affine commutative \( k \)-group scheme of finite type. By Lemma 2.1.7 there is an exact sequence
\[
1 \to H \to G \to U \to 1
\]
with $H$ an almost torus and $U$ split unipotent. This yields a commutative diagram of exact sequences

$$
\begin{array}{cccc}
0 & \rightarrow & H^2(k, H) & \rightarrow & H^2(k, G) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \hat{H}(k)^* & \rightarrow & \hat{G}(k)^* & \rightarrow & 0
\end{array}
$$

The 0’s in the top row are because $H^i(k, U) = 0$ for $i = 1, 2$, by Lemma 2.4.4. The 0’s in the bottom row are because $\hat{U}(k) = 0$ and $H^1(k, \hat{U}) = 0$ by Lemma 2.4.4 again. The first vertical map is an isomorphism by the already-treated case of almost tori, so the other vertical map is also an isomorphism.

4.2 Topology on local cohomology

In this section we begin the proof of Theorem 1.2.4 by establishing some preliminary lemmas. The main issue taken up in this section is to endow the cohomology groups $H^1(k, G)$ and $H^1(k, \hat{G})$ with a non-obvious topology and to prove some refined properties of these topologies. In §4.3 we prove Theorem 1.2.4 in some special cases (such as almost tori). We then digress in §4.4–§4.5 to prove Theorem 1.2.2 (except for some loose ends in characteristic 0). In §4.6 we use all of this work to finally settle Theorem 1.2.4 in general.

For $H$ a $k$-group scheme locally of finite type, [Čes1] describes a topology on $H^1(k, H)$, and shows it to have various desirable properties. The topology is defined as follows. A subset $U \subset H^1(k, H)$ is open if for every locally finite type $k$-scheme $X$, and every $H$-torsor sheaf $\mathcal{X} \rightarrow X$ for the fppf topology, the set $\{ x \in X(k) \mid \mathcal{X}_x \in U \} \subset X(k)$ is open, where $X(k)$ is endowed with its usual topology inherited from that on $k$. By [Čes1] Prop. 3.6(c), Prop. 3.7(c), Prop. 3.9], for commutative $H$ this topology makes $H^1(k, H)$ into a locally compact and Hausdorff topological group, and if $G$ is smooth then it is discrete [Čes1 Prop. 3.5(a)]. In particular, if char($k$) = 0 then the topology is discrete.

For any locally finite type $k$-group scheme $G$, we endow $G(k)$ with its usual topology (inherited from that on $k$), and $H^2(k, G)$ with the discrete topology. For any short exact sequence of locally finite type commutative $k$-group schemes

$$
1 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 1,
$$

the maps in the long exact cohomology sequence up to the $H^2$ level are continuous [Čes1 Prop. 4.2], the map $H^1(k, G) \rightarrow H^1(k, G'')$ is open [Čes1, 4.3(d)]; and if $G$ is smooth then $G''(k) \rightarrow H^1(k, G')$ is also open [Čes1 4.3(b)].

**Proposition 4.2.1.** Let $G$ be an affine commutative $k$-group scheme of finite type. Then the locally compact Hausdorff group $H^1(k, G)$ is second countable and locally profinite. The same holds for $H^1(k, \hat{G})$ if $G$ is an almost torus.
Here, by "locally profinite" we mean that there is a profinite open subgroup (and hence a base of such around the identity). In the final assertion of this proposition we must limit ourselves (for now) to the case of almost tori since beyond that case \( \hat{G} \) is not representable (Proposition 2.2.7), and we therefore haven’t yet defined a topology on \( H^1(k, \hat{G}) \). The proof of Proposition 4.2.1 requires a few auxiliary lemmas.

**Lemma 4.2.2.** Let \( \kappa \) be a finite field and \( G \) an affine commutative \( \kappa \)-group scheme of finite type. Then the groups \( H^1(\kappa, G) \) and \( H^1(\kappa, \hat{G}) \) are finite.

**Proof.** First we show that \( H^1(\kappa, G) \) is finite. We are reduced by the usual arguments to the cases in which \( G \) either is finite or is smooth and connected. If \( G \) is smooth and connected, then \( H^1(\kappa, G) = 0 \) by Lang’s Theorem. We may therefore assume that \( G \) is finite, and by further filtering \( G \), we may assume that it is either infinitesimal or étale.

If \( G = I \) is infinitesimal, then \( H^1(\kappa, I) = H^1_{\text{ét}}(\kappa, I) = 0 \), the first equality holding because \( \kappa \) is perfect (Remark 2.2.20) and the second because \( I \) vanishes as an étale sheaf on \( \text{Spec}(\kappa) \). If \( G = E \) is finite étale, then by splitting \( E \) into its \( l \)-primary parts for various primes \( l \), we may assume that \( E \) is \( l \)-primary. If \( l \neq p := \text{char}(\kappa) \), let \( \kappa'/\kappa \) be a finite separable extension over which \( E \) becomes a product of various \( \mu_p \)'s. If \( l = p \), then choose \( \kappa' \) so that \( E_{\kappa'} \) is constant.

The canonical inclusion \( E \hookrightarrow \mathbb{R}_{\kappa'/\kappa}(E_{\kappa'}) \) defines an exact sequence

\[
1 \rightarrow E \rightarrow \mathbb{R}_{\kappa'/\kappa}(E_{\kappa'}) \rightarrow A \rightarrow 1
\]

with \( A \) finite. Now \( A(\kappa) \) is finite, and so we only need to show that \( H^1(\kappa, \mathbb{R}_{\kappa'/\kappa}(F')) \) is finite for a finite commutative \( \kappa' \)-group scheme \( F' \) that is split multiplicative if \( l \neq p \) and is constant if \( l = p \). But \( H^1(\kappa, \mathbb{R}_{\kappa'/\kappa}(F')) \simeq H^1(\kappa', F') \), so we may replace \( \kappa \) by \( \kappa' \).

Hence, we want to show that \( H^1(\kappa, F) \) is finite if \( F = \mu_l \) or if \( F \) is \( p \)-primary constant. Filtering \( F \) further, we may assume that \( F = \mu_l \) or \( \mathbb{Z}/p\mathbb{Z} \). But \( H^1(\kappa, \mu_l) \simeq \kappa^x/(\kappa^x)^l \) and \( H^1(\kappa, \mathbb{Z}/p\mathbb{Z}) \simeq \kappa/\varphi(\kappa) \), where \( \varphi : \kappa \rightarrow \kappa \) is the Artin-Schreier map \( x \mapsto x^p - x \), so the first assertion is proved.

Now let us prove the second assertion, i.e., that \( H^1(\kappa, \hat{G}) \) is finite. Lemma 2.1.7 and Proposition 2.4.2 reduce us to the case in which \( G \) is an almost torus. Then Lemma 2.1.3 allows us to assume that we have an exact sequence

\[
1 \rightarrow A \rightarrow B \times \mathbb{R}_{\kappa'/\kappa}(T') \rightarrow G \rightarrow 1
\]

with \( A, B \) finite, \( \kappa'/\kappa \) a finite (automatically separable) extension, and \( T' \) a split \( \kappa' \)-torus. Since \( \hat{A}(\kappa) \) is finite, we are reduced to showing that \( H^1(\kappa, \hat{B}) \) and \( H^1(\kappa, \mathbb{R}_{\kappa'/\kappa}(G_{\mathbb{m}})) \) are finite. The first group is finite by the already-proved first assertion of the lemma. For the second, we note that \( \mathbb{R}_{\kappa'/\kappa}(G_{\mathbb{m}}) \simeq \mathbb{R}_{\kappa'/\kappa}(G_{\mathbb{m}}) \simeq \mathbb{R}_{\kappa'/\kappa}(\mathbb{Z}) \) by Lemma 2.2.1. Therefore, \( H^1(\kappa, \mathbb{R}_{\kappa'/\kappa}(G_{\mathbb{m}})) \simeq H^1(\kappa, \mathbb{R}_{\kappa'/\kappa}(\mathbb{Z})) \simeq H^1(\kappa', \mathbb{Z}) = 0 \). This completes the proof. \( \square \)
Following [Ces1], it will be useful to topologize the group \( H^1(\mathcal{O}_k, \mathcal{G}) \) for \( \mathcal{G} \) a commutative locally finite type \( \mathcal{O}_k \)-group scheme, and we do so in a manner analogous to how we topologized \( H^1(k, G) \): declare a subset \( U \subset H^1(\mathcal{O}_k, \mathcal{G}) \) to be open if for every locally finite type \( \mathcal{O}_k \)-scheme \( X \), and every \( \mathcal{G} \)-torsor sheaf \( \mathcal{F} \to X \) for the fppf topology, the set \( \{ x \in X(\mathcal{O}_k) \mid \mathcal{F}_x \subset U \} \subset X(\mathcal{O}_k) \) is open.

**Lemma 4.2.3.** Let \( G \) be an affine commutative finite type \( k \)-group scheme. Then \( H^1(k, G) \) contains a second countable profinite open subgroup. The same holds for \( H^1(k, \hat{G}) \) if \( G \) is an almost torus.

**Proof.** Choose an affine flat finite type \( \mathcal{O}_k \)-group scheme \( \mathcal{G} \) with \( k \)-fiber \( G \) (e.g., schematic closure in \( \text{GL}_n, \mathcal{O}_k \) relative to some \( k \)-subgroup inclusion \( G \hookrightarrow \text{GL}_n, \mathcal{O}_k \)). By [Ces1] Cor. B.7 and the first assertion in Lemma 4.2.2 there is an affine \( \mathcal{O}_k \)-scheme \( X \) and a smooth morphism \( X \to B\mathcal{G} \) such that the induced map \( X(\mathcal{O}_k) \to (B\mathcal{G})(\mathcal{O}_k)/\text{isom} \) is surjective. The map \( X(\mathcal{O}_k) \to (B\mathcal{G})(\mathcal{O}_k)/\text{isom} \) is continuous by definition, and it is open by [Ces1] Prop. 2.9(a)].

Since \( X(\mathcal{O}_k) \) is second countable and has a base of quasi-compact open neighborhoods of every point, it follows that \( (B\mathcal{G})(\mathcal{O}_k)/\text{isom} \) has the same properties. By [Ces1] Prop. 2.9(e)], the map \( (B\mathcal{G})(\mathcal{O}_k) \to (BG)(k)/\text{isom} = H^1(k, G) \) is open, hence its image is a second countable open subgroup of the locally compact Hausdorff abelian group \( H^1(k, G) \) in which every point has a base of compact open neighborhoods. By the Hausdorff condition, such compact open neighborhoods are also closed, and hence \( H^1(k, G) \) is totally disconnected. Thus, any compact open subgroup must be profinite (as profinite groups are exactly the Hausdorff topological groups that are compact and totally disconnected). This proves the first assertion of the lemma.

The second follows by the same argument, using the second assertion of Lemma 4.2.2.

**Proof of Proposition 4.2.1.** By Lemma 4.2.3 to prove the first assertion in Proposition 4.2.1 it suffices to show that for some (equivalently, any) open subgroup \( U \subset H^1(k, G) \), the quotient \( H^1(k, G)/U \) is countable. We will first show that if we have a short exact sequence

\[
1 \rightarrow G' \rightarrow \overset{j}{G} \rightarrow \overset{\pi}{G''} \rightarrow 1
\]

of affine commutative \( k \)-group schemes of finite type and the proposition holds for \( G' \) and \( G'' \) then it also holds for \( G \).

Let \( U \subset H^1(k, G) \) be an open subgroup. The map \( \pi : H^1(k, G) \to H^1(k, G'') \) is open, so \( H^1(k, G'')/\pi(U) \) is countable. The same holds for \( H^1(k, G')/j^{-1}(U) \). The exact sequence

\[
\begin{array}{c}
H^1(k, G') \rightarrow H^1(k, G) \rightarrow H^1(k, G'') \\
j^{-1}(U)
\end{array}
\]

then shows that \( H^1(k, G)/U \) is countable, as desired. The usual arguments therefore reduce us to the cases in which \( G \) is either \( G_a \), a torus, or finite.
The case $G = \mathbb{G}_a$ is trivial, since $H^1(k, \mathbb{G}_a) = 0$. If $G = T$ is a torus, then $H^1(k, T)$ is even finite. Indeed, in the usual manner, Lemma 2.1.3 allows us to assume that there is an exact sequence

$$1 \longrightarrow A \longrightarrow R_{k'/k}(T') \longrightarrow T \longrightarrow 1$$

for some finite separable extension $k'/k$ and some split $k'$-torus $T'$. But $H^1(k, R_{k'/k}(T')) \simeq H^1(k', T') = 0$, so it suffices to show that $H^2(k, A)$ is finite. By Tate local duality for finite commutative group schemes, this is dual to the finite group $\hat{A}(k)$, hence finite.

So we are reduced to the case in which $G$ is finite. Filtering $G$ further, we may assume that $G$ is either étale, multiplicative, or local-local. If $G = E$ is finite étale, then we want to show that $H^1(k, E)$ is countable. Choose a finite separable extension $k'/k$ splitting $E$. Then via the canonical inclusion $E \hookrightarrow R_{k'/k}(E_{k'})$, we obtain an exact sequence

$$1 \longrightarrow E \longrightarrow R_{k'/k}(E_{k'}) \longrightarrow A \longrightarrow 1$$

with $A$ finite. Then $A(k)$ is finite, so it suffices to show that $H^1(k, R_{k'/k}(F'))$ is countable for a finite constant $k'$-group $F'$. But $H^1(k, R_{k'/k}(F')) \simeq H^1(k', F')$, so by renaming $k'$ as $k$ and further filtering $F'$ it is enough to show that $H^1(k, \mathbb{Z}/l\mathbb{Z})$ is countable for prime $l$.

If $l \neq \text{char}(k)$ then $H^1(k, \mathbb{Z}/l\mathbb{Z})$ is even finite since $k$ has (up to isomorphism) only finitely many extensions of a given degree not divisible by $\text{char}(k)$. We may now assume (for the rest of the treatment of finite $G$) that $p = \text{char}(k) > 0$. Then $H^1(k, \mathbb{Z}/p\mathbb{Z}) \simeq k/\wp(k)$, where $\wp : k \to k$ is the Artin-Schreier map $x \mapsto x^p - x$; the quotient $k/\wp(k)$ is second countable and discrete (since $\wp$ is a smooth map, it is open), hence countable. If $G$ is multiplicative, then by local duality for finite group schemes the group $H^1(k, G)$ is the Pontryagin dual of a discrete countable group, hence is second countable.

Finally, suppose that $G = I$ is local-local. Filtering $I$ further, we may assume its relative Frobenius and Verschiebung morphisms vanish. It follows that $I \simeq \alpha_p^n$ for some $n$. Indeed, this holds over $\overline{k}$, and since $\text{Aut}_{\alpha_p^n/k} = \text{GL}_n$ and $H^1(k, \text{GL}_n) = 1$ it follows that $\alpha_p^n$ has no nontrivial $k$-forms for the fppf topology. We may therefore assume that $I = \alpha_p$. But then we have the exact sequence

$$1 \longrightarrow \alpha_p \longrightarrow \mathbb{G}_a \overset{F}{\longrightarrow} \mathbb{G}_a \longrightarrow 1$$

with $F$ the Frobenius $k$-isogeny, defined on $k$-algebras by $X \mapsto X^p$. Since $\mathbb{G}_a$ is smooth, the map $\mathbb{G}_a(k) \to H^1(k, \alpha_p)$ is open, hence we have a topological isomorphism

$$H^1(k, \alpha_p) \simeq k/k^p \quad (4.2.1)$$

and this latter group is second countable. This completes the proof that $H^1(k, G)$ is second countable for any affine commutative $k$-group scheme of finite type.

Now suppose that $G$ is an almost torus. It remains to show that $H^1(k, \hat{G})$ is second countable. A similar argument as the one given near the start of this proof shows that if we have an exact sequence

$$1 \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow 1$$
of almost tori, and if \( H^1(k, \mathcal{G}') \) and \( H^1(k, \mathcal{G}'') \) are second countable, then so is \( H^1(k, \mathcal{G}) \). We may therefore assume that \( G \) is either finite or a torus.

If \( G \) is finite, then the result follows by the already-proved first assertion of the proposition. If \( G = T \) is a torus, then \( H^1(k, T) \) is in fact finite. Indeed, as usual, Lemma 2.1.3 allows us to assume that there is an exact sequence

\[
1 \rightarrow A \rightarrow R_{k'/k}(T') \rightarrow T \rightarrow 1
\]

with \( A \) finite, \( k'/k \) a finite separable extension, and \( T' \) a split \( k' \)-torus. Then \( \mathcal{A}(k) \) is finite, and as we’ve seen several times already, \( H^1(k, R_{k'/k}(T')) \approx H^1(k', \mathbb{Z}^n) = 0 \).

The last general fact we will need concerning the topology on \( H^1(k, G) \) for a commutative \( k \)-group scheme \( G \) locally of finite type is that it is compatible with the so-called Čech topology, defined as follows. For any finite extension \( L/k \), both \( L \) and \( L \otimes_k L \) come equipped with natural topologies as \( k \)-algebras and therefore so do \( G(L) \) and \( G(L \otimes_k L) \) as topological groups. We endow the Čech cohomology group \( \check{H}^1(L/k, G) \) with the subquotient topology, and the Čech cohomology group \( H^1(k, G) = \lim_{\to} H^1(L/k, G) \) with the direct limit topology.

Via the natural isomorphism \( H^1(k, G) \approx H^1(k, G) \approx H^1(k, \mathbb{Z}^n) \), this Čech topology is identified with the topology on \( H^1(k, G) \) given above [Čes1, Thm. 5.11].

This topology on \( H^1(k, G) \) for commutative \( k \)-group schemes \( G \) locally of finite type applies in particular to the dual of an almost torus (see Proposition 2.2.7). We shall later require a topology on \( H^1(k, \mathcal{G}) \) for arbitrary affine commutative \( k \)-group schemes \( G \) of finite type (not just for \( G \) an almost torus), a topic we address in \( \S 4.6 \).

## 4.3 Special cases of duality between \( H^1(k, G) \) and \( H^1(k, \mathcal{G}) \)

We first settle Theorem 1.2.4 in the special cases \( G = \mathbb{G}_a \) and \( G = R_{k'/k}(\mathbb{G}_m) \) for some finite separable extension \( k'/k \) by showing that all of the relevant cohomology groups vanish. In the former case, we know \( H^1(k, \mathbb{G}_a) = 0 \) and we have \( H^1(k, \mathcal{G}_a) = 0 \) by Proposition 2.4.2. In the case \( G = R_{k'/k}(\mathbb{G}_m) \), we have \( H^1(k, R_{k'/k}(\mathbb{G}_m)) \approx H^1(k', \mathbb{G}_m) = 0 \). On the other hand, \( R_{k'/k}(\mathbb{G}_m) = R_{k'/k}(\mathbb{G}_m) = R_{k'/k}(\mathbb{Z}) \) by Lemma 2.2.1 and \( H^1(k, R_{k'/k}(\mathbb{Z})) \approx H^1(k', \mathbb{Z}) = 0 \).

Next, we show that the relevant cohomology groups always have finite exponent:

**Lemma 4.3.1.** Let \( k \) be a field, \( G \) an affine commutative \( k \)-group scheme of finite type. Then \( H^1(k, G) \) and \( H^1(k, \mathcal{G}) \) have finite exponent.

**Proof.** If \( G = \mathbb{G}_a \) or \( R_{k'/k}(\mathbb{G}_m) \) for some finite separable extension \( k'/k \), then we have already shown that these cohomology groups vanish. Now suppose that \( G \) is an almost torus. By Lemma 2.1.3(iv), we may assume that there is an exact sequence

\[
1 \rightarrow B \rightarrow A \times R_{k'/k}(T') \rightarrow G \rightarrow 1
\]
for some finite commutative $k$-group schemes $A, B$, some finite separable extension $k'/k$, and some split $k'$-torus $T'$. Since the cohomology groups of $A, B$, and their duals are clearly of finite exponent, and both $H^1(k, R_{k'/k}(T'))$ and $H^1(k, \hat{R}_{k'/k}(T'))$ vanish, we see that $H^1(k, G)$ and $H^1(k, \hat{G})$ are of finite exponent.

For general $G$, by Lemma 2.1.7 there is an exact sequence
\[ 1 \to H \to G \to U \to 1 \]
with $H$ an almost torus and $U$ split unipotent. Now $H^1(k, U)$ and $H^1(k, \hat{U})$ vanish, the latter due to Proposition 2.4.2. Since we have already shown that $H^1(k, H)$ and $H^1(k, \hat{H})$ are of finite exponent, the same holds for $H^1(k, G)$ and $H^1(k, \hat{G})$.

It follows from Lemma 4.3.1 that the Pontryagin dual of both cohomology groups in the Lemma coincides with its group of continuous homomorphisms into $\mathbb{Q}/\mathbb{Z}$ viewed discretely. Likewise, the pairing
\[ H^1(k, G) \times H^1(k, \hat{G}) \to H^2(k, G_m) \]  
lands in $(1/N)\mathbb{Z}/\mathbb{Z}$ for some integer $N \geq 1$. In particular, continuity of the pairing is the same whether we view $\mathbb{Q}/\mathbb{Z}$ discretely or with the subspace topology from $\mathbb{R}/\mathbb{Z}$.

The rest of this section is devoted to proving:

**Lemma 4.3.2.** Theorem 1.2.4 holds for almost tori.

Let $G$ be an almost torus. First, we prove continuity of the above cup product pairing. This amounts to proving two statements:

(i) There are neighborhoods $U_G \subset H^1(k, G)$, $U_{\hat{G}} \subset H^1(k, \hat{G})$ of $0$ such that $\langle U_G, U_{\hat{G}} \rangle = \{0\}$.

(ii) For any $\alpha \in H^1(k, G)$, there exists a neighborhood $U \subset H^1(k, \hat{G})$ of $0$ such that $\langle \alpha, U \rangle = \{0\}$, and the same holds if we switch the roles of $G$ and $\hat{G}$.

In order to prove (i) and (ii), we use the fact that the topologies on $H^1(k, G), H^1(k, \hat{G})$ agree with the Čech topologies (as reviewed near the end of §4.2). Both assertions therefore follow easily from the continuity of the map $G(L \otimes_k L) \times G(L \otimes_k L) \to (L \otimes_k L \otimes_k L)^\times$ (using the explicit formula defining this cup product) combined with the openness of the subset $(O_L \otimes_{O_k} O_L \otimes_{O_k} O_L)^\times \subset (L \otimes_k L \otimes_k L)^\times$ and the fact that $\text{Br}(O_k) = 0$.

This continuity result provides a map $H^1(k, G) \to H^1(k, \hat{G})^D$, and we want to show that this map is a topological isomorphism. By Proposition 4.2.1 and Lemma 4.3.3 below, it suffices to prove bijectivity (i.e., this map is an isomorphism of groups, disregarding their topologies).
Lemma 4.3.3. Let $H_1, H_2$ be locally compact topological groups, with $H_1$ second countable and $H_2$ Hausdorff. Any continuous surjective homomorphism $f : H_1 \to H_2$ is open. In particular, if $f$ is also bijective then it is a homeomorphism.

Proof. This is [Bou, Ch. IX, §5, Prop. 6].

As usual, we may assume by Lemma 2.1.3 that there is an exact sequence

$$1 \rightarrow A \xrightarrow{j} B \times R_{k'/k}(T') \xrightarrow{\pi} G \rightarrow 1$$

for some finite commutative $k$-group schemes $A, B$, some finite separable extension $k'/k$, and some split $k'$-torus $T'$. For notational simplicity, let us denote $B \times R_{k'/k}(T')$ by $X$. Using Proposition 2.2.3 and the $\delta$-functoriality of cup products (for fppf abelian sheaves), we obtain a commutative diagram

$$
\begin{array}{cccccc}
H^1(k, A) & \rightarrow & H^1(k, X) & \rightarrow & H^1(k, G) & \rightarrow & H^2(k, A) & \rightarrow & H^2(k, X) \\
\downarrow \gamma & & \downarrow \gamma & & \downarrow \gamma & & \downarrow \gamma & & \downarrow \gamma \\
H^1(k, \widehat{A})^D & \rightarrow & H^1(k, \widehat{X})^D & \rightarrow & H^1(k, \widehat{G})^D & \rightarrow & \widehat{A}(k)^D & \rightarrow & \text{Hom}(\widehat{X}(k), \mathbb{Q}/\mathbb{Z})
\end{array}
$$

We claim the bottom row is exact (as the top row certainly is) and that the indicated vertical maps are isomorphisms of groups. Granting these properties of the diagram, a simple diagram chase then shows that the map $H^1(k, G) \rightarrow H^1(k, \widehat{G})^D$ is an isomorphism of groups, which is all we need.

The first and fourth vertical arrows are isomorphisms by local duality for finite commutative group schemes. The second vertical arrow is an isomorphism by local duality for finite commutative group schemes and the already-treated case of separable Weil restrictions of split tori. The fifth vertical arrow is an isomorphism by Theorem 1.2.1.

To see that the bottom row is exact, we first note that all of the groups except for $\widehat{X}(k)$ are torsion, so the other $\mathbb{Q}/\mathbb{Z}$-dual groups coincide with Pontryagin duals. Since exactness at $\widehat{A}(k)^D$ is unaffected by replacing $\text{Hom}(\widehat{X}(k), \mathbb{Q}/\mathbb{Z})$ with $\widehat{X}(k)^D$, and the map at the left end of the bottom row arises from applying Pontryagin duality to an exact sequence of continuous maps and the sequence before dualizing extends to an exact sequence of continuous maps

$$H^1(k, \widehat{X}) \rightarrow H^1(k, \widehat{A}) \rightarrow H^2(k, \widehat{G})$$

exactness of the bottom row is reduced by Proposition 4.2.1 to showing that if

$$M' \xrightarrow{f} M \xrightarrow{g} M'' \xrightarrow{h} M'''$$

is an algebraically exact sequence of continuous maps between locally compact second-countable Hausdorff topological abelian groups then the diagram of Pontryagin duals

$$M'''^D \rightarrow M^D \rightarrow M''^D$$

is exact.
is algebraically exact. For this, it suffices to check that the map $M/f(M') \hookrightarrow M''$ is a homeomorphism onto a closed subgroup. By Lemma 4.3.3, it suffices to check that the image $g(M) \subset M''$ is closed, and this is clear, since $\text{im}(g) = \ker(h)$, which is closed because $M''$ is Hausdorff. This completes the proof of Theorem 1.2.4 for almost tori. Before we can finish the proof in general, to push through some exact sequence arguments we need to prove Theorem 1.2.2, so we now turn to that task.

### 4.4 Preliminaries on the pairing between $H^2(k, \hat{G})$ and $G(k)$

As the first step, we show $H^2(k, \hat{G}) \times G(k) \to \mathbb{Q}/\mathbb{Z}$ is continuous where we recall that the $H^2$-term is discrete by definition and $G(k)$ is given its usual locally profinite topology.

It suffices to show that if $\alpha \in H^2(k, \hat{G})$, then there is an open set $U \subset G(k)$ containing $0 \in G(k)$ such that for all $g \in U$, we have $g(\alpha) = 0$. By Proposition 3.5.1, $\alpha$ is represented by a cocycle $\hat{\alpha} \in \hat{G}(L \otimes_k L \otimes_k L)$ for some finite extension $L/k$. Clearly the natural pairing $G(k) \times \hat{G}(L \otimes_k L \otimes_k L) \to (L \otimes_k L \otimes_k L)^\times$ is continuous. Since $(\mathcal{O}_L \otimes_{\mathcal{O}_k} \mathcal{O}_L \otimes_{\mathcal{O}_k} \mathcal{O}_L)^\times \subset (L \otimes_k L \otimes_k L)^\times$ is open, it follows that there is a neighborhood $U$ of $0 \in G(k)$ such that $g(\hat{\alpha}) \in \mathbb{Z}^2(\mathcal{O}_L/\mathcal{O}_k, \mathbb{G}_m)$ for all $g \in U$. Therefore $g(\alpha) \in \text{Br}(\mathcal{O}_k) = 0$ for all $g \in U$, so this $U$ does the job.

**Lemma 4.4.1.** Let $k$ be a field, $G$ an affine commutative $k$-group scheme of finite type. Then $H^2(k, \hat{G})$ is a torsion group.

**Proof.** Suppose first that $G$ is an almost torus. By Lemma 2.1.1, there is a finite $k$-subgroup scheme $A \subset G$ such that $T := G/A$ is a torus. Proposition 2.2.3 yields an exact sequence

$$H^2(k, \hat{T}) \to H^2(k, \hat{G}) \to H^2(k, \hat{A})$$

Since $\hat{T}$ is representable by a smooth group scheme, $H^2(k, \hat{T}) = H^2_{\text{ét}}(k, \hat{T})$ is a torsion group since higher Galois cohomology is torsion. Further, $H^2(k, \hat{A})$ is clearly torsion, so $H^2(k, \hat{G})$ is torsion as well.

Next suppose that $G = U$ is split connected unipotent. If $\text{char}(k) = 0$ then $H^2(k, \hat{U}) = H^2_{\text{ét}}(k, \hat{U})$ by Remark 2.2.20, and this vanishes since $\hat{U}$ is isomorphic as an étale sheaf. If $\text{char}(k) = p > 0$ then $U$ has finite exponent, so $\hat{U}$ does as well, and hence the lemma is clear.

Finally, in the general case, by Lemma 2.1.7 there is an exact sequence

$$1 \to H \to G \to U \to 1$$

with $H$ an almost torus and $U$ split unipotent. We therefore have an exact sequence

$$H^2(k, \hat{U}) \to H^2(k, \hat{G}) \to H^2(k, \hat{H})$$

Since $H^2(k, \hat{U})$ is torsion and $H^2(k, \hat{H})$ is torsion, so is $H^2(k, \hat{G})$. \qed
It follows that we have an equality \( H^2(k, \hat{G}) = \text{Hom}_{cts}(H^2(k, \hat{G}), Q/Z) \) as profinite groups, so the continuous pairing \( G(k) \times H^2(k, \hat{G}) \to Q/Z \) yields a continuous pairing \( G(k)_{\text{pro}} \times H^2(k, \hat{G}) \to Q/Z \) of locally compact Hausdorff abelian groups, and if the induced continuous homomorphism \( G(k)_{\text{pro}} \to H^2(k, \hat{G}) \) is an algebraic isomorphism then it is automatically a homeomorphism since the source is compact and the target Hausdorff. Hence, the desired perfectness is largely an algebraic problem (but we do need to pay attention to the topological aspects, in view of the somewhat abstract intervention of the profinite completion of \( G(k) \)).

**Proposition 4.4.2.** Let \( k \) be a local field of characteristic \( p > 0 \). For any affine commutative \( k \)-group scheme \( G \) of finite type, the continuous restriction map \( (G(k)_{\text{pro}})^D \to \text{Hom}_{cts}(G(k), Q/Z) \) is an (algebraic, not necessarily topological) isomorphism.

**Remark 4.4.3.** Proposition 4.4.2 is false in characteristic 0. Indeed, if \( G = G_a \) then \( k_{\text{pro}} = 0 \) (because \( k \) is divisible), yet \( \text{Hom}_{cts}(k, Q/Z) \neq 0 \), as may already be seen in the case \( k = Q_p \) (for which we have the canonical continuous isomorphism between \( Q_p/Z_p \) and the \( p \)-primary part of \( Q/Z \)). Proposition 4.4.2 does remain true in characteristic 0, however, if we assume that \( G \) is an almost torus, as the proof given below goes through in that case.

Turning to the proof of Proposition 4.4.2, \( G(k)_{\text{pro}}^D \to \text{Hom}_{cts}(G(k), Q/Z) \) is clearly injective. To see that it is surjective, it suffices to show that any continuous homomorphism \( \phi : G(k) \to Q/Z \) (where \( Q/Z \) is given the subspace topology from \( R/Z \)) factors through \( (1/n)Z/Z \) for some positive integer \( n \). Indeed, then \( \ker \phi \) is an open subgroup of finite index and hence \( \phi \) factors (continuously) through \( G(k)_{\text{pro}} \).

We first reduce to the case in which \( G \) is an almost torus. Given an arbitrary affine commutative \( k \)-group scheme \( G \) of finite type, we have by Lemma 2.1.7 an exact sequence

\[ 1 \to H \to G \to U \to 1 \]

with \( H \) an almost torus and \( U \) split unipotent. Let \( \phi : G(k) \to Q/Z \) be a continuous homomorphism. If \( \phi|_{H(k)} \) factors through some \( (1/m)Z/Z \) then since \( G(k)/H(k) \to U(k) \) has finite exponent (as \( U \) does, since \( \text{char}(k) > 0 \); this is the only place in the proof of Proposition 4.4.2 where we use avoidance of characteristic 0), it follows that \( \phi \) factors through \( (1/n)Z/Z \) for some positive multiple \( n \) of \( m \). So we may indeed assume \( G \) is an almost torus. We need the following two lemmas, the first of which is very well-known.

**Lemma 4.4.4.** Let \( k \) be a non-archimedean local field, and \( G \) an affine \( k \)-group scheme of finite type. Then \( G(k) \) if locally profinite, and in particular has a fundamental system of open neighborhoods of the identity consisting of profinite subgroups.

**Proof.** A closed \( k \)-subgroup inclusion \( G \hookrightarrow \text{GL}_n \) reduces us to the case \( G = \text{GL}_n \) that is obvious.

**Lemma 4.4.5.** Let \( k \) be a non-archimedean local field, \( G \) an almost torus over \( k \). For any open subgroup \( U \subset G(k) \), the quotient \( G(k)/U \) is finitely generated.
Granting Lemma 4.4.5, it follows immediately with the aid of Lemma 4.4.4 and the “no small subgroups” argument that for any almost torus $G$, any continuous homomorphism $\phi : G(k) \to \mathbb{Q}/\mathbb{Z}$ (where $\mathbb{Q}/\mathbb{Z}$ is viewed with its subspace topology from $\mathbb{R}/\mathbb{Z}$) factors through some $(1/n)\mathbb{Z}/\mathbb{Z}$. This completes the proof of Proposition 4.4.2 in general, conditional on Lemma 4.4.5 which we now establish:

**Proof of Lemma 4.4.5.** First, by replacing $G$ with its maximal smooth $k$-subgroup scheme (see [CGP, Lemma C.4.1]), we may assume that $G$ is smooth. Suppose that we have a short exact sequence $1 \to G' \xrightarrow{j} G \xrightarrow{\pi} G'' \to 1$ of smooth almost tori over $k$, and that the lemma holds for $G'$ and $G''$. The map $G \to G''$ is smooth, so $G(k) \to G''(k)$ is open. Hence, $\pi(U)$ is an open subgroup of $G''(k)$, so the exact sequence

$$
\frac{G'(k)}{j^{-1}(U)} \to \frac{G(k)}{U} \to \frac{G''(k)}{\pi(U)}
$$

in which both ends are finitely generated by hypothesis shows that $G(k)/U$ is finitely generated as well.

Since $G/G^0$ is finite étale, and the lemma is clear for finite group schemes, we may therefore assume that $G$ is smooth and connected; i.e., $G$ is a torus. If we have a closed $k$-subgroup inclusion $G \hookrightarrow H$ and if the lemma holds for $H$, then it also holds for $G$. Indeed, this follows by the same method as at the start of the proof since $G(k) \to H(k)$ is a homeomorphism onto its closed image. Since every torus embeds into the finite separable Weil restriction of a split torus, we may assume that $G = \mathbb{R}_{k'/k}(\mathbb{G}_m^n)$ for some finite separable extension $k'/k$ and some $n \geq 1$. The lemma is easy in this case. 

### 4.5 Duality between $H^2(k, \hat{G})$ and $G(k)_{\text{pro}}$

For any affine commutative $k$-group scheme $G$ of finite type, the continuous map of profinite groups $G(k)_{\text{pro}} \to H^2(k, \hat{G})^D$ is an isomorphism (algebraically, or equivalently topologically) if and only if the dual map $H^2(k, \hat{G}) \to (G(k)_{\text{pro}})^D$ (between discrete groups) is bijective. Since $(G(k)_{\text{pro}})^D \cong \text{Hom}_{\text{cts}}(G(k), \mathbb{Q}/\mathbb{Z})$ as groups (not necessarily as topological groups) by Proposition 4.4.2, we see that Theorem 1.2.2 is equivalent to bijectivity of the natural map $H^2(k, \hat{G}) \to \text{Hom}_{\text{cts}}(G(k), \mathbb{Q}/\mathbb{Z})$. This latter reformulation will be used below without comment.

We shall build up to the proof of Theorem 1.2.2 in a series of special cases.

**Lemma 4.5.1.** Let $k'/k$ be a finite separable extension. Then Theorem 1.2.2 holds for $G = \mathbb{R}_{k'/k}(\mathbb{G}_m)$. 

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We have by Proposition C.1.3 a commutative diagram of continuous pairings
\[
\begin{array}{ccc}
H^2(k, \widehat{R_{k'/k}}(\mathbb{G}_m)) \times \ R_{k'/k}(\mathbb{G}_m)(k) & \longrightarrow & \mathbb{Q}/\mathbb{Z} \\
\downarrow & & \downarrow \\
H^2(k', \widehat{\mathbb{G}_m}) \times \ \mathbb{G}_m(k') & \longrightarrow & \mathbb{Q}/\mathbb{Z}
\end{array}
\]

To prove Lemma 4.5.1, it thus suffices to treat the case \( G = \mathbb{G}_m \). We have \( H^2(k, \widehat{\mathbb{G}_m}) = H^2(k, \mathbb{Z}) \). Since \( \mathbb{Z} \) is smooth, \( H^2(k, \mathbb{Z}) = H^2_{\text{ét}}(k, \mathbb{Z}) \). Therefore, thanks to the exact sequence of étale sheaves
\[
0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0
\]
and the fact that \( \mathbb{Q} \) is uniquely divisible, we have a canonical isomorphism
\[
H^2(k, \mathbb{Z}) \simeq H^1_{\text{ét}}(k, \mathbb{Q}/\mathbb{Z}) = (\mathfrak{g}_k^{ab})^D
\]
onto the Pontryagin dual of the profinite topological abelianization of the absolute Galois group \( \mathfrak{g}_k \) of \( k \). (This is a topological isomorphism, because the compactness of \( \mathfrak{g}_k^{ab} \) implies that its dual is discrete.)

We obtain a continuous map \( \phi : \mathbb{G}_m(k) = k^\times \to H^2(k, \widehat{\mathbb{G}_m})^D = (\mathfrak{g}_k^{ab})^{DD} \simeq \mathfrak{g}_k^{ab} \) due to Pontryagin double duality. But there is another natural map \( k^\times \to \mathfrak{g}_k^{ab} \), namely the local reciprocity map, and it is natural to ask whether these two maps agree (at least up to a sign). If they do, then by local class field theory \( \phi \) induces a topological isomorphism \( (k^\times)_{\text{pro}} \simeq \mathfrak{g}_k^{ab} \), which is what we want to show. The following lemma therefore completes the proof of Lemma 4.5.1.

**Lemma 4.5.2.** The map \( \phi : k^\times \to \mathfrak{g}_k^{ab} \) above is the local reciprocity map.

**Proof.** Let \( L/k \) be a finite Galois extension with Galois group \( \Gamma \). The local reciprocity map induces an isomorphism \( \text{rec}_{L/k} : k^\times/N_L(k^\times) \xrightarrow{\sim} \Gamma^{ab} \); the inverse \( \text{rec}_{L/k}^{-1} \) of this map is the more natural one to define. In order to show that \( \phi \) is the local reciprocity map, therefore, we will show that \( \phi \circ \text{rec}_{L/k}^{-1} : \Gamma^{ab} \to \Gamma^{ab} \) is the identity map. Of course, a priori this doesn’t make sense unless we show that \( \phi(N_L(k^\times)) = 0 \) in \( \Gamma^{ab} \), but our argument will show that the composition is the identity regardless of which lift to \( k^\times \) of \( \text{rec}_{L/k}^{-1}(\sigma) \) we take for any \( \sigma \in \Gamma \), and it will then follow that \( \phi \) factors through a map \( k^\times/N_L(k^\times) \to \Gamma^{ab} \).

Let us recall how \( \text{rec}_{L/k}^{-1} \) is defined; cf. [CF, Ch.VI, §2.2]. Let \( n = [L : k] \). We have an isomorphism \( H^2(\Gamma, L^\times) \simeq (1/n)\mathbb{Z}/\mathbb{Z} \) by taking invariants. Let \( u \in H^2(\Gamma, L^\times) \) be the element with invariant \( 1/n \). Letting \( \widehat{H}^\bullet \) denote Tate cohomology groups, \( \widehat{H}^{-2}(\Gamma, \mathbb{Z}) := H_1(\Gamma, \mathbb{Z}) \simeq \Gamma^{ab} \) and by definition \( \text{rec}_{L/k}^{-1} \) is the composition
\[
\Gamma^{ab} = \widehat{H}^{-2}(\Gamma, \mathbb{Z}) \xrightarrow{\cup u} \widehat{H}^0(\Gamma, L^\times) = k^\times/N_L(k^\times).
\]

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Choose $\sigma \in \Gamma^{ab}$. We want to check that $\phi \circ \rec_{L/k}^{-1}(\sigma) = \sigma$. Let $\delta : \Gamma^D = \hat{H}^1(\Gamma, \mathbb{Q}/\mathbb{Z}) \rightarrow \hat{H}^2(\Gamma, \mathbb{Z})$ be the connecting map in the long exact sequence obtained from the exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$$

Then for any $c \in k^\times$ with image $[c] \in \hat{H}^0(\Gamma, L^\times)$, $\phi(c)$ is by definition the element $\tau \in \Gamma^{ab}$ such that $\inv([c] \cup \delta \beta) = \beta(\tau)$ for all $\beta \in \hat{H}^1(\Gamma, \mathbb{Q}/\mathbb{Z})$; note that $[c]$ has even degree, so $[c] \cup \delta \beta = \delta \beta \cup [c]$. So what we need to check is that $\inv(\sigma \cup u \cup \delta \beta) = \beta(\sigma)$.

If we once again think of $\sigma$ as an element of $\hat{H}^2(\Gamma, \mathbb{Z})$, then by the definition of the cup product, $\beta(\sigma) = \sigma \cup \beta \in \hat{H}^1(\Gamma, \mathbb{Q}/\mathbb{Z}) \simeq \ker(N_\Gamma : \mathbb{Q}/\mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z}) = (1/n)\mathbb{Z}/\mathbb{Z}$, where for any $\Gamma$-module $A$ the map $N_\Gamma : A \rightarrow A$ is $a \mapsto \sum_{\tau \in \Gamma} \tau a$. We need to show, therefore, that

$$\inv(\sigma \cup u \cup \delta \beta) = \sigma \cup \beta \quad (4.5.1)$$

Now $\sigma \cup u \cup \delta \beta = (\sigma \cup \delta \beta) \cup u$ since $u$ has even degree, and this in turn is equal to $\delta(\sigma \cup \beta) \cup u$ by the $\delta$-functoriality of cup products (see [CF, Ch.IV, §7, Thm.4(iv)] with $p = -2$). Since $\delta(\sigma \cup \beta) \in \hat{H}^0(\Gamma, \mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$, the left side of (4.5.1) equals $\delta(\sigma \cup \beta)\inv(u) = \delta(\sigma \cup \beta) \cdot (1/n) \in (1/n)\mathbb{Z}/\mathbb{Z}$.

What we need to show, finally, is that the following diagram commutes:

$$\begin{array}{ccc}
\hat{H}^{-1}(\Gamma, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\sim} & (1/n)\mathbb{Z}/\mathbb{Z} \\
\downarrow\delta & & \downarrow x \mapsto nx \\
\hat{H}^0(\Gamma, \mathbb{Z}) & \xrightarrow{\sim} & \mathbb{Z}/n\mathbb{Z}
\end{array}$$

This is simple, going back to how $\delta$ is defined in low degrees as made explicit in [CF, IV, (6.2)]. Explicitly, an element $\xi \in (1/n)\mathbb{Z}/\mathbb{Z}$ lifts to an element $a/n \in \mathbb{Q}$ with $a \in \mathbb{Z}$, and $N_\Gamma : \mathbb{Q} \rightarrow \mathbb{Q}$ (which underlies the connecting map in Tate cohomology from degree $-1$ to degree $0$) carries $a/n$ to $a$. The class of $\delta(\xi) \in \hat{H}^0(\Gamma, \mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$ is therefore represented by $a \mod n\mathbb{Z}$, which comes from $a/n \mod \mathbb{Z}$ under $n : (1/n)\mathbb{Z}/\mathbb{Z} \simeq \mathbb{Z}/n\mathbb{Z}$. \hfill $\Box$

Now we note that Theorem [1.2.2] holds for $G = G_a$. Indeed, this follows from Proposition [3.3.2.2] when $\text{char}(k) = p > 0$.

**Proposition 4.5.3.** Theorem [1.2.2] holds for almost tori.

**Proof.** Let $G$ be an almost torus. After harmlessly modifying $G$, we may assume by Lemma [2.1.3 (iv)] that there is an isogeny $R_{k'/k}(T') \times B \rightarrow G$ for some finite $k$-group scheme $B$, some finite separable extension $k'/k$, and some split $k'$-torus $T'$. For notational convenience, denote $R_{k'/k}(T') \times B$ by $X$. Let $A := \ker(X \rightarrow G)$. Then we have an exact sequence

$$1 \rightarrow A \rightarrow X \rightarrow G \rightarrow 1$$

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This yields a commutative diagram

\[
\begin{array}{ccc}
H^1(k, \hat{X}) & \longrightarrow & H^1(k, \hat{A}) \\
\downarrow & & \downarrow \\
H^1(k, X)^D & \longrightarrow & H^1(k, A)^D \longrightarrow \text{Hom}_{cts}(G(k), \mathbb{Q}/\mathbb{Z}) \\
\downarrow & & \downarrow \\
\longrightarrow & & \longrightarrow \\
\longrightarrow & & \longrightarrow \\
\longrightarrow & & \longrightarrow \\
\longrightarrow & & \longrightarrow \\
\longrightarrow & & \longrightarrow \\
\text{Hom}_{cts}(X(k), \mathbb{Q}/\mathbb{Z}) & \longrightarrow & \text{Hom}_{cts}(A(k), \mathbb{Q}/\mathbb{Z}) \\
\end{array}
\]

(4.5.2)

for which we claim that the indicated maps are isomorphisms and that the bottom row is (algebraically) exact at \(H^1(k, A)^D\) and \(\text{Hom}_{cts}(G(k), \mathbb{Q}/\mathbb{Z})\). A diagram chase would then give the (algebraic) isomorphism property for the middle vertical map, establishing Proposition 4.5.3.

The second and fifth vertical arrows in (4.5.2) are (topological) isomorphisms by local duality for finite commutative \(k\)-group schemes. The fourth is an algebraic (perhaps not topological) isomorphism by: local duality for finite commutative \(k\)-group schemes and Lemma 4.5.1. Finally, the first vertical arrow is an isomorphism by Lemma 4.3.2.

To see that the bottom side of (4.5.2) is exact at \(H^1(k, A)^D\), by the exactness of Pontryagin duality it suffices to show that the map \(H^1(k, \text{im}(G(k))) \hookrightarrow H^1(k, X)\) is a homeomorphism onto a closed subgroup. The image is closed because it is the kernel of the continuous map \(H^1(k, X) \rightarrow H^1(k, G)\). By the same reasoning, the image of \(G(k) \rightarrow H^1(k, A)\) is closed (so the cokernel of this inclusion is locally compact, Hausdorff, and second-countable). Thus, by Proposition 4.2.1 and Lemma 4.3.3, the map \(H^1(k, \text{im}(G(k))) \hookrightarrow H^1(k, X)\) is a homeomorphism onto its image. This proves the exactness of (4.5.2) at \(H^1(k, A)^D\). A similar argument shows exactness at \(\text{Hom}_{cts}(G(k), \mathbb{Q}/\mathbb{Z})\).

To treat the general case of Theorem 1.2.2, we will need the following result.

**Lemma 4.5.4.** Consider a short exact sequence

\[1 \longrightarrow H \longrightarrow G \longrightarrow U \longrightarrow 1\]

of affine commutative \(k\)-group schemes of finite type such that \(U\) is split unipotent. If Theorem 1.2.2 holds for \(H\) and \(U\), and Theorem 1.2.4 holds for \(H\), then Theorem 1.2.2 holds for \(G\).

We will apply this lemma with \(H\) an almost torus, a case of Theorem 1.2.4 that has already been completely proved in §4.3.
Proof. Using Proposition 2.2.3, we have a commutative diagram

\[
\begin{array}{ccccccccc}
H^1(k, \hat{H}) & \longrightarrow & H^2(k, \hat{U}) & \longrightarrow & H^2(k, \hat{G}) & \longrightarrow & H^2(k, \hat{H}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
H^1(k, H)^D & \longrightarrow & \text{Hom}_{cts}(U(k), Q/Z) & \longrightarrow & \text{Hom}_{cts}(G(k), Q/Z) & \longrightarrow & \text{Hom}_{cts}(H(k), Q/Z) & \\
\end{array}
\]

with exact top row (by Proposition 3.3.1) and indicated vertical arrows isomorphisms by the hypotheses.

We claim that the bottom row is exact. For exactness at \(\text{Hom}_{cts}(U(k), Q/Z)\), it suffices by exactness properties of Pontryagin duality to show that the continuous inclusion \(j : U(k)/\text{im}(G(k)) \hookrightarrow H^1(k, H)\) is homeomorphism onto a closed subgroup. The image is closed because it is the kernel of the continuous map \(H^1(k, H) \rightarrow H^1(k, G)\), and by similar reasoning \(G(k)\) has closed image in \(U(k)\) (so \(U(k)/\text{im}(G(k))\) is locally compact, Hausdorff, and second-countable). It then follows from Lemma 4.3.3 that \(j\) is a homeomorphism onto its closed image, as desired.

The proof of exactness at \(\text{Hom}_{cts}(G(k), Q/Z)\) is similar, since \(\text{Hom}_{cts}(U(k), Q/Z)\) coincides with the Pontryagin dual of \(U(k)\), because \(U\) is of finite exponent (since we are assuming that \(\text{char}(k) = p > 0\)). \(\square\)

With Lemma 4.5.4 in hand, the proof of Theorem 1.2.2 is now simple. The case in which \(G = U\) is split unipotent follows immediately by induction and the settled case \(G = G_a\), since the groups \(H^1(k, U)\) and \(H^1(k, \hat{U})\) both vanish when \(U\) is split unipotent, by Lemma 2.4.4.

Now let \(G\) be an arbitrary affine commutative \(k\)-group scheme of finite type. By Lemma 2.1.7, we have an exact sequence

\[
1 \longrightarrow H \longrightarrow G \longrightarrow U \longrightarrow 1
\]

with \(H\) an almost torus and \(U\) split unipotent. By Lemma 4.5.4, to conclude Theorem 1.2.2 for \(G\) we just need to note that Theorem 1.2.2 is settled for the almost torus \(H\) and the split unipotent \(U\) and that, moreover, Theorem 1.2.4 has been settled for \(H\) (Lemma 4.3.2).

### 4.6 Duality between \(H^1(k, G)\) and \(H^1(k, \hat{G})\)

Recall that Theorem 1.2.4 is only well-posed so far for almost tori, since if \(G\) is not an almost torus then \(\hat{G}\) is not representable (Proposition 2.2.7) and so we have not yet defined a topology on \(H^1(k, \hat{G})\). In the case of almost tori, Theorem 1.2.4 has been proved in Lemma 4.3.2. To go beyond that, the first order of business is to define a reasonable topology on \(H^1(k, \hat{G})\) for arbitrary affine commutative group schemes of finite type over the local function field \(k\).
Let $G$ be an affine commutative $k$-group scheme of finite type, so Lemma 2.1.7 furnishes an exact sequence
\[ 1 \to H \to G \to U \to 1 \] (4.6.1)
with $H$ an almost torus and $U$ split unipotent. By Proposition 2.2.3, we have an fppf-exact sequence of dual sheaves
\[ 1 \to \hat{U} \to \hat{G} \to \hat{H} \to 1 \]
Since $H^1(k, \hat{U}) = 0$ (Lemma 2.4.4), the map $H^1(k, \hat{G}) \to H^1(k, \hat{H})$ is injective.

Recall that $\hat{H}$ is represented by a locally finite type commutative $k$-group scheme (Proposition 2.2.7), so we have a natural locally compact Hausdorff topology on $H^1(k, \hat{H})$ (by the method in [Čes1]) that is even second-countable and locally profinite (Proposition 4.2.1). We wish to give $H^1(k, \hat{G})$ the subspace topology from its inclusion into $H^1(k, \hat{H})$.

In the commutative diagram of pairings
\[
\begin{array}{ccc}
H^1(k, G) \times H^1(k, \hat{G}) & \longrightarrow & \mathbb{Q}/\mathbb{Z} \\
\uparrow & & \downarrow \\
H^1(k, H) \times H^1(k, \hat{H}) & \longrightarrow & \mathbb{Q}/\mathbb{Z}
\end{array}
\]
the continuous surjection along the left side is a topological quotient map (due to Proposition 4.2.1 and Lemma 4.3.3). Thus, if we define the topological group structure on $H^1(k, \hat{G})$ as a subgroup of $H^1(k, \hat{H})$ for a fixed choice of (4.6.1), then continuity of the pairing $H^1(k, H) \times H^1(k, \hat{H}) \to \mathbb{Q}/\mathbb{Z}$ implies that $H^1(k, G) \times H^1(k, \hat{G}) \to \mathbb{Q}/\mathbb{Z}$ is continuous. (It does not matter if we view $\mathbb{Q}/\mathbb{Z}$ discretely or with its subspace topology from $\mathbb{R}/\mathbb{Z}$ for this latter continuity of a pairing because the cohomologies involved have finite exponent, due to Lemma 4.3.1.)

There are two immediate problems:

(i) Is $H^1(k, \hat{G}) \hookrightarrow H^1(k, \hat{H})$ a closed subgroup (and therefore locally profinite and second-countable)?

(ii) Is this topology independent of the choice of sequence (4.6.1)?

To obtain an affirmative answer to (ii), we will use the topology defined by a fixed choice of such sequence to make sense of Theorem 1.2.4 for $G$ and to actually prove the Theorem for $G$. Since the topology on $H^1(k, G)$ is intrinsic, it will then follow that the topology just defined on $H^1(k, \hat{G})$ using a choice of (4.6.1) is the Pontryagin dual topology, and hence this topology is independent of the choice of (4.6.1)! Note in particular that if $G$ is an almost torus, then this method of defining an intrinsic topology on $H^1(k, \hat{G})$ would have to recover the topology as already defined earlier in such cases, since we can use the choice $H = G$ and $U = 0$. We now settle problem (i):
Lemma 4.6.1. The subgroup \( H^1(k, \hat{G}) \subset H^1(k, \hat{H}) \) is closed (and hence is locally profinite and second-countable, by Proposition 4.2.1 applied to \( H \)).

Proof. It suffices to show that for any \( \alpha \in H^1(k, \hat{H}) \) such that \( \phi(\alpha) = 0 \) for every \( \phi \in H^1(k, \hat{H})^D \) satisfying \( \phi|_{H^1(k, \hat{G})} = 0 \), necessarily \( \alpha \in H^1(k, \hat{G}) \). (Indeed, the common kernel of all such elements \( \phi \) is exactly the closure of \( H^1(k, \hat{G}) \) inside \( H^1(k, \hat{H}) \).)

We want to show that \( \alpha \) maps to 0 in \( H^2(k, \hat{U}) \). By Theorem 1.2.2, the map \( H^2(k, \hat{U}) \to U(k)^* \) is an inclusion, so it suffices to show that \( \alpha \) (more precisely, its image inside \( H^2(k, \hat{U}) \)) pairs trivially with \( U(k) \). But \( U(k) \) pairs trivially with \( H^1(k, \hat{G}) \), which is to say that the image of the natural map \( U(k) \to H^2(k, \hat{U})^* \to H^1(k, \hat{H})^* \) consists of elements \( \phi \) vanishing on \( H^1(k, \hat{G}) \). Hence, \( U(k) \) pairs trivially with \( \alpha \) by the hypothesis on \( \alpha \).

Before turning to the proof of Theorem 1.2.4, let us make a couple more observations about the topology on \( H^1(k, \hat{G}) \). (We will never use these, so the uninterested reader may skip ahead to the proof of Theorem 1.2.4.) First, lest the reader become complacent, we note that it is not in general \( \delta \)-functorial. That is, given an exact sequence

\[
1 \to G' \to G \to G'' \to 1
\]

of affine commutative \( k \)-groups of finite type, the induced map \( H^1(k, \hat{G'}) \to H^2(k, \hat{G''}) \) is not in general continuous, cf. Remark 1.2.3 (Of course, it is when all of the groups are almost tori, since then the dual sheaves are represented by locally finite type group schemes, by Proposition 2.2.7)

On the other hand, we observe that \( H^1(k, \hat{G}) \) is functorial.

Proposition 4.6.2. Suppose that we have a homomorphism \( G \to G' \) of affine commutative group schemes of finite type over the local field \( k \). Then the induced map \( H^1(k, \hat{G'}) \to H^1(k, \hat{G}) \) is continuous.

Proof. The key is to show that we have a commutative diagram

\[
\begin{array}{cccccc}
1 & \to & H & \to & G & \to & U & \to & 1 \\
| & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & H' & \to & G' & \to & U' & \to & 1
\end{array}
\]

with \( H, H' \) almost tori and \( U, U' \) split unipotent. Then the continuity follows (just by definition of the topologies) from the continuity of the map \( H^1(k, \hat{H'}) \to H^1(k, \hat{H}) \) between cohomology groups of representable sheaves.

To see that we have such a commutative diagram, first choose an almost torus \( H \subset G \) such that \( U := G/H \) is split unipotent. Let \( T \subset H \subset G \), \( T' \subset G' \) be the maximal tori. Then certainly \( T \) maps into \( T' \), so we get a map \( H/T \to G'/T' \) with finite image. Let \( F' \subset G' \) be the preimage of this finite image, so \( H \) maps into \( F' \), which is an almost torus. Now choose an almost torus \( \overline{H'} \subset \overline{G'} := G'/F' \) such that \( U' := \overline{G'}/\overline{H'} \) is split unipotent. Then we take \( H' \subset G' \) to be the preimage of \( \overline{H'} \subset \overline{G'} \).
We may now complete the proof of Theorem 1.2.4. Since we have already established that the group $H^1(k, \hat{G})$ is locally compact, Hausdorff, and second-countable, it suffices by Lemma 4.3.3 to show that the map $H^1(k, \hat{G}) \to H^1(k, G)^D$ is an algebraic isomorphism.

We claim that we have a commutative diagram

\[
\begin{array}{ccccccc}
0 & \longrightarrow & H^1(k, \hat{G}) & \longrightarrow & H^1(k, \hat{H}) & \longrightarrow & H^2(k, \hat{U}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^1(k, G)^D & \longrightarrow & H^1(k, H)^D & \longrightarrow & U(k)^D
\end{array}
\]

in which the top row is exact and the bottom row is exact at $H^1(k, G)^D$. Assuming this, a simple diagram chase shows that the first vertical arrow is an isomorphism. As we have already seen, the top row is exact because $H^1(k, \hat{U}) = 0$ by Lemma 2.4.4. The second vertical arrow is an isomorphism by Lemma 4.3.2. The third vertical arrow is an inclusion by Theorem 1.2.2. Finally, to show the bottom row is exact at $H^1(k, G)^D$ we note that by the vanishing of $H^1(k, U)$ we have an algebraically exact sequence

\[0 \to H(k) \to G(k) \to U(k) \xrightarrow{\delta} H^1(k, H) \to H^1(k, G) \to 0.\]

The maps in this diagram are all continuous for the natural second-countable locally compact Hausdorff topologies on each term (see the summary of relevant results from [Čes1] near the start of §4.2, and use Proposition 4.2.1 for the second-countability of the degree-1 cohomologies). Thus, by the exactness properties of Pontryagin duality and Lemma 4.3.3 we see that the natural injection

\[H^1(k, G)^D \to \ker(H^1(k, H)^D \to U(k)^D)\]

is bijective, as desired.
Chapter 5

Local Integral Cohomology

In this chapter we prove the integral annihilator aspects of local Tate duality (Theorems 1.2.5, 1.2.6, and 1.2.7), generalizing the analogous classical results for finite discrete Galois modules (often stated in terms of unramified cohomology classes). Since there is no good structure theory for arbitrary affine commutative flat group schemes of finite type over discrete valuation rings, we are not able to prove results of such precision, only obtaining the classical results at all but finitely many places (by choosing a model for our group scheme over some dense open subscheme of the scheme of integers of our global field \( k \), by which we mean the scheme \( \text{Spec}(O_k) \) when \( k \) is a number field, and the smooth proper curve \( X \) of which \( k \) is the function field in the function field setting). These results - and especially the injectivity statements for the maps from cohomology of \( O_v \) to cohomology of \( k_v \) for almost all places \( v \) of \( k \) - while interesting in their own right, will also play an important role in relating the cohomology of \( A \) to that of the local fields \( k_v \) in §6.1. Perhaps somewhat surprisingly, it seems that the injectivity of the maps \( H^2(O_v, \mathcal{G}) \to H^2(k_v, \hat{G}) \) for almost all \( v \) (see remark 5.5.2) cannot be proved directly, but actually requires one to prove Theorem 1.2.6 in its entirety, and the proof of this result in turn depends upon first proving Theorem 1.2.7.

5.1 Integral annihilator aspects of duality between \( H^2(k_v, G) \) and \( \hat{G}(k_v) \)

We will first show that \( H^2(O_v, \mathcal{G}) = 0 \) for almost every \( v \). Actually, we will prove the following somewhat stronger result for later use.

**Proposition 5.1.1.** Let \( k \) be a global field, \( G \) a commutative \( k \)-group scheme of finite type. Then there is a finite set \( S \) of places of \( k \) and an \( O_S \)-model \( \mathcal{G} \) of \( G \) such that \( H^i(\prod_{v \in S} O_v, \mathcal{G}) = 0 \) for \( i > 1 \). The same holds for \( i > 0 \) if \( G \) is smooth and connected.

We only need this result for affine \( G \), in which case the \( O_S \)-model \( \mathcal{G} \) can be taken to
be affine. To make the proof work we will have to pass to the wider generality of algebraic space groups. Thus, the following lemma is required in the generality of algebraic spaces:

**Lemma 5.1.2.** Let \( \{ R_i \}_{i \in I} \) be a set of rings, let \( A := \prod_{i \in I} R_i \), and suppose that \( X \) is a quasi-compact quasi-separated algebraic space over \( A \). Then the natural map \( X(A) \to \prod_{i \in I} X(R_i) \) is surjective.

**Proof.** If \( X \) is an affine scheme then the result is clear. If \( X \) is a separated algebraic space then we can pick an étale cover \( U \to X \) by an affine scheme and then \( U \times_X U \) is also an affine scheme fitting into a commutative diagram

\[
\begin{array}{ccc}
X(A) & \to & U(A) \\
\downarrow & & \downarrow \\
\prod_{i \in I} X(R_i) & \to & \prod_{i \in I} U(R_i)
\end{array}
\]

whose rows are equalizer kernels. Thus, the result follows for such \( X \). In particular, the case of quasi-compact separated schemes is settled. But then we can run through the same argument for \( X \) any quasi-compact quasi-separated algebraic space since by taking \( U \) to be affine we see that \( U \times_X U \) is a quasi-compact separated scheme (as \( U \times_X U \to U \times U \) is a quasi-compact immersion into an affine scheme).

**Proof of Proposition 5.1.1.** (B. Conrad) First let us temporarily assume that the desired vanishing holds for all \( i > 0 \) and smooth connected \( G \). (In fact, something much more precise will be proved below in those cases.) We shall deduce the vanishing assertion for general \( G \) (with \( i > 1 \)).

Consider finite \( G \). By [Me, II, 3.2.5] (as we saw in the proof of Lemma 3.3.12 by methods that work over any base scheme) \( G \) fits into an exact sequence

\[
1 \to G \to U \to H \to 1
\]

with commutative smooth connected affine \( k \)-groups \( U \) and \( H \). This may be spread out to an exact sequence of commutative flat affine \( \mathcal{O}_S \)-group schemes of finite type

\[
1 \to \mathcal{G} \to \mathcal{U} \to \mathcal{H} \to 1.
\]

The desired refined vanishing assertion for the general finite \( G \) therefore follows from the one being temporarily assumed in the smooth connected case (applied to \( U \) and \( H \)).

Now consider an arbitrary commutative finite type \( k \)-group scheme \( G \). Let \( G^0 \) be its identity component, and let \( E := G/G^0 \) be the finite étale component group. Then the exact sequence

\[
1 \to G^0 \to G \to E \to 1
\]

spreads out to a short exact sequence of commutative flat affine \( \mathcal{O}_S \)-groups of finite type, so by the settled finite case we are reduced to the connected case.
By [SGA3 VII A, Prop. 8.3], there is an infinitesimal subgroup scheme \( I \subset G \) such that \( H := G/I \) is smooth (and connected). Once again spreading out and applying the smooth connected case would finish the argument. We are thereby reduced to the case in which \( G \) is smooth and connected (but now we must prove the vanishing for \( i > 0 \)).

The smooth connected case will proceed by induction on \( i \), but to make the induction work we need to vastly enlarge the scope of the assertion. Note that by enlarging \( S \) if necessary at the start (after making a first choice of \( \mathcal{G} \), we can arrange that \( \mathcal{G} \to \text{Spec}(\mathcal{O}_S) \) is smooth with geometrically connected fibers. Since étale and fppf cohomology over \( \mathcal{O}_S \) with coefficients in the \( \mathcal{O}_S \)-smooth \( \mathcal{G} \) agree by [BrIII Thm. 11.7], we may and do consider cohomology to be étale rather than fppf. Let \( \mathcal{O}_S := \prod_{v \in S} \mathcal{O}_v \) (the profinite completion of \( \mathcal{O}_S \)).

To carry out the induction, we shall prove the following more precise assertion in which we do not change \( S \) (as will be essential for the success of the induction): if \( \mathcal{G} \) is a smooth commutative finitely presented algebraic space group over \( \hat{\mathcal{O}}_S \) such that the special fiber of \( \mathcal{G} \otimes \mathcal{O}_v \) is connected for every \( v \notin S \), then we claim that \( H^i(\mathcal{O}_S, \mathcal{G}) = 0 \) for all \( i > 0 \). In the case \( i = 1 \), we need to show that if we have an algebraic space \( \mathcal{G} \)-torsor \( \mathcal{X} \) over \( \hat{\mathcal{O}}_S \) then \( \mathcal{X}(\hat{\mathcal{O}}_S) \neq \emptyset \). Since \( \mathcal{X} \) is an algebraic space of finite presentation over \( \hat{\mathcal{O}}_S \) (inherited from \( \hat{\mathcal{O}}_S \)), so it is quasi-compact and quasi-separated, by Lemma 5.1.2 it suffices to show that \( \mathcal{X}(\mathcal{O}_v) \neq \emptyset \) for each \( v \notin S \). Now \( \mathcal{X}_v := \mathcal{X} \otimes \mathcal{O}_v \) is an algebraic space \( \mathcal{G}_v \)-torsor, and the special fiber of \( \mathcal{G}_v \) is geometrically connected. But over fields, torsors for group schemes of finite type are necessarily schemes (by considerations with quasi-projectivity descending through finite extensions of fields), so by Lang’s Theorem \( \mathcal{X}(\kappa_v) \neq \emptyset \), where \( \kappa_v \) is the finite residue field of \( \mathcal{O}_v \).

Since \( \mathcal{X}_v \) is a quasi-compact and quasi-separated algebraic space, any monic map \( x : \text{Spec}(F) \to \mathcal{X}_v \) for a field \( F \) factors through an étale map \( U \to \mathcal{X}_v \) for some affine scheme \( U \) [Kn II, 6.4]. Taking \( x \) with \( F = \kappa_v \), we get a \( \kappa_v \)-point of \( U \). But \( U \) is \( \mathcal{O}_v \)-smooth, so its \( \kappa_v \)-point lifts to an \( \mathcal{O}_v \)-point since \( \mathcal{O}_v \) is henselian. That provides an \( \mathcal{O}_v \)-point of \( \mathcal{X}_v \) as desired, so the case \( i = 1 \) is settled.

Next suppose \( i > 1 \) and that the Proposition holds for \( i - 1 \). Choose \( \alpha \in H^i(\hat{\mathcal{O}}_S, \mathcal{G}) \). On any scheme, any higher étale cohomology class vanishes under pullback to some étale cover. But for the scheme Spec(\( \hat{\mathcal{O}}_S \)), the finite étale covers are cofinal among all étale covers [Con Lemma 7.5.5]. Hence, there exists a finite étale cover \( E \to \text{Spec}(\hat{\mathcal{O}}_S) \) such that \( \alpha|_E = 0 \) in \( H^i(E, \mathcal{G}) \). By exactness of finite pushforward for the étale topology, we have a natural isomorphism \( H^i(E, \mathcal{G}) \simeq H^i(\hat{\mathcal{O}}_S, R_{E/\hat{\mathcal{O}}_S}(\mathcal{G}_E)) \), and this identifies \( E \)-pullback with the effect of composition with the natural map \( \mathcal{G} \to R_{E/\hat{\mathcal{O}}_S}(\mathcal{G}_E) \). Since \( E \) is finite étale over \( \hat{\mathcal{O}}_S \), it is easy to check that \( R_{E/\hat{\mathcal{O}}_S}(\mathcal{G}_E) \) is an algebraic space that is moreover smooth with geometrically connected fibers over Spec(\( \hat{\mathcal{O}}_S \)).

The quotient \( \mathcal{H} := R_{E/\hat{\mathcal{O}}_S}(\mathcal{G}_E)/\mathcal{G} \) is also a quasi-compact and quasi-separated algebraic space, by [Ab Cor. 6.3] (this is the entire reason that we needed to set up the induction with algebraic spaces: at this step if we had worked with schemes then we would have had.
to shrink $S$ to arrange that $\mathcal{H}$ is a scheme, but $\mathcal{H}$ depends on $E$ that in turn depends on the original cohomology class $\alpha$, and the induction would be destroyed if we had to shrink $S$ depending on $\alpha$). Clearly $\mathcal{H}$ is smooth with geometrically connected fibers over $\text{Spec}(\hat{\mathcal{O}}_S)$, and in the cohomology sequence associated to

$$1 \rightarrow \mathcal{G} \rightarrow R_{E/\hat{\mathcal{O}}_S}(\mathcal{G}_E) \rightarrow \mathcal{H} \rightarrow 1$$

the class of $\alpha$ in degree $i$ is forced to lie in the image of $H^{i-1}(\hat{\mathcal{O}}_S, \mathcal{H})$. But $i-1 > 0$, so by induction this latter cohomology group vanishes (no shrinking of $S$!), and thus $\alpha = 0$ as desired. \hfill $\square$

It remains to show that the map $\hat{\mathcal{G}}(\mathcal{O}_v) \rightarrow \hat{G}(k_v)$ is an isomorphism for almost all $v$. Let us first treat the core cases.

**Lemma 5.1.3.** Theorem 1.2.5 holds in the following cases:

(i) $G$ is finite;

(ii) $G = R_{k'/k}(\mathbb{G}_m)$, with $k'/k$ a finite separable extension.

**Proof.** We only need to check that the map $\hat{\mathcal{G}}(\mathcal{O}_v) \rightarrow \hat{G}(k_v)$ is an isomorphism for almost all $v$.

(i) Spread $G$ out to a finite flat commutative $\mathcal{O}_S$-group scheme $\mathcal{G}$. Then $\hat{\mathcal{G}}$ is finite, hence proper, so the valuative criterion for properness yields that $\hat{\mathcal{G}}(\mathcal{O}_v) = \hat{\mathcal{G}}(k_v)$.

(ii) By Lemma 2.2.1, $R_{k'/k}(\mathbb{G}_m)(k_v) = \prod_{v' \mid v} \mathbb{Z}$, where the product is over the places $v'$ of $k'$ lying above $v$. On the other hand, we may spread $G$ out to $\mathcal{G} := R_{\mathcal{O}_S/\mathcal{O}_S}(\mathbb{G}_m)$, where $S'$ is the set of places of $k'$ lying above $S$. If we choose $S$ to consist of all places ramified in $k'$, then for each $v \notin S$, Lemma 2.2.1 again yields $R_{\mathcal{O}_{S'/\mathcal{O}_S}}(\mathbb{G}_m)(\mathcal{O}_v) = \prod_{v' \mid v} \mathbb{Z}$, so the map $\hat{\mathcal{G}}(\mathcal{O}_v) \rightarrow \hat{G}(k_v)$ is an isomorphism. \hfill $\square$

Before proceeding to the proof in general, we prove a lemma which will be crucial in order to push through various dévissage arguments.

**Lemma 5.1.4.** Suppose that we have a short exact sequence

$$1 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 1$$

of affine commutative group schemes of finite type over the global function field $k$. Then there is a finite set $S$ of places of $k$ such that this sequence spreads out to an exact sequence

$$1 \rightarrow \mathcal{G}' \rightarrow \mathcal{G} \rightarrow \mathcal{G}'' \rightarrow 1$$

of $\mathcal{O}_S$-group schemes such that the corresponding sequence

$$1 \rightarrow \hat{\mathcal{G}}'' \rightarrow \hat{\mathcal{G}} \rightarrow \hat{\mathcal{G}}' \rightarrow 1$$

of fppf dual sheaves is exact. The same holds with $\mathcal{O}_S$ replaced by $\prod_{v \notin S} \mathcal{O}_v$. 

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Proof. We will prove the lemma for $O_S$; the proof for $\prod O_v$ is exactly the same. Since faithful flatness of a map spreads out, the only thing that is not clear is that we may obtain surjectivity of the map $\mathcal{G} \to \mathcal{H}$. For this, it is enough to show that $\mathcal{E}xt^1_{O_S}(\mathcal{G}''', G_m) = 0$ for sufficiently large $S$. We know that $G'''$ admits a filtration by finite group schemes, tori, and $G_a$. This filtration spreads out to one by finite flat group schemes, tori, and $G_a$ over some $O_S$. It therefore suffices to show that each of these group schemes has vanishing $\mathcal{E}xt^1(\cdot, G_m)$. For tori and finite flat group schemes, this follows from [SGA7, VIII, Prop. 3.3.1], so we are reduced to the case of $G_a$, which follows from Proposition 2.2.11. □

Now we proceed to the proof of Theorem 1.2.5. First suppose that $G$ is an almost torus. By Lemma 2.1.3(iv), we may harmlessly modify $G$ and thereby assume that there is an exact sequence

$$1 \longrightarrow B \longrightarrow C \times R_{k'/k}(T') \longrightarrow G \longrightarrow 1$$

where $B, C$ are finite commutative $k$-group schemes, $k'/k$ is a finite separable extension, and $T'$ is a split $k'$-torus. Let $X := C \times R_{k'/k}(T')$. We may spread this out and apply Lemma 5.1.4 to obtain an exact sequence

$$1 \longrightarrow \hat{G} \longrightarrow \hat{X} \longrightarrow \hat{B} \longrightarrow 1$$

Consider the following commutative diagram:

\[
\begin{array}{llll}
0 & \longrightarrow & \hat{G}(O_v) & \longrightarrow & \hat{X}(O_v) & \longrightarrow & \hat{B}(O_v) \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \hat{G}(k_v) & \longrightarrow & \hat{X}(k_v) & \longrightarrow & \hat{B}(k_v)
\end{array}
\]

The rows are clearly exact, and the second and third vertical arrows are isomorphisms by Lemma 5.1.3. A simple diagram chase now shows that the first vertical arrow is an isomorphism.

Now suppose that $G$ is an arbitrary affine commutative $k$-group scheme of finite type. By Lemma 2.1.7 there is an exact sequence

$$1 \longrightarrow H \longrightarrow G \longrightarrow U \longrightarrow 1$$

with $H$ an almost torus and $U$ split unipotent. This spreads out to yield an exact sequence

$$1 \longrightarrow \hat{H} \longrightarrow \hat{G} \longrightarrow \hat{U} \longrightarrow 1$$

Consider the following commutative diagram:

\[
\begin{array}{lll}
\hat{G}(O_v) & \sim & \hat{H}(O_v) \\
& \downarrow & \downarrow \\
\hat{G}(k_v) & \sim & \hat{H}(k_v)
\end{array}
\]

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The top horizontal arrow is injective for almost all \( v \) because \( \widehat{\mathcal{U}}(\mathcal{O}_v) = 0 \) for almost all \( v \), since \( \mathcal{U} \) admits a filtration by \( G_a \)'s (after enlarging \( S \)). This arrow is surjective because \( H^1(\mathcal{O}_v, \widehat{\mathcal{U}}) = 0 \) for almost all \( v \) (by filtering \( \mathcal{U} \), this reduces to showing that \( H^1(\mathcal{O}_v, \widehat{G}_a) = 0 \); that in turn follows from Lemma 2.4.3. The right vertical arrow is an isomorphism by the already-treated case of almost tori. Finally, the bottom arrow is clearly injective. A simple check now shows that the left vertical arrow is an isomorphism.

### 5.2 Injectivity of \( H^1(\mathcal{O}_v, \widehat{\mathcal{G}}) \to H^1(k_v, \widehat{G}) \) for almost all \( v \)

The purpose of this section is to prove the following result.

**Lemma 5.2.1.** Let \( G \) be an affine commutative group scheme of finite type over the global field \( k \), and \( \mathcal{G} \) an \( \mathcal{O}_S \)-model of \( G \) for a non-empty finite set \( S \) of places of \( k \). Then for almost every place \( v \) of \( k \), the map \( H^1(\mathcal{O}_v, \mathcal{G}) \to H^1(k_v, \widehat{G}) \) is injective.

**Proof.** When \( G \) is finite, then \( \mathcal{G} \) is a finite flat commutative \( \mathcal{O}_S \)-group, so any torsor \( \mathcal{X} \) for this group over \( \mathcal{O}_v \) is a finite flat \( \mathcal{O}_v \)-scheme. If such a scheme has a \( k_v \)-point then it clearly has an \( \mathcal{O}_v \)-point by the valuative criterion for properness. That settles injectivity when \( G \) is finite.

Next suppose that \( G = R_{k'/k}(G_m) \) for some finite separable extension \( k'/k \). Choose any \( S \) that contains all places of \( k \) that are ramified in \( k' \), and let \( S' \) be the set of places of \( k' \) lying above \( S \). For the \( \mathcal{O}_S \)-model \( \mathcal{G} = R_{\mathcal{O}_{S'}/\mathcal{O}_S}(G_m) \) of \( G \), we claim that \( H^1(\mathcal{O}_v, \mathcal{G}) = 0 \) for almost all \( v \not\in S \).

By Lemma 2.2.1 we have \( \mathcal{G} = R_{\mathcal{O}_{S'}/\mathcal{O}_S}(\mathbb{Z}) \), so we just need to show

\[
H^1(\mathcal{O}_v, \prod_{v' \mid v} R_{\mathcal{O}_{v'}/\mathcal{O}_v}(\mathbb{Z})) = 0
\]

for all \( v \not\in S \). The group scheme \( R_{\mathcal{O}_{v'}/\mathcal{O}_v}(\mathbb{Z}) \) is smooth, so we may take our cohomology to be étale. Since finite pushforward is an exact functor between categories of étale sheaves, it suffices to show that \( H^1(\mathcal{O}_v, \mathbb{Z}) = 0 \). But \( \mathcal{O}_v \) is a normal noetherian domain, so \( H^1(\mathcal{O}_v, \mathbb{Z}) = \text{Hom}_{\mathcal{O}_v}(\pi_1(\mathcal{O}_v), \mathbb{Z}) = 0 \), since \( \pi_1(\mathcal{O}_v) \) is profinite.

Now suppose that \( G = U \) is split unipotent. We shall prove that \( H^1(\mathcal{O}_v, \mathcal{U}) = 0 \) for almost all \( v \). Lemma 5.1.4 reduces us to the case \( U = G_a \), so we need to show \( H^1(\mathcal{O}_v, \widehat{G}_a) = 0 \). That vanishing in turn follows from Lemma 2.4.3.

Next, assume \( G \) is an almost torus. By Lemma 2.1.3(iv), after harmlessly modifying \( G \) we may assume that there is an exact sequence

\[
1 \longrightarrow B \longrightarrow X \longrightarrow G \longrightarrow 1
\]

where \( X = C \times R_{k'/k}(T') \) for a finite separable extension field \( k'/k \), \( B \) and \( C \) are commutative finite \( k \)-group schemes, and \( T' \) is a split \( k' \)-torus. Spreading out, we obtain by Lemma
an exact sequence

\[ 1 \longrightarrow \widehat{G} \longrightarrow \widehat{X} \longrightarrow \widehat{B} \longrightarrow 1 \]

We therefore obtain for almost all \( v \) a commutative diagram with exact rows:

\[
\begin{array}{cccccc}
\widehat{X}(\mathcal{O}_v) & \longrightarrow & \widehat{B}(\mathcal{O}_v) & \longrightarrow & H^1(\mathcal{O}_v, \mathcal{G}) & \longrightarrow & H^1(\mathcal{O}_v, \widehat{X}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\widehat{X}(k_v) & \longrightarrow & \widehat{B}(k_v) & \longrightarrow & H^1(k_v, \mathcal{G}) & \longrightarrow & H^1(k_v, \widehat{X})
\end{array}
\]

in which the first two vertical arrows are isomorphisms by Lemma 5.1.3. The last vertical arrow is an inclusion by the already-treated cases of finite group schemes and separable Weil restrictions of split tori. A simple diagram chase now shows that the third vertical arrow is an inclusion.

Now consider the general case; that is, let \( G \) be an affine commutative \( k \)-group scheme of finite type. By Lemma 2.1.7 there is an exact sequence

\[ 1 \longrightarrow H \longrightarrow G \longrightarrow U \longrightarrow 1 \]

with \( H \) an almost torus and \( U \) split unipotent. This spreads out to yield an exact sequence

\[ 1 \longrightarrow \widehat{H} \longrightarrow \widehat{G} \longrightarrow \widehat{U} \longrightarrow 1. \]

We therefore obtain a commutative diagram for almost every \( v \)

\[
\begin{array}{ccc}
H^1(\mathcal{O}_v, \widehat{G}) & \longrightarrow & H^1(\mathcal{O}_v, \widehat{H}) \\
\downarrow & & \downarrow \\
H^1(k_v, \widehat{G}) & \longrightarrow & H^1(k_v, \widehat{H})
\end{array}
\]

in which the top arrow is an inclusion because \( H^1(\mathcal{O}_v, \widehat{G}) = 0 \) for almost all \( v \) and the right arrow is an inclusion because of the already-treated case of almost tori. It follows that the left vertical arrow is an inclusion.

5.3 Injectivity of \( H^1(\mathcal{O}_v, \mathcal{G}) \to H^1(k_v, G) \) for almost all \( v \)

The goal of this section is to prove the following result.

**Proposition 5.3.1.** Let \( k \) be a global field, \( G \) a commutative \( k \)-group scheme of finite type, \( S \) a finite set of places of \( k \), and \( \mathcal{G} \) an \( \mathcal{O}_S \)-model for \( G \). Then for all but finitely many \( v \), the map \( H^1(\mathcal{O}_v, \mathcal{G}) \to H^1(k_v, G) \) is injective.
Note that we do not assume $G$ to be affine, though we only need this result in the affine case. The following lemma is the special case of Lemma 5.2.1 for finite commutative $k$-group schemes by Cartier duality (or by repeating the same argument as given earlier in that finite case):

**Lemma 5.3.2.** Proposition 5.3.1 holds if $G$ is a finite $k$-group scheme.

Here is the first dévissage step towards the general case of Proposition 5.3.1:

**Lemma 5.3.3.** For $k$ and $G$ as in Proposition 5.3.1, suppose that we have an exact sequence

$$1 \rightarrow H \rightarrow G \rightarrow B \rightarrow 1$$

with $B$ a finite commutative $k$-group scheme. If Proposition 5.3.1 holds for $H$ then it also holds for $G$.

**Proof.** We have an exact sequence

$$1 \rightarrow H \rightarrow G \rightarrow B \rightarrow 1$$

and this spreads out to an exact sequence

$$1 \rightarrow \mathcal{H} \rightarrow \mathcal{I} \rightarrow \mathcal{B} \rightarrow 1$$

over some $\mathcal{O}_S$. At the cost of enlarging $S$, for $v \notin S$ the resulting commutative diagram of exact sequences

$$
\begin{array}{cccc}
\mathcal{B}(O_v) & \rightarrow & H^1(O_v, \mathcal{H}) & \rightarrow & H^1(O_v, \mathcal{I}) & \rightarrow & H^1(O_v, \mathcal{B}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
B(k_v) & \rightarrow & H^1(k_v, H) & \rightarrow & H^1(k_v, G) & \rightarrow & H^1(k_v, B)
\end{array}
$$

has the first vertical arrow an isomorphism because $\mathcal{B}$ is finite, the second vertical arrow an inclusion by hypothesis on $H$, and the last vertical arrow an inclusion by Lemma 5.3.2. A simple diagram chase now shows that the third vertical arrow is injective.

**Lemma 5.3.4.** Proposition 5.3.1 holds for smooth $G$.

**Proof.** By Lemma 5.3.3, we may replace $G$ with $G^0$ to arrange that $G$ is connected. By enlarging $S$, we may thereby assume that $\mathcal{I}$ is smooth with (geometrically) connected fibers. In this case we claim $H^1(O_v, \mathcal{I}) = 0$ for $v \notin S$. The classes in this $H^1$ are algebraic spaces (they are schemes if $\mathcal{I}$ is affine), so we can argue as in the treatment of the case $i = 1$ for smooth connected $G$ in the proof of Proposition 5.1.1.

Note that this completes the proof for number fields. The argument for function fields is more difficult, beginning with some more lemmas:
Lemma 5.3.5. Let $k'$ be a finite extension of the global field $k$, $B'$ a finite commutative $k'$-group scheme, and $G \subset R_{k'/k}(B')$ a closed $k$-subgroup scheme. Then Proposition 5.3.1 holds for $G$.

Proof. We first treat the case $G = R_{k'/k}(B')$. Let $B'$ be a finite flat commutative $\mathcal{O}_{S'}$-model for $B'$, where $S'$ is the set of places of $k'$ above a non-empty finite set $S$ of places of $k$, so $\mathcal{G} := R_{\mathcal{O}_{S'}/\mathcal{O}_S}(B')$ is an $\mathcal{O}_S$-model of $G$. The Leray spectral sequence associated to the morphism $\text{Spec}(\prod_{v' \mid v} \mathcal{O}_{v'}) \to \text{Spec} \mathcal{O}_v$ yields an inclusion $H^1(\mathcal{O}_v, \mathcal{G}) \hookrightarrow \prod_{v' \mid v} H^1(\mathcal{O}_{v'}, B')$, and similarly with $\mathcal{O}_v$ replaced by $k_v$. By functoriality of the spectral sequence, therefore, we obtain a commutative diagram

$$
\begin{array}{ccc}
H^1(\mathcal{O}_v, \mathcal{G}) & \hookrightarrow & \prod_{v' \mid v} H^1(\mathcal{O}_{v'}, B') \\
\downarrow & & \downarrow \\
H^1(k_v, G) & \rightarrow & \prod_{v' \mid v} H^1(k_{v'}, B')
\end{array}
$$

where the second vertical arrow is an inclusion by Lemma 5.3.2. It follows that the first vertical arrow is an inclusion.

Now we turn to the general case. Let $H := R_{k'/k}(B')/G$; note that $H$ is affine. For sufficiently big $S$ (and $S'$ the set of places over it in $k'$) the exact sequence

$$1 \rightarrow G \rightarrow R_{k'/k}(B') \rightarrow H \rightarrow 1$$

spreads out to an exact sequence

$$1 \rightarrow \mathcal{G} \rightarrow R_{\mathcal{O}_{S'}/\mathcal{O}_S}(B') \rightarrow \mathcal{H} \rightarrow 1$$

with $\mathcal{H}$ a finite flat commutative $\mathcal{O}_{S'}$-group scheme, and $\mathcal{H}$ a commutative flat affine $\mathcal{O}_S$-group of finite type. For each $v \notin S$, in the commutative diagram with exact rows

$$
\begin{array}{ccc}
R_{\mathcal{O}_{S'}/\mathcal{O}_S}(B')(\mathcal{O}_v) & \rightarrow & \mathcal{H}(\mathcal{O}_v) \\
\downarrow & & \downarrow \\
R_{k'/k}(B')(k_v) & \rightarrow & \mathcal{H}(k_v)
\end{array}
$$

the first vertical arrow is an isomorphism because $\mathcal{G}(\mathcal{O}_{v'}) = B(k_{v'})$ for each $v' \mid v$ (as $\mathcal{H}$ is finite), the second vertical arrow is an inclusion by affineness of $\mathcal{H}$, and the last vertical arrow is an inclusion by the settled case “$G = R_{k'/k}(B')$”. A simple diagram chase now shows that the third vertical arrow is an inclusion.

Lemma 5.3.6. Let $k'$ be a finite extension of the global field $k$, $B'$ a finite commutative $k'$-group scheme, and $G \subset R_{k'/k}(B')$ a closed $k$-subgroup scheme. Let $\mathcal{G}$ be an $\mathcal{O}_S$-model for $G$ (where, as usual, $S$ denotes a non-empty finite set of places of $k$).

For all but finitely many $v$, the map $\mathcal{G}(\mathcal{O}_v) \rightarrow G(k_v)$ is an isomorphism.
Proof. Let \( H := R_{k'/k}(B')/G \). We may spread out the exact sequence

\[
1 \rightarrow G \rightarrow R_{k'/k}(B') \rightarrow H \rightarrow 1
\]

to obtain an exact sequence

\[
1 \rightarrow \mathcal{G} \rightarrow R_{O_{S'/S}(/B')} \rightarrow \mathcal{H} \rightarrow 1
\]

over \( O_S \), where \( \mathcal{B}' \) is finite flat commutative over \( O_{S'} \) and \( \mathcal{H} \) is a commutative flat affine \( O_S \)-group of finite type. Then we obtain a commutative diagram with exact rows

\[
\begin{array}{ccccccc}
0 & \rightarrow & \mathcal{G}(O_v) & \rightarrow & R_{O_{S'/S}(/B')}(O_v) & \rightarrow & \mathcal{H}(O_v) \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & G(k_v) & \rightarrow & R_{k'/k}(B')(k_v) & \rightarrow & H(k_v)
\end{array}
\]

in which the middle vertical arrow is an isomorphism because \( \mathcal{B}' \) is finite and the last vertical arrow is an inclusion because \( \mathcal{H} \) is affine. An easy diagram chase now shows that the first vertical arrow is an isomorphism.

Lemma 5.3.7. For a global field \( k \), consider a short exact sequence of commutative \( k \)-group schemes of finite type

\[
1 \rightarrow H \rightarrow G \rightarrow R_{k'/k}(B')
\]

with \( k'/k \) a finite extension and \( B' \) a finite commutative \( k' \)-group scheme. If Proposition 5.3.1 holds for \( H \), then it also holds for \( G \).

Proof. Let \( C := \text{im}(G \rightarrow R_{k'/k}(B')) \). Then we have a short exact sequence

\[
1 \rightarrow H \rightarrow G \rightarrow C \rightarrow 1
\]

which we may spread out to a short exact sequence

\[
1 \rightarrow \mathcal{H} \rightarrow \mathcal{G} \rightarrow \mathcal{C} \rightarrow 1
\]

After enlarging \( S \), the commutative diagram with exact rows (for \( v \notin S \))

\[
\begin{array}{ccccccc}
\mathcal{C}(O_v) & \rightarrow & H^1(O_v, \mathcal{H}) & \rightarrow & H^1(O_v, \mathcal{G}) & \rightarrow & H^1(O_v, \mathcal{C}) \\
& \downarrow & \downarrow & \downarrow & & & \\
C(k_v) & \rightarrow & H^1(k_v, H) & \rightarrow & H^1(k_v, G) & \rightarrow & H^1(k_v, C)
\end{array}
\]

has the first vertical arrow an isomorphism by Lemma 5.3.6, the second an inclusion by hypothesis, and the last an inclusion by Lemma 5.3.5. A simple diagram chase now shows that the third vertical arrow is an inclusion.

\[\blacksquare\]

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Here is the crucial lemma that allows us to go beyond the smooth case.

**Lemma 5.3.8.** Let $k$ be a field, $G$ a connected commutative $k$-group scheme of finite type. Suppose that the underlying reduced scheme $G_{\text{red}} \subset G$ is not a smooth $k$-subgroup scheme. Then there is an exact sequence

$$1 \to H \to G \to R_{k'/k}(I')$$

for some finite purely inseparable extension $k'/k$ and some infinitesimal commutative $k'$-group scheme $I'$, such that $\dim(H) < \dim(G)$.

**Proof.** By descent from the perfect closure, there exists a finite purely inseparable extension $k'/k$ such that $(G_{k'})_{\text{red}} \subset G_{k'}$ is a smooth $k'$-subgroup scheme. Let $I' := (G_{k'})_{\text{red}}$, so $I'$ is infinitesimal. Let $H$ be the kernel of the composition $G \to R_{k'/k}(G_{k'}) \to R_{k'/k}(I')$. We need to show that $\dim(H) < \dim(G)$.

Suppose to the contrary that $\dim(H) = \dim(G)$. The composition $H_{k'} \to G_{k'} \to I'$ vanishes by definition of $H$. Thus, by definition of $I'$, we have $H_{k'} \subset (G_{k'})_{\text{red}}$. But $(G_{k'})_{\text{red}}$ is smooth and connected, so its only closed $k'$-subgroup scheme of the same dimension is $(G_{k'})_{\text{red}}$ itself. It follows that $H_{k'}$ is smooth, hence so is $H$. Therefore we must have $H = G_{\text{red}}$ (since $G$ is connected and $\dim(H) = \dim(G)$), so $G_{\text{red}}$ is a smooth $k$-subgroup scheme, violating our assumption. This contradiction shows that $\dim(H) < \dim(G)$. \qed

We may now complete the proof of Proposition 5.3.1 by induction on $\dim(G)$. Since the component group $G/G^0$ is finite, by Lemma 5.3.3 it suffices to prove Proposition 5.3.1 for $G^0$; i.e., we may assume $G$ is connected. If $G_{\text{red}} \subset G$ is a smooth $k$-subgroup scheme then by finiteness of $G/G_{\text{red}}$ we are done by Lemmas 5.3.3 and 5.3.4. If $G_{\text{red}}$ is not a smooth $k$-subgroup scheme then we apply Lemmas 5.3.8 and 5.3.7 together with induction, to conclude.

### 5.4 Integral annihilator aspects of duality between $H^1(k_v, G)$ and $H^1(k_v, \hat{G})$

We will prove the following reformulation of Theorem 1.2.7.

**Theorem 5.4.1.** Let $k$ be a global function field, $G$ an affine commutative $k$-group scheme of finite type, $\mathscr{G}$ an $O_S$-model of $G$. Then the map $H^1(O_v, \mathcal{F}) \to H^1(k_v, G)/H^1(O_v, \mathcal{F})^D$ induced by the composition $H^1(O_v, \mathcal{F}) \to H^1(k_v, \hat{G}) \to H^1(k_v, G)^D$ (the last map being induced by cup product) is a topological isomorphism for almost all $v$.

Let us review the topology on integral cohomology and how such integral cohomology groups map to the cohomologies over the local fields (to see that expressions such as $H^1(k_v, G)/H^1(O_v, \mathcal{F})$ make good sense and are suitable for Pontryagin duality for all but...
finitely many \( v \). The topology on \( H^1(O_v, \mathcal{F}) \) is defined in a manner analogous to the topology on \( H^1(k_v, \hat{G}) \): it is the subspace topology coming from spreading out the dual of an exact sequence

\[
1 \rightarrow H \rightarrow G \rightarrow U \rightarrow 1
\]

with \( H \) an almost torus and \( U \) split unipotent. This yields a closed subgroup of the compact second-countable group \( H^1(O_v, \mathcal{F}) \), by a similar argument to that given in §4.6. In particular, due to compactness of the source and Hausdorffness of the target, it suffices in order to prove Theorem 5.4.1 to show that the map is an algebraic isomorphism.

In view of Lemma 5.2.1 and Proposition 5.3.1 we have for all but finitely many \( v \) continuous inclusions \( H^1(O_v, \mathcal{F}) \leftrightarrow H^1(k_v, G) \) and \( H^1(O_v, \mathcal{F}) \leftrightarrow H^1(k_v, \hat{G}) \) with Hausdorff target and compact source (compactness due to [Čes1, Prop. 2.9(d)] and Lemma 4.2.2 except for the case of \( H^1(O_v, \mathcal{F}) \) that follows from the fact – already noted in the preceding paragraph – that for all but finitely many \( v \) it is a closed subgroup of the compact group \( H^1(O_v, \mathcal{F}) \) for some almost torus \( H \) and \( O_S \)-model \( \mathcal{F} \) of \( H \). Hence, these subgroups are closed. That permits us to use Pontryagin duality to deduce from Theorem 5.4.1 that the cohomologies over \( O_v \) are orthogonal complements of each other under the local duality pairing over \( k_v \), as desired.

To prove Theorem 5.4.1 we first check that it holds for finite \( G \). When \( \text{char}(k) \nmid \#G \), this is part of classical Tate local duality. The general case is due to Milne [Mi, Ch. 3, Cor. 7.2] (which shows that the integral cohomology groups are orthogonal complements; that the map in the proposition is a topological isomorphism then follows from the topological aspects of local duality over \( k_v \)).

To go beyond finite \( G \), we next check that Theorem 5.4.1 holds for \( G = R_{k'/k}(G_m) \) by noting that \( H^1(k_v, \mathcal{F}) \) and \( H^1(k_v, \hat{G}) \) vanish for all but finitely many \( v \) (as we have already seen a few times).

Next we turn to the case when \( G \) is an almost torus over \( k \). By Lemma 2.1.3 iv), after harmlessly modifying \( G \) we may assume that there is an exact sequence

\[
1 \rightarrow B \rightarrow X \rightarrow G \rightarrow 1
\]

where \( X = C \times R_{k'/k}(T') \) for a finite separable extension field \( k'/k \) and split \( k' \)-torus \( T' \), and \( B \) and \( C \) are finite commutative \( k \)-group schemes. We may spread this out to obtain exact sequences of affine flat commutative group schemes of finite type over some \( O_S \):

\[
1 \rightarrow \mathcal{B} \rightarrow \mathcal{X} \rightarrow \mathcal{F} \rightarrow 1,
\]

\[
1 \rightarrow \hat{\mathcal{F}} \rightarrow \hat{\mathcal{X}} \rightarrow \hat{\mathcal{B}} \rightarrow 1.
\]

Define the covariant functor \( Q^i = H^i(k_v, (\cdot)_{k_v})/H^i(O_v, \cdot) \) on affine flat commutative \( O_v \)-
group schemes of finite type. In the commutative diagram

\[
\begin{array}{c}
\hat{X}(O_v) \\
\downarrow \\
Q^2(\hat{X})^D
\end{array} \quad \begin{array}{c}
\hat{B}(O_v) \\
\downarrow \\
Q^2(\hat{B})^D
\end{array} \quad \begin{array}{c}
H^1(O_v, \hat{G}) \\
\downarrow \\
H^1(O_v, \hat{F})
\end{array} \quad \begin{array}{c}
H^1(O_v, \hat{F}) \\
\downarrow \\
H^1(O_v, \hat{B})
\end{array} \quad \begin{array}{c}
H^1(O_v, \hat{G}) \\
\downarrow \\
H^1(O_v, \hat{B})
\end{array}
\]

(5.4.1)

the top row is clearly exact, the first two vertical arrows are isomorphisms by Theorem 1.2.5, and the last two are isomorphisms by the already-treated cases of finite group schemes and separable Weil restrictions of split tori. We claim that the bottom row is exact at the second and third entries for almost all \(v\). (It is actually exact everywhere for almost all \(v\), but we will not need this.) Assuming that, a simple diagram chase shows that the middle vertical arrow is an isomorphism, as desired.

We shall now check exactness at the second entry along the bottom of (5.4.1) for almost all \(v\). Consider the 3-term complex of continuous maps

\[
\begin{array}{c}
H^1(k_v, G) \\
H^1(O_v, \hat{G}) \\
H^1(O_v, \hat{F})
\end{array} \rightarrow \begin{array}{c}
H^2(k_v, B) \\
H^2(O_v, \hat{B}) \\
H^2(O_v, \hat{F})
\end{array} \rightarrow \begin{array}{c}
H^2(k_v, X) \\
H^2(O_v, \hat{X})
\end{array}
\]

in which the last 2 terms are discrete (the \(k_v\)-cohomologies are discrete in degree 2). This is exact for almost all \(v\) because \(H^2(O_v, \hat{B})\) and \(H^2(O_v, \hat{X})\) vanish for all but finitely many \(v\) (by Theorem 1.2.5). Thus, the map

\[
Q^2(\hat{B})/\text{im} Q^1(\hat{F}) \rightarrow Q^2(\hat{X})
\]

between discrete groups is an inclusion, so we have an exact sequence of \(\mathbb{Q}/\mathbb{Z}\)-duals of discrete groups

\[
Q^2(\hat{X})^* \rightarrow Q^2(\hat{B})^* \rightarrow (\text{im} Q^1(\hat{F}))^*.
\]

This yields exactness at the second term along the bottom of (5.4.1).

To complete the proof of Theorem 5.4.1 for almost tori, it remains to check exactness at the third entry along the bottom of (5.4.1). It suffices to show that the complex of continuous maps

\[
Q^1(\hat{X}) \rightarrow Q^1(\hat{F}) \rightarrow Q^2(\hat{B})
\]

between locally compact Hausdorff groups with discrete third term is algebraically exact for almost all \(v\). The algebraic exactness is immediate for almost all \(v\) because \(H^2(O_v, \hat{B}) = 0\) for almost all \(v\) (by Theorem 1.2.5).

Before turning to the general case of Theorem 5.4.1 we prove Theorem 1.2.6 for \(G_a\):

**Lemma 5.4.2.** For a local function field \(k\), the map \(H^2(O_v, \hat{G}_a) \rightarrow (G_a(k_v)/G_a(O_v))^D = (k_v/O_v)^D\) is an isomorphism.

Equivalently, the map \(H^2(O_v, \hat{G}_a) \rightarrow H^2(k_v, \hat{G}_a)\) is injective with image that is the exact annihilator of \(G_a(O_v)\) under the cup product duality pairing.
The equivalence between the two formulations is immediate from the local duality (in the case of \( G_a \)) that we have proved for commutative affine group schemes of finite type, since the degree-2 cohomology is discrete.

**Proof.** First we check that the map \( H^2(\mathcal{O}_v, \hat{G}_a) \to H^2(k_v, \hat{G}_a) \) is injective. By Propositions 2.4.6 and 2.4.8 it suffices to check that the map \( Br(G_a, \mathcal{O}_v) \to Br(G_a, k_v) \) is injective, and this in turn holds because \( G_a, \mathcal{O}_v \) is a regular scheme.

Next we check that \( H^2(\mathcal{O}_v, \hat{G}_a) \) is the exact annihilator of \( \mathcal{O}_v \). By Propositions 2.4.6 and 2.4.8, Proposition 2.6.1, and Lemma 2.4.9, it suffices to prove that for a uniformizer \( \pi \) of \( \mathcal{O}_v \), if \( \lambda \in k_v \) satisfies \( \lambda \alpha d\pi = 0 \) in \( Br(k_v) \) for every \( \alpha \in \mathcal{O}_v \), then \( \lambda \in \mathcal{O}_v \). That is, we will show that if \( \lambda \notin \mathcal{O}_v \), then \( \lambda \alpha d\pi \neq 0 \) for some \( \alpha \in \mathcal{O}_v \).

Let \( F_v \) denote the finite residue field of \( \mathcal{O}_v \). Choose \( c \in F_v \) such that \( \text{Tr}_{F_v/F_p}(c) \neq 0 \). Write \( \lambda = \sum_{n \geq -N} b_n \pi^n \) with \( b_n \in F_v, N > 0 \), and \( b_{-N} \neq 0 \). Choose \( \alpha := c \pi^{-1}/b_{-N} \), so \( \text{Res}(\lambda \alpha d\pi) = c \). Then \( \text{inv}(\lambda \alpha d\pi) = \text{Tr}_{F_v/F_p}(\text{Res}(\lambda \alpha d\pi)) \neq 0 \) by construction. We therefore have \( \lambda \alpha d\pi \neq 0 \) as a Brauer class, as desired. \( \square \)

We may now complete the proof of Theorem 5.4.1. Let \( G \) be an affine commutative \( k \)-group scheme of finite type. By induction on the dimension of the unipotent radical of \( (G^0)_{\text{red}} \) (the 0-dimensional case being the already-treated case of almost tori), we may assume (by Lemma 2.1.7) that we have an exact sequence

\[
1 \to H \to G \to G_a \to 1
\]

such that Theorem 5.4.1 holds for \( H \). We may then spread this out to obtain exact sequences over some \( \mathcal{O}_S \):

\[
1 \to \mathcal{H} \to \mathcal{G} \to G_a \to 1,
1 \to \hat{G}_a \to \hat{G} \to \hat{H} \to 1.
\]

In the commutative diagram of homomorphisms

\[
\begin{array}{c}
0 \to H^1(\mathcal{O}_v, \mathcal{G}) \to H^1(\mathcal{O}_v, \mathcal{H}) \to H^2(\mathcal{O}_v, \hat{G}_a) \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
0 \to Q^1(\mathcal{G})^D \to Q^1(\mathcal{H})^D \to (k_v/\mathcal{O}_v)^D
\end{array}
\]

the top row is exact because \( H^1(\mathcal{O}_v, \hat{G}_a) = 0 \) (by Lemma 2.4.3), the middle vertical arrow is a topological isomorphism by hypothesis, and the last vertical arrow is an algebraic isomorphism by Lemma 5.4.2. The bottom row is exact at \( Q^1(\mathcal{G})^D \) because the map \( H^1(k_v, H) \to H^1(k_v, G) \) is surjective (since \( H^1(k_v, G_a) = 0 \)). A simple diagram chase now shows that the first vertical arrow is an algebraic isomorphism. This completes the proof of Theorem 5.4.1.
5.5 Integral annihilator aspects of duality between $H^2(k_v, \hat{G})$ and $G(k_v)$

Let us first note that for any affine commutative $G$ of finite type, the injectivity of the map $G(\mathcal{O}_v) \to G(k_v)$ is trivial for almost all $v$. Indeed, we may spread $G$ out to some affine $\mathcal{O}_S$-model $\mathcal{G}$, and the injectivity for affine $\mathcal{G}$ is obvious.

Much more interesting is the fact that $H^2(\mathcal{O}_v, \mathcal{G})$ is the exact annihilator of $\mathcal{G}(\mathcal{O}_v)$ for almost all $v$. We will prove this together with the (also much deeper) injectivity of the map $H^2(\mathcal{O}_v, \mathcal{G}) \to H^2(k_v, \hat{G})$ by proving the following slight reformulation of Theorem 1.2.6.

**Theorem 5.5.1.** Let $k$ be a global function field, $G$ an affine commutative $k$-group scheme of finite type, $\mathcal{G}$ an $\mathcal{O}_S$-model of $G$. Then the continuous map $H^2(\mathcal{O}_v, \mathcal{G}) \to (G(k_v)/\mathcal{G}(\mathcal{O}_v))^*$ is an isomorphism for almost all $v$.

**Remark 5.5.2.** Note that the injectivity of the map $H^2(\mathcal{O}_v, \mathcal{G}) \to H^2(k_v, \hat{G})$ for almost all $v$ follows immediately, since the map $H^2(\mathcal{O}_v, \mathcal{G}) \to G(k_v)^*$ factors as a composition $H^2(\mathcal{O}_v, \mathcal{G}) \to H^2(k_v, \hat{G}) \to G(k_v)^*$. That $H^2(\mathcal{O}_v, \mathcal{G}) \subset H^2(k_v, \hat{G})$ is the exact annihilator of $\mathcal{G}(\mathcal{O}_v)$ also follows immediately from Theorem 5.5.1 in conjunction with local duality (more precisely, Theorem 1.2.2).

Theorem 5.5.1 holds for finite $G$ by Theorem 1.2.5 and Cartier duality, and it was proved for $\mathbb{G}_a$ in Lemma 5.4.2. We next prove it for separable Weil restrictions of $G_m$:

**Lemma 5.5.3.** For a global function field $k$ and finite separable extension $k'/k$, Theorem 5.5.1 holds for $G = \text{R}_{k'/k}(G_m)$.

**Proof.** We may assume that $S$ contains the set of places of $k$ unramified in $k'$, and that $\mathcal{G} := \text{R}_{\mathcal{O}_{S'}}/\mathcal{O}_S(G_m)$. By Lemma 2.1.4, $\text{R}_{\mathcal{O}_{S'}/\mathcal{O}_S}(G_m) = \text{R}_{\mathcal{O}_{S'}/\mathcal{O}_S}(\mathbb{Z})$. Since the constant group $\mathbb{Z}$ is smooth, we may consider cohomology relative to the étale topology. For each $v \notin S$ we have by Proposition C.1.3 a commutative diagram with vertical isomorphisms

$$
\begin{array}{ccc}
H^2(\mathcal{O}_v, \text{R}_{\mathcal{O}_{S'}/\mathcal{O}_S}(\mathbb{Z})) & \longrightarrow & (\text{R}_{\mathcal{O}_{S'}/\mathcal{O}_S}(G_m)(k_v)/\text{R}_{\mathcal{O}_{S'}/\mathcal{O}_S}(G_m)(\mathcal{O}_v))^* \\
\downarrow & & \downarrow \\
\prod_{v'|v} H^2(\mathcal{O}_{v'}, \mathbb{Z}) & \longrightarrow & \prod_{v'|v} (k_v^\times/\mathcal{O}_{v'}^\times)^*
\end{array}
$$

in which the products are taken over all places $v'$ of $k'$ lying above $v$. We are therefore reduced to the case $k' = k$.

Our task now is to show that the map $H^2(\mathcal{O}_v, \mathbb{Z}) \to (k_v^\times/\mathcal{O}_v^\times)^*$ is an isomorphism. Via the exact sequence of étale sheaves

$$
0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0
$$
and the fact that $H^i(O_v, \mathbb{Q}) = 0$ for all $i > 0$ (this actually holds for any noetherian normal scheme), we see that $H^2(O_v, \mathbb{Z}) = \text{Hom}_{\text{cts}}(\pi_1(O_v), \mathbb{Q}/\mathbb{Z}) = (\text{Gal}(k_{nr_v}/k_v))^*$, where $k_{nr_v}$ is the maximal unramified extension of $k_v$. Further, the map $(\text{Gal}(k_{nr_v}/k_v))^* \to (k_v^\times/O_v^\times)^*$ is compatible with the map $\text{Gal}(k_{ab_v}/k_v)^* \to (k_v^\times)^*$ that is the Pontryagin dual of the local reciprocity map (see Lemma 4.5.2). What we want to show, therefore, is that $O_v^\times$ is the kernel of the composition $k_v^\times \to \text{Gal}(k_{ab_v}/k_v) \to \text{Gal}(k_{nr_v}/k_v)$, and this follows from local class field theory.

Now let $G$ be an almost torus over $k$. To prove Theorem 5.5.1 for $G$, by Lemma 2.1.3(iv) we may harmlessly modify $G$ so that there is an exact sequence

$$1 \rightarrow B \rightarrow X \rightarrow G \rightarrow 1$$

where $B$ is a finite commutative $k$-group scheme and $X = C \times R_{k'/k}(T')$ for a finite commutative $k$-group scheme $C$, a finite separable extension $k'/k$, and a split $k'$-torus $T'$. This spreads out to yield exact sequences over some $O_S$:

$$1 \rightarrow \mathcal{B} \rightarrow \mathcal{X} \rightarrow \mathcal{G} \rightarrow 1,$$

$$1 \rightarrow \widehat{\mathcal{B}} \rightarrow \widehat{\mathcal{X}} \rightarrow \widehat{\mathcal{G}} \rightarrow 1.$$

After possibly enlarging $S$ we can arrange that in the resulting commutative diagram (for $v \notin S$)

$$
\begin{array}{cccccc}
H^1(O_v, \widehat{\mathcal{X}}) & \rightarrow & H^1(O_v, \widehat{\mathcal{B}}) & \rightarrow & H^2(O_v, \mathcal{B}) & \rightarrow & H^2(O_v, \mathcal{X}) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
Q^1(\mathcal{X})^* & \rightarrow & Q^1(\mathcal{B})^* & \rightarrow & Q^0(\mathcal{G})^* & \rightarrow & Q^0(\mathcal{X})^* & & \\
\end{array}
$$

the top row is exact (because $H^2(O_v, \widehat{\mathcal{B}}) = 0$ for almost all $v$ by Theorem 1.2.5), the first two vertical arrows are isomorphisms (by Theorem 5.4.1), and the last vertical arrow is an isomorphism by the already-treated cases of finite group schemes and separable Weil restrictions of split tori. We claim that the bottom row is exact for almost all $v \notin S$. Assuming this, a simple diagram chase shows that the third vertical arrow is an isomorphism, as required.

We now prove exactness at the second entry in the bottom of (5.5.1); exactness at the third term goes similarly. It suffices to show (for almost all $v$) exactness of the complex

$$
\begin{array}{cccccc}
G(k_v) & \rightarrow & H^1(k_v, B) & \rightarrow & H^1(k_v, X) & \rightarrow & H^1(k_v, G) \\
\mathcal{G}(O_v) & \rightarrow & H^1(O_v, \mathcal{B}) & \rightarrow & H^1(O_v, \mathcal{X}) & \rightarrow & H^1(O_v, \mathcal{G}) \\
\end{array}
$$

indeed, this implies that the map $H^1(k_v, B)/\text{im} \left( \frac{G(k_v)}{\mathcal{G}(O_v)} \right) \rightarrow H^1(k_v, X)/\text{im} \left( \frac{G(k_v)}{\mathcal{G}(O_v)} \right)$ is an inclusion with closed image, in which case that inclusion is a homeomorphism onto a closed subgroup by
Lemma 4.3.3. Pontryagin duality would then be applicable to yield the exactness along the bottom of (5.5.1).

Let us check exactness at $H_1(k_v, B) / H_1(O_v, \mathcal{B})$; exactness at $H_1(k_v, X) / H_1(O_v, \mathcal{X})$ is similar. In the commutative diagram with exact rows

$$
\begin{array}{cccc}
\mathcal{G}(O_v) & \rightarrow & H^1(O_v, \mathcal{B}) & \rightarrow & H^1(O_v, \mathcal{X}) & \rightarrow & H^1(O_v, \mathcal{G}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
G(k_v) & \rightarrow & H^1(k_v, B) & \rightarrow & H^1(k_v, X) & \rightarrow & H^1(k_v, G)
\end{array}
$$

the last vertical arrow is an inclusion for almost all $v$ by Proposition 5.3.1. A simple diagram chase now shows that (5.5.2) is exact at $H_1(k_v, B) / H_1(O_v, \mathcal{B})$. This completes the proof of Theorem 5.5.1 for almost tori.

Now we turn to the general case. Let $G$ be an affine commutative $k$-group scheme of finite type. By induction on the dimension of the unipotent radical of $(G_k)_0^0$ (the 0-dimensional case corresponding to the settle case of almost tori), we may assume by Lemma 2.1.7 that there is an exact sequence

$$
1 \rightarrow H \rightarrow G \rightarrow G_a \rightarrow 1
$$
such that Theorem 5.5.1 holds for $H$. We may then spread this out to obtain exact sequences over some $O_S$:

$$
1 \rightarrow \mathcal{H} \rightarrow \mathcal{G} \rightarrow G_a \rightarrow 1, \\
1 \rightarrow \hat{G}_a \rightarrow \hat{\mathcal{G}} \rightarrow \hat{\mathcal{H}} \rightarrow 1.
$$

In the resulting commutative diagram

$$
\begin{array}{cccc}
H^1(O_v, \hat{\mathcal{H}}) & \rightarrow & H^2(O_v, \hat{G}_a) & \rightarrow & H^2(O_v, \mathcal{G}) & \rightarrow & H^2(O_v, \hat{\mathcal{H}}) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
Q^1(\mathcal{H})^* & \rightarrow & Q^0(G_a)^* & \rightarrow & Q^0(\mathcal{G})^* & \rightarrow & Q^0(\hat{\mathcal{H}})^*
\end{array}
$$

the top row is exact because $H^3(O_v, \hat{G}_a) = 0$ (Proposition 3.3.2), the first vertical arrow is an isomorphism for almost all $v$ by Theorem 5.4.1, the second is an isomorphism for all $v$ by Lemma 5.4.2, and the last vertical arrow is an isomorphism for almost all $v$ by hypothesis. We claim that the bottom row is exact for almost all $v$; assuming this, an easy diagram chase shows that the third vertical arrow is an isomorphism for such $v$, as required.

We check exactness at $Q^0(G_a)^*$; exactness at $Q^0(\mathcal{G})^*$ is similar. It suffices to show that for almost all $v$, the following complex is exact:

$$
\begin{array}{cccc}
G(k_v) & \rightarrow & G_a(k_v) & \rightarrow & H^1(k_v, H) & \rightarrow & H^1(k_v, G)
\end{array}
$$

(5.5.3)
Indeed, exactness for the first three terms gives that \( \left( \frac{G_a(k_v)}{G_a(O_v)} \right) / \text{im} \left( \frac{G(k_v)}{G_a(O_v)} \right) \to H^1(k_v, H) \) is an inclusion, and exactness for the last three terms gives that the image is closed, so it is a homeomorphism onto a closed subgroup (by Lemma 4.3.3), permitting us to conclude via Pontryagin duality. We will check exactness at \( G_a(k_v)/G_a(O_v) \); the proof of exactness at \( H^1(k_v, H)/H^1(O_v, H) \) is similar.

For almost all \( v \), in the commutative diagram with exact rows

\[
\begin{array}{cccccc}
\mathcal{O}_v & \longrightarrow & G_a(O_v) & \longrightarrow & H^1(O_v, H) & \longrightarrow & H^1(O_v, \mathcal{G}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
G(k_v) & \longrightarrow & G_a(k_v) & \longrightarrow & H^1(k_v, H) & \longrightarrow & H^1(k_v, G)
\end{array}
\]

the last vertical arrow is an inclusion by Proposition 5.3.1. A simple diagram chase thereby yields that (5.5.3) is exact at \( G_a(k_v)/G_a(O_v) \). This completes the proof of Theorem 5.5.1.
Chapter 6

Global Fields

In this chapter, we establish the main global theorems stated in §1.2. That is, we prove Theorems 1.2.8, 1.2.9, and 1.2.10. We begin by describing the relationship between the cohomology of $\mathbb{A}_k$ and that of the fields $k_v$ (§6.1). Among other things, this explains the relationship between our results stated in terms of adelic cohomology and the classical results of Poitou and Tate stated in terms of various products and restricted products of local cohomology groups. The first half of the chapter (as well as §6.12) proves the exactness of the nine-term exact sequence in Theorem 1.2.8 at various places. The second half is primarily concerned with the pairings between Tate-Shafarevich groups given by Theorem 1.2.9. Unlike the rest of the results in this manuscript, these results are proven from scratch, without using the case of finite commutative group schemes as a black box. This is done for two reasons: (i) it is convenient for the reader; and (ii) it allows us to avoid checking compatibility between our pairings and the pairings given in [Čes2] (defined in a totally different manner). Finally, the chapter concludes with a proof of the “dual nine-term exact sequence” (§6.16), obtained by dualizing the sequence in Theorem 1.2.8 and then applying local duality.

6.1 Relation between adelic and local cohomology

Let $\mathcal{F}$ be an fppf abelian sheaf on $\text{Spec}(\mathbb{A})$. Then the projection maps $\mathbb{A} \to k_v$ induce maps $H^i(\mathbb{A}, \mathcal{F}) \to \prod_v H^i(k_v, \mathcal{F})$. The purpose of this section is to study those maps. Here are the two main results.

**Proposition 6.1.1.** Let $k$ be a global field, $G$ a commutative $k$-group scheme of finite type. The map $H^i(\mathbb{A}, G) \to \prod_v H^i(k_v, G)$ is an inclusion for all $i$, and induces an isomorphism onto $\bigoplus_v H^i(k_v, G)$ for $i > 1$. If $G$ is smooth and connected, then the same holds for $i > 0$.

**Proposition 6.1.2.** Let $k$ be a global function field, $G$ an affine commutative $k$-group scheme of finite type. The map $H^i(\mathbb{A}, \hat{G}) \to \prod_v H^i(k_v, \hat{G})$ is an inclusion for $i \leq 2$.  

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Proof of Proposition 6.1.1. Let \( \mathcal{G} \) be an \( \mathcal{O}_S \)-model for \( G \), for some suitably large finite set \( S \) of places of \( k \). Since \( H^i(A, G) = \lim_{\to_S} \prod_{v \in S} H^i(k_v, G) \times H^i(\prod_{v \in S} \mathcal{O}_v, \mathcal{G}) \), the latter assertion of the proposition (about the isomorphism onto a direct sum) follows immediately from Proposition 5.1.1. The \( H^0 \) injectivity is clear. To prove the \( H^1 \) injectivity, it suffices to show that the map \( H^1(\prod_{v \in S} \mathcal{O}_v, \mathcal{G}) \to \prod_{v \in S} H^1(k_v, G) \) is an inclusion provided \( S \) is sufficiently large. For this, we factor the map as a composition \( H^1(\prod_{v \in S} \mathcal{O}_v, \mathcal{G}) \to \prod_{v \in S} H^1(k_v, G) \to H^1(k_v, G) \). The second map is an inclusion for suitably large \( S \) by Proposition 5.3.1. To show that the first map is an inclusion, let \( X \) be a \( \mathcal{G} \)-torsor over \( \hat{\mathcal{O}}_S := \prod_{v \in S} \mathcal{O}_v \) such that \( X(\mathcal{O}_v) \neq \emptyset \) for all \( v \notin S \). Then since \( X \) is a finitely presented algebraic space over \( \hat{\mathcal{O}}_S \), we have \( X(\hat{\mathcal{O}}_S) \neq \emptyset \) by Lemma 5.1.2.

The rest of this section will be occupied with the proof of Proposition 6.1.2. Let us first note that the proposition is clear for \( i = 0 \), so we may concentrate on the cases \( i = 1, 2 \). We once again note that \( H^i(A, \hat{G}) = \lim_{\to_S} \prod_{v \in S} H^i(k_v, \hat{G}) \times H^i(\hat{\mathcal{O}}_S, \mathcal{G}) \). It therefore suffices to show that the composition \( H^i(\hat{\mathcal{O}}_S, \mathcal{G}) \to \prod_{v \in S} H^i(\mathcal{O}_v, \mathcal{G}) \to \prod_{v \in S} H^i(k_v, \hat{G}) \) is injective for suitably large \( S \) and \( i \leq 2 \). The second map is an inclusion for large \( S \) by Theorem 1.2.6 for \( i = 1 \) and Theorem 1.2.7 for \( i = 2 \). It remains to show that the first map is an inclusion. This is encoded in the following two results:

**Proposition 6.1.3.** Let \( k \) be a global field, \( G \) an affine commutative \( k \)-group scheme of finite type. There exists a non-empty finite set \( S \) of places of \( k \) containing the archimedean places and an \( \mathcal{O}_S \)-model \( \mathcal{G} \) of \( G \) such that the map \( H^1(\hat{\mathcal{O}}_S, \mathcal{G}) \to \prod_{v \notin S} H^1(\mathcal{O}_v, \mathcal{G}) \) is an isomorphism (and so likewise for any \( S' \supset S \)).

**Proposition 6.1.4.** Let \( k \) be a global function field, \( G \) an affine commutative \( k \)-group scheme of finite type. There exists a non-empty finite set \( S \) of places of \( k \) and an \( \mathcal{O}_S \)-model \( \mathcal{G} \) of \( G \) such that the map \( H^2(\hat{\mathcal{O}}_S, \mathcal{G}) \to \prod_{v \notin S} H^2(\mathcal{O}_v, \mathcal{G}) \) is injective (and so likewise for any \( S' \supset S \)).

We begin by handling various special cases.

**Lemma 6.1.5.** Proposition 6.1.3 holds if \( G \) is a finite commutative \( k \)-group scheme.

**Proof.** We may replace \( \hat{G} \) with \( G \) (by Cartier duality) to reduce to finding \( S \) so that the map \( H^1(\hat{\mathcal{O}}_S, \mathcal{G}) \to \prod_{v \notin S} H^1(\mathcal{O}_v, \mathcal{G}) \) is an isomorphism when \( \mathcal{G} \) is finite flat over \( \mathcal{O}_S \) with rank \( n > 0 \). Injectivity follows by Lemma 5.1.2. For surjectivity, we consider for each \( v \notin S \) a \( \mathcal{G} \)-torsor \( \mathcal{X}_v \) over \( \mathcal{O}_v \) and seek to make a \( \mathcal{G} \)-torsor \( \mathcal{X} \) over \( \prod_{v \notin S} \mathcal{O}_v \). Each \( \mathcal{X}_v \) has the form \( \text{Spec}(B_v) \) with \( B_v \) a finite free \( \mathcal{O}_v \)-module of rank \( n \). Then \( B := \prod_{v \notin S} B_v \) is a finite free \( \hat{\mathcal{O}}_S \)-module of rank \( n \) and the co-action maps \( \alpha_v : B_v \to \mathcal{O}_v(\mathcal{G}) \otimes_{\mathcal{O}_v} B_v \) define a map

\[
\alpha = \prod \alpha_v : B \to \prod_{v} (\mathcal{O}_v(\mathcal{G}) \otimes_{\mathcal{O}_v} B_v) = \hat{\mathcal{O}}_S(\mathcal{G}) \otimes_{\hat{\mathcal{O}}_S} B
\]
are done because the direct product of the analogous maps on coordinate rings for each \( X_v \) have
\[ f \in \mathcal{O}_X \]
de\( \mathcal{O}_S \). It is easily checked that \( \alpha \) is a co-action map, which is to say that the \( \mathcal{O}_S \)-scheme \( X := \text{Spec}(B) \) is equipped with a \( \mathcal{G} \)-action and \( \mathcal{G} \)-compatible isomorphisms \( X_{\mathcal{O}_S} \cong \mathcal{X}_v \) for all \( v \not\in S \). We therefore just need to prove that \( X \) is a \( \mathcal{G} \)-torsor for the fppf topology. But \( X \) is an fppf \( \mathcal{O}_S \)-scheme, so it suffices to show that the natural map
\[ \mathcal{G} \times_{\mathcal{O}_S} X \to X \times_{\mathcal{O}_S} X \]
defined by \((g, x) \mapsto (g \cdot x, x)\) is an isomorphism. This map of affine schemes corresponds to the direct product of the analogous maps on coordinate rings for each \( \mathcal{X}_v \) over \( \mathcal{O}_v \), so we are done because \( \mathcal{X}_v \) is a \( \mathcal{G} \)-torsor over \( \mathcal{O}_v \) for each \( v \not\in S \).

**Lemma 6.1.6.** Let \( \{R_i\}_{i \in I} \) be a set of local rings, and let \( R := \prod_{i \in I} R_i \). For a subset \( J \subset I \), let \( R_J := \prod_{i \in J} R_i \). Then any open cover of \( \text{Spec}(R) \) may be refined by one of the form \( \{\text{Spec}(R_{I_1}), \ldots, \text{Spec}(R_{I_n})\} \), where \( I = I_1 \amalg \cdots \amalg I_n \) is a finite partition of \( I \).

*Proof.* Any open cover of \( \text{Spec}(R) \) may be refined by \( \{\text{Spec}(R_{I_1}), \ldots, \text{Spec}(R_{I_n})\} \) for some \( f_i \in R \) that generate the unit ideal. Since each \( R_i \) is local, it follows that for each \( i \in I \), we have \( (f_j)_i \in R_i^\times \) for some \( 1 \leq j \leq m \). Choose one such \( j := j(i) \) for each \( i \in I \). Then, for \( 1 \leq m \leq n \), we may take \( I_m := \{i \in I \mid j(i) = m\} \).

**Lemma 6.1.7.** Let \( k'/k \) be a finite separable extension of global fields, let \( S \) be a non-empty finite set of places containing all archimedean places and all places of \( k \) that ramify in \( k' \), and let \( S' \) be the set of places of \( k' \) lying above \( S \). Then \( H^1(\mathcal{O}_S, R_{\mathcal{O}_{S'}/\mathcal{O}_S}(G_m)) = 0 \), and \( H^1(\mathcal{O}_v, R_{\mathcal{O}_{S'}/\mathcal{O}_S}(G_m)) = 0 \) for \( v \not\in S \).

*Proof.* By Lemma 6.2.1, we have \( R_{\mathcal{O}_{S'}/\mathcal{O}_S}(G_m) = R_{\mathcal{O}_{S'}/\mathcal{O}_S}(G_m) = R_{\mathcal{O}_{S'}/\mathcal{O}_S}(Z) \). The Leray spectral sequence for the finite morphism \( \text{Spec}(\mathcal{O}_{S'}) \to \text{Spec}(\mathcal{O}_S) \) yields an inclusion \( H^1(\mathcal{O}_v, R_{\mathcal{O}_{S'}/\mathcal{O}_S}(Z)) \hookrightarrow H^1(\prod_{v \not\in S} \mathcal{O}_v, Z) \). We therefore see that it suffices to treat the case \( k' = k \), and a similar argument applies to the first assertion of the lemma.

Since the constant \( \mathcal{O}_S \)-group \( Z \) is smooth, we may take our cohomology to be étale. That \( H^1(\mathcal{O}_v, Z) = 0 \) now follows because \( \mathcal{O}_v \) is noetherian and normal. To show \( H^1(\mathcal{O}_S, Z) = 0 \), we first show that \( H^1((\mathcal{O}_S)_p, Z) = 0 \) for each prime \( p \in \text{Spec}(\mathcal{O}_S) \).

By Lemma 6.2.1, \((\mathcal{O}_S)_p \) is a normal domain, so it suffices to show that \( H^1(A, Z) = 0 \) for any normal domain \( A \). But \( A \) is the direct limit of its finitely generated \( Z \)-algebras, and because \( A \) is a normal domain, the normalization of any such subalgebra (which is a module-finite extension, due to excellence) is also contained in \( A \). It follows that \( A \) is the direct limit of its noetherian normal subrings, so we may assume that \( A \) is noetherian. Then we have the equality \( H^1(A, Z) = \text{Hom}_{cts}(\pi_1(\text{Spec}(A)), Z) \) where \( Z \) is discrete, and this vanishes because \( \pi_1(\text{Spec}(A)) \) is profinite and \( Z \) contains no nontrivial finite subgroup.
We conclude that $H^1((\mathcal{O}_S)_p, \mathbb{Z}) = 0$ for every prime $p$ of $\mathcal{O}_S$. By spreading out, it then follows that for any $\alpha \in H^1(\mathcal{O}_S, \mathbb{Z})$ there is a Zariski-open cover $\{U_i\}$ of $\text{Spec}(\mathcal{O}_S)$ such that $\alpha|_{U_i} = 0$. By Lemma 6.1.6, this cover may be refined to the open cover arising from a partition of the index set $\mathcal{I}$ (the set of places outside $S$) into finitely many pairwise disjoint subsets, so clearly $\alpha = 0$.

**Lemma 6.1.8.** We have $H^1(\mathcal{O}_v, \widehat{\mathbb{G}}_a) = 0$ and $H^1(\mathcal{O}_S, \widehat{\mathbb{G}}_a) = 0$.

The first assertion is an immediate consequence of Lemma 2.4.3. The second follows from the same lemma once we prove:

**Proposition 6.1.9.** Let $\{R_i\}_{i \in I}$ be a set of valuation rings, and let $A := \prod_{i \in I} R_i$. Then $\text{Pic}(A[X_1, \ldots, X_n]) = 0$.

To prove this result, we shall use the criterion in the following lemma.

**Lemma 6.1.10.** Let $X$ be a reduced scheme such that $\text{Pic}(X) = 0$. If $\text{Pic}(A^n_{\mathcal{O}_{X,x}}) = 0$ for every $x \in X$ then $\text{Pic}(A^n_X) = 0$; here, $A^n_Y$ denotes affine $n$-space over a scheme $Y$ (no relationship with adele rings).

**Proof.** Let $\mathcal{L}$ be a line bundle on $A^n_X$, and let $0 : X \to A^n_X$ denote the zero section. By our assumptions, there is a trivialization of $0^*\mathcal{L}$. Fix one such trivialization $\phi$. Given an open subset $U \subset X$ such that $\mathcal{L}|_{A^n_U}$ is trivial, there is a unique trivialization of $\mathcal{L}|_{A^n_U}$ such that its pullback along the 0 section is compatible with $\phi$. Indeed, existence is clear and uniqueness follows from the equality $\Gamma(A^n_U, \mathbb{G}_m) = \Gamma(U, \mathbb{G}_m)$, which holds because $X$ is reduced.

Thus, if there exists an open cover $\{U_i\}_{i \in I}$ of $X$ such that each $\mathcal{L}|_{A^n_{U_i}}$ is trivial then once we modify these trivializations to be compatible with $\phi$ we see that they must glue, hence yield a trivialization of $\mathcal{L}$. But there exists such an open cover because for each $x \in X$ the pullback $\mathcal{L}|_{A^n_{\mathcal{O}_{X,x}}}$ is trivial by hypothesis and such triviality spreads out over some open neighborhood of $x$ in $X$.

To prove Proposition 6.1.9 it remains to verify the hypotheses of Lemma 6.1.10 for the visibly reduced scheme $X := \text{Spec}(A)$. To see that $\text{Pic}(A) = 0$, we note for a $\mathbb{G}_m$-torsor $\mathcal{V}$ over $A$ we have $\mathcal{V}(R_i) \neq \emptyset$ for each $i$ (as each $R_i$ is local), so $\mathcal{V}(A) \neq \emptyset$ by Lemma A.2.2. (This argument shows more generally that for an arbitrary collection of commutative rings $\{R_i\}$, $\text{Pic}(\prod R_i) \to \prod \text{Pic}(R_i)$ is injective.) Finally, we have to check that $\text{Pic}(A_p[X_1, \ldots, X_n]) = 0$ for every prime $p$ of $A$. Each local ring of $A$ is a valuation ring by Lemma A.2.2, so we only need to prove:

**Proposition 6.1.11.** If $R$ is a valuation ring, then $\text{Pic}(R[X_1, \ldots, X_n]) = 0$. 

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Proof. First, we may write \( R \) as the direct limit of its valuation subrings that have the same fraction field as a finitely generated \( \mathbb{Z} \)-subalgebra (by writing \( K = \text{Frac}(R) \) as the direct limit of the subfields that are finitely generated (as fields) over the prime field). Since any such valuation ring is of finite height [Tem, Cor. 2.1.3], we may assume \( R \) is of finite height.

We now recall a standard method for obtaining new valuation rings from old ones. Given a valuation ring \( R_0 \) with residue field \( \kappa \), let \( \overline{R} \subset \kappa \) be the valuation subring associated to a valuation on \( \kappa \). The preimage \( R \) of \( \overline{R} \) inside \( R_0 \) is a valuation ring (and its fraction field obviously coincides with that of \( R_0 \)). One says that \( R \) is composed from the valuation rings \( R_0, \overline{R} \). Any valuation ring with finite positive height on a field may be obtained via repeated compositions beginning with \( R_0 \) of height 1 with each residual \( \overline{R} \) also of height 1 [Tem, Lemma 2.1.4(iii)].

Thus, if we show that the lemma for \( R_0, \overline{R} \) implies it for \( R \) then by induction we will be reduced to the case in which \( R \) is of height 1 (since the case of a trivial valuation ring, which is to say a field, is obvious). Suppose \( \text{Pic}(R_0[X_1, \ldots, X_n]) \) and \( \text{Pic}(\overline{R}[X_1, \ldots, X_n]) \) vanish. Observe that

\[
R[X_1, \ldots, X_n] = R_0[X_1, \ldots, X_n] \times_{\kappa[X_1, \ldots, X_n]} \overline{R}[X_1, \ldots, X_n],
\]

and it is well-known that \( \text{Pic}(\kappa[X_1, \ldots, X_n]) = 0 \). By [CCO, Prop.1.4.11(1)] applied to the module \( M \) of global sections of a line bundle on \( R[X_1, \ldots, X_n] \), we have \( M \simeq M_0 \times_{M_\kappa} \overline{M} \) where \( M_0 := M \otimes_{R[X_1, \ldots, X_n]} R_0[X_1, \ldots, X_n] \), and similarly for \( M_\kappa \) and \( \overline{M} \). Thus, if we choose a trivialization of \( \overline{M} \), pull this back to a trivialization of \( M_\kappa \), and choose an arbitrary trivialization of \( M_0 \) we can use the obvious fact that \( R_0[X_1, \ldots, X_n]^\times \rightarrow \kappa[X_1, \ldots, X_n]^\times \) is surjective (it is the same as \( R_0^\times \rightarrow \kappa^\times \)) to modify the \( M_0 \)-trivialization to be compatible with the one already chosen for \( M_\kappa \). This compatibility yields a trivialization of \( M \), so \( \text{Pic}(R[X_1, \ldots, X_n]) = 0 \) as desired.

We therefore may and do assume that \( R \) is a valuation ring of height 1. Let \( Y := \text{Spec}(R[X_1, \ldots, X_n]) \), \( U := \text{Spec}(K[X_1, \ldots, X_n]) \). Let \( j : U \rightarrow Y \) be the obvious map. Let \( m \) be the maximal ideal of \( R \), \( \kappa = R/m \) the residue field, and \( Z := \text{Spec}(\kappa[X_1, \ldots, X_n]) \), so the closed immersion \( i : Z \hookrightarrow Y \) is complementary to \( U \). Let \( \Gamma = K^\times/R^\times \) be the value group of \( R \) and \( v : K \rightarrow \Gamma \) the valuation on \( K \).

We will define a surjective morphism of sheaves \( \phi : j_* \mathcal{O}_Y^\times \rightarrow i_* \Gamma \), where by abuse of notation, \( \Gamma \) also denotes the constant sheaf on \( Z \) associated to the group \( \Gamma \). It suffices to define \( \phi \) on distinguished open subsets \( D(f) := \text{Spec}(R[X_1, \ldots, X_n]_f) \) for \( f \in R[X_1, \ldots, X_n] \). If \( f \in m[X_1, \ldots, X_n] \), then \( Z \cap D(f) = \emptyset \), so \( (i, \Gamma)(D(f)) = 0 \) and the map is then just the zero map. If \( f \in R[X_1, \ldots, X_n] - m[X_1, \ldots, X_n] \), then \( (i, \Gamma)(D(f)) = \Gamma \), so \( \phi \) on \( D(f) \)-sections has to correspond to a map \( \phi_f : (K[X_1, \ldots, X_n]_f)^\times \rightarrow \Gamma \). For nonzero \( h \in K[X_1, \ldots, X_n] \), we define \( v(h) \) to be the minimal \( v \)-valuation of the coefficients of \( h \). As an example, note that \( v(f) = 0 \) since \( f \in R[X_1, \ldots, X_n] - m[X_1, \ldots, X_n] \). Thus, for nonzero \( g \in K[X_1, \ldots, X_n] \), we choose \( n \geq 0 \) so that \( g f^n \in K[X_1, \ldots, X_n] \) and are motivated to define \( \phi_f(g) := v(g f^n) \).
To show \( \phi_f \) is a well-defined homomorphism, we first claim \( v : K[X_1, \ldots, X_n] - \{0\} \to \Gamma \) satisfies \( v(gh) = v(g) + v(h) \). The proof is the same as the classical Gauss' Lemma. Indeed, it is clear that \( v(ch) = v(c) + v(h) \) for \( c \in K^\times \) and \( h \in K[X_1, \ldots, X_n] - \{0\} \), so we reduce to the case \( v(g) = 0 \) and \( v(h) = 0 \), which is to say \( g, h \in R[X_1, \ldots, X_n] \) with nonzero reductions in \( \kappa[X_1, \ldots, X_n] \). Since \( \kappa[X_1, \ldots, X_n] \) is an integral domain, the reduction of \( gh \) is also nonzero, so \( v(gh) = 0 \), as desired. Thanks to the identity \( v(gh) = v(g) + v(h) \), the vanishing of \( v(f) \) ensures that \( \phi_f \) is well-defined and moreover is a homomorphism.

Finally, to make a sheaf homomorphism \( \phi \) it suffices to check that if \( D(g) \subset D(f) \) then \( \phi_f \) and \( \phi_g \) are compatible on \( D(g) \)-sections. If \( g \in m[X_1, \ldots, X_n] \) then this is trivial, so we may assume that \( g, f \in R[X_1, \ldots, X_n] - m[X_1, \ldots, X_n] \). Since \( f|_{D(g)} \) is a unit (as \( D(g) \subset D(f) \)), we have \( g^n = fh \) for some \( n > 0 \) and some \( h \in R[X_1, \ldots, X_n] \) (and obviously \( h \not\in m[X_1, \ldots, X_n] \) since the same holds for \( g \) and hence for \( g^n \)). This makes explicit that any fraction \( a/f^m \) with \( m \geq 1 \) and nonzero \( a \in K[X_1, \ldots, X_n] \) can be written as \( ah^m/g^m \) with nonzero \( ah^m \in K[X_1, \ldots, X_n] \). Our task is to check that \( \phi_f(a/f^m) = \phi_g(ah^m/g^m) \).

By definition we have \( \phi_f(a/f^m) = v(a) \) and \( \phi_g(ah^m/g^m) = v(ah^m) \), so since \( v(h) = 0 \) we are done; i.e., we have the desired map of sheaves \( \phi : j_* O_U^\times \to i_* \Gamma \).

To show that \( \phi \) is surjective, it suffices to check that each \( \phi_f \) is surjective. But that is trivial if \( f \in m[X_1, \ldots, X_n] \), whereas if \( f \in R[X_1, \ldots, X_n] - m[X_1, \ldots, X_n] \), then it follows from the fact that the map \( v : K^\times \to \Gamma \) is surjective. Going a step further, we claim that the diagram of sheaves

\[
1 \longrightarrow O_Y^\times \longrightarrow j_* O_U^\times \xrightarrow{\phi} i_* \Gamma \longrightarrow 1
\]

is a short exact sequence. It is easily seen to be a complex, so all that remains to be checked is the exactness in the middle. That is, we need to check that for each nonzero \( f \in R[X_1, \ldots, X_n] \), the sequence

\[
(R[X_1, \ldots, X_n]/f)^\times \longrightarrow (K[X_1, \ldots, X_n]/f)^\times \xrightarrow{\phi_f} (i_* \Gamma)(D(f))
\]

is exact. If \( f \in m[X_1, \ldots, X_n] - \{0\} \) then we need to check that \( (R[X_1, \ldots, X_n]/f)^\times = (K[X_1, \ldots, X_n]/f)^\times \). For a nonzero coefficient \( \pi \in m - \{0\} \) of \( f \) with minimal valuation, clearly \( \pi/f \) in \( R[X_1, \ldots, X_n] \) and so \( \pi \in (R[X_1, \ldots, X_n]/f)^\times \). But \( R[1/\pi] = K \) by the height-1 hypothesis (this is the only place where we use this), so \( R[X_1, \ldots, X_n]/f = K[X_1, \ldots, X_n]/f \) and thus \((6.1.2)\) is exact for such \( f \).

Next suppose \( f \in R[X_1, \ldots, X_n] - m[X_1, \ldots, X_n] \). We need to check that a unit \( g \in (K[X_1, \ldots, X_n]/f)^\times \) lies in \((R[X_1, \ldots, X_n]/f)^\times \) when \( \phi_f(g) = 0 \) (the converse being obvious). For this, it suffices to check that \( g \in R[X_1, \ldots, X_n] \) precisely when \( \phi_f(g) \geq 0 \), and in order to show that it suffices to check that for \( g \in K[X_1, \ldots, X_n] \), we have \( g \in R[X_1, \ldots, X_n] \) precisely when \( v(g) \geq 0 \). But that is obvious, so \((6.1.2)\) is exact in general and hence \((6.1.1)\) is short exact.

Taking the associated long exact cohomology sequence, we get an exact sequence

\[
K^\times \xrightarrow{v} \Gamma \longrightarrow \text{Pic}(R[X_1, \ldots, X_n]) \longrightarrow H^1(Y, j_* O_U^\times)
\]
Since \( v : K^\times \to \Gamma \) is surjective, the proof will be complete if we show that \( H^1(Y, j_* \mathcal{O}_Y^\times) = 0 \). Thanks to the Leray spectral sequence \( E_2^{p,q} = H^p(Y, R^q j_* \mathcal{O}_Y^\times) \Rightarrow H^{p+q}(U, \mathcal{O}_U^\times) \), this follows from the fact that \( H^1(U, \mathcal{O}_U^\times) = \text{Pic}(K[X_1, \ldots, X_n]) = 0 \). \( \square \)

**Lemma 6.1.12.** The map \( H^2(\hat{\mathcal{O}}_S, \hat{\mathcal{G}}_a) \to \prod_v \mathcal{H}^2(\mathcal{O}_v, \hat{\mathcal{G}}_a) \) is injective.

**Proof.** We have a natural inclusion \( H^2(\hat{\mathcal{O}}_S, \hat{\mathcal{G}}_a) \to \text{Br}(\mathcal{G}_a, \hat{\mathcal{O}}_S)_{\text{prim}} \) by Proposition [2.4.7][2.4.7] and Proposition [6.1.11][6.1.11] and likewise for cohomology over each \( \mathcal{O}_v \). It therefore suffices to show that the map \( \text{Br}(\mathcal{G}_a, \hat{\mathcal{O}}_S)[p] \to \prod_v \text{Br}(\mathcal{G}_a, \mathcal{O}_v)[p] \) is injective. For this, we shall use the Azumaya algebra interpretation of Brauer groups (and the fact that the torsion subgroup of the cohomological Brauer group agrees with the Azumaya Brauer group for affine schemes [Gal Ch. II, Thm. 1]).

Let us show more generally that for a product \( R := \prod_{i \in I} R_i \) of local rings \( R_i \) and an Azumaya algebra \( A \) over \( R \) that splits over each \( R_i \), \( A \) splits over \( R \). We first reduce to the case when \( A \) has constant fiber rank. Indeed, for each positive integer \( n \), let \( U_n \subset \text{Spec}(R) \) be the open subset on which \( A \) has rank \( n^2 \). By Lemma [6.1.6][6.1.6] the open cover \( \{U_n\} \) may be refined by one obtained by a finite partition of the index set \( I \). By restricting our attention to each open subset defined by this partition, we are reduced to the case in which \( A \) has constant fiber rank \( n^2 \). That is, we know (since each \( R_i \) is local, hence every vector bundle on it is trivial) that \( A \otimes_R R_i \simeq M_n(R_i) \) for each \( i \), and we want to show that \( A \simeq M_n(R) \) as \( R \)-algebras. It suffices to show that the natural map of \( R \)-algebras \( A \to \prod_{i \in I} (A \otimes_R R_i) \) is an isomorphism. But more generally, the natural map of \( R \)-modules \( M \to \prod_{i \in I} (M \otimes_R R_i) \) is an isomorphism for any finite projective \( R \)-module \( M \) (by realizing \( M \) as a direct factor of a finite free module to infer the result from the easy case of free \( M \)). \( \square \)

We may now prove Proposition [6.1.3][6.1.3]. First suppose that \( G \) is an almost torus. By Lemma [2.1.3][2.1.3](iv), after harmlessly modifying \( G \), we obtain an exact sequence

\[
1 \to B \to X \to G \to 1
\]

where \( X = C \times R_{k'/k}(T') \), \( k'/k \) is a finite separable extension, \( T' \) is a split \( k' \)-torus, and \( B \) and \( C \) are finite commutative \( k' \)-group schemes. We may then spread this out to obtain a “dualized” exact sequence (for the fppf topology on the category of all \( \mathcal{O}_S \)-schemes)

\[
1 \to \hat{\mathcal{B}} \to \hat{\mathcal{X}} \to \hat{\mathcal{B}} \to 1
\]

with \( \mathcal{B} \) a finite flat commutative \( \mathcal{O}_S \)-group scheme. In the commutative diagram with exact rows

\[
\begin{array}{c}
\hat{\mathcal{B}}(\hat{\mathcal{O}}_S) \to H^1(\hat{\mathcal{O}}_S, \hat{\mathcal{B}}) \to H^1(\hat{\mathcal{O}}_S, \hat{\mathcal{X}}) \to H^1(\hat{\mathcal{O}}_S, \hat{\mathcal{B}}) \\
\downarrow \downarrow \downarrow \\
\prod_v \hat{\mathcal{B}}(\mathcal{O}_v) \to \prod_v H^1(\mathcal{O}_v, \hat{\mathcal{B}}) \to \prod_v H^1(\mathcal{O}_v, \hat{\mathcal{X}}) \to \prod_v H^1(\mathcal{O}_v, \hat{\mathcal{B}})
\end{array}
\]
the last two vertical arrows are isomorphisms by Lemmas 6.1.5 and 6.1.7, and the first vertical arrow is an isomorphism because the Cartier dual \( \mathcal{B} \) is an affine (even finite) \( \mathcal{O}_S \)-group scheme. A simple diagram chase shows that the second vertical arrow is surjective, and it remains to show that this arrow is injective.

Letting \( N \) be the exponent of \( \mathcal{B} \), it suffices to show that in the related commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
\hat{\mathcal{X}}(\mathcal{O}_S)/N\hat{\mathcal{X}}(\mathcal{O}_S) & \longrightarrow & \hat{\mathcal{B}}(\mathcal{O}_S) & \longrightarrow & H^1(\hat{\mathcal{O}}_S, \hat{\mathcal{B}}) & \longrightarrow & H^1(\hat{\mathcal{O}}_S, \hat{\mathcal{X}}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
(\prod_v \hat{\mathcal{X}}(\mathcal{O}_v))/N\prod_v \hat{\mathcal{X}}(\mathcal{O}_v) & \longrightarrow & \prod_v \hat{\mathcal{B}}(\mathcal{O}_v) & \longrightarrow & \prod_v H^1(\mathcal{O}_v, \hat{\mathcal{B}}) & \longrightarrow & \prod_v H^1(\mathcal{O}_v, \hat{\mathcal{X}})
\end{array}
\]

the first vertical arrow is surjective after possibly enlarging \( S \).

By increasing \( S \) we may assume \( \mathcal{X} = \mathcal{C} \times R_{\mathcal{O}_{S'}/\mathcal{O}_S}(G_m) \) with \( \mathcal{C} \) finite flat over \( \mathcal{O}_S \) and \( \mathcal{O}_{S'} \) finite étale over \( \mathcal{O}_S \). It suffices to treat the analogous surjectivity questions for \( \mathcal{C} \) and \( R_{\mathcal{O}_{S'}/\mathcal{O}_S}(G_m) \) separately in place of \( \mathcal{X} \). The case of \( \mathcal{C} \) is trivial since then \( \hat{\mathcal{C}} \) is the Cartier dual that is represented by an affine scheme. The case of \( R_{\mathcal{O}_{S'}/\mathcal{O}_S}(G_m) \) is reduced by Lemma 2.2.1 to the surjectivity of the natural map

\[
Z(\hat{\mathcal{O}}_{S'}/NZ(\hat{\mathcal{O}}_{S'}) \to \prod_{v' \notin S'} (Z/NZ).
\]

Pick \( \alpha \in \prod_{v' \notin S'} (Z/NZ) \). Partition the index set \( I \) of the product (which is just the set of places of \( k' \) not in \( S' \)) into \( N \) pieces: \( I = \Pi_{n \in \mathbb{Z}/NZ} I_n \), where \( I_n := \{ i \in I \mid \alpha_i = n \} \). For \( J \subset S \), let \( \hat{\mathcal{O}}_{J} := \prod_{v \in J} \mathcal{O}_v \). We have \( Z(\hat{\mathcal{O}}_{S'}) = \Pi_{n \in \mathbb{Z}/NZ} Z(\hat{\mathcal{O}}_{I_n}) \). Clearly \( \prod_{i \in I_n} n \) comes from \( Z(\hat{\mathcal{O}}_{I_n}) \), so we are done.

Now we turn to the general case of Proposition 6.1.3. Let \( G \) be an affine commutative \( k \)-group scheme of finite type. By induction on the dimension of the unipotent radical of \( (G_{k'})_{\text{red}} \) (the dimension-0 case corresponding to the case of almost tori that we just settled), together with Lemma 2.1.7 we may assume that there is an exact sequence

\[
1 \to H \to G \to G_a \to 1
\]

such that Proposition 6.1.3 holds for \( H \). We may then spread this out to obtain an exact sequence

\[
1 \to \hat{G}_a \to \hat{\mathcal{G}} \to \hat{\mathcal{H}} \to 1
\]

for the fppf topology on the category of \( \mathcal{O}_S \)-schemes for some \( S \).

After possibly increasing \( S \), in the commutative diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & H^1(\hat{\mathcal{O}}_S, \hat{\mathcal{B}}) & \longrightarrow & H^1(\hat{\mathcal{O}}_S, \hat{\mathcal{X}}) & \longrightarrow & H^2(\hat{\mathcal{O}}_S, \hat{G}_a) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \prod_v H^1(\mathcal{O}_v, \hat{\mathcal{B}}) & \longrightarrow & \prod_v H^1(\mathcal{O}_v, \hat{\mathcal{X}}) & \longrightarrow & \prod_v H^2(\mathcal{O}_v, \hat{G}_a)
\end{array}
\]

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the rows are exact by Lemma 6.1.8, the middle vertical arrow is an isomorphism by assumption, and the last vertical arrow is an inclusion by Lemma 6.1.12. A simple diagram chase now shows that the first vertical arrow is an isomorphism. This completes the proof of Proposition 6.1.3.

Before we turn to the proof of Proposition 6.1.4, we need some preliminary lemmas.

Lemma 6.1.13. Let \( X \) be a noetherian normal scheme. Then \( H^i(X, \mathbb{Q}) = 0 \) for \( i > 0 \).

Proof. Since the constant commutative \( X \)-group \( \mathbb{Q} \) is smooth, we may take our cohomology to be étale. We may assume that \( X \) is connected, hence (because of normality) irreducible. Let \( \eta \) be its generic point, \( f : \eta \to X \) the canonical inclusion. We have a Leray spectral sequence

\[
E_2^{i,j} = H^i(X, R^j f_* \mathbb{Q}) \Longrightarrow H^{i+j}(\eta, \mathbb{Q})
\]

We first claim that \( R^j f_* \mathbb{Q} = 0 \) for \( j > 0 \). Indeed, it is the sheafification of the presheaf \( U \mapsto H^j(U, \mathbb{Q}) \). But \( U_\eta \) is a finite disjoint union of spectra of fields, so because of the vanishing of the higher Galois cohomology of \( \mathbb{Q} \) (due to its unique divisibility), we see that \( H^j(U_\eta, \mathbb{Q}) = 0 \) for \( j > 0 \). For the same reason, we have \( H^i(\eta, \mathbb{Q}) = 0 \) for \( i > 0 \). The lemma will follow, therefore, if we show that the natural map \( \mathbb{Q} \to f_* \mathbb{Q} \) is an isomorphism.

For any étale \( X \)-scheme \( U \) we have \( (f_* \mathbb{Q})(U) = \mathbb{Q}(U_\eta) \) by definition. Since \( U \) is a finite disjoint union of connected components, it suffices to show that \( U_\eta \) is connected when \( U \) is connected. (We follow the convention that connected spaces are non-empty by definition.) Because \( U \) is étale over \( X \), \( U_\eta \) is the disjoint union of the generic points of \( U \), so it suffices to show that \( U \) is irreducible. But \( U \) is connected (so non-empty), noetherian, and normal (because it is étale over the noetherian normal scheme \( X \)), so it is indeed irreducible.

Lemma 6.1.14. \( H^i(\widehat{\mathcal{O}}_S, \mathbb{Q}) = 0 \) for \( i > 0 \).

Proof. We will first show that \( H^i((\widehat{\mathcal{O}}_S)_p, \mathbb{Q}) = 0 \) for \( i > 0 \) and for every prime ideal \( p \) of \( \widehat{\mathcal{O}}_S \). By Lemma A.2.1, \((\widehat{\mathcal{O}}_S)_p \) is a normal domain and hence is the direct limit of its finite-type \( \mathbb{Z} \)-subalgebras \( A_i \). Replacing \( A_i \) with its normalization (which is finite over \( A_i \) due to excellence), we see that \((\widehat{\mathcal{O}}_S)_p \) is the direct limit of noetherian normal subrings \( A_i \). We therefore have \( H^i((\widehat{\mathcal{O}}_S)_p, \mathbb{Q}) = \lim_{\rightarrow} H^i(A_i, \mathbb{Q}) = 0 \) for \( i > 0 \) by Lemma 6.1.13.

The vanishing for Zariski-local rings for each fixed \( i > 0 \) provides an open cover \{\( U_j \)\} of \( \text{Spec}(\widehat{\mathcal{O}}_S) \) depending on \( i \) such that \( H^j(U_j, \mathbb{Q}) = 0 \) for each \( j \) (and the fixed choice of \( i \)). By Lemma 6.1.6, any open cover of \( \text{Spec}(\widehat{\mathcal{O}}_S) \) may be refined by one obtained by a finite partition of the index set \( I \) for the product \( \widehat{\mathcal{O}}_S = \prod_{v \in S} \mathcal{O}_v \). That is, there is a partition \( I = I_1 \Pi \cdots \Pi I_n \) (depending on \( i \)) such that \( H^j(\prod_{v \in I_m} \mathcal{O}_v, \mathbb{Q}) = 0 \) for \( 1 \leq m \leq n \). It follows that \( H^i(\widehat{\mathcal{O}}_S, \mathbb{Q}) = \prod_{m=1}^n H^i(\prod_{v \in I_m} \mathcal{O}_v, \mathbb{Q}) = 0 \) for our choice of \( i > 0 \).

Lemma 6.1.15. Let \( k'/k \) be a finite separable extension, \( T' \) a split \( k' \)-torus, and \( G := R_{k'/k}(T') \). Then Proposition 6.1.4 holds for \( G \).
Proof. We may assume $T' = G_m$. Let $S$ be a non-empty finite set of places of $k$ containing the places that ramify in $k'$, and let $S'$ be the set of places of $k'$ lying above $S$. We use $\mathcal{G} := R_{\hat{\mathcal{O}}_{S'}/\hat{\mathcal{O}}_S}(G_m)$ as an $\mathcal{O}_S$-model of $G$. By Lemma 2.1.3(iv), we may harmlessly modify the first two vertical arrows are isomorphisms by Proposition 6.1.3, and the last vertical arrow is an inclusion on the $R_{k'/k}(T')$-factor by Lemma 6.1.15 and an inclusion on the $C$-factor after possibly increasing $S$ because $\text{H}^2(\hat{\mathcal{O}}_S, \mathcal{G}) = 0$ (by Proposition 5.1.1). A simple
Now consider the general case. That is, suppose that $G$ is a commutative affine $k$-group scheme of finite type. By induction on the dimension of the unipotent radical of $(G_{\text{red}})^{\text{red}}$ (the dimension-0 case corresponding to the settled case of almost tori), we may (by Lemma 2.1.7) suppose that there is an exact sequence

$$1 \rightarrow H \rightarrow G \rightarrow G' \rightarrow 1$$

such that Proposition 6.1.4 holds for $H$. We spread this out to get an exact sequence for the fppf topology on the category of schemes over some $O_S$:

$$1 \rightarrow \widehat{G}_a \rightarrow \widehat{G} \rightarrow \widehat{H} \rightarrow 1.$$ 

After possibly enlarging $S$, in the commutative diagram with exact rows

$$
\begin{array}{cccccc}
H^1(O_S, \widehat{H}) & \rightarrow & H^2(O_S, \widehat{G}_a) \\
\downarrow & & \downarrow & & \downarrow \\
\prod_v H^1(O_v, \widehat{H}) & \rightarrow & \prod_v H^2(O_v, \widehat{G}_a) & \rightarrow & \prod_v H^2(O_v, \widehat{G}) & \rightarrow & \prod_v H^2(O_v, \widehat{H})
\end{array}
$$

the first vertical arrow is an isomorphism by Proposition 6.1.3, the second vertical arrow is an inclusion by Lemma 6.1.12, and the last vertical arrow is an inclusion by hypothesis. A simple diagram chase now shows that the third vertical arrow is an inclusion, so the proof of Proposition 6.1.4 is complete!

### 6.2 Exactness properties for profinite completions of rational points

The goal of this section is to prove the following two results.

**Proposition 6.2.1.** Let $k$ be a local function field, and suppose that we have an exact sequence

$$1 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 1$$

of affine commutative $k$-group schemes of finite type. Then the sequence

$$1 \rightarrow G'(k)_{\text{pro}} \rightarrow G(k)_{\text{pro}} \rightarrow G''(k)_{\text{pro}}$$

is exact.

Recall that for a global field $k$ and a $k$-group scheme $G$, we endow $G(k)$ with the discrete topology.
Proposition 6.2.2. Let $k$ be a global function field, and suppose that we have an exact sequence

$$1 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 1$$

of affine commutative $k$-group schemes of finite type. Then the sequences

$$1 \rightarrow G'(k)_{\text{pro}} \rightarrow G(k)_{\text{pro}} \rightarrow G''(k)_{\text{pro}}$$

and

$$1 \rightarrow G'(A)_{\text{pro}} \rightarrow G(A)_{\text{pro}} \rightarrow G''(A)_{\text{pro}}$$

are exact.

Proof of Proposition 6.2.1. Perhaps one can prove this directly, but it is easier at this stage to simply use local duality. It suffices to check that the discrete Pontryagin dual sequence is algebraically exact. By local duality (in particular, Theorem 1.2.2), this is equivalent to the exactness of the sequence

$$H^2(k, \hat{G}''_v) \rightarrow H^2(k, \hat{G}_v) \rightarrow H^2(k, \hat{G}_v') \rightarrow 0$$

This exactness follows from Proposition 3.4.1 \qed

The exactness of the second sequence in Proposition 6.2.2 follows immediately from Proposition 6.2.1 and the following result.

Proposition 6.2.3. Let $k$ be a global field, $G$ a $k$-group scheme of finite type. Then the natural map

$$G(A)_{\text{pro}} \rightarrow \prod_v G(k_v)_{\text{pro}}$$

is a topological isomorphism.

Proof. It suffices to show that the map is a bijection, because it is clearly continuous, the source is compact, and the target is Hausdorff. First, the map clearly has dense image, so, since the image is compact, the map is surjective. To see that it is injective, let $X \subset G(A)$ be a closed subgroup of finite index. We will show that there is a finite set $S$ of places of $k$, and closed subgroups $X_v \subset G(k_v)$ of finite index for each $v \in S$ such that $\prod_{v \in S} X_v \times G(A^S) \subset X$, where $A^S$ denotes the ring of $S$-adeles (i.e. the projection of $A$ onto factors outside $S$). This will suffice.

For each place $v$, let $X_v := X \cap G(k_v)$ (via the obvious inclusion $G(k_v) \hookrightarrow G(A)$, which is the 0 map on each factor other than $v$). Then for any finite set $S$ of places, $\prod_{v \in S} X_v \times \prod_{v \notin S} 0 \subset X$, so (since $X$ is closed) $G(A) \cap \prod_v X_v \subset X$. Now $[G(A) : X] \geq \prod_{v \in I}[G(k_v) : X_v]$ for every finite set $I$ of places of $k$, so $X_v = G(k_v)$ for all but finitely many $v$. This proves the claim and the proposition. \qed

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Before turning to the proof of exactness of the first sequence in Proposition 6.2.2, we need a lemma. Recall that for an abelian group $A$, $A_{\text{div}} := \bigcap_{n \in \mathbb{Z}_+} nA$ denotes the group of divisible elements, and $A_{\text{tors}}$ the group of torsion elements.

**Lemma 6.2.4.** Let $k$ be a global function field, $G$ a commutative $k$-group scheme of finite type. Then $G(k)_{\text{div}} = 0$ and $G(k)_{\text{tors}}$ has finite exponent.

This lemma will only be used for affine $G$,

**Remark 6.2.5.** This lemma is false for number fields (and in fact for any characteristic 0 field). For example, $G_a(k) = k$ is divisible. The result remains true, however, for groups whose reduced geometric identity component is a semi-abelian variety. The proof is basically the same as the one below.

**Proof.** By Lemma [B.2.8] if we have an exact sequence

$$1 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 1$$

and the lemma holds for $G'$, $G''$, then it also holds for $G$. Note also that the lemma is clear for groups of finite exponent, so in particular for finite group schemes.

By [SGA3, VII$_A$, Prop. 8.3], there is a normal infinitesimal $k$-subgroup scheme $I \subset G$ such that $G/I$ is smooth, so we may assume that $G$ is smooth. Filtering $G$ by $G^0$ and $G/G^0$, we may also assume that $G$ is connected. There is therefore an anti-affine smooth connected $k$-group $G_{\text{ant}} < G$ (anti-affine means that $\mathbb{H}^0(G_{\text{ant}}, \mathcal{O}_{G_{\text{ant}}}) = k$) such that $G/G_{\text{ant}}$ is affine [CGP, Thm. A.3.9]. We may therefore assume that $G$ is either affine or anti-affine. By [CGP, Thm. A.3.9] again, any anti-affine $k$-group is a semi-abelian variety. We may therefore assume that $G$ is either an abelian variety or affine. If $G$ is an abelian variety then $G(k)$ is finitely-generated by the Mordell–Weil theorem, so the lemma is clear. If $G$ is affine, then for the maximal $k$-torus $T \subset G$ the quotient $G/T$ is unipotent, so we may assume that $G$ is either a torus or unipotent. In the latter case, $G$ has finite exponent (this is where we use $\text{char}(k) > 0$!), hence the lemma is clear. So we may assume that $G = T$ is a torus.

Let $k'/k$ be a finite separable extension that splits $T$. Then thanks to the inclusion $T(k) \hookrightarrow T(k')$, we may (renaming $k'$ as $k$) assume that $T$ is split. That is, we need to show that $(k^*)_{\text{div}} = 0$ and that $(k^*)_{\text{tors}}$ has finite exponent. If $\lambda \in (k^*)_{\text{div}}$, then $\text{ord}_v(\lambda) = 0$ for all non-archimedean places $v$ of $k$. Thus, $\lambda$ is a global unit. Since the group of global units is finitely generated, and an element of $k^*$ is a global unit whenever its $n$th power is for any fixed $n > 0$, we deduce that $\lambda = 1$. Finally, as is well-known, $(k^*)_{\text{tors}} = \mu_\infty(k)$ is finite.

**Proof of Proposition 6.2.2.** It only remains to prove exactness of the first sequence. Let us first show that the map $G'(k)_{\text{pro}} \rightarrow G(k)_{\text{pro}}$ is injective. By Lemma [B.2.2] applied to the discrete groups of rational points over the global function field $k$, it suffices to show that
\[(G(k)/G'(k))_{\text{div}} = 0\] and that \((G(k)/G'(k))_{\text{tors}}\) has finite exponent. Since \(G(k)/G'(k) \hookrightarrow G''(k)\), this follows from Lemma 6.2.4.

Next we check exactness in the middle. Let \(f\) denote the map \(G \to G''\). We have an exact sequence
\[1 \to G'(k) \to G(k) \to f(G(k)) \to 1\]
Since profinite completion is right-exact (Proposition B.1.1), this yields an exact sequence
\[G'(k)_{\text{pro}} \to G(k)_{\text{pro}} \to f(G(k))_{\text{pro}} \to 1\]
If we show that the map \(f(G(k))_{\text{pro}} \to G''(k)_{\text{pro}}\) is injective, it will follow that the sequence
\[G'(k)_{\text{pro}} \to G(k)_{\text{pro}} \to G''(k)_{\text{pro}}\]
is exact, and the proof will be complete.

To see this injectivity, we note that we have an inclusion \(G''(k)/f(G(k)) \hookrightarrow H^1(k, G')\), hence by Lemma 4.3.1 \(G''(k)/f(G(k))\) has finite exponent. The desired injectivity therefore follows from Corollary B.2.3 (applied to discrete topological groups).

\[\square\]

### 6.3 Exactness at \(G(k)_{\text{pro}}\)

The goal of this section is to prove the following proposition.

**Proposition 6.3.1.** Let \(k\) be a global function field, \(G\) an affine commutative \(k\)-group scheme of finite type. Then the map \(G(k)_{\text{pro}} \to G(\mathbb{A})_{\text{pro}}\) is injective.

This also holds for number fields, but the proof must be slightly modified. We briefly indicate the required changes in Remark 6.3.6.

First let us prove Proposition 6.3.1 when \(G = \mathbb{G}_m\):

**Lemma 6.3.2.** For a global function field \(k\), the map \((k^\times)_{\text{pro}} \to (\mathbb{A}^\times)_{\text{pro}}\) is injective.

**Proof.** Choose a finite set \(S\) of places of \(k\) such that \(\text{Pic}(\mathcal{O}_S) = 1\). Then we have an evident short exact sequence
\[0 \to \mathcal{O}_S^\times \to k^\times \to \bigoplus_{v \notin S} \mathbb{Z} \to 0.\]
For each place \(v \notin S\), choose an element \(\pi_v \in k^\times\) having valuation 1 at \(v\) and 0 at all other places \(v' \notin S\). These elements yield a splitting of the sequence, hence an isomorphism \(k^\times \simeq \mathcal{O}_S^\times \times \bigoplus_{v \notin S} \mathbb{Z}\). It suffices to show that for any subgroups \(X \subset \mathcal{O}_S^\times\), \(Y \subset \bigoplus_{v \notin S} \mathbb{Z}\) of finite index, there exist closed subgroups \(X' \subset X\) and \(Y' \subset Y\) of finite index such that \(X' \cap k^\times \subset X \times \bigoplus_{v \notin S} \mathbb{Z}\) and \(Y' \cap k^\times \subset \mathcal{O}_S^\times \times Y\) (as then \(X' \cap Y'\) is closed of finite index in \(\mathcal{O}_S^\times \times Y\)). We first treat the case of subgroups of \(\bigoplus_{v \notin S} \mathbb{Z}\).
The map $k^\times \to \bigoplus_{v \in S} \mathbb{Z}$ factors as a composition $k^\times \to A^\times \to \bigoplus_{v \in S} \mathbb{Z}$ whose second map has open kernel. Given a subgroup $Y \subset \bigoplus_{v \in S} \mathbb{Z}$ of finite index, we simply let $Y'$ be its preimage in $A^\times$.

Now we treat finite-index subgroups of $\mathcal{O}_{S}^\times$. This group is finitely generated, so cofinal among these subgroups are the groups $(\mathcal{O}_{S}^\times)^n$ for $n$ a positive integer. We are free to replace $n$ with some positive multiple of itself, hence we may choose some $v_0 \notin S$ and assume that $(\mathcal{O}_{S}^\times)^n \subset 1 + \pi_{v_0} \mathcal{O}_{v_0}$. Let $C$ be the smooth proper geometrically connected curve (over a finite field $F$) such that $k$ is the function field of $C$, and let $N$ be the exponent of the finite group $\text{Pic}^0(C)$. Let $X' \subset A^\times$ be the closed subgroup of ideles $a \in A^\times$ such that $\text{ord}_w(a/\prod_{v \in S} \pi_v^{\text{ord}_v(a)}) \equiv 0 \pmod{n}$ for all $w \in S$ and $a/\prod_{v \notin S} \pi_v^{\text{ord}_v(a)} \equiv 1 \pmod{\pi_{v_0}}$; this is clearly of finite index.

We claim that $X' \cap k^\times \subset (\mathcal{O}_{S}^\times)^n \times \bigoplus_{v \in S} \mathbb{Z}$. Indeed, given $f \in X' \cap k^\times$, let $g := f/\prod_{v \in S} \pi_v^{\text{ord}_v(f)} \in \mathcal{O}_{S}^\times$. Then $\text{div}(g) \in nN \text{Pic}^0(C)$, and $g \equiv 1 \pmod{\pi_v}$. By design, $N$ kills $\text{Pic}^0(C)$, so $\text{div}(g) = \text{div}(h^n)$ for some $h \in k^\times$, and $h$ must be an $S$-unit since $h^n$ is one (as $g$ is an $S$-unit). Since $h^n \equiv 1 \pmod{\pi_{v_0}}$ by our choice of $n$, we deduce that $g/h^n \equiv 1 \pmod{\pi_{v_0}}$. But $g/h^n$ is a global unit of $C$, hence lies in $F^\times$, and the map $F \to \mathcal{O}_v/\pi_v \mathcal{O}_v$ is an inclusion, so $g = h^n$, and we’re done.

Remark 6.3.3. If $k$ is a number field, one proceeds similarly but must use a theorem of Chevalley [Che, Thm. 1] that every finite-index subgroup of $\mathcal{O}_k^\times$ is a congruence subgroup; that is, it contains a subgroup of the form $\{x \in \mathcal{O}_k^\times \mid x \equiv 1 \pmod{\alpha}\}$ for some nonzero $\alpha \in \mathcal{O}_k$.

Lemma 6.3.4. Let $k$ be a global function field. Suppose that we have an exact sequence

$$1 \to G' \to G \to G'' \to 1$$

of affine commutative $k$-group schemes of finite type. If Proposition 6.3.1 holds for $G'$ and $G''$ then it also holds for $G$.

Proof. Suppose that Proposition 6.3.1 holds for $G'$ and $G''$. In the commutative diagram

$$
\begin{array}{ccc}
G'(k)_{\text{pro}} & \longrightarrow & G(k)_{\text{pro}} \\
\downarrow & & \downarrow \\
1 & \longrightarrow & G'(A)_{\text{pro}}
\end{array}
\begin{array}{ccc}
& \longrightarrow & G''(k)_{\text{pro}} \\
& \downarrow & \downarrow \\
& G(A)_{\text{pro}} & \longrightarrow & G''(A)_{\text{pro}}
\end{array}
$$

the rows are exact by Proposition 6.2.2, and the first and third vertical arrows are injective by hypothesis. A simple diagram chase shows that the middle vertical arrow is injective.

We are now reduced as usual to the cases when $G$ is either finite, $G_a$, or a torus. The finite case is immediate from the following lemma.

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Lemma 6.3.5. Let $k$ be a global field, $G$ a finite $k$-group scheme. Then $G(k)$ and $G(A)$ are profinite.

Proof. For $G(k)$, this is obvious because it is finite. For $G(A)$, we claim that the natural map $G(A) \rightarrow \prod_v G(k_v)$ is a topological isomorphism where the target is given the product topology, hence is profinite because each $G(k_v)$ is finite discrete. To see that this map is a topological isomorphism, we spread $G$ out to a finite $\mathcal{O}_S$-group scheme $\mathcal{G}$ for some non-empty finite set $S$ of places of $k$. For $v \not\in S$, we have $\mathcal{G}(\mathcal{O}_v) = G(k_v)$ by the valuative criterion for properness. Thus, topologically,

$$\prod_v G(k_v) = \left( \prod_{v \in S} G(k_v) \right) \times \left( \prod_{v \not\in S} \mathcal{G}(\mathcal{O}_v) \right) = G(\prod_{v \in S} k_v) \times \mathcal{G}(\prod_{v \not\in S} \mathcal{O}_v).$$

It follows that the map is indeed an isomorphism of topological groups.

Next, for $G = G_a$ we need to show that the map $k_{\text{pro}} \rightarrow A_{\text{pro}}$ is injective. It is equivalent to show that the associated map $(A_{\text{pro}})^D \rightarrow (k_{\text{pro}})^D$ between discrete Pontryagin duals is surjective. But for any locally compact Hausdorff abelian topological group $X$ with finite exponent, the map $(X_{\text{pro}})^D \rightarrow X^D$ is an (algebraic, not necessarily topological) isomorphism because the kernel of any continuous homomorphism $X \rightarrow \mathbb{R}/\mathbb{Z}$ is closed of finite index. We therefore want to show that the map $A^D \rightarrow k^D$ is surjective, and this holds because $k$ is closed inside $A$.

Finally, we come to the case when $G = T$ is a torus. First note that given an inclusion $T \hookrightarrow T'$ of $k$-tori, if Proposition 6.3.1 holds for $T'$, then it also holds for $T$ due to the commutative diagram

$$
\begin{array}{ccc}
T(k)_{\text{pro}} & \hookrightarrow & T'(k)_{\text{pro}} \\
\downarrow & & \downarrow \\
T(A)_{\text{pro}} & \longrightarrow & T'(A)_{\text{pro}}
\end{array}
$$

in which the top horizontal arrow is an inclusion by Proposition 6.2.2 and the right vertical arrow is an inclusion by hypothesis. Let $k'/k$ be a finite separable extension splitting $T$. Using the inclusion $T \hookrightarrow R_{k'/k}(T_{k'})$, and renaming $k'$ as $k$, we are reduced to showing that $(k^\times)_{\text{pro}} \rightarrow (A^\times)_{\text{pro}}$ is injective, and this is the content of Lemma 6.3.2. The proof of Proposition 6.3.1 is now complete.

Remark 6.3.6. We have throughout assumed that $k$ is a global function field. This is because we have made essential use of the fact that any unipotent $k$-group $U$ has finite exponent, a property that is completely false for number fields. However, one may modify the arguments in the case that $k$ is a number field by utilizing the fact that any connected $k$-group scheme is the product of a torus and $G^n_a$. Then one uses the fact that $G_a(k)_{\text{pro}} = k_{\text{pro}} = 0$ because $k$ is divisible, and so one reduces everything to the case of tori, for which the proofs are essentially the same as in the function field case. We leave the details to the interested reader.
6.4 Exactness at $G(A)_{\text{pro}}$

The goal of this section is to prove the following result.

**Proposition 6.4.1.** Let $k$ be a global function field, $G$ an affine commutative $k$-group scheme of finite type. Then the complex

$$G(k)_{\text{pro}} \rightarrow G(A)_{\text{pro}} \rightarrow H^2(k, \hat{G})^*$$

is exact.

The case in which $G$ is finite follows from Poitou–Tate for finite commutative $k$-group schemes, together with Lemma 6.3.5. To handle the general case, we first prove the following lemma.

**Lemma 6.4.2.** Suppose that we have an exact sequence

$$1 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 1$$

of affine commutative $k$-group schemes of finite type such that $H^1(k, G') = 0$. If Proposition 6.4.1 holds for $G'$ and $G''$, then it also holds for $G$.

**Proof.** Consider the following commutative diagram:

$$
\begin{array}{ccc}
G'(k)_{\text{pro}} & \rightarrow & G(k)_{\text{pro}} & \rightarrow & G'(k)_{\text{pro}} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
G'(A)_{\text{pro}} & \rightarrow & G(A)_{\text{pro}} & \rightarrow & G''(A)_{\text{pro}} & \rightarrow & \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & H^2(k, \hat{G}')^* & \rightarrow & H^2(k, \hat{G})^* & \rightarrow & H^2(k, \hat{G}'')^* \\
\end{array}
$$

The first and third columns are exact by hypothesis. Proposition 6.4.1 implies exactness of the first two rows except at $G'(k)_{\text{pro}}$, where exactness holds because $H^1(k, G') = 0$ (which implies that $G(k) \rightarrow G'(k)$ is surjective, hence so is the map on profinite completions). Finally, the bottom row is exact by Proposition 3.4.1. A simple diagram chase now shows that the middle column is exact. \hfill \square

**Lemma 6.4.3.** Proposition 6.4.1 holds for split tori and split unipotent groups.

**Proof.** First we treat split tori, so we may assume $G = G_m$. Due to the right-exactness of profinite completion (Proposition B.1.1), we have $(A^\times)_{\text{pro}}/(k^\times)_{\text{pro}} = (A^\times/k^\times)_{\text{pro}}$, so we want to show that the map $A^\times/k^\times \rightarrow H^2(k, \mathbb{Z})^*$ induces an isomorphism from the profinite
completion of the idele class group. As we have seen several times before, via the exact sequence of Galois modules

\[ 0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0 \]

and the unique divisibility of \( \mathbb{Q} \), we have \( H^2(k, \mathbb{Z}) = (\mathfrak{g}_k^{ab})^D \) as torsion discrete topological groups, so \( H^2(k, \mathbb{Z}) = \mathfrak{g}_k^{ab} \) where \( \mathfrak{g}_k \) is the absolute Galois group of \( k \). Thus we have a map \( \mathbb{A}^\times /k^\times \to \mathfrak{g}_k^{ab} \), and we would like to say that it induces an isomorphism from the profinite completion of the idele class group. If this map were the reciprocity map of global class field theory, then the result would follow from class field theory. But it is indeed the reciprocity map, thanks to Lemma 4.5.2 and the compatibility of local and global class field theory.

Next we treat split unipotent groups. Lemma 6.4.2 reduces us to the case of \( G_\alpha \). Proposition 3.2.2 shows that the map \( \mathbb{A}/k \to H^2(k, \hat{G}_\alpha)^* \) is a topological isomorphism. To complete the proof we must show that \( \mathbb{A}/k \) is profinite. It is compact, and it is Hausdorff because \( k \) is closed in \( \mathbb{A} \). It only remains to check that \( \mathbb{A}/k \) is totally disconnected. By translation, it suffices to build a totally disconnected open and closed neighborhood of 0. But \( \mathbb{A} \) contains the profinite subgroup \( \prod_v \mathcal{O}_v \) as an open and closed subgroup, so we are done.

**Lemma 6.4.4.** If we have an inclusion \( G \to G' \) of affine commutative \( k \)-group schemes of finite type and Proposition 6.4.1 holds for \( G' \), then it holds for \( G \).

**Proof.** Let \( G'' := G'/G \). Consider the commutative diagram

\[
\begin{array}{cccccc}
0 & \to & G(k)_{\text{pro}} & \longrightarrow & G(\mathbb{A})_{\text{pro}} & \longrightarrow & H^2(k, \hat{G})^* \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
G'(k)_{\text{pro}} & \longrightarrow & G'(\mathbb{A})_{\text{pro}} & \longrightarrow & H^2(k, \hat{G'})^* \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & G''(k)_{\text{pro}} & \longrightarrow & G''(\mathbb{A})_{\text{pro}} & \\
\end{array}
\]

The second row is exact by hypothesis, the bottom row is exact by Proposition 6.3.1 and the columns are exact by Proposition 6.2.2. A simple diagram chase now shows that the first row is exact.

**Lemma 6.4.5.** Let \( k'/k \) be a finite extension of global function fields, and let \( G' \) be an affine commutative \( k' \)-group scheme of finite type. If Proposition 6.4.1 holds for \( G' \), then it also holds for \( G := R_{k'/k}(G') \).
Proof. We have a commutative diagram of abelian groups (the topological aspects being irrelevant here)

\[
\begin{array}{ccc}
G(k)_{\text{pro}} & \longrightarrow & G(A_k)_{\text{pro}} \\
\downarrow & & \downarrow \\
G'(k')_{\text{pro}} & \longrightarrow & G'(A_{k'})_{\text{pro}}
\end{array}
\]

in which the last vertical arrow is induced by the map \( N_{k'/k} : \hat{G}' \to \hat{G} \) (see the beginning of §2.2); this diagram commutes by Proposition C.1.2. The bottom row is exact by hypothesis, hence so is the top.

Lemma 6.4.6. Let \( k'/k \) be a finite extension of global function fields, and let \( G \) be an affine commutative \( k \)-group scheme of finite type. If Proposition 6.4.1 holds for \( G_{k'} \), then it holds for \( G \).

Proof. We have the canonical embedding \( G \hookrightarrow R_{k'/k}(G_{k'}) \), so by Lemma 6.4.4 the Proposition for \( G \) follows from the Proposition for \( R_{k'/k}(G_{k'}) \). But by Lemma 6.4.5 this in turn follows from the Proposition for \( G_{k'} \).

Now we are free to extend scalars to any finite extension of \( k \). After a suitable such extension, \( G^0_{\text{red}} \) is a smooth connected \( k \)-subgroup scheme that is a product of a split torus and a split unipotent group. By Lemma 6.4.2 (applied with \( G' = G^0_{\text{red}} \)) and the already-treated finite case, we may therefore assume that \( G \) is either a split torus or split unipotent. But then Proposition 6.4.1 holds for \( G \) by Lemma 6.4.3. This completes the proof of Proposition 6.4.1.

6.5 Exactness at \( G(A) \)

In this section, we will use Proposition 6.4.1 to deduce the following result, which we shall require later (and is of interest in its own right).

Proposition 6.5.1. Let \( k \) be a global function field, \( G \) an affine commutative \( k \)-group scheme of finite type. The complex

\[
G(k) \longrightarrow G(A) \longrightarrow H^2(k, \hat{G})^*
\]

is exact.

Remark 6.5.2. Unlike most of the results that we prove for global fields, this one really only holds for function fields. Indeed, if \( k \) is a number field, then Proposition 6.5.1 fails for \( G_a \), since \( H^2(k, \hat{G}_a) = 0 \) because \( k \) is perfect (Lemma 2.4.1), but clearly \( A/k \neq 0 \).
In order to prove Proposition 6.5.1, we first make a definition. Given a global field $k$, we have the norm map $|·| : \mathbb{A}^\times \to \mathbb{R}_{>0}$ given by $a \mapsto \prod_v |a|_v$. By the product formula, $|\lambda| = 1$ for any $\lambda \in k^\times$. Now given a finite type $k$-group scheme $G$, we obtain for any character $\chi \in \hat{G}(k)$ a continuous norm map $||\chi|| : G(\mathbb{A}) \to \mathbb{R}_{>0}$ via $a \mapsto ||\chi(a)||$. We then make the following definition.

**Definition 6.5.3.** $G(\mathbb{A})_1 := \bigcap_{\chi \in \hat{G}(k)} \ker(||\chi||)$.

Since $|k^\times| = \{1\}$, we have $G(k) \subset G(\mathbb{A})_1$. Further, the group $\hat{G}(k)$ is finitely generated; let $\chi_1, \ldots, \chi_n$ be elements that freely generate $\hat{G}(k)$ modulo torsion. Then $G(\mathbb{A})_1$ is the kernel of the map $(||\chi_1||, \ldots, ||\chi_n||) : G(\mathbb{A}) \to (\mathbb{R}_{>0})^n$. If $k$ is a function field then the image of this map is a lattice in $(\mathbb{R}_{>0})^n$ (i.e., a discrete subgroup that is cocompact; i.e., with full rank). If $k$ is a number field, then the image is $(\mathbb{R}_{>0})^n$ ([Oes Ch.I, Prop.5.6]). In particular, if $k$ is a function field then $G(\mathbb{A})_1$ is open and $G(\mathbb{A})/G(\mathbb{A})_1$ is a discrete free abelian group of finite rank, whereas if $k$ is a number field then this quotient is divisible.

We need some lemmas.

**Lemma 6.5.4.** If we have an exact sequence

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

of locally compact second-countable Hausdorff abelian groups such that $A', A''$ are compact, then so is $A$.

**Proof.** It suffices by Pontryagin duality to check that $A^D$ is discrete. First, we claim that the sequence

$$0 \longrightarrow A''^D \longrightarrow A^D \longrightarrow A'^D \longrightarrow 0$$

is exact. Exactness on the left is clear. Exactness on the right holds because $A' \hookrightarrow A$ is an embedding onto a closed subgroup (since the source is compact and the target Hausdorff). For exactness in the middle, we need to check that the map $A/A' \to A''$ is a homeomorphism, and this follows from Lemma 4.3.3. Since $A'^D$ is discrete, it follows that the image of the map $A'^D \to A''$ is open. If we check that this map is a homeomorphism onto its image, then we will be done, since $A'^D$ is discrete. The image is the kernel of the continuous map $A'' \to A'^D$, hence closed, so the result once again follows from Lemma 4.3.3. \qed

**Proposition 6.5.5.** Let $k$ be a global field, $G$ an affine commutative $k$-group scheme of finite type. Then $G(\mathbb{A})_1/G(k)$ is compact Hausdorff.

**Remark 6.5.6.** This compactness assertion is made in [Oes Ch.IV, Thm.1.3] for any solvable smooth affine $k$-group $G$. The proof requires that $G$ be connected (in order to apply the Lie–Kolchin theorem to $G$), so the correct statement of [Oes Ch.IV, Thm.1.3] is that if $G$ is a smooth connected affine solvable group scheme over the global field $k$ then $G(\mathbb{A})_1/G(k)$
is compact. It is false without connectedness, as the following example due to Brian Conrad illustrates.

Let $G = G_m \times \mathbb{Z}/2\mathbb{Z}$ where the semi-direct product structure is through inversion. As we saw in Remark 2.2.6 any $\chi \in \hat{G}(k)$ restricts to the trivial character on $G_m$, hence has finite order. It follows that $G(A)_1 = G(A)$. But we claim that $G(A)/G(k)$ is not compact.

The quotient group $G_m(A)/G_m(k) = A^\times/k^\times$ is non-compact (due to non-compactness of its image in $\mathbb{R}_{>0}$ under the idelic norm), so it suffices to show that the natural continuous injective map $j : G_m(A)/G_m(k) \hookrightarrow G(A)/G(k)$ is a homeomorphism onto a closed subspace. Since $G(k) \rightarrow \mathbb{Z}/2\mathbb{Z}$ is surjective, this image is easily seen to coincide with the fiber over 1 for the continuous map $G(A)/G(k) \rightarrow (\mathbb{Z}/2\mathbb{Z})(A)/(\mathbb{Z}/2\mathbb{Z})$, which is closed because $(\mathbb{Z}/2\mathbb{Z})(A)/(\mathbb{Z}/2\mathbb{Z})$ is Hausdorff. The homeomorphism property onto the image is now a local problem on $G_m(A)/G_m(k)$, so by translation considerations it suffices to analyze it near the identity. But $G_m(k)$ is discrete in $G_m(A)$ and $G(k)$ is discrete in $G(A)$, so since $G_m(A) \rightarrow G(A)$ is a topological (even closed) embedding we are done.

**Proof of Proposition 6.5.5** Since $k \hookrightarrow A$ is a closed subgroup, by choosing local coordinate charts for $G$ we can see that $G(k) \hookrightarrow G(A)$ is closed, hence $G(A)_1/G(k)$ is Hausdorff, so we concentrate on compactness. Replacing $G$ with its maximal smooth $k$-subgroup scheme (see [CGP] Lemma C.4.1, Remark C.4.2) has no effect on its adelic or rational points. We claim that it also has no effect on the norm-1 points. That is, if $g \in G(A)$ satisfies $\|\chi(g)\| = 1$ for every $\chi \in \hat{G}(k)$, then $\|\chi(g)\| = 1$ for every $\chi \in \hat{G_{\text{sm}}}(k)$, where $G_{\text{sm}} \subset G$ is the maximal smooth subgroup. For this, it suffices to show that for every $\chi \in \hat{G_{\text{sm}}}(k)$, $\chi^n$ extends to $\hat{G}(k)$ for some $n > 0$. This follows from Proposition 2.2.5. So we may assume that $G$ is smooth. By Remark 6.5.6 we know the result when $G$ is smooth and connected. We will deduce the general case from this one.

Let $G^0$ be the identity component of $G$, and let $E := G/G^0$ be the finite étale quotient. Then $E(A) = E(A)_1$ is compact by Lemma 6.3.5 hence so is $E(A)/E(k)$. Let $X$ be defined by the following exact sequence

$$0 \rightarrow X \rightarrow G(A)_1/G(k) \rightarrow E(A)/E(k)$$

Then we claim that the map $G(A)_1/G(k) \rightarrow E(A)/E(k)$ has closed image and that $X$ is compact. Lemma 6.5.4 will then complete the proof.

First we check that the map $G(A)_1/G(k) \rightarrow E(A)/E(k)$ has closed image. Since $E(k)$ is finite, it suffices to check that the map $G(A)_1 \rightarrow E(A)$ has closed image. We note that the map $G(A) \rightarrow E(A)$ has closed image, since it is the kernel of the continuous map $E(A) \rightarrow H^1(A, G^0)$, in which the target group is Hausdorff. It follows from Lemma 4.3.3 that the map $G(A)/G^0(A) \rightarrow E(A)$ induces a homeomorphism onto a closed subgroup. Therefore, if we show that the map $G(A)_1 \rightarrow G(A)/G^0(A)$ has closed image, then the claim will follow. That is, we need to show that the subgroup $G(A)_1G^0(A) \subset G(A)$ is closed.
First suppose that $k$ is a function field. Then the quotient $G(A)/G(A)_1$ is discrete, hence the quotient $G(A)/G(A)_1G^0(A)$ is discrete, so $G(A)_1G^0(A) \subset G(A)$ is closed. Next suppose that $k$ is a number field. Then the quotient $G(A)/G(A)_1$ is divisible, and the quotient $G(A)/G^0(A) \hookrightarrow E(A)$ is of finite exponent, so the quotient $G(A)/G(A)_1G^0(A)$ is divisible and of finite exponent, thus trivial.

Next we need to show that $X$ is compact. We will show that $[X : G^0(A)_1/G^0(k)]$ is finite, and the desired compactness will then follow from the known compactness of $G^0(A)_1/G^0(k)$. We prove finiteness in two steps: first that $[X : (G^0(A) \cap G(A)_1)/G^0(k)]$ is finite, and then that $G^0(A) \cap G(A)_1 = G^0(A)_1$.

Define an inclusion
\[
\phi : \frac{X}{(G^0(A) \cap G(A)_1)/G^0(k)} \hookrightarrow \Pi^1(G^0)
\]
as follows. We have a commutative diagram with exact rows:
\[
\begin{aligned}
G^0(k) & \longrightarrow G(k) \longrightarrow E(k) \longrightarrow \Pi^1(k,G^0) \\
\downarrow & \quad \downarrow \quad \downarrow \quad \downarrow \\
G(A)_1 \cap G^0(A) & \longrightarrow G(A)_1 \longrightarrow E(A) \longrightarrow \Pi^1(A,G^0)
\end{aligned}
\]
Given $g \in G(A)_1$ representing an element of $X$, $f(g)$ lifts to an element $e \in E(k)$ and we define $\phi(g) := \delta(e)$. A straightforward diagram chase then shows that $\phi$ is indeed an inclusion into $\Pi^1(G^0)$. By [Con Thm.1.3.3(i)], $\Pi^1(k,H)$ is finite for all affine $k$-group schemes $H$ of finite type (even without commutativity hypotheses). This therefore proves the finiteness of $[X : (G^0(A) \cap G(A)_1)/G^0(k)]$. It remains to show $G^0(A) \cap G(A)_1 = G^0(A)_1$.

Suppose that $g \in G^0(A) \cap G(A)_1$. We need to show that $\|\chi(g)\|_1 = 1$ for $\chi \in \hat{G}^0(k)$. By Corollary 2.2.5, $\chi^n$ extends to a character of $\hat{G}(k)$ for some positive integer $n$. Therefore, $\|\chi^n(g)\|_1 = 1$, so $\|\chi(g)\|_1 = 1$, as desired. \qed

**Proposition 6.5.7.** Let $k$ be a global function field, $G$ an affine commutative $k$-group scheme of finite type. Then $G(A)_1/G(k)$ is profinite.

**Remark 6.5.8.** This is false if $k$ is a number field. Indeed, if $G = G_a$ then $G(A)_1 = A$, but $A/k$ is divisible and so not profinite.

**Proof.** By Proposition 6.5.5, $G(A)_1/G(k)$ is compact Hausdorff, so we only need to check that $G(A)/G(k)$ is totally disconnected. In order to do this, it suffices to show that if $g \in G(A) - G(k)$, then there is a compact open set $U \subset G(A)$ containing $g$ and disjoint from $G(k)$. Since $G(k) \subset G(A)$ is closed, there exists an open such $U$. Since any neighborhood of $g$ contains a compact open neighborhood of $g$ (as $k$ is a function field, so it has no archimedean places), we are done. \qed
Proof of Proposition 6.5.1. Thanks to Proposition 6.4.1 and the right-exactness of profinite completion (Proposition B.1.1), we only have to show that the map \( G(A)/G(k) \to (G(A)/G(k))_{\text{pro}} \) is injective. Consider the following commutative diagram:

\[
\begin{array}{c}
G(A)/G(k) \\
\downarrow \quad \downarrow \quad \downarrow \\
(G(A)/G(k))_{\text{pro}} \\
0 \\
\end{array}
\]

The top row is clearly exact. The bottom row is exact by Proposition B.2.1, since the quotient \( G(A)/G(A)_1 \) is discrete and finitely generated. The first vertical arrow is an isomorphism by Proposition 6.5.7, and the third vertical arrow is an inclusion because \( G(A)/G(A)_1 \) is discrete and finitely generated. A simple diagram chase now shows that the middle vertical arrow is an inclusion. \( \square \)

6.6 Exactness at \( H^2(k, \hat{G})^* \)

The goal in this section is to prove the following result.

Proposition 6.6.1. Let \( k \) be a global function field, \( G \) a commutative affine \( k \)-group scheme of finite type. The complex

\[
G(A)_{\text{pro}} \to H^2(k, \hat{G})^* \to H^2(\hat{G})^*
\]

is exact.

Before taking up the proof, we explain the definition of the first map (the second map being the obvious one via restriction). As usual, for the group \( H^2(k, \hat{G}) \) that is torsion (Lemma 4.4.1), we give its Pontryagin dual \( H^2(k, \hat{G})^* \) the (profinite! dual topology in which an element is near 0 when on some large finite subset of \( H^2(k, \hat{G}) \) it takes values near 0 in \( \mathbb{Q}/\mathbb{Z} \) equipped with the “archimedean” topology from \( \mathbb{R}/\mathbb{Z} \). To define \( G(A)_{\text{pro}} \to H^2(k, \hat{G})^* \) as a continuous homomorphism amounts to defining a continuous homomorphism \( G(A) \to H^2(k, \hat{G})^* \) (with profinite target). This latter homomorphism is defined by cupping everywhere locally and summing the resulting invariants (all but finitely many of which vanish since \( H^2(O_v, \mathbb{G}_m) = 0 \)). To check that this map is continuous, we need to show that for any finite subset \( T \subset H^2(k, \hat{G}) \) and any \( \epsilon > 0 \), there is a neighborhood \( U \) of 0 \( \in G(A) \) such that \( |\langle g, \alpha \rangle| < \epsilon \) for all \( g \in U, \alpha \in T \).

Fix \( T \) and \( \epsilon \). We may and do pick a non-empty finite set \( S \) of places of \( k \) for which \( G \) is the \( k \)-fiber of a commutative flat affine \( O_S \)-group scheme \( \mathcal{G} \) of finite type and each of the finitely many \( \alpha \in T \) arises from \( H^2(O_v, \mathbb{G}_m) = 0 \). For each \( v \in S \), the continuity of the local cup product pairing (see Theorem 1.2.2) yields a neighborhood \( U_v \) of 0 \( \in G(k_v) \) such that \( \langle g, \alpha_v \rangle \in \mathbb{Q}/\mathbb{Z} \) is represented by a rational number in the interval \((-\epsilon/\#S', \epsilon/\#S)\) for all
\[ g \in U_v,\; \alpha \in T. \] We claim that the open subset \( U := \prod_{v \in S} U_v \times \prod_{v \notin S} \mathcal{O}_v \subset G(A) \) does what we want. By design \( \sum_{v \in S} \langle g_v, \alpha_v \rangle \) is represented by a rational number in the interval \((-\epsilon, \epsilon)\) for \( g \in U, \alpha \in T \). We further have \( \langle g_v, \alpha_v \rangle = 0 \) for all \( v \notin S \) for such \( g, \alpha \) since the cup product factors through \( H^2(\mathcal{O}_v, G_m) = 0 \).

Hence, it suffices to show that \( R/\mathbb{Z} \) has dense image inside \( G(A) \) has dense image. That is, given a finite subset \( T \subset H^2(k, \hat{G}) \) and \( \phi \in H^2(k, \hat{G})^* \) vanishing on \( H^2(\hat{G}) \), we seek \( g \in G(A) \) such that \( \langle g, \alpha \rangle \) is close to \( \phi(\alpha) \) inside \( R/\mathbb{Z} \) for every \( \alpha \in T \). Since \( H^2(k, \hat{G}) \) is torsion, we may replace \( T \) with the finite group that it generates and thereby assume that \( T \) is a subgroup.

Note that \( \phi|_T \) factors through the quotient \( T/(T \cap H^2(\hat{G})) \) due to where \( \phi \) comes from. We claim that for some non-empty finite set \( S \) of places of \( k \), the map \( T/(T \cap H^2(\hat{G})) \to \prod_{v \in S} H^2(k_v, \hat{G}) \) is injective, so \( \phi|_T \) would be induced by an element of \( (\prod_{v \in S} H^2(k_v, \hat{G}))^* \).

To find such an \( S \), we shall use that the two definitions of \( H^2(\hat{G}) \), as the kernel of either of the two maps \( H^2(k, \hat{G}) \to H^2(A, \hat{G}) \) or \( H^2(k, \hat{G}) \to \prod_v H^2(k_v, \hat{G}) \), coincide (due to the injectivity statement in Theorem 1.2.6). Thus, for each \( \alpha \in T \setminus (T \cap H^2(\hat{G})) \) there exists a place \( v = v(\alpha) \) of \( k \) such that \( \alpha_v \neq 0 \). We may choose \( S \) to consist of the \( v(\alpha) \).

We will construct the desired \( g \in G(A) \) “approximating” \( \phi \) on \( T \) to satisfy \( g_v = 0 \) for \( v \notin S \). Our task is to appropriately choose \( g_v \in G(k_v) \) for each \( v \in S \) (note that \( g = (g_v) \in \prod G(k_v) \) must then lie in \( G(A) \)). It has been shown that \( \phi|_T \) lifts to \( (\prod_{v \in S} H^2(k_v, \hat{G}))^* \), so to find the required \( (g_v)_{v \in S} \in \prod_{v \in S} G(k_v) \), it suffices to show that the natural map \( \prod_{v \in S} G(k_v) \to (\prod_{v \in S} H^2(k_v, \hat{G}))^* \) has dense image, or equivalently that the natural continuous homomorphism \( G(k_v) \to H^2(k_v, \hat{G})^* \) has dense image for each \( v \in S \). By local duality (see Theorem 1.2.2), the target of this latter map is topologically identified with \( G(k_v)_{\text{pro}} \).

The desired density follows, completing the proof of Proposition 6.6.1.

### 6.7 The fundamental exact sequence

The goal of this section is to prove that the last three terms in Theorem 1.2.8 form an exact sequence. We refer to this 3-term sequence as the fundamental exact sequence since when \( G = G_m \) it forms (most of) the fundamental exact sequence of class field theory (due to Proposition 6.1.1 for degree-2 cohomology of \( G_m \)).

**Proposition 6.7.1.** Let \( k \) be a global function field, \( G \) an affine commutative \( k \)-group scheme of finite type. The following sequence is exact:

\[ H^2(k, G) \longrightarrow H^2(A, G) \longrightarrow \hat{G}(k)^* \longrightarrow 0. \]
Note that we may replace $H^2(A, G)$ with $\bigoplus_v H^2(k_v, G)$, by Proposition 6.1.1.

When $G$ is finite, this is just part of the Poitou-Tate sequence for finite group schemes. Further, Proposition 6.7.1 is trivial for smooth connected unipotent groups $U$, since all of the terms appearing in the sequence are 0 in that case. Indeed, for the $H^2$ terms this follows from Lemma 2.4.4 and we also have $\hat{U}(k) = 0$, since unipotent groups have no nontrivial characters over a field. Next we treat separable Weil restrictions of split tori.

**Lemma 6.7.2.** Let $k'/k$ be a finite separable extension of global fields. Then Proposition 6.7.1 holds for $R_{k'/k}(G_m)$.

**Proof.** By Proposition 6.1.1, we have a commutative diagram

$$
\begin{array}{cccccc}
H^2(k, R_{k'/k}(T')) & \to & \bigoplus_v H^2(k_v, R_{k'/k}(T')) & \to & R_{k'/k}(T')(k)^* & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
H^2(k', T') & \to & \bigoplus_v H^2(k'_v, T') & \to & \hat{T}'(k)^* & \to & 0 \\
\end{array}
$$

(commutativity by Proposition C.1.2), where the bottom row is exact by class field theory, and hence so is the top row.

Next we prove Proposition 6.7.1 when $G$ is an almost torus. By Lemma 2.1.3(iv), we may modify $G$ in order to assume that there is an exact sequence

$$
1 \to B \to X \to G \to 1
$$

with $X = C \times R_{k'/k}(T')$, $B, C$ finite commutative $k$-group schemes, $k'/k$ a finite separable extension, and $T'$ a split $k'$-torus. Consider the commutative diagram

$$
\begin{array}{cccccc}
H^2(k, X) & \to & H^2(k, G) & & & \\
\downarrow & & \downarrow & & & \\
\bigoplus_v H^2(k_v, B) & \to & \bigoplus_v H^2(k_v, X) & \to & \bigoplus_v H^2(k_v, G) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\hat{B}(k)^* & \to & \hat{X}(k)^* & \to & \hat{G}(k)^* & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & & \\
\end{array}
$$

The third row is clearly exact (due to the injectivity of the group $\mathbb{Q}/\mathbb{Z}$), and the second row is exact by Proposition 3.1.1. The first column is exact because Proposition 6.7.1 holds.
for the finite $B$, and the middle column is exact due to the already-settled cases of finite group schemes and separable Weil restrictions of split tori. We want to show that the last column, which we know to be a complex, is exact. This is a diagram chase which we leave to the reader.

Now we prove Proposition 6.7.1 in the general case. Let $G$ be an affine commutative $k$-group scheme of finite type. By Lemma 2.1.7, there is an exact sequence

$$1 \rightarrow H \rightarrow G \rightarrow U \rightarrow 1$$

with $H$ an almost torus and $U$ split unipotent. We then have a commutative diagram

$$
\begin{array}{cccccc}
H^2(k, H) & \rightarrow & \bigoplus_v H^2(k_v, H) & \rightarrow & \tilde{H}(k)^* & \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & \\
H^2(k, G) & \rightarrow & \bigoplus_v H^2(k_v, G) & \rightarrow & \tilde{G}(k)^* & \rightarrow 0
\end{array}
$$

in which the top row is exact by the already treated case of almost tori; the second vertical arrow is surjective because $H^2(k_v, U) = 0$ by Lemma 2.4.4(i); and the last vertical arrow is an isomorphism because $\tilde{U}(k) = 0$ (since the unipotent group $U$ has no nontrivial characters), and $H^1(k, \tilde{U}) = 0$ (Lemma 2.4.4(iii)). A simple diagram chase now shows that the second row is exact. This completes the proof of Proposition 6.7.1.

### 6.8 Topology on cohomology of the adeles

Let $k$ be a global function field, $G$ a commutative $k$-group scheme locally of finite type. We have made $H^1(k_v, G)$ into a locally compact, Hausdorff, second countable topological group (as reviewed near the start of §4.2). We have similarly endowed the groups $H^2(k_v, G)$ with the discrete topology (and $H^0(k_v, G) = G(k_v)$ with its usual topology coming from an embedding into affine space). We will now use these topologies to topologize the groups $H^i(A, G)$ for $i \leq 2$. First choose some $\mathcal{O}_S$-model $\mathcal{G}$ of $G$ (the topology will be independent of this model), where $S$ is a finite set of places of $k$ containing the archimedean places. Then a fundamental system of neighborhoods of $0 \in H^i(A, G)$ is given by the sets

$$\prod_{v \in S'} U_v \times \text{im} \left( H^i\left( \prod_{v \notin S'} \mathcal{O}_v, \mathcal{G} \right) \rightarrow H^i(A^S, G) \right)$$

where $S' \supset S$ is a finite set of places of $k$, and $U_v \subset H^i(k_v, G)$ is a neighborhood of $0 \in H^i(k_v, G)$. By Proposition 6.1.1 and the Hausdorffness of $H^i(k_v, G)$, we deduce that $H^i(A, G)$ is Hausdorff. The local compactness and second countability of the groups $H^i(k_v, G)$ imply the corresponding properties for the groups $H^i(A, G)$. 

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These adelic cohomology groups inherit the same functorial properties as $H^1(k_v, G)$. For example, if we have a map $f : G \to G'$ between two (locally finite type commutative) $k$-groups, then the induced map $H^1(A, G) \to H^1(A, G')$ is continuous (as we can find a suitable $S$ for which there exists a homomorphism $\mathcal{G} \to \mathcal{G}'$ between their $\mathcal{O}_S$-models inducing $f$ over $k$).

We may similarly topologize the groups $H^i(A, \hat{G})$ ($i \leq 2$) by using the restricted product topology as above. That these groups are second countable and locally compact follows from the analogous properties for the groups $H^1(k_v, G)$. Further, they are Hausdorff by Proposition 6.1.2 and the Hausdorffness of the groups $H^1(k_v, \hat{G})$.

For $0 \leq i \leq 2$, we have a pairing $H^i(A, G) \times H^{2-i}(A, \hat{G}) \to \mathbb{Q}/\mathbb{Z}$ defined by cupping everywhere locally and then summing the invariants (all but finitely many of which vanish, since $Br(\mathcal{O}_v) = 0$ for all non-archimedean $v$). This pairing is continuous when the target $\mathbb{Q}/\mathbb{Z}$ is viewed discretely, due to the continuity of the local pairings $H^i(k_v, G) \times H^{2-i}(k_v, \hat{G}) \to \mathbb{Q}/\mathbb{Z}$ (with discrete target) and the fact that $H^2(\mathcal{O}_v, \mathbb{G}_m) = 0$. We then have the following result, which is the adelic analogue of Theorems 1.2.1 and 1.2.4.

**Proposition 6.8.1.** Let $G$ be an affine commutative group scheme of finite type over the global function field $k$. Then the adelic pairing induces perfect pairings

\[
H^2(A, G) \times \hat{G}(A)_{\text{pro}} \to \mathbb{Q}/\mathbb{Z}
\]

\[
H^1(A, G) \times H^1(A, \hat{G}) \to \mathbb{Q}/\mathbb{Z}
\]

**Remark 6.8.2.** The reader may be wondering if the corresponding pairing between $H^2(A, \hat{G})$ and either $G(A)$ or $G(A)_{\text{pro}}$ is perfect. The answer is no in either case. Indeed, restricting to a single place $v$ of $k$ shows that the pairing with $G(A)$ cannot be perfect, as the Pontryagin dual of $H^2(k_v, \hat{G})$ is $G(k_v)_{\text{pro}}$, not (in general) $G(k_v)$. To see that the pairing with $G(A)_{\text{pro}}$ is not perfect, we note that if it were, then $H^2(A, \hat{G})$ would have compact dual and would therefore be discrete. This is not true in general. Indeed, for this group to be discrete exactly says that there is a finite set $S$ of places of $k$ such that $H^2(\prod_{v \notin S} \mathcal{O}_v, \hat{G}) \to H^2(A, \hat{G})$ is the zero map. This would imply in particular that $H^2(\mathcal{O}_v, \hat{G}) \to H^2(k_v, \hat{G})$ is the zero map for all $v \notin S$. This is false, as may be seen by, for example, invoking Theorem 1.2.6.

Let us prove that the first pairing is perfect. By Proposition 6.1.1, $H^2(A, G) = \bigoplus_v H^2(k_v, G)$, hence $H^2(A, G)^D = \prod_v H^2(k_v, G)^D = \prod_v G(k_v)_{\text{pro}}$, the last equality by Theorem 1.2.1. It therefore suffices to prove the following lemma.

**Lemma 6.8.3.** Let $G$ be an affine commutative group scheme of finite type over the global field $k$. Then the natural map $\hat{G}(A)_{\text{pro}} \to \prod_v \hat{G}(k_v)_{\text{pro}}$ is a topological isomorphism.

**Proof.** The proof is exactly the same as for Proposition 6.2.3. □

Next we prove that the $H^1$ pairing is perfect. By Lemma 4.3.3 we only need to show that the induced map $H^1(A, G) \to H^1(A, \hat{G})^D$ is an algebraic isomorphism. Since $H^1(k_v, \cdot)$
may be split off from both sides for any desired $v$, the injectivity follows from Proposition 6.1.1 and Theorem 1.2.4. Thus, we concentrate on proving surjectivity.

The key will be relating the adelic cohomology groups to restricted products of cohomology groups over local fields. More precisely, given a finite-type commutative $k$-group scheme $G$, we may define the restricted direct product

$$\prod_v H^1(k_v, G)$$

of the cohomology groups $H^1(k_v, G)$ with respect to the integral cohomology groups as follows. Choose some $\mathcal{O}_S$-model $\mathcal{G}$ of $G$. (As usual, $S$ denotes a non-empty finite set of places of $k$.) We define the above “restricted product” to be the set of all elements of $\alpha \in \prod_v H^1(k_v, G)$ such that $\alpha_v \in H^1(\mathcal{O}_v, \mathcal{G})$ for all but finitely many $v$ (which makes sense since $H^1(\mathcal{O}_v, \mathcal{G}) \to H^1(k_v, G)$ is injective for all but finitely many $v$, by Proposition 5.3.1). We give this group the restricted product topology coming from the topologies that we have already defined on the groups $H^1(k_v, G)$. Note that this definition is independent of our choice of $\mathcal{G}$, since any two such models become isomorphic over some $\mathcal{O}_S'$. We may similarly (using Lemma 5.2.1) define the restricted product

$$\prod_v H^1(k_v, \widehat{G})$$

with respect to the subgroups $H^1(\mathcal{O}_v, \mathcal{G}) \subset H^1(k_v, \widehat{G})$ for all but finitely many $v$. The natural map $H^1(A, G) \to \prod_v H^1(k_v, G)$ lands inside of the restricted product, since $H^1(A, G) = \lim_{\to} \prod_v H^1(k_v, G) \times H^1(\prod_{v \in S'} \mathcal{O}_v, \mathcal{G})$, and similarly for $\widehat{G}$. The key point now is the following result:

**Proposition 6.8.4.** Let $G$ be an affine commutative group scheme of finite type over the global field $k$. Then the maps $H^1(A, G) \to \prod_v H^1(k_v, G)$ and $H^1(A, \widehat{G}) \to \prod_v H^1(k_v, \widehat{G})$ are topological isomorphisms.

Before giving the proof, we require a lemma.

**Lemma 6.8.5.** Let $k$ be a global field, $G$ a commutative affine $k$-group scheme of finite type, $\mathcal{G}$ an $\mathcal{O}_S$-model of $G$. Then, after suitably enlarging $S$, the map $H^1(\prod_{v \notin S} \mathcal{O}_v, \mathcal{G}) \to \prod_{v \notin S} H^1(\mathcal{O}_v, \mathcal{G})$ is surjective.

This lemma is valid without the affineness hypothesis, with essentially the same proof; we indicate below the one extra argument required to handle non-affine $G$.

**Proof.** As we have done before, let $\widehat{\mathcal{O}}_S$ denote $\prod_{v \notin S} \mathcal{O}_v$. By Lemma 2.1.1 (and Remark 2.1.2 if one wishes to remove the affineness assumption), there is a finite subgroup scheme $B \subset G$ such that $H := G/B$ is smooth and connected. The exact sequence

$$1 \to B \to G \to H \to 1$$

...
spreads out to yield an exact sequence

\[ 1 \rightarrow B \rightarrow G \rightarrow H \rightarrow 1 \]

of \( O_S \)-models (after possibly enlarging \( S \)).

In the commutative diagram

\[
\begin{array}{ccc}
H^1(O_S, B) & \rightarrow & H^1(O_S, G) \\
\downarrow & & \downarrow \\
\prod_{v \in S} H^1(O_v, B) & \rightarrow & \prod_{v \in S} H^1(O_v, G)
\end{array}
\]

the left arrow is surjective by Proposition 6.1.3 and Cartier duality. The bottom arrow is surjective after possibly enlarging \( S \) because, by Proposition 5.1.1, we have \( H^1(O_v, H) = 0 \) for all but finitely many \( v \notin S \). It follows that the right vertical arrow is surjective after possibly enlarging \( S \).

Proof of Proposition 6.8.4. It suffices to show that these manifestly continuous maps are algebraic isomorphisms, by Lemma 4.3.3. Injectivity follows from Propositions 6.1.1 and 6.1.2. Surjectivity for \( G \)-cohomology follows from the equality

\[
H^1(A, G) = \lim_{\rightarrow} \prod_{v \in S'} H^1(k_v, G) \times H^1(O_{S'}, G)
\]

and Lemma 6.8.5. We may similarly deduce surjectivity for \( \hat{G} \)-cohomology from Proposition 6.1.3.

The proof of Proposition 6.8.1 is now simple. It is easy to see that we have an equality

\[
(\prod_v' H^1(k_v, \hat{G}))^D = \prod_v' H^1(k_v, \hat{G})^D
\]

where the second product is the restricted direct product with respect to the subgroups of \( H^1(k_v, G)^D \) consisting of those homomorphisms that kill \( H^1(O_v, G) \) for \( v \notin S \). But by our results on local duality – more precisely, Theorem 1.2.4 (proved in Chapter 4) and Theorem 1.2.7 (proved in [5.4]) – this last subgroup is exactly (via the local duality pairing) the group \( H^1(O_v, G) \subset H^1(k_v, G) = H^1(k_v, \hat{G})^D \) for all but finitely many \( v \notin S \). So the restricted product on the right is \( H^1(A, G) \) by Proposition 6.8.4, which is what we want.

## 6.9 Defining the III-pairings

In this section we will define the pairings arising in Theorem 1.2.9. Our definition imitates Tate’s original definition in terms of cocycles. First, we need a lemma.
Lemma 6.9.1. If $k$ is a global field, then $\hat{H}^3(k, G_m) = 0$.

This is the Čech-fppf variant of the well-known derived-functor result $H^3(k, G_m) = 1$ [CF, Ch. VII, §11.4].

Proof. We have the Čech-to-derived functor spectral sequence

$$E_2^{p,q} = \check{H}^p(k, \mathcal{H}^q(G_m)) \Rightarrow H^{p+q}(k, G_m)$$

Since $H^3(k, G_m) = 0$, it is enough to show that $E_2^{1,1}$ and $E_2^{0,2}$ vanish. First,

$$E_2^{0,2} = \check{H}^0(k, \mathcal{H}^2(G_m)) = 0$$

because for any finite extension $L/k$ and any $\alpha \in H^2(L, G_m) = Br(L)$, there is a finite extension $L'/L$ such that $\alpha$ dies in $Br(L') = H^2(L', G_m)$. Next, to show that $E_2^{1,1} = \check{H}^1(k, \mathcal{H}^1(G_m))$ vanishes, it is enough to show that $Pic(L \otimes_k L) = 0$ for any finite extension $L/k$. But this is clear, because $Spec(L \otimes_k L)$ is topologically a disjoint union of points. \(\square\)

Now we will define the pairings. Let us first define the pairing

$$\langle \cdot, \cdot \rangle_{\mathbb{W}_G^2} : \mathbb{W}^2(k, G) \times \mathbb{W}^1(k, \hat{G}) \rightarrow \mathbb{Q}/\mathbb{Z}$$

By Proposition [3.5.1] it suffices to define the pairing on Čech cohomology. Let $k$ be a global function field, and choose

$$\xi \in \mathbb{W}^2(k, G) \subset H^2(k, G) = \hat{H}^2(k, G), \quad \xi' \in \mathbb{W}^1(k, \hat{G}) \subset H^1(k, \hat{G}) = \hat{H}^1(k, \hat{G}).$$

Let $\alpha \in \hat{Z}^2(k, G), \alpha' \in \hat{Z}^1(k, \hat{G})$ be respective representative Čech cocycles, so they are each everywhere locally coboundaries. That is, for every place $v$ of $k$, there exists a 1-cocohain $\beta_v \in \check{C}^1(k_v, G)$ and a 0-cocohain $\beta'_v \in \check{C}^0(k_v, \hat{G})$ such that $\alpha_v = d\beta_v, \alpha'_v = d\beta'_v$.

Since $\hat{H}^3(k, G_m) = 0$, there exists a 2-cocohain $h \in \check{C}^2(k, G_m)$ such that $\alpha \cup \alpha' = dh$. Then $d(\beta_v \cup \beta'_v) = dh = d(\alpha_v \cup \beta'_v)$. It follows that $(\alpha_v \cup \beta'_v) - h$ and $(\beta_v \cup \alpha'_v) - h_v$ are 2-cocohains. Further, they yield the same class in $\hat{H}^2(k_v, G_m) = H^2(k_v, G_m)$, because $d(\beta_v \cup \beta'_v) = (\alpha_v \cup \beta'_v) - (\beta_v \cup \alpha'_v)$. We then define $\langle \xi, \xi' \rangle_{\mathbb{W}_G^2}$ to be the sum over all $v$ of the invariants of these elements of $H^2(k_v, G_m)$.

It is easy to check that this pairing is bilinear and independent of all choices. (The independence of choice of $h$ uses the fact that the sum of the local invariants of a global Brauer class is 0.) It only remains to check that the sum appearing in the definition is finite. But by assumption, $\alpha'$ maps to a coboundary in $\hat{H}^1(A, \hat{G})$, which implies that we may choose $\beta'_v$ to actually come from $\check{C}^0(O_v, \hat{G})$ for almost every $v$. Since $\alpha$ and $h$ both extend to cocohains over some $O_v$, it follows that the cocyce $\langle \alpha \cup \beta'_v \rangle - h \in \hat{Z}^2(k_v, G_m)$ extends to an element of $\hat{Z}^2(O_v, G_m)$ for almost every $v$, hence represents the trivial cohomology class, since $H^2(O_v, G_m) = 0$. The sum is therefore finite.
The definition of the other $\Theta$-pairing
\[ \langle \cdot, \cdot \rangle_{\Theta}^{G_1} : \Theta^1(k, G) \times \Theta^2(k, \hat{G}) \rightarrow \mathbb{Q}/\mathbb{Z} \]
is entirely analogous, once again using Proposition 3.5.1.

Remark 6.9.2. Suppose that $G$ is a finite commutative $k$-group scheme. We have now defined pairings between $\Theta^1(G)$ and $\Theta^2(\hat{G})$, and between $\Theta^1(G^{\wedge \wedge})$ and $\Theta^2(\hat{G})$. Via Cartier duality, of course, we have, canonically, $G^{\wedge \wedge} = G$, and we would like to know that these two pairings are compatible with Cartier duality. This follows immediately from their definitions in terms of cocycles.

6.10 Injectivity of $\Theta^1(G) \to \Theta^2(\hat{G})^*$

The goal of this section is to prove the following result.

Proposition 6.10.1. Let $G$ be an affine commutative group scheme of finite type over the global function field $k$. Then the map $\Theta^1(G) \to \Theta^2(\hat{G})^*$ induced by $\langle \cdot, \cdot \rangle_{\Theta}^{G_1}$ is injective.

Remark 6.10.2. Unlike the other results of this manuscript, we will not take the finite case of Theorem 1.2.9 as a black box. This is for two reasons: (i) to do so would require us to check the compatibility between our pairing and the one defined in [Čes2]; and (ii) we thought this approach would be more convenient for the reader.

We will concentrate on the case of function fields. The result is still true for number fields, but the proof must be modified. The crux of the proof lies in the following lemma.

Lemma 6.10.3. Let $k$ be a global function field. Suppose that we have an inclusion $G \hookrightarrow G'$ of affine commutative $k$-group schemes of finite type such that Proposition 6.10.1 holds for $G'$. Then it also holds for $G$.

Let us assume this for the moment and use it to complete the proof of Proposition 6.10.1. We will need a couple of lemmas.

Lemma 6.10.4. Let $k$ be a field, $L/k$ a (not necessarily algebraic) separable extension field of $k$, and let $I$ be an infinitesimal commutative $k$-group scheme. Then the map $\Theta^1(k, I) \to \Theta^1(L, I)$ is injective.

Proof. Suppose that we have a $I$-torsor $E$ over $k$ such that $E(L) \neq \emptyset$. We want to show that $E(k) \neq \emptyset$. Any such $E$ is finite radiciel over $k$ by fpqc descent due to the same property holding for $I$, so it suffices to show that for any finite radiciel $k$-scheme $X$ we have $X(L) \neq \emptyset \Rightarrow X(k) \neq \emptyset$. The pullbacks of $x \in X(L)$ under the two projections $L \otimes_k L \rightarrow L$ must yield the same point in $X(L \otimes_k L)$ since this latter set has at most one point (as $L \otimes_k L$ is reduced due to the separability of $L/k$, and $X$ is finite radiciel over $k$). By fpqc descent, $x$ therefore descends to a point in $X(k)$, as desired. \qed
Lemma 6.10.5. Let $G$ be an affine commutative $k$-group scheme of finite type. Then for some finite extension $k'/k$, we have $\Pi^1(G_{k'}) = 0$.

Proof. After such a base change, we may assume that $G_{\text{red}}^0$ is a subgroup scheme and is in fact the product of a split torus and a split unipotent group. Thus, $H^1(k', G_{\text{red}}^0) = 0$, so we only need to show that $\Pi^1(B) = 0$, where $B = G/G_{\text{red}}^0$. By extending scalars further, we may assume that $B$ is the product of an infinitesimal and a finite étale group scheme. We know that $\Pi^1$ vanishes for infinitesimal groups (by Lemma 6.10.4 applied to the separable extension $k_{\text{red}}/k$), so we are left to treat the case when $B = E$ is étale. Extending scalars further, we may assume that $E$ is constant. Since $\Pi^1(Z/nZ) = 0$ by the Chebotarev Density Theorem, we are done.

Now let us prove Proposition 6.10.1 conditional on Lemma 6.10.3. Let $k'/k$ be a finite extension such that $\Pi^1(G_{k'}) = 0$. Using the inclusion $G \hookrightarrow R_{k'/k}(G_{k'})$ and Lemma 6.10.3 it suffices to prove Proposition 6.10.1 for $R_{k'/k}(G_{k'})$. But we have a Leray spectral sequence associated to the morphism $j : \text{Spec}(k') \to \text{Spec}(k)$, namely

$$E^2_{ij} = H^i(k, R^j f_* G_{k'}) \Rightarrow H^{i+j}(k', G_{k'}) ,$$

and similarly for the morphism $\text{Spec}(A_{k'}) \to \text{Spec}(A_k)$. By compatibility of these spectral sequences, we get an inclusion $\Pi^1(k, R_{k'/k}(G_{k'})) \hookrightarrow \Pi^1(k', G_{k'})$. Since the latter group vanishes, so does the former, hence Proposition 6.10.1 holds trivially for $R_{k'/k}(G_{k'})$.

It remains to prove Lemma 6.10.3. Let $H := G'/G$, so we have an exact sequence

$$1 \to G \xrightarrow{j} G' \xrightarrow{\pi} H \to 1 .$$

Consider the following commutative diagram with exact rows:

$$\begin{array}{ccc}
G'(k) & \xrightarrow{\pi} & H(k) \\
\downarrow & & \downarrow \\
G(A) & \xrightarrow{j} & G'(A)
\end{array}$$

$$\begin{array}{ccc}
\xrightarrow{\delta} & H^1(k, G) & \xrightarrow{j} & H^1(k, G') \\
\downarrow & & \downarrow & \downarrow \\
\xrightarrow{\delta} & H^1(A, G) & \xrightarrow{\pi} & H^1(A, G')
\end{array}$$

Suppose that $\alpha \in \Pi^1(G)$ annihilates $\Pi^2(G)$ under $\langle \cdot, \cdot \rangle_{\Pi^2_G}$. We want to show that $\alpha = 0$. By functoriality of $\langle \cdot, \cdot \rangle_{\Pi^2_G}$ and our hypothesis that Proposition 6.10.1 holds for $G'$, we deduce that $j(\alpha) = 0$, hence $\alpha = \delta(h)$ for some $h \in H(k)$. Then $\delta(h_A) = 0$ in $H^1(A, G)$, so $h_A = \pi(g')$ for some $g' \in G'(A)$. We are going to show that by adjusting $g'$ by an element of $G'(k)$ (which has the effect of changing $h$ by an element of $\pi(G'(k))$, as we are free to do for the purpose of studying $\alpha = \delta(h)$), we can arrange that $g'$ comes from $G(A)$, so $h_A \in \pi(j(G(A))) = 0$, forcing $h = 0$, so $\alpha = 0$, as desired.

The natural map $G'(A) \to H^2(k, G')^*$ assigns to $g'$ a linear form $H^2(k, G') \to Q/Z$. Consider the subgroup $\widehat{j}^{-1}(\Pi^2(G)) \subset H^2(k, G')$, where $\widehat{j} : H^2(k, G') \to H^2(k, G)$ is the map induced by $j$. The key point for controlling the choice of $g'$ is the following lemma.
Lemma 6.10.6. The point $g'$ annihilates $\hat{j}^{-1}(\Pi^2(\hat{G})) \subset H^2(k, \hat{G}')$.

Proof. Since $\alpha$ annihilates $\Pi^2(\hat{G})$ by assumption, it suffices to show (without any hypotheses on $\alpha$!) that for any $z \in \hat{j}^{-1}(\Pi^2(\hat{G}))$ we have

$$\langle g', z \rangle = \langle \alpha, \hat{j}(z) \rangle_{\Pi^2}. \quad (6.10.1)$$

Lift $h \in H(k)$ to some $x \in \hat{C}^0(k, G') = G'(\bar{k})$. Then $\pi(dx) = dh = 0$, so $dx = j(\alpha')$ for some 1-cochain $\alpha' \in \hat{C}^1(k, G)$ that is a 1-cocycle (as $d(dx) = 0$ and $j$ is injective). The Čech cohomology class of $\alpha'$ coincides with the derived functor cohomology class of $\delta(h) = \alpha$ since the identification of Čech and derived functor cohomology in degrees $\leq 1$ is $\delta$-functorial by Proposition E.2.1 so we may and do denote the 1-cocycle $\alpha'$ as $\alpha$.

Since $\hat{j}(z) \in \Pi^2(\hat{G})$ by the way we chose $z$, if we also write $z$ to denote a representative class in $\hat{Z}^2(k, \hat{G})$ (as we may do by Proposition 3.5.1), then for each place $v$ of $k$ we have

$$\hat{j}(z) = dy_v \text{ for some } y_v \in \hat{C}^1(k_v, \hat{G}).$$

We have $\alpha \cup \hat{j}(z) = dm$ for some $m \in \hat{C}^2(k, \Gamma_m)$ since $H^3(k, \Gamma_m) = 0$ (Lemma 6.9.1). Then by definition of the two pairings, (6.10.1) is equivalent to

$$\sum_v \text{inv}_v(g'_v \cup z) = \sum_v \text{inv}_v((\alpha \cup y_v) - m) \quad (6.10.2)$$

Since $\pi(g'_v) = h = \pi(x)$, we deduce that $g'_v = x + j(f_v)$ for some $f_v \in \hat{C}^0(k_v, G)$. Applying $d$ to both sides yields (since $g'_v$ is a cocycle, representing an element of $H^0(k_v, G')$)

$$0 = dx + dj(f_v) = j(\alpha) + j(df_v),$$

so $df_v = -\alpha$. Now we compute at the cochain level

$$g'_v \cup z = (x + j(f_v)) \cup z = (x \cup z) + (f_v \cup \hat{j}(z)) = (x \cup z) + (f_v \cup dy_v) = (x \cup z) - (df_v \cup y_v) + df_v \cup y_v$$

and the final expression is cohomologous to

$$(x \cup z) - (df_v \cup y_v) = (x \cup z) + (\alpha \cup y_v).$$

The way we have arrived at this ensures that the final expression is a cocycle, and likewise the differences $(\alpha \cup y_v) - m$ appearing on the right side of (6.10.2) are cocycles, so taking difference gives that the global cochain $(x \cup z) + m$ is a cocycle (as this can be checked locally at any single place). Thus, the desired identity (6.10.2) is equivalent to the vanishing of the sum

$$\sum_v \text{inv}_v((x \cup z) + m)$$

of local invariants of a global Brauer class. Such sums always vanish. \qed
Via the inclusion (actually isomorphism, but we do not need this)

\[ H^2(k, \hat{G})/\hat{j}^{-1}(\mathbb{II}^2(\hat{G})) \to H^2(k, \hat{G})/\mathbb{II}^2(\hat{G}) \]

we can extend the homomorphism induced by \( g' \) to a homomorphism on \( H^2(k, \hat{G}) \) that kills \( \mathbb{II}^2(\hat{G}) \). We would like to say that this homomorphism is induced by an element of \( G(A) \). Thanks to Propositions 6.4.1 and 6.5.1 this follows from the following lemma.

**Lemma 6.10.7.** Let \( k \) be a global function field. Suppose that we have an inclusion \( G \hookrightarrow G' \) of affine commutative \( k \)-group schemes of finite type. Then the following diagram is Cartesian:

\[
\begin{array}{ccc}
G(A)/G(k) & \longrightarrow & G'(A)/G'(k) \\
\downarrow & & \downarrow \\
(G(A)/G(k))_{\text{pro}} & \longrightarrow & (G'(A)/G'(k))_{\text{pro}}
\end{array}
\]

Let us assume this for the moment, and use it to complete the proof of Lemma 6.10.3. Modifying \( g' \) by a suitable element of \( G(A) \) (as we are free to do), we may assume that it annihilates all of \( H^2(k, \hat{G'}) \). By Proposition 6.5.1 it follows that it lifts to an element of \( G'(k) \). Modifying \( h \) by this element of \( G'(k) \) (as we may do), we obtain \( h_A = 0 \), hence \( h = 0 \), so \( \alpha = \delta(h) = 0 \), and the proof is complete.

It only remains to prove Lemma 6.10.7. Before giving the proof, we require the following easy lemma.

**Lemma 6.10.8.** For any inclusion \( A \hookrightarrow A' \) of discrete finitely generated abelian groups, the following diagram is Cartesian:

\[
\begin{array}{ccc}
A & \longrightarrow & A' \\
\downarrow & & \downarrow \\
A_{\text{pro}} & \longrightarrow & A'_{\text{pro}}
\end{array}
\]

**Proof.** By the structure theorem for finitely generated abelian groups it is clear that the canonical map \( \hat{\mathbb{Z}} \otimes \mathbb{Z} A \to A_{\text{pro}} \) is an isomorphism. If we define \( A'' = A'/A \) then by the \( \mathbb{Z} \)-flatness of \( \hat{\mathbb{Z}} \) we have a commutative diagram of short exact sequences

\[
\begin{array}{cccc}
0 & \longrightarrow & A & \longrightarrow & A' & \longrightarrow & A'' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_{\text{pro}} & \longrightarrow & A'_{\text{pro}} & \longrightarrow & A''_{\text{pro}} & \longrightarrow & 0
\end{array}
\]

Thus, the Cartesian assertion reduces to showing that \( M \to M_{\text{pro}} \) is injective for any finitely generated abelian group \( M \) (applied to \( M = A'' \)). Such injectivity is immediate either from the faithful flatness of \( \hat{\mathbb{Z}} \) over \( \mathbb{Z} \) or by using the structure theorem to reduce to the easy cases when \( M = \mathbb{Z} \) and \( M = \mathbb{Z}/n\mathbb{Z} \).

\[ \square \]
Proof of Lemma 6.10.7. The vertical maps are inclusions (by Proposition 6.5.1). For ease of notation, let $B := G(A)/G(k)$, $B' := G'((A)/G'(k)$. Also let $B_1 := G(A)_1/G(k)$ and $B'_1 := G'(A)_1/G'(k)$ (for the definition of $G(A)_1$, see Definition 6.5.3). Then $B/B_1$ and $B'/B'_1$ are discrete and finitely generated, and $B_1, B'_1$ are profinite (Proposition 6.11.1).

We claim that $B/B_1 \to B'/B'_1$ is an inclusion. Indeed, pick $g \in G(A) \cap G'(A)_1$ and $\chi \in \hat{G}(k)$. We need to show that $\|\chi(g)\| = 1$. By Corollary 2.2.5, $\chi^n$ extends to a character in $\hat{G}(k)$ for some positive integer $n$. We then have that $|\chi^n(g)| = 1$, so $|\chi(g)| = 1$, hence $g \in G(A)_1$, as desired.

Now consider $b' \in B'$ and $b \in B_{\text{pro}}$ such that $b$ maps to the image of $b'$ in $B'_{\text{pro}}$. We want to show that $b$ comes from an element of $B$. The diagram

$$
\begin{array}{c}
B/B_1 & \xrightarrow{\cdot \equiv b} & B'/B'_1 \\
\downarrow & & \downarrow \\
(B/B_1)_{\text{pro}} & \xrightarrow{\cdot \equiv b} & (B'/B'_1)_{\text{pro}}
\end{array}
$$

is Cartesian by Lemma 6.10.8 so by modifying $b'$ by an element of $B$, we may assume that $b' \in B'_1$. Since the map $B_{\text{pro}}/(B_1)_{\text{pro}} = (B/B_1)_{\text{pro}} \to (B'/B'_1)_{\text{pro}} = B'_{\text{pro}}/(B'_1)_{\text{pro}}$ is an inclusion (e.g. by Proposition 2.2.1), we may assume that $b$ lies in the image of the map $(B_1)_{\text{pro}} = B_1 \to B_{\text{pro}}$. We are done, therefore, provided that the map $B_1 = (B_1)_{\text{pro}} \to B'_{\text{pro}}$ is injective, and this injectivity follows from Proposition 2.2.1. 

6.11 Injectivity of $\mathbb{III}^2(G) \to \mathbb{III}^1(\hat{G})^*$ for finite $G$

The purpose of this section is to prove the following result.

**Proposition 6.11.1.** Let $G$ be a finite commutative group scheme over the global function field $k$. Then the map $\mathbb{III}^2(G) \to \mathbb{III}^1(\hat{G})^*$ induced by $\langle \cdot , \cdot \rangle_{\mathbb{III}_G^2}$ is injective.

Combining this with Proposition 6.10.1 and Remark 6.9.2 we obtain the following result once it is known that $\mathbb{III}^1(H)$ is finite all finite commutative $k$-group schemes $H$ (as then by Cartier duality the same holds for $\mathbb{III}^2(H)$ due to Proposition 6.11.1):

**Corollary 6.11.2.** Theorem 1.2.9 holds for finite group schemes.

To prove that $\mathbb{III}^1(H)$ is finite for all finite commutative $k$-group schemes $H$, we first observe that if $H_{\text{sm}}$ is the maximal smooth closed $k$-subgroup of $H$ (or equivalently the maximal étale closed $k$-subgroup of the finite $k$-group scheme $H$) then $\mathbb{III}^1(H_{\text{sm}}) \to \mathbb{III}^1(H)$ is bijective due to the separability of $k_v/k$ for all places $v$ of $k$; see [CGP, Ex. C.4.3] (with $S = \emptyset$ there) for the details. Thus, it suffices to treat étale $H$, so in effect finite discrete Galois modules (without requiring $\#H \in k^\times$). If $k'/k$ is a finite Galois extension that splits $H$ then by finiteness of $H^1(k'/k, H(k'))$ we see via via inflation-restriction that finiteness
of $\Pi_1(H)$ reduces to that of $\Pi_1(H_{k'})$. In other words, we may assume $H$ is the constant $k$-group associated to a finite abelian group $A$. Hence, $H^1(F, H) = \text{Hom}_{\text{cts}}(\text{Gal}(F_s/F), A)$ for any field $F/k$, so $\Pi_1(H) = 0$ by the Chebotarev Density Theorem.

We begin the proof of Proposition 6.11 with the following lemma.

**Lemma 6.11.3.** Let $G$ be a finite commutative $k$-group scheme. There exists a finite separable extension $k'/k$ such that $\Pi_1(G_{k''}) = 0$ for all finite extensions $k''/k'$.

**Proof.** We have the connected-étale sequence:

$$1 \longrightarrow I \longrightarrow G \longrightarrow E \longrightarrow 1$$

with $I$ infinitesimal and $E$ étale. Replacing $k$ with a finite separable extension, we may assume that $E$ is constant. Of course, this constancy is preserved under any further extension of scalars. It suffices to show, then, that if $E$ is constant, then $\Pi_1(G) = 0$. We have a commutative diagram with exact rows

$$
\begin{array}{c}
E(k) \\ \downarrow \\
\prod_v E(k_v) \\
\end{array} \longrightarrow 
\begin{array}{c}
H^1(k, I) \\ \downarrow \\
\prod_v H^1(k_v, I) \\
\end{array} \longrightarrow 
\begin{array}{c}
H^1(k, G) \\ \downarrow \\
\prod_v H^1(k_v, G) \\
\end{array} \longrightarrow 
\begin{array}{c}
H^1(k, E) \\ \downarrow \\
\prod_v H^1(k_v, E) \\
\end{array}
$$

in which the last vertical arrow is an inclusion because $\Pi_1(E) = 0$ by the Chebotarev density Theorem. Now suppose that $\alpha \in \Pi_1(G)$. Then looking at the diagram, we see that $\alpha$ lifts to some $\beta \in H^1(k, I)$. Further, $\beta_v \mapsto 0 \in H^1(k_v, G)$, hence lifts to some $e_v \in E(k_v)$. Since $E$ is constant, we may lift $e_v$ to $e \in E(k)$. Then modifying $\beta$ by $e$, we see that $\beta_v = 0$, hence $\beta = 0$ by Lemma 6.10.4. Therefore $\alpha = 0$ as well, so $\Pi_1(G) = 0$. \qed

The following degree-2 variant of Lemma 6.11.3 lies deeper.

**Proposition 6.11.4.** Let $G$ be a finite commutative $k$-group scheme. There exists a finite separable extension $k'/k$ such that $\Pi_2(G_{k''}) = 0$ for every finite extension $k''/k'$.

The proof requires several preparatory lemmas.

**Lemma 6.11.5.** Let $k$ be a global field, $G$ an affine commutative $k$-group scheme of finite type. Then $H^i(k, G)$ and $H^i(k, \hat{G})$ are countable for $i \leq 2$.

**Remark 6.11.6.** Actually, this holds for all $i$, but we will only be interested in the case $i \leq 2$. In fact, the main case of interest for us is $i = 1$.

**Proof.** As usual, Lemma 2.1.7 reduces us to the case when $G$ is either an almost torus or $\mathbf{G}_a$. In fact, Lemma 2.1.3 reduces the almost torus case to the case when $G$ is either finite or a torus. So we may assume that $G$ is either finite, a torus, or $\mathbf{G}_a$.\hfill158
For $G = G_a$, all of the relevant cohomology groups vanish except for $H^0(k, G) = k$, which is countable, and possibly $H^2(k, G)$ (Proposition 2.4.2 and Lemma 2.4.4(ii)). So we are left to show that $H^1(k, G)$ is countable. If $\text{char}(k) = 0$, then this group vanishes (Proposition 2.4.1), while if $\text{char}(k) > 0$, then it is a one-dimensional $k$-vector space (Corollary 2.6.2), hence countable.

We next treat the case when $G$ is finite, for which we only need to treat $G$ (and not also $\hat{G}$) by Cartier duality. We proceed by induction on $i$. The assertion is trivial for $i = 0$. If it holds for $i - 1$, then to deduce it for $i$, we first note that we may filter $B$ and therefore assume that it is either étale, multiplicative, or local-local. If $B$ is local-local, then we may filter it further to assume that its relative Frobenius and Verschiebung morphisms vanish, assume that it is either étale, multiplicative, or local-local. If $\hat{G}$ is finite, then for which we only need to treat $\hat{G}$ (Corollary 2.6.2), hence countable.

By the spectral sequence $\text{Br}(k, B)$, we proceed by induction on $i$. We may therefore assume that either $B = \alpha_p$, $B = B/k$, or $B = Z/pB$, with $p = \text{char}(k) > 0$ in the first and third cases.

If $p = \text{char}(k) > 0$, then $H^1(k, \alpha_p) = k/k^p$ is countable and $H^2(k, \alpha_p) = 0$. For $\mu_n$ and arbitrary characteristic, we have $H^1(k, \mu_n) = k^1/(\kappa^n)$ (which is obviously countable) and $H^2(k, \mu_n) = \text{Br}(k)[n]$ (which is countable by global class field theory). Finally, if $p = \text{char}(k) > 0$ then $H^1(k, Z/pZ) = k/\phi(k)$ (clearly countable) with $\phi : k \to k$ the Artin-Schreier map $x \mapsto x^p - x$, and $H^2(k, Z/pZ) = 0$.

Next we treat the case when $G = T$ is a torus. The cases with $i = 0$ are clear, so we may assume that $i > 0$. Since $T$ and $\hat{T}$ are represented by smooth $k$-group schemes, the cohomology may be taken to be étale. Since higher Galois cohomology is torsion, it suffices to show that the groups $H^i(k, T)[n]$ are countable for all $n > 0$, and ditto for $\hat{T}$. For this, note that we have an exact sequence

$$1 \to T[n] \to T \xrightarrow{[n]} T \to 1$$

with $T[n]$ a finite $k$-group scheme. Indeed, this may be checked fppf locally, so we may assume that $T$ is split, and the assertion is then obvious. We therefore obtain for all $i$ a surjective map $H^i(k, T[n]) \to H^i(k, T)[n]$, so the already-treated finite case implies that the latter group is countable. The groups $H^i(k, \hat{T})[n]$ are treated similarly, by using the exact sequence

$$1 \to \hat{T} \xrightarrow{[n]} \hat{T} \to \hat{T}/[n]\hat{T} \to 1$$

in which $\hat{T}/[n]\hat{T}$ is a finite $k$-group scheme.

\[\square\]

Remark 6.11.7. Before continuing with the proof, it will help to review some input concerning exactness at certain already-defined stages of the 9-term Poitou-Tate sequence in Theorem 1.2.8 in the special case of finite commutative $k$-group schemes, avoiding the stages that involve III-pairings (whose perfectness in Theorem 1.2.9 we have not yet established). Upon removing from the 9-term sequence the maps coming out of the groups $H^i(k, \hat{G})^*$
(i = 1, 2) (so what remains does not depend on III-pairings, nor on Theorem 1.2.9), we know that the three resulting smaller sequences are exact when G is finite due to prior work of others (see Remark 1.2.12). Lemma 6.11.8 below extends one of these exactness properties by a useful small amount.

**Lemma 6.11.8.** Let G be a finite group scheme over the global function field k. Then the following sequence is exact:

\[ H^1(k, G) \rightarrow H^1(A, G) \rightarrow H^1(k, \hat{G})^* \rightarrow \text{III}^1(\hat{G})^* \]

**Proof.** Given what we are assuming about finite commutative group schemes (cf. Remark 6.11.7), we only have to check exactness at \( H^1(k, \hat{G})^* \). Consider the exact sequence

\[ \text{III}^1(\hat{G}) \rightarrow H^1(k, \hat{G}) \rightarrow H^1(A, \hat{G}) \]  

(6.11.1)

We claim that applying \((\cdot)^D\) to this sequence preserves exactness (where we are endowing \( H^1(k, \hat{G}) \) and \( \text{III}^1(\hat{G}) \) with the discrete topology). Note that the map \( H^1(A, G) \rightarrow H^1(k, \hat{G})^* \) is continuous, by the same argument as the one given several paragraphs after the statement of Theorem 1.2.8 to show that the map \( G(A) \rightarrow H^2(k, \hat{G})^* \) is continuous. Assuming such preservation of exactness under \((\cdot)^D\) and recalling that \( H^1(k, \hat{G})^* = H^2(k, \hat{G})^D \) by Lemma 4.3.1, we get an exact sequence

\[ H^1(A, \hat{G})^D \rightarrow H^1(k, \hat{G})^* \rightarrow \text{III}^1(\hat{G})^*. \]

By Proposition 6.8.1 (and the obvious compatibility of the pairings \( H^1(A, G) \times H^1(A, \hat{G}) \rightarrow \mathbb{Q}/\mathbb{Z} \) and \( H^1(A, G) \times H^1(k, \hat{G}) \rightarrow \mathbb{Q}/\mathbb{Z} \)), the proof would be done. Thus, it only remains to prove the asserted exactness of the Pontryagin dual of (6.11.1).

For this, it suffices to show that the continuous inclusion \( H^1(k, \hat{G})/\text{III}^1(\hat{G}) \rightarrow H^1(A, \hat{G}) \) is a homeomorphism onto a closed subgroup. By Lemmas 4.3.3 and 6.11.5 it suffices to simply show that the image is closed. This follows from exactness of the sequence

\[ H^1(k, \hat{G}) \rightarrow H^1(A, \hat{G}) \rightarrow H^1(k, G)^* \]

which is part of the Poitou-Tate sequence for finite commutative group schemes (and even of the part that we are assuming to be exact!), since this expresses this image as the kernel of a continuous map to a Hausdorff group.

**Lemma 6.11.9.** Suppose we have an exact sequence

\[ 1 \rightarrow G' \xrightarrow{j} G \xrightarrow{\pi} G'' \rightarrow 1 \]

of finite commutative k-group schemes, and suppose that \( \alpha \in \text{III}^2(G) \) annihilates \( \text{III}^1(\hat{G}) \) under \( \langle \cdot, \cdot \rangle_{\text{III}^2(\hat{G})} \) and lifts to an element of \( H^2(k, G') \). Then \( \alpha \) lifts to an element of \( \text{III}^2(G') \).
Proof. Consider the following commutative diagram with exact rows:

\[
\begin{array}{cccccc}
\text{H}^1(k, G) & \longrightarrow & \text{H}^1(k, G'') & \longrightarrow & \text{H}^2(k, G') & \longrightarrow & \text{H}^2(k, G) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{H}^1(A, G) & \longrightarrow & \text{H}^1(A, G'') & \longrightarrow & \text{H}^2(A, G') & \longrightarrow & \text{H}^2(A, G) \\
\downarrow & & \downarrow & & \downarrow & & \\
\text{H}^1(k, \hat{G})^* & \longrightarrow & \text{H}^1(k, \hat{G}'')^* & \longrightarrow & \text{H}^2(A, \hat{G})^* & \longrightarrow & \text{H}^2(A, \hat{G})^* \\
\downarrow & & \downarrow & & \downarrow & & \\
\text{III}^1(\hat{G})^* & \longrightarrow & \text{III}^1(\hat{G}'')^* & \longrightarrow & \text{III}^1(\hat{G})^* & \longrightarrow & \text{III}^1(\hat{G}'')^*
\end{array}
\]

The two leftmost columns are exact by previously known exactness for parts of the Poitou-Tate sequence for finite commutative \( k \)-group schemes as discussed in Remark 6.11.7. Lift \( \alpha \) to \( \beta \in \text{H}^2(k, G') \), so \( \beta_A \in \text{H}^2(A, G') \) lifts to some \( \gamma \in \text{H}^1(A, G'') \). Via the usual adelic pairing, \( \gamma \) yields an element \( \phi_\gamma \in \text{H}^1(k, \hat{G}'')^* \). The key point is then Lemma 6.11.10 below. Once we have that, we see that \( \phi_\gamma \) lifts to an element \( \psi \in (\text{H}^1(k, \hat{G})/\text{III}^1(\hat{G}))^* \) due to the injectivity of the map

\[
\text{H}^1(k, \hat{G}'')/\hat{\pi}^{-1}(\text{III}^1(\hat{G})) \rightarrow \text{H}^1(k, \hat{G})/\text{III}^1(\hat{G})
\]

and the injectivity of the abelian group \( \mathbb{Q}/\mathbb{Z} \). Again appealing to already-known exactness properties at stages of the Poitou-Tate sequence in the finite case as discussed in Remark 6.11.7, there exists \( g \in \text{H}^1(A, G) \) inducing \( \psi \). Modifying \( \gamma \) by \( g \) (as we may), we then obtain \( \phi_\gamma = 0 \), hence \( \gamma \) lifts to an element \( g'' \in \text{H}^1(k, G'') \). Modifying \( \beta \) by \( g'' \) (as we may), we then obtain that \( \beta \in \text{III}^2(G') \), as desired. \( \Box \)

Lemma 6.11.10. With notation as above, \( \phi_\gamma \) annihilates \( \hat{\pi}^{-1}(\text{III}^1(\hat{G})) \subset \text{H}^1(k, \hat{G}'') \) under the adelic pairing.

Proof. The key point is to prove the following equality: for \( z \in \hat{\pi}^{-1}(\text{III}^1(\hat{G})) \), we have

\[
\langle \gamma, z \rangle = \langle \alpha, \hat{\pi}(z) \rangle_{\text{III}^2_{\hat{G}}}
\] (6.11.2)

Since \( \alpha \) annihilates \( \text{III}^1(\hat{G}) \) by assumption, this would imply what we want.

Let us first note that the lemma is independent of this choice of lift \( \gamma \in \text{H}^1(A, G'') \). Indeed, if we change \( \gamma \) by \( \pi(\zeta) \) for some \( \zeta \in \text{H}^1(A, G) \), then for \( c \in \hat{\pi}^{-1}(\text{III}^1(\hat{G})) \), \( \langle \gamma, c \rangle \) changes by \( \langle \pi(\zeta), c \rangle = \langle \zeta, \hat{\pi}(c) \rangle = 0 \), since \( \zeta \) pairs trivially with \( \text{III}^1(\hat{G}) \).

We know that \( \delta(\gamma) = \beta_A \) as cohomology classes, where \( \delta \) is the connecting map in derived-functor cohomology. For the rest of this proof we shall use the symbols \( \gamma, \alpha \), etc. to denote fixed Čech cocycles representing the corresponding cohomology classes (e.g.,
\(\check{H}^2(k,G') = H^2(k,G')\) by Proposition 3.5.1, so we can pick \(\beta \in \hat{Z}^2(k,G')\). Consider the Čech cocycle \(\gamma'\) defined as follows. We know that the image of \(j(\beta_A)\) in \(H^2(A,G)\) vanishes, so its image in the Čech cohomology group \(\check{H}^2(A,G)\) vanishes since the edge map \(\check{H}^2 \to H^2\) is always injective (due to \(\hat{H}^1 \to H^1\) being an isomorphism). Hence, \(j(\beta_A) = dw\) in \(\hat{C}^2(A,G)\). The Čech 1-cochain \(\delta := \pi(w) \in \check{C}^1(A,G')\) is a cocycle since \(d\pi(w) = \pi(dw) = \pi(j(\beta_A)) = 0\) in \(\check{C}^2(A,G')\) due to the vanishing of \(\pi \circ j\). Hence, \(\delta(\gamma') = \beta_A\) as Čech cohomology classes (where \(\delta\) is the partially-defined connecting map in Čech cohomology: defined on the set of classes in a certain image on which the snake-lemma procedure makes sense). We may then replace \(\gamma\) with the cocycle \(\gamma'\), and thereby assume that \(\gamma = \pi(w)\) and \(dw = j(\beta_A)\) as Čech cochains.

The left side of (6.11.2) is by definition

\[
\sum_v \text{inv}_v(\gamma \cup z)
\]

Since \(\hat{\pi}(z)\) represents an element of \(\check{\Pi}^1(\hat{G}) = \ker(H^1(k,\hat{G}) \to H^1(A,\hat{G}))\) with \(\hat{H}^1 = H^1\), we have \(\hat{\pi}(z) = dx\) for some \(x \in \check{C}^0(A,\hat{G})\). Further, we have \(\alpha \cup \hat{\pi}(z) = dh\) for some \(h \in \check{C}^2(k,\hat{G}_m)\) since \(\check{H}^3(k,\hat{G}_m) = 0\) by Lemma 6.9.1. The right side of (6.11.2) is then by definition

\[
\sum_v \text{inv}_v((\alpha \cup x_v) - h)
\]

Now we have as adelic Čech 2-cocycles

\[
\gamma \cup z = \pi(w) \cup z = w \cup \hat{\pi}(z) = w \cup dx = dw \cup x - d(w \cup x)
\]

and the last expression is cohomologous to

\[
dw \cup x = j(\beta) \cup x = \alpha \cup x.
\]

Equation (6.11.2) is therefore equivalent to the assertion

\[
\sum_v \text{inv}_v(h) = 0
\]

that in turn holds because the sum of the invariants of a global Brauer class is 0. \(\square\)

**Proof of Proposition 6.11.4** First suppose that we have an exact sequence

\[
1 \to G' \to G \to G'' \to 1
\]

of finite commutative \(k\)-group schemes, and that Proposition 6.11.4 holds for \(G',G''\). Then we claim that it also holds for \(G\). Indeed, we may by hypothesis extend scalars and thereby

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assume that $III^2(G') = III^2(G'') = 0$ (and that this remains true after any further finite extension of scalars on $k$). By Lemma 6.11.3, we may further assume that $III^1(\hat{G}) = 0$ and that this remains true after any further finite extension of scalars on $k$.

We will show under these hypotheses that $III^2(G) = 0$ (so we can reduce to treating $G'$ and $G''$ in place of $G$). Pick $\alpha \in III^2(G)$. Since $III^2(G'') = 0$, $\alpha$ lifts to $H^2(k, G')$. Then by Lemma 6.11.3 $\alpha$ lifts to $III^2(G') = 0$, so $\alpha = 0$.

Now we run a dévissage based on the preceding argument: by taking the preceding short exact sequence to be the connected-étale sequence of $G$ it suffices to treat separately the cases that $G$ is either infinitesimal or étale, and in the infinitesimal case we may use the connected-étale sequence for $\hat{G}$ to arrange that $G$ is either multiplicative or local-local. After possibly replacing $k$ with a finite separable extension, therefore, we may assume that $G$ is either $\mu_n$ for some $n > 0$ or is local-local or $p$-primary étale. In the last two cases we even have $H^2(k, G) = 0$ (by Lemma 3.5.4), and in the first case we are done because $III^2(\mu_n) = 0$ by class field theory.

Lemma 6.11.11. Let $G$ be a finite commutative $k$-group scheme. There is a surjection $G' \twoheadrightarrow G$ with $G'$ a finite commutative $k$-group scheme such that $III^2(G') = 0$.

Proof. We will prove this for $\hat{G}$ instead of $G$ (which of course suffices due to Cartier duality). By Proposition 6.11.4, there is a finite separable extension $k'/k$ such that $III^2(\hat{G}_{k'}) = 0$. Consider the canonical inclusion $G \rightarrow R_{k'/k}(\hat{G}_{k'})$. Since $k'/k$ is separable, $R_{k'/k}(\hat{G}_{k'})$ is a finite $k$-group scheme. By dualizing this inclusion, therefore, we obtain the desired surjection, since $III^2(R_{k'/k}(\hat{G}_{k'})) = III^2(R_{k'/k}(\hat{G}_{k'}')) = III^2(\hat{G}_{k'}) = 0$, where the first equality is due to Lemma 2.2.1 and the second is because finite separable Weil restriction is an exact functor on fppf abelian sheaves (since such exactness may be checked after scalar extension to a Galois closure $K/k$ of $k'/k$, which causes the Weil restriction to become a product of copies of pullbacks by the various $k$-embeddings of $k'$ in to $K$).

Lemma 6.11.12. Suppose that we have a surjection $G' \twoheadrightarrow G$ between affine commutative $k$-group schemes of finite type, and that an element $\alpha \in III^2(G)$ annihilates $III^1(\hat{G})$ under $\langle \cdot, \cdot \rangle_{III^2_G}$. Suppose also that the sequence

$$H^1(A, G) \rightarrow H^1(k, \hat{G})^* \rightarrow III^1(\hat{G})^*$$

is exact. Then $\alpha$ lifts to an element of $III^2(G')$.

Proof. The proof is very similar to that of Lemma 6.11.9. Let $H := \ker(G' \rightarrow G)$. Then we have an exact sequence

$$1 \rightarrow H \overset{j}{\rightarrow} G' \overset{\pi}{\rightarrow} G \rightarrow 1$$

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We claim that the commutative diagram

\[
\begin{array}{cccccc}
H^2(k, H) & \longrightarrow & H^2(k, G') & \longrightarrow & H^2(k, G) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
H^1(A, G) & \longrightarrow & H^2(A, H) & \longrightarrow & H^2(A, G') & \longrightarrow & H^2(A, G) \\
\downarrow & & \downarrow & & \downarrow & & \\
H^1(k, \hat{G})^* & \longrightarrow & \hat{H}(k)^* & & & & \\
\downarrow & & & & & & \\
\Pi^1(\hat{G})^* & & & & & & \\
\end{array}
\]

has exact rows and columns. Indeed, the first row is exact because \(H^3(k, H) = 0\) by Proposition 3.1.1 and the second row is clearly exact. The first column is exact by hypothesis, and the second column is exact by Proposition 6.7.1. We deduce that \(\alpha\) lifts to some \(\beta \in H^2(k, G')\) and that the adelic class \(\beta_A \in H^2(A, G')\) lifts to some \(\gamma \in H^2(A, H)\).

The dual sequence \(0 \to \hat{G} \to \hat{G}' \to \hat{H} \to 0\) is exact by Proposition 2.2.3. Let \(\hat{\delta} : \hat{H}(k) \to H^1(k, \hat{G})\) denote the connecting map. By Lemma 6.11.13 below, the homomorphism \(\phi_\gamma \in \hat{H}(k)^*\) induced by \(\gamma\) under the adelic pairing kills \(\hat{\delta}^{-1}(\Pi^1(\hat{G})) \subset \hat{H}(k)\). It therefore extends to an element of \((H^1(k, \hat{G})/\Pi^1(\hat{G}))^*\). By exactness of the first column of the above diagram, this latter homomorphism is induced by an element \(g \in H^1(A, G)\). Modifying \(\gamma\) by \(g\) (as we may), we may assume that \(\phi_\gamma = 0\). Therefore, \(\gamma\) lifts to some element \(h \in H^2(k, H)\).

Modifying \(\beta\) by \(h\) (as we may), we may assume that \(\beta \in \Pi^2(G')\), as desired.

**Lemma 6.11.13.** With notation as above, \(\gamma\) annihilates \(\hat{\delta}^{-1}(\Pi^1(\hat{G})) \subset \hat{H}(k)\) under the adelic pairing.

**Proof.** Choose \(z \in \hat{\delta}^{-1}(\Pi^1(\hat{G}))\). We will show that

\[
\langle \gamma, z \rangle = -\langle \alpha, \hat{\delta}(z) \rangle_{\Pi^1_H^G} \tag{6.11.3}
\]

This will prove the lemma because \(\alpha\) annihilates \(\Pi^1(\hat{G})\) by hypothesis.

Since \(H^2(k, G') = H^2(k, G')\) by Proposition 3.5.1 we can represent \(\beta\) by a class in \(\hat{Z}^2(k, G')\) and then take \(\pi(\beta) \in \hat{Z}^2(k, G)\) as our representative for \(\alpha\). In what follows, we use the symbols \(\alpha, \beta\), etc. to denote Čech cocycles representing the corresponding cohomology classes (so in particular we have arranged that \(\pi(\beta) = \alpha\) as Čech cochains). The left side of (6.11.3) is by definition

\[
\sum_v \text{inv}_v(\gamma_v \cup z)
\]

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The cohomology class \( \delta(z) \in H^1(k, \hat{G}) = \hat{H}^1(k, \hat{G}) \) is represented by a Čech 1-cocycle computed as follows. We may write \( z = j(x) \) for some \( x \in \hat{C}^0(k, \hat{G}^\prime) \) (by Corollary 2.2.4 since \( \hat{C}^0(k, \hat{G}^\prime) = G^\prime(\hat{k}) \)). Then we have \( dx = \pi(y) \) for some \( y \in \hat{C}^0(k, \hat{G}) \), and \( y \) represents the cohomology class \( \delta(z) \) by Proposition E.2.1.

Since \( \hat{H}^1 = H^1 \), we have \( j(\gamma) = \beta + df \) for some \( f \in \hat{C}^1(A, G^\prime) \). Therefore, \( \alpha = \pi(\beta) = -d\pi(f) \) since \( \pi \circ j = 0 \). We also have \( \alpha \cup y = dh \) for some \( h \in \hat{C}^2(k, G_m) \) since \( \hat{H}^3(k, G_m) = 0 \) by Lemma 6.9.1 The right side of (6.11.3) is by definition

\[- \sum_v \text{inv}_v(-(\pi(f_v) \cup y) - h)\]

Now in \( \hat{C}^2(A, \hat{G}) \) we have

\[\pi(f) \cup y = f \cup \hat{\pi}(y) = f \cup dx = df \cup x - d(f \cup x)\]

and this is cohomologous to \( df \cup x \). Equation (6.11.3) is therefore equivalent to

\[\sum_v \text{inv}_v((\gamma_v \cup z) - (df_v \cup x) - h) = 0\]  (6.11.4)

But

\[\gamma_v \cup z = \gamma_v \cup j(x) = j(\gamma_v) \cup x = (\beta + df_v) \cup x = (\beta \cup x + df_v \cup x)\]

so (6.11.4) is equivalent to

\[\sum_v \text{inv}_v((\beta \cup x) - h) = 0\]

and this is true because the sum of the invariants of a global Brauer class is 0. \( \square \)

It is now easy to complete the proof of Proposition 6.11.1. Suppose that \( \alpha \in \text{III}^2(G) \) annihilates \( \text{III}^1(\hat{G}) \). By Lemma 6.11.11, there is a surjection \( G^\prime \rightarrow G \) for some finite commutative \( k \)-group scheme \( G^\prime \) such that \( \text{III}^2(G^\prime) = 0 \). By Lemma 6.11.12 \( \alpha \) lifts to an element of \( \text{III}^2(G^\prime) = 0 \), hence \( \alpha = 0 \).

### 6.12 Exactness at \( H^1(A, G), H^1(A, \hat{G}), H^1(k, \hat{G}^\ast), \text{and } H^1(k, G^\ast) \)

In this section we will show that the Poitou-Tate sequence is exact at both \( H^1(A, G) \) and \( H^1(A, \hat{G}) \), and will use this to establish exactness at the global \( \mathbb{Q}/\mathbb{Z} \)-dual terms \( H^1(k, \hat{G}^\ast) \) and \( H^1(k, G^\ast) \) as well. The first of these results is the following proposition.

**Proposition 6.12.1.** Let \( k \) be a global function field, \( G \) an affine commutative \( k \)-group scheme of finite type. Then the following sequence is exact:

\[H^1(k, G) \rightarrow H^1(A, G) \rightarrow H^1(k, \hat{G})^\ast\]

where the maps are as in Theorem 1.2.8.

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As usual, in Proposition [6.12.1] we take the finite case as known, due to the work of Poitou, Tate, and Česnavicius. We also know that the sequence is always a complex (because the sum of the invariants of a global Brauer class is 0). To prove exactness in general, we need some preparatory lemmas.

**Lemma 6.12.2.** Let $k'/k$ be a finite extension of fields of characteristic $p > 0$. Then we have a functorial inclusion $H^2(k, R_{k'/k}(\mu_{p^n})) \hookrightarrow H^2(k', \mu_{p^n})$.

**Proof.** If $k'/k$ is a finite separable extension, and $f : \text{Spec}(k') \to \text{Spec}(k)$ the corresponding map, then $f_\ast$ is an exact functor between the categories of fppf abelian sheaves. Indeed, if $k''$ is a Galois closure of $k'/k$ and $g : \text{Spec}(k'') \to \text{Spec}(k)$ the corresponding map, then $g^\ast f_\ast = \prod_{\sigma}^\ast$, where the product is over all $k$-embeddings $\sigma : k' \hookrightarrow k''$. Let $L/k$ be the maximal separable subextension of $k'/k$. Thanks to the transitivity of Weil restriction, by first treating $L/k$ and then $k'/L$, and using the exactness just observed for separable pushforwards, we may assume that $k'/k$ is purely inseparable.

Consider the Kummer sequence

$$1 \to \mu_{p^n} \to G_m \xrightarrow{p^n} G_m \to 1$$

We claim that $R^1 f_\ast G_m = 0$. Assuming this, we obtain an exact sequence

$$f_\ast G_m \xrightarrow{p^n} f_\ast G_m \to R^1 f_\ast \mu_{p^n} \to 0 \quad (6.12.1)$$

To prove the claim, we note that $R^1 f_\ast G_m$ is the sheafification of the presheaf $X \to H^1(X \otimes_k k', G_m)$, since $G_m$ is smooth, we may take our cohomology to be étale. Since $X \otimes_k k'$ is radical over $X$, we have $H^1(X \otimes_k k', G_m) = H^1(X, G_m)$. Since the sheafification of the presheaf $X \to H^1(X, G_m)$ is trivial, we see that $R^1 f_\ast G_m = 0$, as claimed.

Let $k''$ be an intermediate field between $k'$ and $k$. Then we have a natural inclusion $R_{k''/k}(G_m) \hookrightarrow R_{k'/k}(G_m)$, given on $R$-points ($R$ a $k$-algebra) by the inclusion $(R \otimes_k k''^\times) \hookrightarrow (R \otimes_k k')^\times$. Let $k'' : = (k')^{p^n} k$ (i.e., the compositum of the fields $(k')^{p^n}$ and $k$). Then we claim that

$$\text{im}(f_\ast G_m \xrightarrow{p^n} f_\ast G_m) = R_{k''/k}(G_m) \quad (6.12.2)$$

Indeed, the direction $\subseteq$ is clear: we need to check that $((R \otimes_k k')^\times)^{p^n} \subset R \otimes_k k''$, and this may be checked on pure tensors where it is obvious. Since every element of $R$ is fppf-locally a $p$th power, we also see that if $\lambda \in k''$, then $\lambda r \in \text{im}(f_\ast G_m \xrightarrow{p^n} f_\ast G_m)$. Combining (6.12.1) and (6.12.2), we obtain an exact sequence

$$0 \to R_{k''/k}(G_m) \to R_{k'/k}(G_m) \to R^1 f_\ast \mu_{p^n} \to 0$$

Thanks to the Leray spectral sequence, we have $H^1(k, R_{k''/k}(G_m)) \hookrightarrow H^1(k'', G_m) = 0$, hence $H^0(k, R^1 f_\ast \mu_{p^n}) \simeq (k')^\times \otimes (k'')^\times$. We also have $H^1(k', \mu_{p^n}) \simeq (k')^\times / (k'')^{p^n}$. The Leray spectral sequence $E_2^{i,j} = H^i(k, R^j f_\ast \mu_{p^n}) \Longrightarrow H^{i+j}(k', \mu_{p^n})$ yields a map $H^1(k', \mu_{p^n}) \to
H^0(k, R^1 f_* \mu_{p^n}), and to prove the lemma, it is equivalent to show that this map is surjective. But under the isomorphisms obtained above, this map corresponds to the obvious map \((k')^\times/(k')^\times p^n \to (k')^\times/(k')^\times\), which is clearly surjective. That these two maps are the same follows from the fact that the map \(H^1(k', \mu_{p^n}) \to H^0(k, R^1 f_* \mu_{p^n})\) is the natural map from the \(k\)-points of the presheaf \(X \mapsto H^1(X \otimes_k k', \mu_{p^n})\) to the \(k\)-points of the sheafification of this presheaf.

Lemma 6.12.3. Let \(k\) be a global field, \(G\) a finite commutative \(k\)-group scheme. Then there is an inclusion \(G \hookrightarrow G'\) with \(G'\) a finite commutative \(k\)-group scheme such that \(\Pi^2(G') = 0\).

Proof. We will first construct \(G'\) as in the lemma that is almost-unipotent rather than finite. In order to do this, we note that for some finite extension \(k'/k\), the connected étale sequence for \(G_{k'}\) splits, hence \(G_{k'} = E \times I\) for some constant étale \(E\) and infinitesimal \(I\). Enlarging \(k'\) further, the connected-étale sequence for \(\hat{I}\) splits, hence \(I = M \times L\), where \(M\) is split \(p\)-primary multiplicative and \(L\) is local-local. By Lemma 2.1.11, therefore, it suffices to show that \(\Pi^2(R_{k'/k}(G')) = 0\) if \(G' = \mu_n\) (with \(n\) a prime power), \(Z/pZ\), or local-local.

First, if \(G'\) is either \(\mu_n\) with \(p \nmid n\) or \(Z/pZ\), then since it is smooth, étale and fpf cohomology agree for \(R_{k'/k}(G')\). Since finite pushforward is exact for the étale topology, we have \(\Pi^2(k, R_{k'/k}(G')) \approx \Pi^2(k', G')\), and this last group vanishes by class field theory in the \(\mu_n\) case, and it vanishes in the \(Z/pZ\) case because any field of characteristic \(p\) has \(p\)-cohomological dimension \(\leq 1\) ([Ser1 Ch. II, §2.2, Prop. 3]).

Next consider the local-local case. Then by [Oes App. 3, A.3.6], there is an exact sequence

\[1 \longrightarrow U \longrightarrow R_{k'/k}(G')_{k'} \longrightarrow G' \longrightarrow 1\]

with \(U\) split unipotent. It follows that \(R_{k'/k}(G')_{k'}\) is connected, almost-unipotent, and contains no nontrivial local-étale group scheme. By [SGA3] VIIA, Prop.8.3], there is an infinitesimal subgroup scheme \(I \subset R_{k'/k}(G')\) such that \(R_{k'/k}(G')/I\) is smooth, hence unipotent. By what we have just observed, \(I\) must be local-local (otherwise it contains a multiplicative group scheme, by consideration of the connected-étale sequence for \(\hat{I}\)). We therefore have \(H^2(k, R_{k'/k}(G')) = 0\) by Lemmas 2.4.4 and 3.5.4.

Finally, consider the case when \(G' = \mu_{p^n}\). By Lemma 6.12.2, we have an inclusion \(\Pi^2(k, R_{k'/k}(\mu_{p^n})) \hookrightarrow \Pi^2(k', \mu_{p^n})\), and the latter group vanishes by class field theory.

So we have an inclusion \(G \hookrightarrow H\) with \(H\) an almost-unipotent \(k\)-group such that \(\Pi^2(H) = 0\). Let \(\overline{H} := H/G\), and let \(\pi : H \to \overline{H}\) be the projection. Then \(\overline{H}\) is almost-unipotent, so by Lemma 2.1.8(iii), there is a finite \(k\)-subgroup scheme \(\overline{G'} \subset \overline{H}\) such that \(\overline{H}/\overline{G'}\) is split unipotent. Then \(G' = \pi^{-1}(\overline{G'})\) is a finite \(k\)-group scheme containing \(G\), and \(U := H/G'\) is split unipotent. I claim that \(\Pi^2(G') = 0\), which will complete the proof. To
see this, consider the following commutative diagram:

\[
\begin{array}{c}
0 \longrightarrow H^2(k, G') \longrightarrow H^2(k, H) \\
\downarrow \quad \quad \quad \downarrow \\
H^2(A, G') \longrightarrow H^2(A, H)
\end{array}
\]

The first row is exact because $H^1(k, U) = 0$ (Lemma 2.4.4), and the second vertical map is an inclusion because $\Pi^2(H) = 0$, as claimed.

Let us now begin the proof of Proposition 6.12.1. First suppose that $G$ is an almost torus. By Lemma 2.1.3(iv), we may harmlessly modify $G$ and thereby assume that there is an exact sequence

\[
1 \longrightarrow B \longrightarrow R_{k'/k}(T') \times C \longrightarrow G \longrightarrow 1
\]

with $B, C$ finite, $k'/k$ a finite separable extension, and $T'$ a split $k'$-torus. We have $\Pi^2(k, R_{k'/k}(T')) \simeq \Pi^2(k', T') = 0$, the first equality because $T'$ is smooth so the cohomology may be taken to be étale, and the second by the fundamental exact sequence of class field theory. By Lemma 6.11.11 there is a surjection $C' \twoheadrightarrow C$ for some finite commutative $k$-group scheme $C'$ such that $\Pi^2(C') = 0$. The composition $R_{k'/k}(T') \times C' \to R_{k'/k}(T') \times C \to G$ is also an isogeny, so we may replace $C$ with $C'$ and thereby arrange that $\Pi^2(R_{k'/k}(T') \times C) = 0$. For notational convenience let $X := R_{k'/k}(T') \times C$, so we have an exact sequence

\[
1 \longrightarrow B \longrightarrow X \longrightarrow G \longrightarrow 1
\]

with $\Pi^2(X) = 0$.

Let $\alpha \in H^1(A, G)$ be such that $\alpha$ maps to $0$ in $H^1(k, \hat{G})^\ast$. We want $\alpha$ to arise from $H^1(k, G)$. Consider the following commutative diagram:

\[
\begin{array}{cccc}
0 & \longrightarrow & H^1(k, G) & \longrightarrow & H^2(k, B) & \longrightarrow & H^2(k, X) \\
& \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
& H^1(A, X) & \longrightarrow & H^1(A, G) & \longrightarrow & H^2(A, B) & \longrightarrow & H^2(A, X) \\
& & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
& & H^1(k, \hat{G})^\ast & \longrightarrow & \hat{B}(k)^\ast & & &
\end{array}
\]

The third column is exact by Proposition 6.7.1. The fourth column is exact because $\Pi^2(X) = 0$. All of the rows are clearly exact. A simple diagram chase now shows that
upon changing $\alpha$ by the image of an element of $H^1(k, G)$, it lifts to an element in $H^1(A, X)$. We may therefore assume that $\alpha$ lifts to $H^1(A, X)$.

By Lemma 6.12.3 there is an inclusion $B \hookrightarrow B'$ for some finite commutative $k$-group $B'$ such that $\text{III}^2(k') = 0$. Pushing out the sequence

$$1 \rightarrow B \rightarrow X \rightarrow G \rightarrow 1$$

by the inclusion $B \hookrightarrow B'$, we obtain the following commutative diagram of exact sequences:

$$\begin{array}{cccccc}
1 & \rightarrow & B & \rightarrow & X & \rightarrow & G & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \rightarrow & B' & \rightarrow & X' & \rightarrow & G & \rightarrow & 1
\end{array}$$

(6.12.3)


Proof. By definition, $X' = (X \times B')/B$, where $B$ is embedded anti-diagonally in $X \times B'$: $b \mapsto (b, -b)$. Recall that $X = R_{k'/k}(T') \times C$. Let $f : B \rightarrow C$ denote the induced map, and $j : B \hookrightarrow B'$ the inclusion. Let $C' := (C \times B')/B$, where the map $B \rightarrow C \times B'$ is $b \mapsto (f(b), j(b))$. Then $C'$ is finite, and we have an exact sequence

$$1 \rightarrow R_{k'/k}(T') \rightarrow X' \rightarrow C' \rightarrow 1$$

where the first map is given by $r \mapsto (r, 0, 0) \in (R_{k'/k}(T') \times C \times B')/B$ (more precisely, $r$ is an $R$-valued point of $R_{k'/k}(T')$ for some $k$-algebra $R$), and the second map is given by $(r, c, b') \mapsto (c, b')$ for $r \in R_{k'/k}(T'), c \in C, b' \in B'$. One easily check that these maps are well-defined and make the above sequence exact.

Consider the following commutative diagram:

$$\begin{array}{cccc}
\text{H}^1(k, X') & \rightarrow & \text{H}^1(k, C') & \rightarrow & \text{H}^2(k, R_{k'/k}(T')) \\
\downarrow & & \downarrow & & \downarrow \\
\text{H}^1(\hat{A}, X') & \rightarrow & \text{H}^1(\hat{A}, C') & \rightarrow & \text{H}^2(\hat{A}, R_{k'/k}(T')) \\
\downarrow & & \downarrow & & \downarrow \\
\text{H}^1(k, \hat{X}')^* & \rightarrow & \text{H}^1(k, \hat{C}')^*
\end{array}$$

The second column is exact by Poitou-Tate for finite group schemes. The third column is exact because $\text{III}^2(k, R_{k'/k}(T')) \simeq \text{III}^2(k', T') = 0$. The exactness of the first row is clear, and the second row is exact because $\text{H}^1(\hat{A}, R_{k'/k}(T')) = 0$ (by Proposition 6.1.1 and the equality $\text{H}^1(k_v, R_{k_v'/k_v}(T')) = \text{H}^1(k_v', T') = 0$). A simple diagram chase shows that the first column, which we already know to be a complex, is exact.

$\square$
Using the commutative diagram (6.12.3), and the fact that the element \( \alpha \in H^1(A, G) \) lifts to \( H^1(A, X) \), we see that \( \alpha \) lifts to \( H^1(A, X') \). Consider the following commutative diagram:

\[
\begin{array}{c}
H^1(k, X') & \longrightarrow & H^1(k, G) \\
\downarrow & & \downarrow \\
H^1(A, B') & \longrightarrow & H^1(A, X') & \longrightarrow & H^1(A, G) \\
\downarrow & & \downarrow & & \downarrow \\
H^1(k, \hat{B}')^* & \longrightarrow & H^1(k, \hat{X}')^* & \longrightarrow & H^1(k, \hat{G})^* \\
\downarrow & & & & \downarrow \\
0 & & & & 0
\end{array}
\]

The first column is exact by Poitou-Tate for finite group schemes, because \( \text{III}^2(B') = 0 \) (We have already proved Theorem 1.2.9 for finite group schemes, cf. Corollary 6.11.2). The second column is exact by Lemma 6.12.4. The rows are clearly exact. A diagram chase now shows that \( \alpha \) (which we showed lifts to \( H^1(A, X') \)) lifts to \( H^1(k, G) \). This completes the proof of Proposition 6.12.1 for almost tori.

Now suppose that \( G \) is an arbitrary affine commutative \( k \)-group scheme of finite type. We will prove Proposition 6.12.1 by induction on the dimension of the unipotent radical of \( \text{unip}(G) \), the 0-dimensional case corresponding to the already-treated case of almost tori. By Lemma 2.1.7, therefore, we may assume that there is an exact sequence

\[
1 \longrightarrow H \longrightarrow G \longrightarrow G_a \longrightarrow 1
\]

such that Proposition 6.12.1 holds for \( H \). Consider the following commutative diagram:

\[
\begin{array}{c}
H^1(k, H) & \longrightarrow & H^1(k, G) \\
\downarrow & & \downarrow \\
A & \longrightarrow & H^1(A, H) & \longrightarrow & H^1(A, G) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
H^2(k, \hat{G}_a)^* & \longrightarrow & H^1(k, \hat{H})^* & \longrightarrow & H^1(k, \hat{G})^* \\
\downarrow & & & & \downarrow \\
0 & & & & 0
\end{array}
\]

The first column is exact by Proposition 3.2.2, and the second column is exact by hypothesis. The second row is exact because \( H^1(A, G_a) = 0 \), and the third row is clearly exact. An easy diagram chase now shows that the last column, which we already know to be a complex, is exact. This completes the proof of Proposition 6.12.1.
Next, we show the analogue of Proposition 6.12.1 upon switching the roles of $G$ and $\hat{G}$. Although this is part of Theorem 1.2.10, the reason that we are proving it now is that we will use it to prove exactness of the Poitou-Tate sequence at $H^1(k, \hat{G})^*$.

**Proposition 6.12.5.** Let $k$ be a global function field, $G$ an affine commutative $k$-group scheme of finite type. Then the following complex is exact:

$$H^1(k, \hat{G}) \longrightarrow H^1(A, \hat{G}) \longrightarrow H^1(k, G)^*$$

Here the first map is the obvious one, and the second is obtained by cupping everywhere locally and summing the local invariants.

Note that the final term in this complex, defined as the $\mathbb{Q}/\mathbb{Z}$-dual of the torsion group $H^1(k, G)$, coincides with its Pontryagin dual as a discrete group since $H^1(k, G)$ has finite exponent (by Lemma 4.3.1). In this way that final term is naturally a profinite group, and as such the maps in the complex are continuous when the left term is viewed discretely and the middle is given its natural locally compact Hausdorff topology. The same topological observations apply (with the same reasoning) to the complex in Proposition 6.12.1, so the idea of the proof of Proposition 6.12.5 is simply to take the Pontryagin dual of Proposition 6.12.1 and apply Proposition 6.8.1 (along with Pontryagin double duality). In order to make this work, however, we need to first prove the following result, which is of interest in its own right.

**Proposition 6.12.6.** Let $G$ be an affine commutative group scheme of finite type over the global function field $k$. Then the image of the map $H^1(k, G) \rightarrow H^1(A, G)$ is closed, discrete, and cocompact.

Let us assume Proposition 6.12.6 for the moment and see how Proposition 6.12.5 follows. The sequence

$$H^1(k, G) \longrightarrow H^1(A, G) \longrightarrow H^1(k, \hat{G})^*$$

(with continuous maps, viewing the left term as discrete and the right term as profinite via its identification with the Pontryagin dual of the torsion group $H^1(k, \hat{G})$ of finite exponent) is algebraically exact by Proposition 6.12.1. We claim that the Pontryagin dual complex is also algebraically exact. Assuming this, we would be done by Pontryagin double duality and Proposition 6.8.1. To prove that the Pontryagin dual sequence is exact, we note that by Proposition 6.12.6 the quotient $H^1(A, G)/\text{im}(H^1(k, G))$ is compact, so the continuous inclusion $H^1(A, G)/\text{im}(H^1(k, G)) \hookrightarrow H^1(k, \hat{G})^*$ to a Hausdorff group is a homeomorphism onto a closed subgroup.

Before giving the proof of Proposition 6.12.6, we require a simple lemma.

**Lemma 6.12.7.** Let $G$ be an affine commutative group scheme of finite type over the global field $k$. Then there is an inclusion $G \hookrightarrow H$ with $H$ an affine commutative $k$-group scheme of finite type such that $H^1(A, H) = 0$. 

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Proof. By Lemma 2.1.1, there is an exact sequence
\[
1 \rightarrow B \rightarrow G \rightarrow F \rightarrow 1
\]
with \( B \) finite and \( F \) smooth and connected. By [Mc II, 3.2.5], there is an inclusion \( B \hookrightarrow C \) with \( C \) smooth and connected. We may push out the above sequence by this inclusion to obtain an inclusion \( G \hookrightarrow G' \) with \( G' \) smooth and connected, so we may assume that \( G \) is smooth and connected. But then for some finite extension \( k'/k \), we have \( G_{k'} = T' \times U' \) for some split \( k' \)-torus \( T' \) and some split unipotent \( k' \)-group \( U' \). Then the canonical inclusion \( G \hookrightarrow R_{k'/k}(G_{k'}) \) does the job. Indeed, for any fppf abelian sheaf \( \mathcal{F}' \) on \( \mathbf{A}_{k'} \), the Leray spectral sequence yields an inclusion \( H^1(\mathbf{A}_{k'},R_{k'/k}(\mathcal{F}')) \hookrightarrow H^1(\mathbf{A}_{k'},\mathcal{F}') \) (note that the pushforward functor \( R_{k'/k} \) is generally not exact for the fppf topology since \( k'/k \) may not be separable), so it only remains to show that \( H^1(\mathbf{A},G_m) = H^1(\mathbf{A},G_a) = 0 \), and this is easy (or, if you prefer, it follows from Proposition 6.1.1).

Proof of Proposition 6.12.6. To show that the image is discrete and closed, it suffices to prove that the map \( H^1(k,G)/\Pi^1(G) \hookrightarrow H^1(\mathbf{A},G) \) is a homeomorphism onto a closed subgroup. By Lemmas 4.3.3 and 6.11.5, it suffices to just show that the image is closed. This follows immediately from Proposition 6.12.1. It therefore only remains to show that \( H^1(\mathbf{A},G)/\text{im}(H^1(k,G)) \) is compact.

By Lemma 6.12.7, we have an exact sequence
\[
1 \rightarrow G \rightarrow H \rightarrow Q \rightarrow 1
\]
of affine commutative \( k \)-group schemes of finite type such that \( H^1(\mathbf{A},H) = 0 \). We therefore have a continuous surjection \( Q(\mathbf{A})/Q(k) \twoheadrightarrow H^1(\mathbf{A},G)/\text{im}(H^1(k,G)) \). This induces a continuous surjection \( Q(\mathbf{A})/Q(\mathbf{A})_1 \twoheadrightarrow H^1(\mathbf{A},G)/\text{im}(H^1(k,G)),\text{im}(Q(\mathbf{A})_1)) \). (See Definition 6.5.3 for the definition of \( Q(\mathbf{A})_1 \).) The group \( Q(\mathbf{A})/Q(\mathbf{A})_1 \) is free of finite rank (because \( k \) is a function field) while the target is torsion (by Proposition 6.1.1 and Lemma 4.3.1), so the target must be finite. That is, the image of the map \( Q(\mathbf{A})_1/Q(k) \rightarrow H^1(\mathbf{A},G)/\text{im}(H^1(k,G)) \) is of finite index. But the source is compact by Proposition 6.5.5, so the target must be compact also.

Finally, we prove the last main results of this section:

**Proposition 6.12.8.** Let \( G \) be an affine commutative group scheme of finite type over the global function field \( k \). Then the following two sequences are exact:
\[
H^1(\mathbf{A},G) \rightarrow H^1(k,\hat{G})^* \rightarrow \Pi^1(\hat{G})^* \rightarrow 0
\]
\[
H^1(\mathbf{A},\hat{G}) \rightarrow H^1(k,G)^* \rightarrow \Pi^1(G)^* \rightarrow 0
\]
Proof. We prove that the first sequence is exact, and the proof for the second is exactly the same (simply switching the roles of $G$ and $\hat{G}$, and replacing the use of Proposition 6.12.5 below with Proposition 6.12.1). Let us first note that exactness at $\text{III}^1(\hat{G})^*$ is trivial, because $\mathbb{Q}/\mathbb{Z}$ is injective. For exactness at $H^1(k,\hat{G})^*$, we begin with the exact sequence

$$\text{III}^1(\hat{G}) \to H^1(k,\hat{G}) \to H^1(A,\hat{G})$$

We claim that applying Pontryagin duals to this sequence preserves exactness. Assuming this, we obtain an exact sequence

$$H^1(A,\hat{G})^D \to H^1(k,\hat{G})^* \to \text{III}^1(\hat{G})^*$$

(note that $H^1(k,\hat{G})^D = H^1(k,\hat{G})^*$ by Lemma 4.3.1), and we are done by Proposition 6.8.1.

It therefore only remains to prove the exactness of the dual sequence asserted above.

To prove such exactness, it suffices to show the continuous inclusion $H^1(k,\hat{G})^*/\text{III}^1(\hat{G}) \hookrightarrow H^1(A,\hat{G})$ is a homeomorphism onto a closed subgroup. By Lemmas 4.3.3 and 6.11.5, it suffices to show that the image is closed, and this follows from Proposition 6.12.5, which expresses this image as the kernel of a continuous map to a Hausdorff group. \qed

6.13 Injectivity of $\text{III}^2(G) \to \text{III}^1(\hat{G})^*$ for general $G$

In this section we will prove the following result.

**Proposition 6.13.1.** Let $k$ be a global function field, $G$ an affine commutative $k$-group scheme of finite type. Then the map $\text{III}^2(G) \to \text{III}^1(\hat{G})^*$ induced by $\langle \cdot, \cdot \rangle_{\text{III}^2}$ is injective.

**Lemma 6.13.2.** Suppose that we have an exact sequence

$$1 \to H \to G \to U \to 1$$

of affine commutative $k$-group schemes of finite type with $U$ split unipotent. Then the maps $\text{III}^2(H) \to \text{III}^2(G)$ and $\text{III}^1(\hat{H}) \to \text{III}^1(\hat{G})$ are isomorphisms.

**Proof.** By filtering $U$ by $\mathbb{G}_a$'s, it suffices to treat the case $U = \mathbb{G}_a$. First we handle $\text{III}^2$. In the commutative diagram

$$
\begin{array}{ccc}
H^2(k, H) & \to & H^2(k, G) \\
\downarrow & & \downarrow \\
H^2(A, H) & \to & H^2(A, G)
\end{array}
$$

the two horizontal arrows are isomorphisms because $H^i(k, \mathbb{G}_a) = H^i(A, \mathbb{G}_a) = 0$ for $i > 0$ by Lemma 2.4.4(ii), and Proposition 6.1.1. The $\text{III}^2$ assertion follows immediately.

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Now we treat the $\text{III}^1$ assertion. In the commutative diagram

$$
\begin{array}{cccc}
0 & \longrightarrow & H^1(k, \hat{G}) & \longrightarrow & H^1(k, \hat{H}) & \longrightarrow & H^2(k, \hat{G}_a) \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^1(A, \hat{G}) & \longrightarrow & H^1(A, \hat{H}) & \longrightarrow & H^2(A, \hat{G}_a)
\end{array}
$$

the rows are exact by Lemma 2.4.4(iii) and Proposition 6.1.1, and the last vertical arrow is an inclusion by Lemma 2.6.7. A simple diagram chase now shows that the map $\text{III}^1(\hat{G}) \rightarrow \text{III}^1(\hat{H})$ is an isomorphism.

By Lemma 6.13.2, in conjunction with Lemma 2.1.7 and the functoriality of $\langle \cdot, \cdot \rangle_{\hat{G}}$, reduces the proof of Proposition 6.13.1 to the case of almost tori. So let $G$ be an almost torus. By Lemma 2.1.3(iv), there is a surjection $B \times R_{k'/k}(T') \rightarrow G$ with $B$ a finite $k$-group scheme, $k'/k$ a finite separable extension, and $T'$ a split $k'$-torus. By Lemma 6.11.11, we may assume that $\text{III}^2(B) = 0$. Let $X := B \times R_{k'/k}(T')$. Then we claim that $\text{III}^2(X) = 0$. Assuming this, Lemma 6.11.12 and Proposition 6.12.8 finish the proof.

In order to prove that $\text{III}^2(X) = 0$, it suffices to show that $\text{III}^2(R_{k'/k}(G_m)) = 0$. Since $G_m$ and its Weil restriction are smooth, we may take our cohomology to be étale. Since finite pushforward is exact between categories of étale abelian sheaves, we have $\text{III}^2(k, R_{k'/k}(G_m)) = \text{III}^2(k', G_m)$, and this latter group vanishes by class field theory.

### 6.14 Injectivity of $\text{III}^2(\hat{G}) \rightarrow \text{III}^1(G)^*$

In this section we will prove the following result.

**Proposition 6.14.1.** Let $G$ be an affine commutative $k$-group scheme of finite type. The map $\text{III}^2(\hat{G}) \rightarrow \text{III}^1(G)^*$ induced by $\langle \cdot, \cdot \rangle_{\hat{G}}$ is injective.

As usual, we begin with the crucial case of separable Weil restrictions of split tori:

**Lemma 6.14.2.** For a finite separable extension $k'/k$ and a split $k'$-torus $T'$, the group $\text{III}^2(k, R_{k'/k}(\hat{T}'))$ vanishes.

**Proof.** Without loss of generality, $T' = G_m$. By Lemma 2.2.1 it suffices to prove the vanishing of $\text{III}^2(k, R_{k'/k}(Z))$. We may take the cohomology to be étale, since $R_{k'/k}(Z)$ is represented by a smooth $k$-scheme. Since finite pushforward is exact between categories of étale abelian sheaves, we are therefore reduced to showing that $\text{III}^2(Z) = 0$. That is, we claim that the map

$$f : H^2(k, Z) \rightarrow \prod_v H^2(k_v, Z)$$

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is injective. Using the exact sequence of Galois modules

\[ 0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0 \]

over any field and the vanishing (for any field) of higher Galois cohomology with \( \mathbb{Q} \)-coefficients, we have \( H^2(F, \mathbb{Z}) \cong H^1(F, \mathbb{Q}/\mathbb{Z}) \) functorially in any field \( F \). Thus, \( f \) is identified with the natural map \( H^1(k, \mathbb{Q}/\mathbb{Z}) \to \prod_v H^1(k_v, \mathbb{Q}/\mathbb{Z}) \). But \( H^1(F, \mathbb{Q}/\mathbb{Z}) \cong \operatorname{Hom}_{cts}(\operatorname{Gal}(F_s/F), \mathbb{Q}/\mathbb{Z}) \) naturally in any field \( F \) (equipped with a specified separable closure), so the desired injectivity is reduced to the fact that the collection of decomposition groups \( D_v \) for non-archimedean places \( v \) on \( k \) and their lifts \( \overline{v} \) on \( k_s \) generates a dense subgroup; this density in turn follows from the Chebotarev Density Theorem.

Now we make a definition which will be useful below (and rests on the obvious fact that a commutative affine group scheme \( H \) of finite type over a field \( F \) contains a unique \( F \)-torus \( T \) which is maximal in the sense that it contains all others; for an almost torus \( H \) we have \( \dim T = \dim H \)).

**Definition 6.14.3.** An almost torus \( G \) over \( k \) is quasi-trivial if its maximal \( k \)-torus is of the form \( \mathbb{R}_{k'/k}(T') \) for some finite separable extension \( k'/k \) and split \( k' \)-torus \( T' \). This is equivalent to \( G^0_\text{red} \) being such a \( k \)-torus, as well as to the existence of an exact sequence of \( k \)-group schemes

\[ 1 \to \mathbb{R}_{k'/k}(T') \to G \to B \to 1 \]

with \( B \) a finite commutative \( k \)-group scheme, \( k'/k \) finite separable, and \( T' \) a split \( k' \)-torus.


**Proof.** Let \( G \) be a quasi-trivial almost torus. Then we have an exact sequence

\[ 1 \to \mathbb{R}_{k'/k}(T') \to G \to B \to 1 \]

with \( k'/k \) a finite separable extension, \( T' \) a split \( k' \)-torus, and \( B \) a finite commutative \( k \)-group scheme. We will show that the maps \( \Pi^2(\widehat{B}) \to \Pi^2(\widehat{G}) \) and \( \Pi^1(G) \to \Pi^1(B) \) are isomorphisms, so the lemma will follow from the already-known case of finite group schemes (Corollary 6.11.2) and functoriality.

To see that \( \Pi^2(\widehat{B}) \to \Pi^2(\widehat{G}) \) is an isomorphism, consider the following commutative diagram:

\[
\begin{array}{ccc}
0 & \to & H^2(k, \widehat{B}) \\
\downarrow & & \downarrow \\
0 & \to & H^2(A, \widehat{B})
\end{array}
\quad
\begin{array}{ccc}
& & H^2(k, \widehat{G}) \to H^2(k, \mathbb{R}_{k'/k}(T')) \\
H^2(k, \mathbb{R}_{k'/k}(T')) & \cong & H^2(\mathbb{R}_{k'/k}(T'))
\end{array}
\quad
\begin{array}{ccc}
0 & \to & H^2(A, \widehat{G}) \\
\downarrow & & \downarrow \\
0 & \to & H^2(A, \mathbb{R}_{k'/k}(T'))
\end{array}
\]

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The top row is exact because \( H^1(k, R_{k'/k}(T')) = H^1(k, R_{k'/k}(\mathbf{Z})) \cong H^1(k', \mathbf{Z}) = 0 \), and the bottom is because \( H^1(A, R_{k'/k}(T')) = 0 \) (by Proposition 6.1.2). The last vertical arrow is an inclusion by Lemma 6.14.2. A simple diagram chase shows that the map \( \Pi^2(\widehat{B}) \to \Pi^2(\widehat{G}) \) is an isomorphism.

To see that \( \Pi^1(G) \to \Pi^1(B) \) is an isomorphism, consider the following commutative diagram:

\[
\begin{array}{cccccc}
0 & \longrightarrow & H^1(k, G) & \longrightarrow & H^1(k, B) & \longrightarrow & H^2(k, R_{k'/k}(T')) \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^1(A, G) & \longrightarrow & H^1(A, B) & \longrightarrow & H^2(A, R_{k'/k}(T'))
\end{array}
\]

The top row is exact because \( H^1(k, R_{k'/k}(T')) \cong H^1(k', T') = 0 \), and the bottom row is exact because \( H^1(A, R_{k'/k}(T')) = H^1(A_{k'}, T') = 0 \) (first equality by finiteness of \( \text{Spec}(A_{k'}) \to \text{Spec}(A) \) and the smoothness of \( T' \)), which allows us to take étale cohomology, and second equality by Proposition 6.1.1). The last vertical arrow is an inclusion due to the isomorphism \( \Pi^2(k, R_{k'/k}(T')) \cong \Pi^2(k', T') \) whose target vanishes by class field theory. A simple diagram chase now shows that the map \( \Pi^1(G) \to \Pi^1(B) \) is an isomorphism.

Now let us turn to the proof of Proposition 6.14.1 for almost tori, so let \( G \) be an almost torus. By Lemma 2.1.3(iv), modifying \( G \) in a harmless manner allows us to assume that there is an exact sequence

\[ 1 \to B \to X \to \pi \to G \to 1 \quad (6.14.1) \]

with \( B \) a finite commutative \( k \)-group scheme and \( X \) a quasi-trivial almost torus (even split quasi-trivial). By Lemma 6.12.3 there is an inclusion \( B \to B' \) for some finite commutative \( k \)-group scheme \( B' \) such that \( \Pi^2(B') = 0 \). Pushing out the sequence \((6.14.1)\) by the inclusion \( B \to B' \), and renaming, we may therefore assume that \( \Pi^2(B) = 0 \) (and \( X \) is still a quasi-trivial almost torus, no longer necessarily split).

Suppose that \( \alpha \in \Pi^2(\widehat{G}) \) annihilates \( \Pi^1(G) \). We want to show that \( \alpha = 0 \). By functoriality, \( \widehat{\pi}(\alpha) \in \Pi^2(\widehat{X}) \) annihilates \( \Pi^1(X) \), so \( \widehat{\pi}(\alpha) = 0 \) by Lemma 6.14.4. Hence, there exists \( \beta \in H^1(k, \widehat{B}) \) such that \( \delta(\beta) = \alpha \), where \( \delta : H^1(k, \widehat{B}) \to H^2(k, \widehat{G}) \) is the connecting map. Note that we are free to modify \( \beta \) by the image of an element of \( H^1(k, \widehat{X}) \).

Let \( \beta_A \) denote the image of \( \beta \) in \( H^1(A, \widehat{B}) \). Since \( \delta(\beta) \in \Pi^2(\widehat{G}) \), there exists \( x \in H^1(A, \widehat{X}) \) such that \( \widehat{\gamma}(x) = \beta_A \). Note that we are free to modify \( x \) by an element of \( H^1(A, \widehat{X}) \). Recall that we have a natural pairing between \( H^1(A, \widehat{X}) \) and \( H^1(k, X) \) defined as usual by cupping everywhere locally and summing the invariants. Via this pairing, \( x \) defines an element of \( H^1(k, X)^* \). Consider the subgroup

\[ \pi^{-1}(\Pi^1(G)) := \{ \gamma \in H^1(k, X) \mid \pi(\gamma) \in \Pi^1(G) \} \subset H^1(k, X). \]

The crucial point is the following lemma.
Lemma 6.14.5. The element $x$ annihilates $\pi^{-1}(\Pi^1(G))$ under the adelic evaluation pairing.

Proof. Choose $y \in \pi^{-1}(\Pi^1(G))$. The key is to show that

$$\langle x, y_A \rangle = -\langle \alpha, \pi(y) \rangle_{\Pi^1_G},$$

(6.14.2)

where the left side is the adelic evaluation pairing. Since $\alpha$ annihilates $\Pi^1(G)$ by assumption, the lemma would then follow immediately.

The validity or not of (6.14.2) is independent of the choice of $x$ because $H^1(A, \hat{G})$ annihilates $\Pi^1(G)$ under the adelic evaluation pairing. Thus, later in the calculation we may make whatever choices of $x$ is convenient. We may also make whichever choice of $\beta$ is convenient, since modifying $\beta$ by an element of $H^1(k, \hat{X})$ changes $x$ by the same element (more precisely, allows us to modify $x$, since the choice of $x$ is not unique), and this has no effect on the adelic evaluation pairing with $y$, since the sum of the invariants of a global Brauer class is 0.

Choose $x_v \in \hat{Z}^1(k_v, \hat{X})$ representing the localization of $x$ at a place $v$, and choose a cocycle $\tilde{y}_v \in \hat{Z}^1(k, X)$ representing $y$. The left side of (6.14.2) is by definition $\sum_v \text{inv}_v(x_v \cup \tilde{y}_v)$. To evaluate the right side, note that since $\pi(y) \in \Pi^1(G)$, it follows for each place $v$ of $k$, we have $\tilde{y}_v = j(b_v) + dc_v$ in $\hat{Z}^1(k_v, X)$ for some $b_v \in \hat{Z}^1(k_v, B)$ and $c_v \in C^0(k_v, X)$, so $\pi(\tilde{y}_v) = d(\pi(c_v))$. Choose $\tilde{\alpha} \in \hat{Z}^2(k, \hat{G})$ representing $\alpha$, so we may find $h \in \hat{C}^2(k, \mathbb{G}_m)$ such that $\tilde{\alpha} \cup \pi(\tilde{y}_v) = dh$. The right side of (6.14.2) is by definition $\sum_v \text{inv}_v((\tilde{\alpha} \cup \pi(c_v)) - h)$. Thus, what we need to check is:

$$\sum_v \text{inv}_v(x_v \cup \tilde{y}_v) = \sum_v \text{inv}_v(-(\tilde{\alpha} \cup \pi(c_v)) + h)$$

(6.14.3)

Denote by $\tilde{\beta}$ a cocycle in $\hat{Z}^1(k, \hat{B})$ representing a choice for the cohomology class $\beta$ mapping to $\alpha$ under the connecting map. We want to describe the connecting map in terms of Čech cohomology: choose $m \in \hat{C}^1(k, \hat{X})$ lifting $\tilde{\beta}$ through $j$, and then $dm$ lifts to an element of $\hat{Z}^2(k, \hat{G})$ whose cohomology class is $\delta(\tilde{\beta})$. Unfortunately, there is no reason that a general choice of representative $\beta$ should lift to $\hat{C}^1(k, \hat{X})$. This is the usual deficiency of Čech theory for Grothendieck topologies (and even usual topologies with general abelian sheaves): the sequence of Čech complexes associated to a short exact sequence of abelian sheaves need not be short exact. Nevertheless, we can make the idea work as follows. The 2-cocycle $\tilde{\alpha}$ satisfies $\tilde{\pi}(\tilde{\alpha}) = dm$ for some $m \in \hat{C}^1(k, \hat{X})$ since the cohomology class $\alpha$ has vanishing image in $H^2(k, \hat{X})$. Then we may choose $\beta$ to be the 1-cocycle $j(m)$ since we have seen that for our purposes it does not matter which $\beta$ and $x$ we choose (for the given $\alpha$). The connecting map sends the cocycle $\tilde{\beta}$ to $\tilde{\alpha}$ by Proposition E.2.1, so $\beta$ represents a class $\beta$ satisfying $\delta(\beta) = \alpha$. Since $x \mapsto \beta_A$ as cohomology classes, we have $\hat{j}(x_v) = \beta + dc_v$ for some $e_v \in C^0(k_v, \hat{B})$.

Now let us compute the left side of (6.14.3). We have $x_v \cup \tilde{y}_v = x_v \cup j(b_v) + x_v \cup dc_v = \hat{j}(x_v) \cup b_v + dx_v \cup c_v - d(x_v \cup c_v) = \hat{j}(x_v) \cup b_v - d(x_v \cup c_v)$ since $dx_v = 0$. As a cohomology
class, therefore, this equals \( \hat{j}(x_v) \cup b_v = (\hat{\beta} + dc_v) \cup b_v = (\hat{\beta} \cup b_v) + d(e_v \cup b_v) = (\hat{\beta} \cup b_v) + d(e_v \cup b_v) \) since \( db_v = 0 \). As cohomology classes, therefore, this equals \( \hat{\beta} \cup b_v \).

Thus, the left side of (6.14.3) is \( \sum_v \text{inv}_v(\hat{\beta} \cup b_v) \).

Now we compute the right side of (6.14.3). We have \( \hat{\alpha} \cup \pi(c_v) = \hat{\pi}(\hat{\alpha}) \cup c_v = dm \cup c_v = m \cup dc_v + d(m \cup c_v) \) as cochains, and \( m \cup dc_v = (m \cup \hat{y}) - (m \cup j(b_v)) = (m \cup \hat{y}) - (\hat{\beta} \cup b_v) \). The right side of (6.14.3) is therefore \( \sum_v \text{inv}_v(-(m \cup \hat{y}) + (\hat{\beta} \cup b_v) + h) \).

Thus, (6.14.3) is equivalent to the equality

\[
\sum_v \text{inv}_v(h - (m \cup \hat{y})) = 0
\]

which holds because the sum of the local invariants of a global Brauer class is 0. \( \square \)

We claim that by modifying \( x \) by an element of \( H^1(\mathbf{A}, \hat{G}) \) (as we are free to do), we can arrange that \( x \) annihilates \( H^1(k, X) \). It then follows from Proposition 6.12.5 that \( x \) lifts to an element \( \xi \in H^1(k, \hat{X}) \). Replacing \( \beta \) with \( \beta - \hat{j}(\xi) \) (as we are free to do), we may arrange that \( \hat{j}(x) = 0 \). But \( \beta = \hat{j}(x) \), so \( \beta \in \text{III}^1(\hat{B}) \). Since \( \text{III}^2(\hat{B}) = 0 \) by design, the group \( \text{III}^1(\hat{B}) \) vanishes by Corollary 6.11.2. Hence \( \alpha = \delta(\beta) = 0 \), so the proof of Proposition 6.14.1 is complete for almost tori.

To prove the claim concerning modification of \( x \in H^1(\mathbf{A}, \hat{X}) \), let \( \phi_x \in H^1(k, X)^* \) denote the functional induced by \( x \). We have an inclusion

\[
\frac{H^1(k, X)}{\pi^{-1}(\text{III}^1(G))} \hookrightarrow \frac{H^1(k, G)}{\text{III}^1(G)}
\]

so since \( \phi_x \) annihilates \( \pi^{-1}(\text{III}^1(G)) \) by Lemma 6.14.5, we see that \( \phi_x \) lifts to an element of \((H^1(k, G)/\text{III}^1(G))^*\). By Proposition 6.12.8, it follows that \( \phi_x \) agrees with the homomorphism induced by some element of \( H^1(\mathbf{A}, G) \). Modifying \( x \) by this element, we may assume that \( \phi_x = 0 \), as claimed. This completes the proof of Proposition 6.14.1 for almost tori.

Now consider the general case; i.e., let \( G \) be an affine commutative \( k \)-group scheme of finite type. We proceed by induction on the dimension of the unipotent radical of \( (G_k)^0_{\text{red}} \).

By Lemma 2.1.7, we may assume that there is an exact sequence

\[
1 \rightarrow H \rightarrow j \rightarrow G \rightarrow \pi \rightarrow G_a \rightarrow 1
\]

such that Proposition 6.14.1 holds for \( H \). Suppose that \( \alpha \in \text{III}^2(\hat{G}) \) annihilates \( \text{III}^1(G) \). By functoriality, \( \hat{j}(\alpha) \in \text{III}^2(\hat{H}) \) annihilates \( \text{III}^1(H) \), so by hypothesis, we have \( \hat{j}(\alpha) = 0 \).

We therefore have \( \alpha = \hat{\pi}(u) \) for some \( u \in H^2(k, G_a) \). Let \( u_\mathbf{A} \) denote the image of \( u \) in \( H^2(\mathbf{A}, \hat{G}_a) \). Since \( \hat{\pi}(u) = \alpha \in \text{III}^2(\hat{G}) \), we see that \( u_\mathbf{A} = \delta'(w) \) for some \( w \in H^1(\mathbf{A}, \hat{H}) \), where \( \delta': H^1(\mathbf{A}, \hat{H}) \rightarrow H^2(\mathbf{A}, \hat{G}_a) \) is the connecting map. Via the adelic pairing for \( H \), \( w \) defines an element of \( H^1(\mathbf{A}, H)^* \). By analogy with Lemma 6.14.5, we have the following:

**Lemma 6.14.6.** The element \( w \in H^1(\mathbf{A}, \hat{H}) \) annihilates \( \text{III}^1(G) \subset H^1(k, H) \).
Proof. Choose \( x \in j^{-1}(\Theta^1(G)) \). We will show that
\[
\langle w, x_A \rangle = \langle \alpha, j(x) \rangle_{\Theta^1_C}
\] (6.14.5)
where the left side is the adelic pairing for \( H \). Since \( \alpha \) annihilates \( \Theta^1(G) \) by assumption, this will prove the lemma.

Let us first compute the left side of (6.14.5). Let \( \delta : G_\alpha(A) \to H^1(A, H) \) denote the connecting map. Since \( j(x) \in \Theta^1(G) \), for each place \( v \) we have \( x_v = \delta(y_v) \) for some \( y_v \in G_\alpha(k_v) \). We abuse notation and refer to \( w, y_v \) as cocycles (fixed hereafter) representing the cohomology classes that we have been calling \( w, y_v \). Let us recall how \( \delta(y_v) \) is defined. We may choose \( z_v \in \hat{C}^0(k_v, G) \) such that \( \pi(z_v) = y_v \) (since the the sequence of \( \hat{F}_v \)-points associated to the short exact sequence (6.14.4) is still short exact). Then \( \delta(y_v) \in Z^1(k_v, H) \) is the cocycle such that \( j(\delta y_v) = dz_v \), so the left side of (6.14.5) is
\[
\sum_v \text{inv}_v(w_v \cup \delta y_v)
\]

Now we compute the right side of (6.14.5). Let \( \alpha, u, \) and so on denote representative cocycles for cohomology classes denoted above by the same notation. By modifying \( \alpha \), we may assume \( \alpha = \hat{\pi}(u) \) as cocycles rather than merely as cohomology classes. We have \( x_v = \delta y_v + d e_v \) for some \( e_v \in \hat{C}^0(k_v, H) \), so \( j(x) = j(\delta y_v) + dj(e_v) = dz_v + dj(e_v) \). Also, \( \alpha \cup j(x) = (\hat{j} \circ \hat{\pi}(u)) \cup x = 0 \), so the right side of (6.14.5) is by definition
\[
\sum_v \text{inv}_v((\alpha \cup z_v) + (\alpha \cup j(e_v)))
\]
But \( \alpha = \hat{\pi}(u) \), so \( (\alpha \cup z_v) + (\alpha \cup j(e_v)) = (u \cup \pi(z_v)) + (u \cup \pi \circ j(e_v)) = u \cup \pi(z_v) = u \cup y_v \).

We also have \( u_v = \delta' w_v \) as cohomology classes, so the right side of (6.14.5) equals
\[
\sum_v \text{inv}_v(\delta' w_v \cup y_v)
\]
Recall that (perhaps after modifying \( w_v \) so that the required lifting works out) \( \delta' w_v \) is defined to be a cocycle such that \( \hat{\pi}(\delta' w_v) = dm_v \) for some \( m_v \in \hat{C}^1(k_v, \hat{G}) \) satisfying \( \hat{j}(m_v) = w_v \). So \( \delta' w_v \cup y_v = \delta' w_v \cup \pi(z_v) = \hat{\pi}(\delta' w_v) \cup z_v = dm_v \cup z_v = m_v \cup dz_v + d(m_v \cup z_v) \), which is cohomologous to \( m_v \cup dz_v = m_v \cup j(\delta y_v) = \hat{j}(m_v) \cup \delta y_v = w_v \cup \delta y_v \). Thus the right side of (6.14.5) equals
\[
\sum_v \text{inv}_v(w_v \cup \delta y_v),
\]
which is the same as the left side. \( \square \)

We claim that by modifying \( w \) by an element of \( H^1(A, \hat{G}) \) (as we are free to do), we can ensure that it annihilates \( H^1(k, H) \). Assuming this, it follows from Proposition 6.12.3 that
w lifts to some \( h \in H^1(k, \hat{H}) \), so by replacing \( u \) with \( u - \delta'(h) \) we may assume \( u \in \Pi^2(\hat{G}_a) \). But \( \Pi^2(\hat{G}_a) = 0 \) by Lemma 2.6.7, so \( \alpha = \pi(u) = 0 \), as desired.

It remains to prove the claim concerning modifying \( w \). Let \( \phi_w \in H^1(k, H)^* \) be the functional induced by \( w \). We have an inclusion

\[
\frac{H^1(k, H)}{j^{-1}(\Pi^1(G))} \xrightarrow{} H^1(k, G) / \Pi^1(G)
\]

By Lemma 6.14.6, \( \phi_w \) descends to an element of \( (H^1(k, H)/j^{-1}(\Pi^1(G)))^* \), hence extends to an element \( \psi \in (H^1(k, G)/\Pi^1(G))^* \). By Proposition 6.12.8, \( \psi \) is induced by some element of \( H^1(A, \hat{G}) \). Modifying \( w \) by this element, we obtain that \( w \) kills \( H^1(k, H) \), as desired. The proof of Proposition 6.14.1 is complete.

### 6.15 Injectivity of \( \Pi^1(\hat{G}) \to \Pi^2(G)^* \) and finiteness of \( \Pi \)

In this section we will complete the proof of Theorem 1.2.9. First we turn to the following result.

**Proposition 6.15.1.** Let \( k \) be a global function field, \( G \) an affine commutative \( k \)-group scheme of finite type. Then the map \( \Pi^1(\hat{G}) \to \Pi^2(G)^* \) induced by \( \langle \cdot, \cdot \rangle_{\Pi^2_G} \) is injective.

Lemmas 6.13.2 and 2.1.7, together with functoriality of \( \langle \cdot, \cdot \rangle_{\Pi^2_G} \), reduce us to the case when \( G \) is an almost torus, so we will assume that we are in this case. By Lemma 2.1.3(iv), we may modify \( G \) in order to assume that we have an exact sequence

\[
1 \longrightarrow B \xrightarrow{j} R_{k'/k}(T') \times A \xrightarrow{\pi} G \longrightarrow 1
\]

for some finite commutative \( k \)-groups \( A, B \), some finite separable extension \( k'/k \), and some split \( k' \)-torus \( T' \). For notational convenience, let \( X := R_{k'/k}(T') \times A \), so we have an exact sequence

\[
1 \longrightarrow B \xrightarrow{j} X \xrightarrow{\pi} G \longrightarrow 1 \tag{6.15.1}
\]

We begin by showing that \( \langle \cdot, \cdot \rangle_{\Pi^2_G} \) induces a perfect pairing

\[
\frac{\Pi^2(G)}{\pi(\Pi^2(X))} \times \ker(\Pi^1(\hat{G}) \to \Pi^1(\hat{X})) \to Q/Z
\]

by relating both of these groups to more concrete groups equipped with a compatible pairing for which the perfectness will be more straightforward. Since we already know Proposition 6.15.1 for \( X \) (by Corollary 6.11.2 and because \( \Pi^1(R_{k'/k}(\hat{G}_m)) = \Pi^1(R_{k'/k}(\hat{Z})) = \Pi^1(k', Z) = 0 \)), this perfection, in conjunction with functoriality of \( \langle \cdot, \cdot \rangle_{\Pi^2_G} \), will prove the proposition in general.
For an fppf sheaf $\mathcal{F}$ on $\text{Spec}(k)$, define $Q_i(\mathcal{F}) := \text{coker}(H^i(k, \mathcal{F}) \to H^i(A, \mathcal{F}))$. We will first define an isomorphism $f : \ker(Q_2(B) \to Q_2(X))/\delta(Q_1(G)) \to \hat{\mathcal{X}}_2(G)/\pi(\hat{\mathcal{X}}_2(X))$.

Consider the commutative diagram of exact sequences

$$
\begin{array}{cccccc}
H^1(k, G) & \longrightarrow & H^2(k, B) & \longrightarrow & H^2(k, X) & \longrightarrow H^2(k, G) & \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \beta & & \\
H^1(A, G) & \longrightarrow & H^2(A, B) & \longrightarrow & H^2(A, X) & \longrightarrow H^2(A, G)
\end{array}
$$

(0 in the top row via Proposition 3.1.1). For $x \in \ker(Q_2(B) \to Q_2(X))/\delta(Q_1(G))$, choose $y \in H^2(A, B)$ representing $x$. Since $x \in \ker(Q_2(B) \to Q_2(X))$, there exists $z \in H^2(k, X)$ such that $\beta(z) = \alpha(y)$. We define $f(x)$ to be the class of $\gamma(z)$ in $\hat{\mathcal{X}}_2(G)/\pi(\hat{\mathcal{X}}_2(X))$. It is straightforward using the above diagram to show that $f$ is well-defined and an isomorphism.

Thanks to Proposition 6.7.1, for any affine commutative $k$-group scheme $H$ of finite type we have a functorial isomorphism $Q_2(H) \simeq \hat{H}(k)^*$ by forming cup products over every $k_v$ and adding the local invariants. We therefore have an isomorphism $\phi : \ker(\hat{B}(k) \to \hat{X}(k)^*)/\delta(Q_1(G)) \to \hat{\mathcal{X}}_2(G)/\pi(\hat{\mathcal{X}}_2(X))$.

The exact sequence

$$
1 \longrightarrow \hat{G} \longrightarrow \hat{X} \longrightarrow \hat{B} \longrightarrow 1
$$

gives rise to a commutative diagram with exact rows

$$
\begin{array}{cccccc}
\hat{X}(k) & \longrightarrow & \hat{B}(k) & \longrightarrow & H^1(k, \hat{G}) & \longrightarrow H^1(k, \hat{X}) \\
\downarrow & & \downarrow \delta' & & \downarrow & & \\
\hat{X}(A) & \longrightarrow & \hat{B}(A) & \longrightarrow & H^1(A, \hat{G}) & \longrightarrow H^1(A, \hat{X})
\end{array}
$$

where we have abused notation by denoting the pullback map $\hat{X} \to \hat{B}$ on $k$-points and on $A$-points (as well as the map on sheaves) as $\hat{j}$. Using this diagram, it is easy to check that the connecting map $\delta'$ defines an isomorphism

$$
\psi : \{ \chi \in \hat{B}(k) \mid \epsilon(\chi) \in \hat{j}(\hat{X}(A)) \} \to \ker(\hat{\mathcal{X}}_1(\hat{G}) \to \hat{\mathcal{X}}_1(\hat{X}))
$$

Before continuing with the proof of Proposition 6.15.1, let us prove the following result, which, other than Proposition 6.15.1, is all that remains in order to prove Theorem 1.2.9.
Proposition 6.15.2. Let $G$ be an affine commutative $k$-group scheme of finite type over the global function field $k$. Then the groups $\Pi^i(G)$ and $\Pi^i(\hat{G})$ are finite for $i = 1, 2$.

Proof. The finiteness of $\Pi^1(k, G)$ for all affine $k$-group schemes of finite type (even without commutativity hypotheses) is [Con, Thm. 1.3.3(i)], whose proof in the commutative case is much easier than in the general case (as one can reduce to the smooth connected case by essentially elementary arguments [Con, §6.1–6.2], and the smooth connected commutative case is settled in [Oes] IV, 2.6(a)]). The finiteness of $\Pi^2(\hat{G})$ follows from that of $\Pi^1(G)$ and Proposition 6.14.1. If we prove that $\Pi^1(\hat{G})$ is finite, then the finiteness of $\Pi^2(\hat{G})$ would follow from Proposition 6.13.4.

It therefore remains to show that $\Pi^1(\hat{G})$ is finite. Lemmas 6.13.2 and 2.1.7 reduce us to the case in which $G$ is an almost torus. Then, in the notation above, since the group $\hat{B}(k)$ is finite, the isomorphism $\psi$ shows that the group $\ker(\Pi^1(\hat{G}) \to \Pi^1(\hat{X}))$ is finite. It therefore suffices to show that $\Pi^1(\hat{X})$ is finite. As we have already seen, $\Pi^1(k, R_{k'/k}(T')) = 0$, so it remains to show that $\Pi^1(\hat{A})$ is finite for any finite commutative $k$-group scheme $A$. By Cartier duality, we may replace $\hat{A}$ with $A$, and the finiteness then follows from the general finiteness of $\Pi^1(G)$ addressed above.

There is an obvious pairing between the groups on the left sides of (6.15.2) and (6.15.3), namely the one coming from the natural pairing $\hat{B}(k) \times \hat{B}(k)^* \to \mathbb{Q}/\mathbb{Z}$. It is easy to see that this is well-defined, as follows. By definition, $\ker(\hat{B}(k) \to \hat{X}(k)^*)$ kills $\hat{j}(\hat{X}(k))$. To see that $\delta(\Pi^1(G))$ annihilates $\{ \chi \in \hat{B}(k) \mid \epsilon(\chi) \in \hat{j}(\hat{X}(A)) \}$, recall that the pairing $\Pi^2(B) \times \hat{B}(k) \to \mathbb{Q}/\mathbb{Z}$ is defined by applying the cup product and adding the local invariants. Thus, it suffices to show that for $z \in \text{H}^1(A, G)$ and $\chi \in \hat{B}(k)$ that everywhere locally comes from $\hat{X}(k_v)$, $\delta(z)_v \cup \chi_v = 0$ in $\text{H}^2(k_v, G_m)$ for all places $v$. More specifically, it suffices to check that for any place $v$ of $k$, any $z \in \text{H}^1(k_v, G)$, and any $\chi' \in \hat{X}(k_v)$, the cup product $\delta(z) \cup \hat{j}(\chi')$ vanishes. But by the functoriality of cup product, the left side equals $\text{H}^1(j)(\delta(z)) \cup \chi'$, which is 0 because $\text{H}^1(j) \circ \delta = 0$.

Now it is natural to ask whether the pairing between the left sides of (6.15.2) and (6.15.3) is compatible with the one between the right sides. This is almost true:

Lemma 6.15.3. The pairings between the left sides and between the right sides of (6.15.2) and (6.15.3) differ by a sign. That is, the following diagram commutes:

$$
\begin{array}{ccc}
\ker(\hat{B}(k)^*) \to \hat{X}(k)^*/\delta(\Pi^1(G)) & \times & \frac{\{ \chi \in \hat{B}(k) \mid \epsilon(\chi) \in \hat{j}(\hat{X}(A)) \}}{\hat{j}(\hat{X}(A))} \\
\phi \downarrow & & \downarrow \psi \\
\Pi^2(G)/\pi(\Pi^2(X)) & \times & \ker(\Pi^1(\hat{G}) \to \Pi^1(\hat{X})) \\
\end{array}
\longrightarrow \mathbb{Q}/\mathbb{Z}
$$

where the top pairing is the one induced by the natural pairing $\hat{B}(k) \times \hat{B}(k)^* \to \mathbb{Q}/\mathbb{Z}$ and the bottom pairing is the one induced by $\langle \cdot, \cdot \rangle_{\Pi^2_2} : \Pi^2(G) \times \Pi^1(\hat{G}) \to \mathbb{Q}/\mathbb{Z}$.
Proof. To avoid a proliferation of Greek letters, we will simply abuse notation and use \( j, \pi, \) etc. to denote induced maps of group schemes, cohomology groups, Čech cohomology, etc., where we recall that these letters are defined by the exact sequences

\[
1 \to B \xrightarrow{j} X \xrightarrow{\pi} G \to 1
\]

and

\[
1 \to \hat{G} \xrightarrow{\pi} \hat{X} \xrightarrow{\hat{j}} \hat{B} \to 1.
\]

Choose \( \chi \in \hat{B}(k) \) and \( z \in \hat{B}(k)^* \). Assume that \( z|_{\hat{X}(k)} = 0 \) and that for each place \( v \) of \( k \), \( \chi \) extends to a character \( \zeta_v \in \hat{X}(k_v) \). We want to show that

\[
- \langle z, \chi \rangle = \langle \phi(z), \zeta(\chi) \rangle_{\Pi_G^2}
\]

We first lift \( z \) to an element \( y \in \mathbb{Q}^2(B) \). That is, we choose \( y \in H^2(\mathbb{A}, B) \) such that \( z(\chi') = \sum_v \text{inv}_v(\chi'(y_v)) \) for all \( \chi' \in \hat{B}(k) \), where the sum is over all places \( v \) of \( k \). Choose cocycles \( \hat{y}_v \) representing \( y_v \) (using Proposition 3.5.1). Thus, the left side of (6.15.4) equals

\[
- \sum_v \text{inv}_v(\chi(\hat{y}_v)).
\]

Now we compute the right side of (6.15.4). First, \( \zeta(\chi) = \delta'(\chi) \). Now there exists \( w \in H^2(k, X) \) such that \( j(y_v) = w_v \) for all \( v \). Then \( \pi(w) = \phi(z) \), by definition of \( \phi \). To compute \( \langle \pi(w), \delta'(\chi) \rangle_{\Pi_G^2} \), we choose a 2-cocycle \( \hat{w} \in \check{Z}^2(k, B) \) representing \( w \). Then

\[
\hat{w}_v = j(\hat{y}_v) + d(u_v)
\]

for some \( u_v \in \check{C}^1(k_v, X) \). We have \( \pi(\hat{w})_v = \pi(j(\hat{y}_v)) + \pi(d(u_v)) = d\pi(u_v) \), since \( \pi \circ j = 0 \).

We compute \( \delta'(\chi) \) as follows: Choose \( \zeta \in \hat{X}(\hat{k}) \) such that \( \zeta|_B = \chi \), and consider \( \zeta \) as an element of \( \check{C}^0(k, X) \). Then choose a lift \( \gamma \in \check{Z}^1(k, \hat{X}) \) of \( d\zeta \). That is, \( \hat{\pi}(\gamma) = d\zeta \). The cocycle \( \gamma \) represents \( \delta'(\chi) \). Choose \( h \in \check{C}^2(k, \mathbb{G}_m) \) such that \( dh = \pi(w) \cup \gamma \). The right side of (6.15.4) is, by definition,

\[
\sum_v \text{inv}_v(\pi(u_v) \cup \gamma_v) = h).
\]

Thus, we need to show that

\[
- \sum_v \text{inv}_v(\chi(\hat{y}_v)) = \sum_v \text{inv}_v(\pi(u_v) \cup \gamma_v) - h).
\]

We have \( \pi(u_v) \cup \gamma_v = u_v \cup \pi(\gamma_v) = u_v \cup d\zeta_v - d(u_v \cup \zeta_v) + d\zeta_v \cup u_v \). Thus, as cohomology classes, \( \pi(u_v) \cup \gamma_v = du_v \cup \zeta_v = (\hat{w}_v \cup \zeta_v) - (\hat{\gamma}(\hat{y}_v)) = (\hat{w}_v \cup \zeta_v) - \chi_v(\hat{y}_v), \) since \( \hat{j}(\zeta) = \chi \). Therefore, (6.15.6) reduces to

\[
\sum_v \text{inv}_v((\hat{w} \cup \zeta) - h) \equiv 0,
\]

and this follows from the fact that the sum of the local invariants of a global Brauer class is 0. \( \square \)
Now we may prove Proposition 6.15.1 for the almost torus \( G \), which, as we have seen, suffices to complete the proof in general. We know by Corollary 6.11.2 and the already-treated case of separable Weil restrictions of split tori that Proposition 6.15.1 holds for \( X \).

Given \( \alpha \in X_1(\hat{G}) \) that annihilates \( X_2(G) \), therefore, the functoriality of \( X_2(G) \) implies that \( \alpha \in \ker(\Pi^1(\hat{G}) \to \Pi^1(\hat{X})) \). It therefore suffices to show that the map \( \ker(\Pi^1(\hat{G}) \to \Pi^1(\hat{X})) \) is injective. By Lemma 6.15.3, it suffices to show that for any \( \chi \in \hat{B}(k) \) that annihilates \( \ker(\hat{B}(k) \to \hat{X}(k)) \), we have \( \chi \in \hat{j}(\hat{X}(k)) \). Since the groups \( \hat{B}(k) \) is finite, this is a piece of elementary group theory. The proof of Proposition 6.15.1 is therefore complete.

### 6.16 The dual 9-term exact sequence

In this section we deduce Theorem 1.2.10 as the Pontryagin dual of Theorem 1.2.8. This will be easy given what we already know. Consider the exact sequence of Theorem 1.2.8:

\[
0 \longrightarrow G(k)_{\text{pro}} \longrightarrow G(A)_{\text{pro}} \longrightarrow H^2(k, \hat{G})^D \longrightarrow H^1(k, G) \downarrow
\]

\[
0 \longleftarrow \hat{G}(k)^* \longleftarrow H^2(A, G) \longleftarrow H^2(k, G) \longleftarrow H^1(k, \hat{G})^D \longleftarrow H^1(A, G)
\]

(6.16.1)

where the global cohomology groups \( H^i(k, \cdot) \) are endowed with the discrete topology. Since the groups \( H^i(k, \hat{G}) \) (\( i = 1, 2 \)) are discrete torsion (Lemma 4.3.1 and Lemma 4.4.1), their algebraic \( \mathbb{Q}/\mathbb{Z} \)-duals agree with their Pontryagin duals. Further, since \( \hat{G}(k) \) is finitely generated, we may and do replace \( \hat{G}(k)^* \) with \( (\hat{G}(k)_{\text{pro}})^D \).

By arguments similar to those we have used several times, all maps in (6.16.1) are continuous, so it makes sense to pass to the Pontryagin dual complex. Using the replacement of \( \hat{G}(k)^* \) by \( (\hat{G}(k)_{\text{pro}})^D \), this dual complex takes the form:

\[
0 \longleftarrow (G(k)_{\text{pro}})^D \longleftarrow (G(A)_{\text{pro}})^D \longleftarrow H^2(k, \hat{G}) \longleftarrow H^1(k, G)^*
\]

\[
0 \longrightarrow \hat{G}(k)_{\text{pro}} \longrightarrow H^2(A, G)^D \longrightarrow H^2(k, G)^* \longrightarrow H^1(k, \hat{G}) \longrightarrow H^1(A, G)^D
\]

(6.16.2)

where all maps are the evident ones (and the two maps between degree-2 and degree-1 cohomologies rest on the perfect \( \Pi \)-pairings). By Proposition 6.8.1, we may replace the terms \( H^1(A, G)^D \) and \( H^2(A, G)^D \) with \( H^1(A, \hat{G}) \) and \( \hat{G}(A)_{\text{pro}} \), respectively. Finally, \( G(A)_{\text{pro}} = \prod_v G(k_v)_{\text{pro}} \) by Lemma 6.2.3, so

\[
(G(A)_{\text{pro}})^D = \bigoplus_v G(k_v)^D = \bigoplus_v H^2(k_v, \hat{G}),
\]

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the last equality by Theorem \ref{eq:1.2.2}

Making these replacements in \eqref{eq:6.16.2} gives exactly the desired diagram in Theorem \ref{eq:1.2.10} and so completes the proof provided that passage to the Pontryagin dual complex preserves exactness. Such exactness of the dual complex follows immediately from the following general fact: given an exact sequence
\[ A \to B \to C \to D \]
of second-countable, locally compact Hausdorff topological abelian groups, the dual diagram
\[ C^D \to B^D \to A^D \]
is exact. Note that $\text{im}(A)$ is closed in $B$ since it equals $\ker(B \to C)$, so the second-countable locally compact topological abelian group $B/\text{im}(A)$ is Hausdorff.

Consider $\phi \in B^D$ whose restriction to $A$ vanishes, so $\phi \in (B/\text{im}(A))^D$. We want to show that $\phi$ arises from $C^D$. If the continuous inclusion $B/\text{im}(A) \hookrightarrow C$ is a homeomorphism onto a closed subgroup then $\phi$ extends to an element of $C^D$ and so we would be done. By Lemma \ref{eq:4.3.3} it suffices to show that $\text{im}(B)$ is closed in $C$, and such closedness follows from the fact that this image is the kernel of the continuous map $C \to D$ with $D$ Hausdorff.
Appendix A

Products and Ultraproducts

This appendix is concerned with certain rather abstract constructions involving ultraproducts. These seemingly esoteric notions are relevant because such ultraproducts appear as localizations of rings that are described as infinite products, such as $\prod_v O_v$, the product of the rings of integers of the places of a global field $k$. As one may imagine, such localizations play an important role when studying a ring such as the adeles of a global field $k$.

A.1 The basic construction

Let $I$ be a non-empty set, $\{R_i\}_{i \in I}$ a set of rings (commutative with identity) indexed by $I$. The purpose of this section is to undertake a study of $\text{Spec}(\prod_{i \in I} R_i)$, as this will be necessary to prove some important results about the cohomology of the adeles in §6.1. We will be particularly interested in studying the local rings of $\text{Spec}(\prod_{i \in I} R_i)$.

In order to do this, we need to recall the notion of an ultraproduct of commutative rings (though the same can be done for quite general mathematical structures; we focus on rings since it is all we need). Let $U$ be an ultrafilter on the set $I$. That is, $U$ is a proper collection of subsets of $I$ with three properties: (i) if $A \in U$ and $A \subseteq B \subseteq I$ then $B \in U$; (ii) if $A, A' \in U$ then $A \cap A' \in U$; and (iii) for all $A \subseteq I$, one of $A$ or $I - A$ belongs to $U$. (Condition (iii) implies that $U$ is not the empty collection.) In (iii), we cannot have both belonging to $U$ or else by (ii) we would have $\emptyset \in U$ and hence by (i) every subset of $I$ belongs to $U$, contradicting that $U$ is not the entire power set of $I$. Conditions (i) and (ii) say that $U$ is a filter, and (iii) is equivalent to the condition that $U$ is not strictly contained in any larger filter on $I$. The “obvious” ultrafilters are those consisting of all subsets of $I$ containing a fixed element $i_0 \in I$; these are called principal ultrafilters. It is elementary to check that if $I$ is finite, then the only ultrafilters on $I$ are the principal ones.

Remark A.1.1. One should regard an ultrafilter as defining a notion of “bigness” for subsets of $I$ (so one might say that $A \subseteq I$ is $U$-big when $A \in U$); the axioms of an ultrafilter express reasonable conditions on any notion of bigness for subsets of a given set.
We define the ultraproduct $\prod_U R_i$ to be the quotient ring $(\prod_{i \in I} R_i)/\sim$, where $\sim$ is the equivalence relation defined by $(a_i) \sim (b_i) \iff \{i \in I \mid a_i = b_i\} \in U$. The ring structure is induced by that on $\prod R_i$ via coordinate-wise addition and multiplication. (In other words, $\prod_U R_i$ is the quotient of $\prod R_i$ modulo the ideal $J(U)$ consisting of elements $(a_i) \in \prod R_i$ for which $\{i \in I \mid a_i = 0\} \in U$; this is an ideal because $U$ is a filter.) Note that when $U$ is the principal ultrafilter corresponding to an element $i_0 \in I$, the associated ultraproduct $\prod_U R_i$ is the factor ring $R_{i_0}$. The “interesting” ultraproducts are therefore those corresponding to non-principal ultrafilters (and by Zorn’s Lemma such do exist whenever $I$ is infinite, though we will never use this fact; our entire discussion in this chapter is independent of the axiom of choice).

Informally, working in the ultraproduct $\prod_U R_i$ amounts to considering elements $(x_i)$ only for “big” sets of indices $i$ where bigness is defined via $U$. Note that if $x \in \prod_U R_i$ is the class of an $I$-tuple $(x_i)$, then $x \neq 0$ if and only if $\{i \in I \mid x_i = 0\} \notin U$, but the complement of this set of indices is exactly $\{i \in I \mid x_i \neq 0\}$, so by the ultrafilter property of $U$ we conclude that $x \neq 0$ if and only if $\{i \in I \mid x_i \neq 0\} \in U$. In other words, $x = 0$ precisely when $x_i$ vanishes for a “big” set of indices $i \in I$, and $x \neq 0$ precisely when $x_i \neq 0$ for a “big” set of indices $i \in I$! We will use this without comment in some arguments below.

Why do we care about this? For a subset $S \subset I$, define the idempotent $z_S \in \prod_{i \in I} R_i$ by

$$(z_S)_i = \begin{cases} 0, & i \in S \\ 1, & i \notin S \end{cases}$$

To each prime ideal $p$ of $\prod R_i$ we can assign an ultrafilter $U(p)$ on $I$ via

$$U(p) = \{S \subset I \mid z_S \in p\};$$

the primality of $p$ underlies why this is an ultrafilter, since $z_S \cdot z_{I-S} = 0$ for any $S \subset I$, so either $z_S$ or $z_{I-S}$ lies in $p$. (For the uninteresting case that $p$ arises from a factor ring $R_{i_0}$, $U(p)$ is the principal ultrafilter associated to $i_0 \in I$. It is not hard to show (assuming the axioms of choice) that every ultrafilter on $I$ arises as $U(p)$ for some $p$, but we do not need this fact.)

Let $J(p)$ denote the ideal $J(U(p))$ of $\prod U_i$; explicitly, $J(p)$ consists of those elements $a = (a_i)$ for which the set $S$ of $i \in I$ such that $a_i = 0$ belongs to $U(p)$, which is to say $z_S \in p$. But clearly $a$ is a multiple of $z_S$, so in more concrete terms $J(p)$ is generated by the elements $z_S$ that belong to $p$. Conversely, if $S \subset I$ is such that $z_S \in J(p)$ then necessarily $S \in U(p)$. Indeed, by the description of $J(p)$ we have $z_S = x_1 z_{S_1} + \cdots + x_n z_{S_n}$ for some $x_1, \ldots, x_n \in \prod R_i$ and $S_1, \ldots, S_n \in U(p)$, so $S' := S_1 \cap \cdots \cap S_n \in U(p)$. But clearly every $z_{S_j}$ is a multiple of $z_{S_j}$, so $z_S$ is a multiple of $z_{S_j}$, and thus $S \supset S'$, so $S \in U(p)$.

For the closed (but typically not open!) subscheme $\text{Spec}((\prod R_i)/J(p))$ passing through the point $\{p\}$, at all of its points the local ring coincides with that of the ambient scheme $\text{Spec}(\prod R_i)$. We only need this equality at the point $p$ of initial interest, and there it is
very easy to verify: we just need to check that each of the generators \( z_S \in p \) of \( J(p) \) vanish in \( (\prod R_i)_p \), and that is immediate because \( z_{I-S} \not\in p \) (as \( z_{I-S} + z_S = 1 \)) and \( z_{I-S} \cdot z_S = 0 \).

### A.2 Applications

The upshot of the preceding discussion is that every local ring of \( \text{Spec}(\prod R_i) \) is the local ring on one of the ultraproducts \( \prod U R_i \). This is in turn useful because many reasonable (more precisely: “first-order”) properties of commutative rings \( R_i \) are inherited by ultraproducts. Any such property that is also inherited by localization at primes is therefore inherited by every local ring on \( \prod R_i \) when it holds for each \( R_i \)!

Let us consider some examples (sufficient for our needs) for the sake of illustration. If all of the rings \( R_i \) are domains then we claim that so is any ultraproduct \( \prod_U R_i \) (and thus so is the local ring of \( \prod R_i \) at every prime ideal \( p \), which is not so obvious if one doesn’t have the geometric idea to try proving that the closed subscheme \( \text{Spec}(\prod_{U(p)} R_i) \) passing through \( \{p\} \) is actually a domain). Consider \( x = (x_i), y = (y_i) \in \prod U R_i \) (more precisely, \( x, y \) are the classes of \( (x_i), (y_i) \)) such that \( xy = 0 \). Thus, \( \{ i \mid x_i y_i = 0 \} \in U \). Since each \( R_i \) is a domain, it follows that \( \{ i \mid x_i = 0 \} \cup \{ i \mid y_i = 0 \} \in U \), hence either \( \{ i \mid x_i = 0 \} \in U \) or \( \{ i \mid y_i = 0 \} \in U \) (or both); i.e., either \( x = 0 \) or \( y = 0 \) in \( \prod_U R_i \). (Here we have used the fact immediate from the definition of an ultrafilter that if \( S, S' \subseteq I \) satisfy \( S \cup S' \subseteq U \) then \( S \subseteq U \) or \( S' \subseteq U \); indeed, if \( S \not\subseteq U \) and \( S' \not\subseteq U \) then \( I - S, I - S' \subseteq U \) and hence \( I - (S \cup S') = (I - S) \cap (I - S') \subseteq U \), contradicting that \( S \cup S' \subseteq U \).) This proves that \( \prod U R_i \) is a domain when every \( R_i \) is a domain.

Similarly, suppose that each \( R_i \) is a domain such that any element of \( K_i := \text{Frac}(R_i) \) satisfying a monic polynomial of degree 3 over \( R_i \) lies in \( R_i \). Then we claim that the same holds for any ultraproduct \( \prod_U R_i \). Indeed, we already know that \( \prod_U R_i \) is a domain, so suppose that \( x = (x_i), y = (y_i) \in \prod U R_i \) with \( y \neq 0 \) (so \( y_i \neq 0 \) precisely for \( i \) belonging to a set \( S \subseteq U \)) such that

\[
(x/y)^3 + a_2(x/y)^2 + a_1(x/y) + a_0 = 0
\]

in \( \prod_U R_i \) for some \( a_0, a_1, a_2 \in \prod_U R_i \), or equivalently (since \( \prod_U R_i \) is a domain),

\[
x^3 + a_2x^2y + a_1xy^2 + a_0y^3 = 0
\]

in \( \prod U R_i \). The latter vanishing in the ultraproduct says that the set \( S' \) of indices \( i \) such that the corresponding vanishing holds at index \( i \) (without any condition on \( y_i \) vanishing or not!) is a member of \( U \).

But \( S \cap S' \) must also belong to \( U \), and by our assumption about the \( R_i \) it follows that for any \( i \in S \cap S' \) we have \( x_i = z_i y_i \) for some \( z_i \in R_i \). Defining \( z \) be the class in \( \prod U R_i \) of the \( I \)-tuple with \( i \)th component \( z_i \) for \( i \in S \cap S' \) and whatever component we wish (say 0, or 1) for indices not in \( S \cap S' \), we have \( x = yz \) in the domain \( \prod U R_i \), so \( x/y = z \in \prod U R_i \), as desired. A similar argument with 3 replaced by any positive integer shows that if each
If \( R_i \) is an integrally closed domain then so is \( \prod_U R_i \). (Note that being integrally closed is not a “first order” condition, but the more limited analogous property as just considered for monic polynomials of a fixed positive degree is a “first order” condition.)

Since any localization of an integrally closed domain is an integrally closed domain, and every local ring on \( \prod R_i \) is a local ring of an ultraproduct of the \( R_i \)'s, we have proved the following useful result.

**Lemma A.2.1.** Let \( R_i \) be a collection of rings indexed by the non-empty set \( I \). If each \( R_i \) is an integrally closed domain, then the local rings on the scheme \( \text{Spec}(\prod_{i \in I} R_i) \) are integrally closed domains.

In the same spirit, we also have a refinement:

**Lemma A.2.2.** Let \( R_i \) be a collection of valuation rings indexed by a non-empty set \( I \). Each local ring of \( \text{Spec}(\prod_{i \in I} R_i) \) is a valuation ring.

**Proof.** The idea of the proof is the same as for Lemma A.2.1. Using the formulation that a domain \( R \) with fraction field \( K \) is a valuation ring if for any \( x \in K^\times \) either \( x \in R \) or \( x^{-1} \in R \), one shows that \( \prod_U R_i \) is a valuation ring. But any localization of a valuation ring is a valuation ring, so we are done (since every local ring of \( \prod R_i \) is a local ring at a prime for an ultraproduct among the \( R_i \)'s).

As the reader has likely noticed, we have used nothing particularly special about the class of integrally closed domains or of valuation rings, beyond that their definitions are built up in terms of conditions expressible in sufficiently finitistic terms (a notion made precise in terms of first-order logic). Łoś’s Theorem states that any first-order statement that holds for each of the factors \( R_i \) holds for any ultraproduct \( \prod_U R_i \) (and even more generally for ultraproducts of a wide class of mathematical structures). As we have seen, in any special case of interest it is easy to prove by a direct argument that such properties are inherited by ultraproducts, so there is no need for the general Łoś’s Theorem anywhere in our work.
Appendix B

Profinite Completions

This appendix discusses some generalities we require on profinite completions of topological abelian groups. We need to understand such completions because several of the main duality results of this manuscript involve profinite completions of various groups (generally groups of rational points), and several dévissage arguments will require an understanding of how the process of profinite completion behaves with respect to exact sequences.

B.1 Right-exactness of profinite completion

Let us first recall the basic definitions. An abelian topological group $A$ is said to be profinite if it is compact, Hausdorff, and totally disconnected. (One can make this definition for nonabelian groups also, but we won’t need this and therefore won’t bother.) This is equivalent to saying that the canonical map $A \to \lim_{\rightarrow} A/H$ is an isomorphism, where the limit is over all closed subgroups $H \subset A$ of finite index (hence the term “profinite”).

Given an arbitrary abelian topological group $A$, the profinite completion $A_{\text{pro}}$ of $A$ is the initial profinite group with a map from $A$. That is, we have a continuous map $\phi : A \to A_{\text{pro}}$ such that any (continuous) map $A \to B$ with $B$ profinite factors uniquely as a composition of continuous homomorphisms $A \to A_{\text{pro}} \to B$. Then $A_{\text{pro}}$ is clearly unique, and it also exists: we have $A_{\text{pro}} = \lim_{\rightarrow} A/H$, where the limit is over all closed subgroups $H \subset A$ of finite index. If $A$ is an abelian group without any topology, then $A_{\text{pro}}$ is taken to mean the profinite completion of $A$ with the discrete topology.

A sequence

$$0 \to A' \to A \to A'' \to 0$$

of topological abelian groups is said to be short exact if it is exact as a sequence of abelian groups, the map $A' \to A$ is a topological embedding, and the quotient map $A/A' \to A''$ is a topological isomorphism. We say that the sequence

$$A' \xrightarrow{\phi} A \to A'' \to 0$$

is exact.
of continuous maps is right-exact if it is exact as a sequence of groups and if the map \( A/\phi(A') \to A'' \) is a topological isomorphism. Let \( \text{TAb} \) denote the category of abelian topological groups. We say that a functor \( F : \text{TAb} \to \text{TAb} \) is right-exact if it sends short-exact sequences to right-exact sequences.

The main goal of this section is to prove the following result.

**Proposition B.1.1.** The functor \( A \mapsto A_{\text{pro}} \) is right-exact.

**Lemma B.1.2.** Let \( A \) be a profinite abelian group, \( B \subset A \) a compact subgroup. Then \( A/B \) is profinite.

**Proof.** Since \( A \) is compact, so is \( A/B \). Because \( B \) is compact, it is closed, hence \( A/B \) is Hausdorff. It only remains to show that \( A/B \) is totally disconnected. That is, we need to show that if \( a \in A - B \), then there is a clopen subset \( U \subset A \) such that \( a \in U \) and \( U \cap B = \emptyset \). Since \( A \) is totally disconnected, for each \( b \in B \) there exists a clopen set \( U_b \subset A \) such that \( a \in U_b \) and \( b \notin U_b \). We claim that there are finitely many \( b_1, \ldots, b_n \in B \) such that \( U_{b_1} \cap \cdots \cap U_{b_n} \cap B = \emptyset \). Assuming this, we may take \( U := U_{b_1} \cap \cdots \cap U_{b_n} \) as our desired clopen subset. If the claim were false, then the sets \( \{U_b \cap B\}_{b \in B} \) would be a collection of closed subsets of the compact set \( B \) satisfying the finite intersection property, which is impossible since \( B \cap (\cap_{b \in B} U_b) = \emptyset \).

Now we can prove Proposition B.1.1.

**Proof.** Suppose that we have an inclusion of abelian topological groups \( \phi : B \hookrightarrow A \) (where \( B \) has the subspace topology). We need to show that the natural continuous map \( A_{\text{pro}}/\phi(B_{\text{pro}}) \to (A/B)_{\text{pro}} \) is a topological isomorphism. We will do this by constructing an inverse. We have a natural continuous map \( A/B \to A_{\text{pro}}/\phi(B_{\text{pro}}) \). The group \( A_{\text{pro}}/\phi(B_{\text{pro}}) \) is profinite by Lemma B.1.2 so we obtain a map \( (A/B)_{\text{pro}} \to A_{\text{pro}}/\phi(B_{\text{pro}}) \). To see that this is inverse to the other map, we note that it suffices to check that composition in either direction is the identity on the dense image of \( A \) inside both groups, and this is clear.

---

**B.2 Conditions under which profinite completion preserves exactness**

As we saw in the previous section, profinite completion is right-exact. It is not, however, exact. Indeed, consider the inclusion \( \mathbb{Z} \to \mathbb{Q} \). The profinite completion \( \hat{\mathbb{Z}} \) of \( \mathbb{Z} \) is nonzero, but \( \mathbb{Q}_{\text{pro}} = 0 \) because \( \mathbb{Q} \) is divisible. The goal of this section is to give some useful conditions ensuring that exactness is preserved by profinite completion. Here is the first.

**Proposition B.2.1.** Suppose that we have a short exact sequence of abelian topological groups

\[
0 \longrightarrow A' \longrightarrow A \overset{f}{\longrightarrow} A'' \longrightarrow 0
\]
such that $A''$ is discrete and finitely-generated. Then the sequence
\[ 0 \rightarrow A'_\text{pro} \rightarrow A_\text{pro} \rightarrow A''_\text{pro} \rightarrow 0 \]
is also short exact.

**Proof.** Let us note that this is a purely algebraic statement; there is no topology. This is because the map $A'_\text{pro} \rightarrow A_\text{pro}$, if injective, is necessarily a homeomorphism onto its image since the source is compact and the target Hausdorff. By Proposition B.1.1, we only need to check the injectivity of the map $A'_\text{pro} \rightarrow A_\text{pro}$. Let $H' \subset A'$ be an open subgroup of finite index. It will suffice to find a finite index open subgroup $H \subset A$ such that $H \cap A' = H'$. Let $A'' = F \times T$ with $F$ free of finite rank and $T$ finite. Let $a_1, \ldots, a_n \in A$ be such that the elements $f(a_i) \in A''$ form a set of free generators for $F$. We take $H = \langle a_1, \ldots, a_n, H' \rangle$. Then $H$ is open because $H'$ is open in $A$ (since $A' \subset A$ is open). Suppose that $b \in A' \cap H$. Then $b = h' + \sum_{i=1}^n r_i a_i$ for some $h' \in H', r_i \in \mathbb{Z}$. Since $b \in A'$, we have $0 = f(b) = \sum_{i=1}^n r_i f(a_i)$, so $r_i = 0$. Therefore, $b = h' \in H'$, so $H \cap A' = H'$. Finally, $H$ is of finite index inside $A$ because we have an exact sequence
\[ 0 \rightarrow A'/H' \rightarrow A/H \rightarrow A''/f(H) \rightarrow 0 \]
and $A'/H'$ is finite, as is $A''/f(H) \simeq T$. \qed

For an abelian group $A$, $A_{\text{div}} := \bigcap_{n \in \mathbb{Z}^+} nA$ denotes the subgroup of divisible elements, and $A_{\text{tors}}$ the subgroup of torsion elements.

**Proposition B.2.2.** Suppose that we have a short exact sequence
\[ 0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0 \]
of abelian topological groups. Suppose that $A''_{\text{div}} = 0$ and that $A''_{\text{tors}}$ has finite exponent. Finally, suppose also that for each positive integer $n$, the subgroup $nA \subset A$ is open. Then the sequence
\[ 0 \rightarrow A'_\text{pro} \rightarrow A_\text{pro} \rightarrow A''_\text{pro} \rightarrow 0 \]
is also short exact.

Again, we only need to check the injectivity of the map $A'_\text{pro} \rightarrow A_\text{pro}$. If $A''$ has finite exponent, then $A''_{\text{div}} = 0$. Thus, once Proposition B.2.2 is proved, we immediately obtain:

**Corollary B.2.3.** Consider a short exact sequence
\[ 0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0 \]
of abelian topological groups. Suppose that $A''$ has finite exponent, and that the subgroup $nA \subset A$ is open for each integer $n > 0$. Then the sequence
\[ 0 \rightarrow A'_\text{pro} \rightarrow A_\text{pro} \rightarrow A''_\text{pro} \rightarrow 0 \]
is also short exact.
To begin the proof of Proposition B.2.2 we require some preliminary lemmas:

**Lemma B.2.4.** Let $A$ be a finite abelian group, and suppose that $0 \neq a \in A$ lies in every nonzero subgroup of $A$. Then $A \simeq \mathbb{Z}/p^n\mathbb{Z}$ for some prime $p$ and integer $n \geq 0$.

*Proof.* It suffices to show that we can’t have $A \simeq A_1 \times A_2$ with $A_1, A_2$ nontrivial. If such an isomorphism exists then $a \in (A_1 \times 0) \cap (0 \times A_2) = 0$, a contradiction. □

**Lemma B.2.5.** Let $A$ be an abelian group of finite exponent, and suppose that $0 \neq a \in A$ lies in every nonzero subgroup of $A$. Then $A$ is finite (and in fact, it is necessarily of the form $\mathbb{Z}/p^n\mathbb{Z}$ by Lemma B.2.4).

*Proof.* It suffices to show that if $N$ is the exponent of $A$, then for any finite collection $b_1, \ldots, b_m \in A$, the subgroup $B$ generated by $a, b_1, \ldots, b_m$ (which is finite since $A$ has finite exponent) has size at most $N$. But $a$ is contained in every nonzero subgroup of $B$, so by Lemma B.2.4 the finite $B$ must be cyclic of some order $n > 0$. But $N$ kills $B$, so $n | N$. Thus, $\#B = n \leq N$, as desired. □

**Lemma B.2.6.** Let $A$ be an abelian group, $0 \neq a \in A$. Then there exists a subgroup $B \subset A$ maximal with respect to the property of not containing $a$. That is, $a \notin B$, and if $B' \supset B$ is a strictly larger subgroup of $A$, then $a \in B'$.

*Proof.* Consider the set $S$ of all subgroups of $A$ not containing $A$. This set is nonempty, since $a \notin \{0\}$. Further, any chain in $S$ has an upper bound (the union of the elements of the chain). Hence, by Zorn’s Lemma, $S$ has a maximal element. □

**Lemma B.2.7.** Let $A$ be an abelian topological group such that $nA \subset A$ is open for every positive integer $n$, and let $a \in A$. In order for there to exist an open subgroup $B \subset A$ of finite index such that $a \notin B$, it is necessary and sufficient that $a \notin A_{\text{div}}$.

*Proof.* First, if $a$ is divisible and $B \subset A$ is a finite index subgroup, then $\overline{a} := a \mod B \in A/B$ is a divisible element of the finite group $A/B$, hence $\overline{a} = 0$; i.e., $a \in B$.

Conversely, suppose that $a \notin A_{\text{div}}$, so $a \notin nA$ for some $n > 0$. Since $nA$ is open, $A/nA$ is discrete, so by replacing $A$ with $A/nA$ and $a$ with its nonzero image in $A/nA$ we may suppose that $A$ is discrete of finite exponent. By Lemma B.2.6 there is a subgroup $B \subset A$ maximal with respect to the property of not containing $a$. Replacing $A$ with $A/B$, therefore, we may assume that $a \neq 0$ lies in every nonzero subgroup of the group $A$ (which is still of finite exponent). But then $A$ is finite by Lemma B.2.5, so the assertion is immediate. □

**Lemma B.2.8.** Suppose that we have a short exact sequence of abelian groups

$$0 \longrightarrow A' \longrightarrow A \overset{f}{\longrightarrow} A'' \longrightarrow 0$$

with $A'_{\text{div}}, A''_{\text{div}} = 0$, and suppose that $A''_{\text{tors}}$ has finite exponent. Then $A_{\text{div}} = 0$. 193
Proof. Pick a ∈ A_{\text{div}}. Then f(a) ∈ A_{\text{div}}'' = 0, so a ∈ A'. Let N be the exponent of A_{\text{tors}}''. For every positive integer m, we have a = mnB for some b ∈ A. Since f(a) = 0, so f(b) ∈ A_{\text{tors}}'', we have f(Nb) = Nf(b) = 0 and thus Nb ∈ A'. We conclude that a = m(Nb) ∈ mA'. Since m was arbitrary, we deduce that a ∈ A_{\text{div}}'' = 0.

Remark B.2.9. Lemma B.2.8 is false if we omit the assumption that A_{\text{tors}}'' is of finite exponent. Indeed, let A be the free abelian group on generators x_n (n ∈ Z_4) modulo the relations nx_n = x_1. Clearly x_1 is divisible. It is also non-torsion, since we have the map A → Q sending x_n to 1/n. If we let A' = ⟨x_1⟩ ⊂ A, then A' ∼= Z, and A'' := A/A' ∼= \oplus_{n≥2} Z/nZ. So A_{\text{div}}', A_{\text{div}}'' = 0, but A_{\text{div}} ≠ 0.

Proof of Proposition B.2.2. Let B' ⊂ A' be a closed subgroup of finite index. We will show that B' = X ∩ A' for some closed subgroup X ⊂ A of finite index. This will imply that the map A'_{\text{pro}} → A_{\text{pro}} is injective. By Proposition B.2.1, we have an exact sequence of abelian groups

\[ 0 → A'/B' → A/B' → A'' → 0 \]

where the intersection is over all finite index closed subgroups X ⊂ A containing B'. This amounts to showing that for any a ∈ A ∩ B', there exists such X not containing a. That is, we need to show that for \( \bar{a} := a \mod B' ∈ A/B' \), there exists an open finite index subgroup \( \bar{X} ⊂ A/B' \) not containing \( \bar{a} \). The openness of nA is preserved under quotients, hence the hypotheses of Lemma B.2.7 holds for A/B'. It suffices, therefore, to show that \( (A/B')_{\text{div}} = 0 \). We have an exact sequence of abelian groups

\[ 0 → A'/B' → A/B' → A'' → 0 \]

Since A'/B' is finite, so \( (A'/B')_{\text{div}} = 0 \), the hypotheses that A_{\text{div}}'' = 0 and A_{\text{tors}}'' has finite exponent imply \( (A/B')_{\text{div}} = 0 \) by Lemma B.2.8. This completes the proof of (B.2.1).

For any closed subgroup X ⊂ A of finite index containing B', clearly \( [A' : X ∩ A'] ≤ [A' : B'] < ∞ \). Thus, we can choose X which maximizes \( [A' : X ∩ A'] \). We claim that such an X satisfies X ∩ A' = B', so we would be done. If such equality fails, so B' is a proper subgroup of X ∩ A', then by (B.2.1) there exists a closed subgroup Y ⊂ A of finite index containing B' such that the inclusion Y ∩ X ∩ A' ⊂ X ∩ A' is strict. But then

\[ [A' : (X ∩ Y) ∩ A'] > [A' : X ∩ A'] \]

violating the maximality property of X. Thus B' = X ∩ A', so we are done.
Appendix C

Compatibility Between Norms and Invariants

C.1 The setup and results

Let \( k \) be a field, and let \( k'/k \) be a finite separable extension. Suppose that \( G' \) is a commutative \( k' \)-group scheme of finite type, and let \( G := R_{k'/k}(G') \). By Lemma 2.2.1 we have a functorial isomorphism \( N_{k'/k} : R_{k'/k}(\hat{G'}) \cong \hat{G} \). Further, given any morphism \( f : X' \to X \) of schemes, and any fppf sheaf \( \mathcal{F}' \) on \( X' \), we have a canonical morphism \( H^i(X, f_*\mathcal{F}') \to H^i(X', \mathcal{F}') \). We will always, for notational simplicity, simply denote this morphism by \( \phi \).

Remark C.1.1. If \( \mathcal{F}' \) is a smooth group scheme and \( f \) is finite, then \( \phi \) is an isomorphism.

(Actually, if \( X \) is quasi-compact and quasi-separated, then the same conclusion holds with no assumptions on \( \mathcal{F}' \) provided that \( f \) is finite étale, but we will not need this.) Indeed, our assumption on \( \mathcal{F}' \) implies that we may take our cohomology to be étale rather than fppf by [BrIII, Thm.11.7], and then we use the fact that pushforward through a finite map is an exact functor between categories of étale sheaves.

Now suppose that \( k \) is a global field. Then we have the usual adelic pairing (that is, cupping everywhere locally and then adding the invariants)

\[
H^i(A_k, G) \times H^{2-i}(A_k, \hat{G}) \to \mathbb{Q}/\mathbb{Z}
\]

and similarly for \( G' \). We also have the composition \( H^i(A_k, \hat{G}) \cong H^i(A_k, R_{k'/k}(\hat{G}')) \phi \)
$H^i(A_{k'}, \hat{G}')$. For each $i$, therefore, we have a diagram:

$$
\begin{array}{c}
H^i(A_k, G) \times H^{2-i}(A_k, \hat{G}) \longrightarrow Q/Z \\
\downarrow \phi \downarrow \phi \circ N_{k'/k}^{-1} \downarrow \\
H^i(A_{k'}, G') \times H^{2-i}(A_{k'}, \hat{G}') \longrightarrow Q/Z
\end{array}
$$

(C.1.1)

It is now natural to ask whether diagram (C.1.1) commutes. The first main result of this appendix is the following affirmative answer.

**Proposition C.1.2.** Diagram (C.1.1) commutes.

Let us reduce this commutativity to a purely local question. First, we define another norm map. Given a finite extension $L/F$, we have a map $N_{L/F} : R_{L/F}(G_m) \to G_m$ given functorially on $F$-algebras by the norm map $(R \otimes F L)^\times \to R^\times$.

Now for each place $v$ of $k$, we have a diagram:

$$
\begin{array}{c}
H^i(k_v, G) \times H^{2-i}(k_v, \hat{G}) \longrightarrow H^2(k_v, G_m) \longrightarrow \text{inv} \longrightarrow Q/Z \\
\uparrow \downarrow \phi \uparrow \downarrow \phi \circ N_{k'/k}^{-1} \uparrow \\
\prod_{v'|v} H^i(k_{v'}, G') \times \prod_{v'|v} H^{2-i}(k_{v'}, \hat{G}') \longrightarrow \prod_{v'|v} H^2(k_{v'}, G_m) \longrightarrow \text{inv} \longrightarrow \prod_{v'|v} Q/Z
\end{array}
$$

(C.1.2)

Here all products are over the places $v'$ of $k'$ lying above $v$. The first three columns commute because of the functoriality of cup product, $\phi$, and the norm maps. Diagram (C.1.1) breaks up into a diagram for each place $v$, and the above diagram breaks up factor by factor into a diagram for each place $v'$ above $v$. It remains to prove that the last two columns commute. Such commutativity is equivalent to the following local analog of Proposition C.1.2.

**Proposition C.1.3.** Let $k'/k$ be a finite separable extension of local fields. Then the following diagram commutes:

$$
\begin{array}{c}
H^i(k, G) \times H^{2-i}(k, \hat{G}) \longrightarrow H^2(k, G_m) \longrightarrow \text{inv} \longrightarrow Q/Z \\
\downarrow \phi \downarrow \phi \circ N_{k'/k}^{-1} \downarrow \phi \\
H^i(k', G') \times H^{2-i}(k', \hat{G}') \longrightarrow H^2(k', G_m) \longrightarrow \text{inv} \longrightarrow Q/Z
\end{array}
$$

(C.1.3)
As we mentioned above, it only remains to prove commutativity of the second square in Proposition C.1.3. It therefore remains to prove the following result. Note that the map \( \phi \) appearing in the proposition below is an isomorphism by Remark C.1.1.

**Proposition C.1.4.** Let \( k'/k \) be a finite separable extension of local fields. Then the following diagram commutes:

\[
\begin{align*}
\text{H}^2(k, \mathbb{G}_m) & \xrightarrow{\text{inv}} \mathbb{Q}/\mathbb{Z} \\
\text{N}_{k'/k} \circ \phi^{-1} & \downarrow \\
\text{H}^2(k', \mathbb{G}_m) & \xrightarrow{\text{inv}} \mathbb{Q}/\mathbb{Z}
\end{align*}
\]

We will prove this in the next section.

**C.2 Proof of Proposition C.1.4**

First we treat the case of nonarchimedean local fields. Our reference for everything in this paragraph is [CF, Ch.VI, §1.1]. Let \( E \) be a nonarchimedean local field, and let \( E_{\text{nr}} \) be the maximal unramified extension of \( E \). We have an isomorphism \( \text{H}^2(E, E_{\text{nr}}^\times) \cong \text{H}^2(\text{Gal}(E_{\text{nr}}/E), E_{\text{nr}}^\times) \). Further, the map \( \text{H}^2(\text{Gal}(E_{\text{nr}}/E), E_{\text{nr}}^\times) \to \text{H}^2(\text{Gal}(E_{\text{nr}}/E), \mathbb{Z}) \) induced by the valuation map \( v_E : E_{\text{nr}}^\times \to \mathbb{Z} \) (normalized so that \( v_E(\pi_E) = 1 \), where \( \pi_E \in E \) is a uniformizer) is an isomorphism. Next, we have an isomorphism \( \hat{\mathbb{Z}} \to \text{Gal}(E_{\text{nr}}/E) \) induced by \( 1 \mapsto F_E \), where \( F_E \) is the Frobenius element (the unique lift of the Frobenius element in \( \text{Gal}(\kappa/\kappa) \), where \( \kappa = E/\pi_E \) is the residue field of \( E \)). Thus we get an isomorphism \( \text{H}^2(\text{Gal}(E_{\text{nr}}/E), \mathbb{Z}) \cong \text{H}^2(\hat{\mathbb{Z}}, \mathbb{Z}) \).

Finally, \( \text{H}^2(\hat{\mathbb{Z}}, \mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z} \), as follows. Take the long exact cohomology sequence associated to the sequence \( 0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0 \), and use the fact that \( \mathbb{Q} \) is uniquely divisible, hence its higher cohomology vanishes. Therefore, \( \text{H}^2(\hat{\mathbb{Z}}, \mathbb{Z}) \cong \text{H}^1(\hat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z}) = \text{Hom}_{\text{cts}}(\hat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z} \). Summing up, every cohomology class in \( \text{H}^2(E, E_{\text{nr}}^\times) \) is unramified, and the isomorphism \( \text{H}^2(\text{Gal}(E_{\text{nr}}/E), E_{\text{nr}}^\times) \cong \text{H}^2(\hat{\mathbb{Z}}, \mathbb{Z}) \) is induced by functoriality by the pair of maps \( v_E : E_{\text{nr}}^\times \to \mathbb{Z} \) and \( \hat{\mathbb{Z}} \to \text{Gal}(E_{\text{nr}}/E), 1 \mapsto F_E \).

Returning to our situation above, with \( L/K \) a finite separable extension of nonarchimedean local fields, let \( L_{\text{nr}} \) be the maximal unramified extension of \( L \), and let \( K_{\text{nr}} \subset L_{\text{nr}} \) be the maximal unramified extension of \( K \). Then the discussion above implies that the whole situation really descends to one over \( L_{\text{nr}} \). More precisely, we need to show that \( \text{inv}(N_{L/K}(\alpha)) = \text{inv}(\phi(\alpha)) \) for \( \alpha \in \text{H}^2(\text{Gal}(K_{\text{nr}}/K), (L \otimes_K K_{\text{nr}})^\times) \). That is, we need to show that the following diagram commutes

\[
\begin{align*}
\text{H}^2(K_{\text{nr}}/K, (L \otimes_K K_{\text{nr}})^\times) & \xrightarrow{N_{L/K}} \text{H}^2(K_{\text{nr}}/K, K_{\text{nr}}^\times) \\
\phi \downarrow & \\
\text{H}^2(L_{\text{nr}}/L, L_{\text{nr}}^\times) & \xrightarrow{\text{inv}} \mathbb{Q}/\mathbb{Z}
\end{align*}
\]
where $N_{L/K}$ is the map induced by $N_{L/K} : L \otimes_K K_{nr} \to K_{nr}$, and $\phi$ is the map induced by functoriality by the pair of maps $\text{Gal}(L_{nr}/L) \xrightarrow{\text{res}} \text{Gal}(K_{nr}/K)$, $m : (L \otimes_K K_{nr})^\times \to L_{nr}^\times$, where the first map is restriction and the second is defined by $m(\alpha \otimes \beta) = \alpha \beta$. Let $H := \text{Gal}(L_{nr}/L)$, $G := \text{Gal}(K_{nr}/K)$. Note that $H \subset G$ via restriction.

We remark that $\phi$ is an isomorphism because $(L \otimes_K K_{nr})^\times = \text{Ind}^G_H L_{nr}^\times$, and $\phi$ is then the canonical isomorphism between the cohomology of a module and its induced module. These assertions follow immediately from the following elementary lemma.

**Lemma C.2.1.** Let $L/K$ be a finite separable extension of nonarchimedean local fields. Then the map $\psi : L \otimes_K K_{nr} \to \prod_{\sigma \in \text{Hom}_K(L,L_{nr})} L_{nr}$ given by $\alpha \otimes \beta \mapsto \prod_{\sigma \in \text{Hom}_K(L,L_{nr})} \sigma(\alpha) \beta$ is an isomorphism.

*Proof.* One checks that the assertion behaves well in towers. That is, if we have a tower $F/L/K$ of finite separable extensions of local fields, and if the lemma holds for $F/L$ and $L/K$, then it also holds for $F/K$. Indeed, suppose that we have isomorphisms (via the maps in the lemma)

$$L \otimes_K K_{nr} \xrightarrow{\sim} \prod_{\sigma \in \text{Hom}_K(L,L_{nr})} L_{nr}$$

$$F \otimes_L L_{nr} \xrightarrow{\sim} \prod_{\sigma \in \text{Hom}_L(F,F_{nr})} F_{nr}$$

Then applying $F \otimes_L$ to the top isomorphism, and then using the second isomorphism, yields the lemma for the extension $F/K$.

Returning to the notation in the statement of the lemma, by considering the maximal unramified extension of $K$ contained in $L$, we may therefore assume that $L/K$ is either unramified or totally ramified.

First suppose that it is unramified. Then $L_{nr} = K_{nr}$, and every $K$-embedding of $L$ into $K_s = L_s$ has image that is unramified over $K$, hence over $L$, so $\text{Hom}_K(L,L_{nr}) = \text{Hom}_K(L,K_s)$, hence the map becomes an isomorphism upon extending scalars to $K_s$, and is therefore an isomorphism.

Now suppose that $L/K$ is totally ramified. Then the only $K$-embedding of $L$ into $L_{nr}$ is the inclusion, so we just need to check that the map $L \otimes_K K_{nr} \to L_{nr}$ given by multiplication is an isomorphism. Clearly, the compositum of $L$ and $K_{nr}$ is $L_{nr}$, since it is an unramified extension of $L$ with algebraically closed residue field. On the other hand, $L \cap K_{nr} = K$ because it is an unramified extension of $K$ contained in $L$. Thus the map is indeed an isomorphism. \qed
We now break up diagram (C.2.1) as follows:

\[
\begin{array}{ccc}
\text{H}^2(K_{nr}/K, (L \otimes_K K_{nr})^\times) & \xrightarrow{N_L/K} & \text{H}^2(K_{nr}/K, K_{nr}^\times) \\
\downarrow \text{Ind}_H^G v_L & & \downarrow v_K \\
\text{H}^2(K_{nr}/K, \text{Ind}_H^G \mathbb{Z}) & \xrightarrow{\phi_Z} & \text{H}^2(L_{nr}/L, \mathbb{Z}) \\
\downarrow \text{Cor} & & \downarrow \text{log}_{FL} & \downarrow \text{log}_{FK} \\
\text{H}^2(\mathbb{Z}, \mathbb{Z}) & & \downarrow & \downarrow & \downarrow & \downarrow & \mathbb{Q}/\mathbb{Z} \\
& & & (C.2.2)
\end{array}
\]

Here Cor is the corestriction map; \( \text{log}_{FK} \) denotes the map induced by the inverse of the isomorphism \( \hat{\mathbb{Z}} \rightarrow \text{Gal}(K_{nr}/K) \) defined by \( 1 \mapsto F_K \), and similarly for \( \text{log}_{FL} \). The map \( \phi_Z \) is the canonical isomorphism between the cohomology of a module and its induced module; i.e., the isomorphism induced by the pair of maps \( (H \rightarrow G, \psi \mapsto \psi(1)) \) for \( \psi \in \text{Ind}_H^G \mathbb{Z} \). Note that \( \text{log}_{FL} \circ v_L \) and \( \text{log}_{FK} \circ v_K \) (composed with the isomorphism \( \text{H}^2(\hat{\mathbb{Z}}, \mathbb{Z}) \simeq \mathbb{Q}/\mathbb{Z} \)) yield the invariant maps for \( L \) and \( K \), respectively, by our earlier discussion of how the invariant is defined. Commutativity of (C.2.2) therefore does indeed imply that of (C.2.1).

So it remains to show that the subdiagrams (1)–(4) commute. We handle each of these in turn.

**Commutativity of (1):** This is clear, basically by functoriality of the isomorphism \( \text{H}^i(G, \text{Ind}_H^G(\cdot)) \simeq \text{H}^i(H, \cdot) \).

**Commutativity of (2):** The two paths from top left to bottom right of (2) are induced by the two maps \( v_K \circ N_{L/K}, N_G/H \circ \text{Ind}_H^G v_L : (L \otimes_K K_{nr})^\times \rightarrow \mathbb{Z} \), so it suffices to check that these two maps agree. Via the isomorphism in Lemma (C.2.1) these two maps decompose factor by factor, the first given on each factor by \( v_K \circ N_{L_{nr}/K_{nr}} \), and the second by \( v_L \). Thus we only have to prove the following lemma.

**Lemma C.2.2.** \( v_K \circ N_{L_{nr}/K_{nr}} = v_L \).

**Proof.** We want to show that \( v_K(N_{L_{nr}/K_{nr}}(\alpha)) = v_L(\alpha) \) for \( \alpha \in L^\times \). Both sides are multiplicative in \( \alpha \), so we may assume that either \( \alpha \in O_{L_{nr}}^\times \) is a unit, or \( \alpha = \pi_K \), a uniformizer of \( K \). In case \( \alpha \) is a unit, both sides are 0. If \( \alpha = \pi_K \), then the desired equation becomes \( [L_{nr} : K_{nr}] = e := e(L/K) \), which is the content of the following lemma. \( \Box \)
Lemma C.2.3. $[L_{nr} : K_{nr}] = e$.

Proof. Both sides are multiplicative in towers, so by considering the maximal unramified subextension of $L/K$, we may assume that either $L/K$ is unramified or it is totally ramified. The unramified case is clear: then $L_{nr} = K_{nr}$, so both sides equal 1. Now suppose that $L/K$ is totally ramified, so $e = n := [L : K]$.

First note that $L_{nr} = L K_{nr}$. Indeed, the right side is an unramified extension of $L$ with algebraically closed residue field. It follows that a $K$-basis of $L$ spans $L_{nr}$ as a $K_{nr}$-vector space, so $L_{nr} \otimes_K K_{nr} \to L_{nr}$ is surjective. It remains to check that it is a field so that this map is an isomorphism. But this follows from the fact that $L \cap K_{nr} = K$, which holds because it is an unramified extension of $K$ contained in the totally ramified extension $L$ of $K$. \hfill \qed

Commutativity of (3): This is purely a fact about group cohomology (essentially true by definition), cf. [Ser1, Ch.I, §2.5, Rem.(b)].

Commutativity of (4): The key point is the following lemma, in which $f$ denotes the residue degree of $L/K$.

Lemma C.2.4. The following diagram commutes.

\[
\begin{array}{c}
\text{H}^2(K_{nr}/K, \mathbb{Z}) \xrightarrow{\text{Res}} \text{H}^2(L_{nr}/L, \mathbb{Z}) \\
\downarrow \log_{FK} \quad \quad \quad \downarrow \log_{FL} \\
\text{H}^2(\hat{\mathbb{Z}}, \mathbb{Z}) \xrightarrow{f} \text{H}^2(\hat{\mathbb{Z}}, \mathbb{Z})
\end{array}
\]

Proof. First, we claim that the map $\text{H}^2(\hat{\mathbb{Z}}, \mathbb{Z}) \to \text{H}^2(\hat{\mathbb{Z}}, \mathbb{Z})$ induced by the multiplication by $f$ map on $\hat{\mathbb{Z}}$ and the identity on $\mathbb{Z}$ is multiplication by $f$. More generally, this holds with $\hat{\mathbb{Z}}$ replaced by any abelian group $G$. Indeed, taking the long exact cohomology sequence associated to the short exact sequence

\[
0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0
\]

and using the fact that $\mathbb{Q}$ has vanishing higher cohomology (because it is uniquely divisible), we obtain an isomorphism, functorial in $G$, $\text{H}^2(G, \mathbb{Z}) \simeq \text{Hom}(G, \mathbb{Q}/\mathbb{Z})$. The functoriality of this isomorphism implies our claim.

Now we check that the following diagram commutes:

\[
\begin{array}{c}
\text{H}^2(K_{nr}/K, \mathbb{Z}) \xrightarrow{\text{Res}} \text{H}^2(L_{nr}/L, \mathbb{Z}) \\
\downarrow \log_{FK} \quad \quad \quad \downarrow \log_{FL} \\
\text{H}^2(\hat{\mathbb{Z}}, \mathbb{Z}) \xrightarrow{\psi f} \text{H}^2(\hat{\mathbb{Z}}, \mathbb{Z})
\end{array}
\]
Here $\psi_f$ is the map induced by multiplication by $f$ on $\hat{\mathbb{Z}}$, and by the first paragraph this is just multiplication by $f$ on $H^2$, so this commutativity will prove the lemma (but, as we are about to see, it is more convenient to work with $\psi_f$). The map obtained by going right and then down is induced by functoriality by the identity on $\mathbb{Z}$ and by the map $\hat{\mathbb{Z}} \to \text{Gal}(K_{nr}/K)$ induced by $1 \mapsto F_{L|K_{nr}}$. The map obtained by going down and then right is similarly obtained by functoriality by the map $1 \mapsto F_f^{L|K_{nr}}$. The desired commutativity therefore follows from the fact that $F_{L|K_{nr}} = F_{f^{L|K}}$, which follows by looking at the actions on the residue fields, and using that for a degree-$f$ extension $\kappa'/\kappa$ of finite fields, $F_{\kappa'|\kappa} = F_{f^{\kappa'|\kappa}}$. 

With Lemma C.2.4 in hand, the proof that (4) commutes is now easy. We first note that Lemma C.2.4 implies that $\text{Res}: H^2(K_{nr}/K,\mathbb{Z}) \to H^2(L_{nr}/L,\mathbb{Z})$ is surjective, since multiplication by $f$ is surjective on $H^2(\hat{\mathbb{Z}},\mathbb{Z}) \simeq \mathbb{Q}/\mathbb{Z}$. So in order to check that diagram (4) commutes; i.e., that $\log_{F_K} \circ \text{Cor} = \log_{F_L}$, it suffices to check that $\log_{F_K} \circ \text{Cor} \circ \text{Res} = \log_{F_L} \circ \text{Res}$. But $\text{Cor} \circ \text{Res} = [G : H]$ by [Ser1, Ch.I, §2.4]. Therefore, if we can show that $[G : H] = f$, then it will follow that the above equality becomes $f \cdot \log_{F_K} = \log_{F_L} \circ \text{Res}$, which is Lemma C.2.4, and this will complete the proof that (4) commutes, and with it the proof of Proposition C.1.4 in the nonarchimedean case. So all that remains is to prove:

**Lemma C.2.5.** $[G : H] = f$, where $G = \text{Gal}(K_{nr}/K)$, $H = \text{Gal}(L_{nr}/L)$, and we view $H$ as a subgroup of $G$ via restriction.

*Proof.* The two groups are identified with the absolute Galois groups of the residue fields of $K$ and $L$, hence the index is just the degree of the residue field extension associated to $L/K$, i.e., $f(L/K)$. 

It remains to prove Proposition C.1.4 for archimedean local fields. If $L = K$, then the assertion is trivial, so we may assume that $L = \mathbb{C}$, $K = \mathbb{R}$. As in the nonarchimedean case, the desired assertion then reduces to the commutativity of the following diagram

\[
\xymatrix{ \text{H}^2(\mathbb{R}, (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})^\times) \ar[d]^\phi \ar[r]^{N_{C/R}} & \text{H}^2(\mathbb{R}, \mathbb{C}^\times) \ar[d]^\text{inv} \\
\text{H}^2(\mathbb{C}, \mathbb{C}^\times) \ar[r]^\text{inv} & \frac{1}{2} \mathbb{Z}/\mathbb{Z} }
\]

(This is the analogue of diagram (C.2.1).) But this is trivial: $\text{H}^2(\mathbb{R}, (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})^\times) = 0$, since $\phi$ is an isomorphism. This completes the proof of Proposition C.1.4. 

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Appendix D

Cohomology and Direct Limits

The goal of this appendix is to show that fppf cohomology behaves well with respect to
direct limits, a fact that we use repeatedly in various arguments. More precisely, we have
the following result.

**Proposition D.0.1.** Let \( \{ X_\lambda \} \) be a filtered inverse system of quasi-compact quasi-separated
schemes with affine transition maps \( \pi_{\lambda', \lambda} : X_{\lambda'} \to X_\lambda \) whenever \( \lambda' \geq \lambda \). For each \( \lambda \), let \( \mathcal{F}_\lambda \) be an fppf abelian sheaf on
\( X_\lambda \), and suppose that we are given compatible (in the obvious sense) transition maps \( \pi_{\lambda', \lambda}^* \mathcal{F}_\lambda \to \mathcal{F}_{\lambda'} \) for \( \lambda' \geq \lambda \).

For \( X := \lim \leftarrow X_\lambda \), and the direct limit \( \mathcal{F} \) of the pullbacks of all of these sheaves to \( X \),
the natural map
\[
\lim \to H^i(X_\lambda, \mathcal{F}_\lambda) \to H^i(X, \mathcal{F})
\]
of fppf cohomology groups is an isomorphism.

A version of this result is proved in the context of very general topoi in [SGA4\( \text{II} \), Exp. VI, §5]; the proof we give here is a concrete version of that general argument.

**Proof.** We first treat degree 0 by a direct argument. Any fppf cover of a quasi-compact and quasi-separated scheme \( U \) admits a finite refinement \( \{ U_i \} \) with each map \( U_i \to U \) quasi-compact and quasi-separated, and so all finite fiber products among the \( U_i \)’s are quasi-compact over \( U \). This is the key point.

Let \( \mathcal{G} \) be the presheaf \( \mathcal{G}(U) = \lim \to \mathcal{F}_\lambda(U) \) whose sheafification \( \mathcal{G}^+ \) is \( \lim \to \mathcal{F}_\lambda \). Since \( \{ \mathcal{F}_\lambda \} \) is filtered, we see that if a flat locally finitely presented map \( U \to X \) is quasi-compact and quasi-separated, and if \( \mathcal{U} = \{ U_i \} \) is an fppf cover of \( U \) by finitely many \( U \)-schemes \( U_i \) that are quasi-compact and quasi-separated over \( U \), then \( \mathcal{G}(U) = \tilde{H}^0(\mathcal{U}, \mathcal{G}) \). Such covers \( \mathcal{U} \) are cofinal among fppf covers of \( U \), so the first step of the sheafification process for \( \mathcal{G} \) does not change the values on quasi-compact and quasi-separated \( X \)-schemes \( U \). Hence, the second step of the sheafification process does not change the values on such objects \( U \). By taking \( U = X \), we get the desired result in degree 0.
Since filtered direct limits commute with finite products and any fppf cover of $X$ admits a finite refinement $U = \{U_i\}$ such that all finite products among the $U_i$’s are quasi-compact and quasi-separated, it follows from the degree-0 case that for a cofinal system of covers $U$, $\check{H}^\bullet(U, \cdot)$ commutes with filtered direct limits. By the theorem on exchange of iterated filtered limits, we conclude that $\check{H}^\bullet(X, \cdot)$ also commutes with filtered direct limits. It is not true in general that Čech cohomology agrees with fppf cohomology, but we have a spectral sequence
$$E_2^{p,q} = \check{H}^p(X, \check{H}^q(\mathcal{F})) \Rightarrow \check{H}^{p+q}(X, \mathcal{F}).$$
Thus, the problem of moving filtered direct limits through $\check{H}^n(X, F)$ is reduced to that of the $E_2^{p,q}$-terms for $p + q = n$. Since $E_2^{0,n} = 0$, we may suppose $q < n$. We also have $E_2^{n,0} = \check{H}^n(X, \mathcal{F})$, so we may suppose $p > 0$. This settles $n = 1$, and we may then carry out an induction to handle all $n > 1$ (with an inductive hypothesis that is quantified over all quasi-compact and quasi-separated flat and locally finitely presented $X$-schemes).

Let us discuss how Proposition D.0.1 applies in the context of our work. Suppose that we have $\{X_\lambda\}$ and $X$ as in the result, and are given a finitely presented $X$-group scheme $\mathcal{G}$. Then $\mathcal{G}$ descends to compatible finitely presented group schemes $X_\lambda$-group schemes $\mathcal{G}_\lambda$ for all sufficiently large $\lambda$, and any fppf $X$-scheme $U$ descends to a compatible system of fppf $X_\lambda$-schemes $U_\lambda$ for all sufficiently large $\lambda$. Further, any element $g \in \mathcal{G}(U)$ descends to compatible elements $g_\lambda \in \mathcal{G}_\lambda(U_\lambda)$, and any two such $g_\lambda, g_\lambda'$ become the same under pullback to some $U_\lambda''$.

We therefore have $\mathcal{G} = \varinjlim \pi_\lambda^* \mathcal{G}_\lambda$ as sheaves, where $\pi_\lambda : X \to X_\lambda$ is the natural map. In contrast, if we consider the sheaf for the étale topology represented by a non-étale group scheme then the pullback sheaf under a non-étale map is generally not represented by the corresponding base change of the given group scheme. By Proposition D.0.1 it follows that $H^i(X, \mathcal{G}) = \varinjlim H^i(X_\lambda, \mathcal{G}_\lambda)$.

The same reasoning applies in many situations where we will use implicitly that cohomology behaves well with respect to suitable direct limits. For example, by very similar reasoning we have $H^i(X, \mathcal{F}) = \varinjlim H^i(X_\lambda, \mathcal{G}_\lambda)$ where the functor $\widehat{(-)} = \mathcal{H}om(\cdot, \mathbb{G}_m)$ denotes the $\mathbb{G}_m$-dual of an fppf group scheme. This dual is generally not represented by an $X$-scheme, but it is “locally of finite presentation” as a functor, and that is what matters for the preceding limit considerations to be applicable.

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Appendix E

Compatibility Between Čech and Derived Functor Constructions

This appendix proves the compatibility between certain constructions for Čech and derived functor cohomology, and especially between two constructions of connecting maps in cohomology. This compatibility is used in order to prove results on the agreement between Čech and derived functor cohomology (Proposition 3.5.1), as well as to prove Theorem 1.2.9 since the pairings between the Tate-Shafarevich groups are defined in terms of Čech cocycles.

E.1 Edge maps

Let $X$ be a scheme, $\mathcal{F}$ an abelian sheaf for a Grothendieck topology on $X$ (we need the fppf and étale topologies), and $U$ a cover of $X$. For the Čech cohomology groups $\check{H}^i(U, \mathcal{F})$, the Čech-to-derived functor spectral sequence yields canonical edge maps

$$\check{H}^i(U, \mathcal{F}) \to H^i(X, \mathcal{F}) \quad (E.1.1)$$

for all $i$. Our goal in this appendix is twofold: to show that a natural direct way of constructing such maps (without appeal to spectral sequences) coincides with these edge maps, and to show that connecting maps on pieces of Čech cohomology are compatible (via the canonical Čech-to-derived maps $E.1.1$) with connecting maps for derived functor cohomology. Both of these compatibilities are extremely useful, but surprisingly there does not seem to be a discussion of either of them in any standard (or other) reference.

We begin by discussing a useful more direct construction of the edge maps; this will be used in our discussion of compatibility for connecting maps. Let $\mathcal{I}^\bullet$ be an injective resolution of $\mathcal{F}$. The Čech complex of sheaves $\check{C}^\bullet(U, \mathcal{F})$ is also a resolution of $\mathcal{F}$, so there is a map of such resolutions $\check{C}^\bullet(U, \mathcal{F}) \to \mathcal{I}^\bullet$ that is unique up to homotopy. Passing to
global sections, this induces canonical maps (independent of all choices) $f^i_U : \check{H}^i(U, \mathcal{F}) \to H^i(X, \mathcal{F})$ for all $i$.

**Proposition E.1.1.** The map $f^i_U$ coincides with the edge map (E.1.1) for all $i$.

**Proof.** (O. Gabber) Let us keep the notation above. We will suppress the $U$ subscript for notational simplicity. The Čech-to-derived functor spectral sequence is the spectral sequence associated to the double complex

$$K := \Gamma(\mathcal{C}^\bullet(\mathcal{I}^\bullet))$$

where $\Gamma(\cdot)$ denotes the functor $\Gamma(X, \cdot)$. For the reader’s convenience, let us define this “double complex” and state our conventions for the total complex $\text{tot}(K)$ of $K$.

By definition, $K^{p,q} := \Gamma(\mathcal{C}^p(\mathcal{I}^q))$ with horizontal differentials given along the $q$th row by the Čech differentials and the vertical differentials in the $p$th column are induced by applying $\mathcal{C}^p$ to the complex $\mathcal{I}^q$. Note that this is a commuting diagram (so not the standard anti-commuting condition in the definition of “double complex”); i.e., the following diagram commutes for all $p,q$:

$$
\begin{array}{ccc}
\Gamma(\mathcal{C}^p(\mathcal{I}^{q+1})) & \xrightarrow{\partial^{p,q+1}_{\text{hor}}} & \Gamma(\mathcal{C}^{p+1}(\mathcal{I}^{q+1})) \\
\downarrow^{\partial^{p,q}_{\text{ver}}} & & \downarrow^{\partial^{p+1,q}_{\text{hor}}} \\
\Gamma(\mathcal{C}^p(\mathcal{I}^q)) & \xrightarrow{\partial^{p,q}_{\text{hor}}} & \Gamma(\mathcal{C}^{p+1}(\mathcal{I}^q))
\end{array}
$$

The total complex $\text{tot}(K)$ of $K$ is the cochain complex defined by

$$(\text{tot}(K))^n := \bigoplus_{p+q=n} \Gamma(\mathcal{C}^p(\mathcal{I}^q))$$

with differential given on $K^{p,q}$ by $\partial^{p,q}_{\text{hor}} + (-1)^p \partial^{p,q}_{\text{ver}}$.

The edge map from $\check{H}^n(X, \mathcal{F})$ to the $n$th cohomology of $\text{tot}(K)$ (which is the abutment of the spectral sequence) is defined by the edge map

$$g : \Gamma(\mathcal{C}^\bullet(\mathcal{F})) \to \text{tot}(K)$$

defined in degree $n$ by the map

$$\mathcal{C}^n(\mathcal{F})(X) = \prod_{i_0 \ldots i_n} \mathcal{F}(U_{i_0 \ldots i_n}) \to K^{n,0} = \prod_{i_0 \ldots i_n} \mathcal{I}^0(U_{i_0 \ldots i_n})$$

induced by the inclusion $\mathcal{F} \hookrightarrow \mathcal{I}^0$. That this is a map of cochain complexes follows from the fact that the differential $\mathcal{I}^0 \to \mathcal{I}^1$ kills $\mathcal{F}$.
The $n$th homology of the total complex of $K$ is identified with $\text{H}^n(X, \mathcal{F})$ in the following manner. We have an edge map

$$f_2 : \Gamma(\mathcal{F}^\bullet) \rightarrow \text{tot}(K)$$

defined in degree $n$ by the restriction map $\mathcal{F}^n(X) \rightarrow K^{0,n} = \prod_i \mathcal{F}^n(U_i)$. That this is a map of cochain complexes follows from the fact that the $\mathcal{F}^n$ are sheaves. When constructing the Čech-to-derived functor spectral sequence, one shows that this map is a quasi-isomorphism (the main point being that each $\mathcal{C}^p(\mathcal{F})$ is flasque for an injective abelian sheaf $\mathcal{I}$), thereby identifying the homology of $\text{tot}(K)$ with the derived functor cohomology of $\mathcal{F}$.

Now we bring in the map (unique up to homotopy) $h : \mathcal{C}^\bullet(\mathcal{F}) \rightarrow \mathcal{I}^\bullet$ of resolutions of $\mathcal{F}$. Consider the double complex

$$K' := \Gamma(\mathcal{C}^\bullet(\mathcal{C}^\bullet(\mathcal{F})))$$

(same conventions as before). We have a commutative diagram

$$
\begin{array}{ccc}
\Gamma(\mathcal{C}^\bullet(\mathcal{F})) & \xrightarrow{e'} & \Gamma(\mathcal{C}^\bullet(\mathcal{C}^\bullet(\mathcal{F}))) \\
\downarrow & & \downarrow f_2 \\
\Gamma(\mathcal{C}^\bullet(\mathcal{I})) & \xrightarrow{g} & \Gamma(\mathcal{I}^\bullet)
\end{array}
$$

of cochain complexes, where (i) the lower horizontal maps are the “edge maps” defined above and the upper horizontal maps are defined analogously using the Čech resolution of $\mathcal{F}$ in place of the injective resolution $\mathcal{I}^\bullet$, and (ii) the middle and right vertical maps are induced by the map of resolutions $h : \mathcal{C}^\bullet(\mathcal{F}) \rightarrow \mathcal{I}^\bullet$.

Our task is to show that $f_2 \circ f_1$ and $g$ induce the same maps on cohomology in each degree, so by commutativity of the diagram it suffices to show that $e'$ and $e''$ are homotopic. For the reader’s convenience, let us write out formulas for all of the relevant maps.

First, $\Gamma(\mathcal{C}^n(\mathcal{F})) = \prod_{i_0...i_n} \mathcal{F}(U_{i_0...i_n})$, and for $\sigma \in \Gamma(\mathcal{C}^n(\mathcal{F}))$, we have

$$(\partial \sigma)(i_0, \ldots, i_{n+1}) = \sum_{r=0}^{n+1} (-1)^r \sigma(i_0 \ldots \hat{i}_r \ldots i_{n+1})|_{U_{i_0...i_{n+1}}}$$

where the notation means that the index $i_r$ is omitted. The group

$$K^{p,q} = \Gamma(\mathcal{C}^p(\mathcal{C}^q(\mathcal{F})))$$

is naturally identified with $\prod_{i_0...i_p,j_0...j_q} \mathcal{F}(U_{i_0...i_p,j_0...j_q})$. The vertical and horizontal differentials at the $(p,q)$-entry are

$$(\partial^\mathcal{C}_\text{vert} \sigma)(i_0, \ldots, i_{p+1}, j_0, \ldots, j_q) = \sum_{r=0}^{p+1} (-1)^r \sigma(i_0, \ldots, \hat{i}_r, \ldots, i_{p+1}, j_0, \ldots, j_q)|_{U_{i_0...i_p,j_0...j_q}}$$
\[
(\partial_{\text{ver}}^p\sigma)(i_0, \ldots, i_p, j_0, \ldots, j_{q+1}) = \sum_{r=0}^{q+1} (-1)^r \sigma(i_0, \ldots, i_p, j_0, \ldots, \hat{j}_r, \ldots, j_{q+1})|_{U_{i_0 \ldots i_p j_0 \ldots j_{q+1}}}
\]

The total complex \(\text{tot}(K')\) of \(K'\) is therefore given by

\[
(\text{tot}(K'))^n = \bigoplus_{p+q=n} K'^{p,q} \cong \bigoplus_{p+q=n} \Gamma(\mathcal{E}^{n+1}(\mathcal{F}))
\]

with differential given on \(K'^{p,q}\) by \(\partial_{\text{hor}}^p + (-1)^p \partial_{\text{ver}}^p\).

The edge maps are defined as follows for \(\sigma \in \Gamma(\mathcal{E}^{n}(\mathcal{F}))\): \(e'(\sigma)^{p,q} = 0\) for \(q > 0\), \(e''(\sigma)^{p,q} = 0\) for \(p > 0\), and for any \(n \geq 0\) we define

\[
e'(\sigma)^{n,0}(i_0, \ldots, i_n, j_0) = \sigma(i_0, \ldots, i_n)|_{U_{i_0 \ldots i_n j_0}},
\]

\[
e''(\sigma)^{0,n}(i_0, j_0, \ldots, j_n) = \sigma(j_0, \ldots, j_n)|_{U_{i_0 j_0 \ldots j_n}}.
\]

Let us now define our homotopy \(H\). Let \(H : \Gamma(\mathcal{E}^{n+1}(\mathcal{F})) \to (\text{tot}(K'))^n\) be defined to have its component map into the \((p, n - p)\)-factor \((\text{tot}(K'))^{p,n-p} = \Gamma(\mathcal{E}^{n+1}(\mathcal{F}))\) given by multiplication by \((-1)^p\). Then one checks that for \(p, q \geq 0\) with \(p + q = n\), as elements of the factor \(\mathcal{F}(U_{i_0 \ldots i_p j_0 \ldots j_q})\) of \(\Gamma(\mathcal{E}^{n+1}(\mathcal{F}))\) we have

\[
(H \partial \sigma)^{p,q}(i_0, \ldots, i_p, j_0, \ldots, j_q) = (-1)^p \partial \sigma(i_0, \ldots, i_p, j_0, \ldots, j_q)
\]

\[
= (-1)^p \sum_{r=0}^{p} (-1)^r \sigma(i_0, \ldots, \hat{i}_r, \ldots, i_p, j_0, \ldots, j_q)
\]

\[
+ (-1)^p \sum_{r=0}^{q} (-1)^{r+1} \sigma(i_0, \ldots, i_p, \hat{j}_r, \ldots, j_q)
\]

\[
= \sum_{r=0}^{p} (-1)^{p+r} \sigma(i_0, \ldots, \hat{i}_r, \ldots, i_p, j_0, \ldots, j_q)
\]

\[
+ \sum_{r=0}^{q} (-1)^{r+1} \sigma(i_0, \ldots, i_p, \hat{j}_r, \ldots, j_q)
\]

(suppressing the evident restriction to \(U_{i_0 \ldots j_q}\) for all terms in the sums) and

\[
(\partial H \sigma)^{p,q}(i_0, \ldots, i_p, j_0, \ldots, j_q) = \sum_{r=0}^{p} (-1)^r H(\sigma)^{p-1,q}(i_0, \ldots, \hat{i}_r, \ldots, i_p, j_0, \ldots, j_q) +
\]

\[
(-1)^p \sum_{r=0}^{q} (-1)^r H(\sigma)^{p,q-1}(i_0, \ldots, i_p, \hat{j}_r, \ldots, j_q)
\]
(again suppressing notation for restriction to \( U_{i_0\ldots j_q} \)) with the first summand only appearing if \( p > 0 \) and the second appearing only if \( q > 0 \). This final expression is equal to

\[
- \sum_{r=0}^{p} (-1)^{p+r}\sigma(i_0, \ldots, \hat{i}_r, \ldots, i_p, j_0, \ldots, j_q) - \sum_{r=0}^{q} (-1)^{r+1}\sigma(i_0, \ldots, i_p, j_0, \ldots, \hat{j}_r, \ldots, j_q)
\]

with the same suppression of notation for restriction maps and the same caveat that the first (resp. second) summand appears only when \( p > 0 \) (resp. \( q > 0 \)).

It is now a straightforward calculation with the definitions and inspection of the \((p, q)\)-component of the output in each of the four cases (i) \( p, q > 0 \), (ii) \( p > 0, q = 0 \), (iii) \( p = 0, q > 0 \), (iv) \( p = q = 0 \) that (after much cancellation) we have

\[
\partial H + H\partial = e'' - e'.
\]

Hence, \( H \) is a homotopy between \( e' \) and \( e'' \).

E.2 Connecting maps

Now consider a short exact sequence

\[
0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0
\]

of abelian sheaves for a Grothendieck topology on \( X \) (such as the fppf or étale topologies, which are all we shall need). For a cover \( \mathcal{U} \) of \( X \), consider a Čech cocycle \( \check{\alpha} \in \check{Z}^i(\mathcal{U}, \mathcal{F}'') \) that happens to lift to a cochain \( \check{\beta} \in \check{C}^i(\mathcal{U}, \mathcal{F}) \). Of course, in general no such lift will exist; we only consider \( \check{\alpha} \) admitting such a lift, as this situation arises several times in our work.

Define a cocycle \( \check{\gamma} \in \check{Z}^{i+1}(\mathcal{U}, \mathcal{F}') \) via the snake lemma procedure. That is, since \( \pi(d\check{\beta}) = d\check{\alpha} = 0 \), there is a unique cochain \( \check{\gamma} \in \check{C}^{i+1}(\mathcal{U}, \mathcal{F}') \) such that \( j(\check{\gamma}) = d\check{\beta} \). The cochain \( \check{\gamma} \) is a cocycle because \( j(d\check{\gamma}) = d^2\check{\beta} = 0 \) and \( \ker j = 0 \).

**Proposition E.2.1.** Letting \([c]\) denote the derived-functor cohomology class arising from a Čech cocycle \( \check{c} \) via the edge map \((E.1.1)\), for \( \check{\alpha} \) and \( \check{\gamma} \) as considered above we have \( \delta([\alpha]) = [\gamma] \) for the connecting map \( \delta : \check{H}^i(X, \mathcal{F}'') \rightarrow \check{H}^{i+1}(X, \mathcal{F}') \).

The main step in the proof is a general homological lemma. We use the convention that all complexes have ascending degree and that for any complex \( C^\bullet \) concentrated in degree \( \geq 0 \), \( \check{H}^0(C^\bullet) := \ker(d^0 : C^0 \rightarrow C^1) \) denotes the homology of \( C^\bullet \) upon augmenting by 0.

**Lemma E.2.2.** Consider left-exact sequences of complexes

\[
0 \rightarrow C^\bullet \rightarrow C^\bullet \rightarrow C''^\bullet
\]

and

\[
0 \rightarrow I^\bullet \rightarrow I^\bullet \rightarrow I''^\bullet
\]

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in an abelian category, with each complex concentrated in non-negative degree and exact in positive degree. Assume $I^•, I^•, I''^•$ are complexes of injectives.

For any commutative diagram of degree-0 kernels

$$
\begin{array}{ccc}
0 & \rightarrow & H^0(C') \\
\downarrow g' & & \downarrow g \\
0 & \rightarrow & H^0(I')
\end{array}
\rightarrow
\begin{array}{ccc}
H^0(C) & \rightarrow & H^0(C'') \\
\downarrow g'' & & \downarrow g'' \\
H^0(I) & \rightarrow & H^0(I'')
\end{array}
\quad (E.2.1)

in which the horizontal maps are those induced by the sequences of complexes above, there is a commutative diagram of complexes

$$
\begin{array}{ccc}
0 & \rightarrow & C'\quad \rightarrow \quad C \quad \rightarrow \quad C'' \quad \rightarrow \quad C'''
\downarrow f' & & \downarrow f & & \downarrow f'' \\
0 & \rightarrow & I' \quad \rightarrow \quad I \quad \rightarrow \quad I'' \quad \rightarrow \quad I'''
\end{array}
$$

inducing diagram (E.2.1).

In the special case when both given left-exact sequences of complexes are short exact, this lemma is in every reference on homological algebra. But to our surprise, the preceding version with just left-exactness does not seem to be in any reference on homological algebra.

Proof. We use the method of “chasing members” that permits us to make arguments in general abelian categories using notation as if working in the category of abelian groups; see [Mac Ch.VIII, §4] for a discussion of this technique and its application to prove the snake lemma in a general abelian category. This method is used here so that we may chase diagrams and present a proof that is actually comprehensible (to the author, as well as to the reader!).

It is a standard fact in homological algebra that we may construct $f' : C'\rightarrow I$ inducing $g'$ on $H^0$’s, but for the convenience of the reader, we now recall how this is proved, since the method involves arguments that will be adapted to construct the map of complexes $f$ (compatibly with $f'$) and then $f''$ (compatibly with $f'$). First, let $f^0(c') = g'(c') \in H^0(I') \subset I^0$ for $c' \in H^0(C') \subset C^0$. By injectivity of $I^0$, this extend this to a map $f^0 : C^0 \rightarrow I^0$. In general, for $n > 0$, we assume that $f^m$ has already been constructed for $m < n$ so that the diagram of complexes commutes up to degree $n - 1$, and we shall now inductively construct $f^m$ to satisfy $df^{m-1} = f^md$.

First, we define $f^m$ on $dC^m$ by setting $f^m(dc') := df^{m-1}(c')$ for $c' \in C^m$. Assuming for a moment that this is well-defined, we use injectivity of $I^m$ to extend this to a map $f^m : C^m \rightarrow I^m$ that satisfies the desired property by construction. To show well-definedness, we must check that if $c' \in C^{m-1}$ satisfies $dc' = 0$, then $df^{m-1}(c') = 0$. If $n = 1$ then by design $f^0$ carries $c'$ to a cocycle in $I^0$, so we get what we want. Next suppose $n > 1,$
so \( C' \) is exact at \( C^{n-1} \) by hypothesis. Hence, \( c' = db' \) for some \( b' \in C^{n-2} \). But then \( df^{n-1}(c') = df^{n-1}(db') = d^2 f^{n-2}(b') = 0 \), as desired, completing the construction of \( f' \).

Next we construct \( f \). We once again do this inductively. Let us first construct \( f^0 \). We need \( f^0 \) to satisfy \( f^0 j_C(c') = j_I f^0(c') \) for \( c' \in C^0 \) and \( f^0(c) = g(c) \) for \( c \in H^0(C) \). Assuming that these yield a well-defined map on the span of \( H^0(C) \) and \( j_C(C^0) \) inside \( C^0 \), by injectivity of \( I^0 \) it extends to a map \( f^0 : C^0 \to I^0 \) with the desired properties. For well-definedness, we need to check that if \( c' \in C^0 \) satisfies \( d j_C(c') = 0 \) (i.e., \( c := j_C(c') \in H^0(C) \)), so in fact \( c' \in H^0(C') \) because \( j_C dc' = d j_C(c') = 0 \) (and \( \ker j_C = 0 \)), then necessarily \( j_I f^0(c') = gj_C(c') \) inside \( I^0 \supset H^0(I) \). Since \( gj_C(c') = j_I g'(c') \) (as \((E.2.1)\) commutes) and \( g'(c') = f^0(c') \) inside \( I^0 \supset H^0(I) \) by design of \( f^0 \), the well-definedness is proved.

We now construct \( f^n \) for \( n > 0 \) inductively, assuming that the maps \( \{ f^m \}_{m<n} \) have been constructed satisfying all required commutativity properties. The two properties that we need \( f^n \) to satisfy are

- \( f^n(dc) = df^{n-1}(c) \) for \( c \in C^{n-1} \),
- \( f^n j_C(c') = j_I f^n(c') \) for \( c' \in C^n \).

We want to simply declare \( f^n \) to satisfy these two properties on the span of \( j_C(C^m) \) and \( d(C^{m-1}) \) inside \( C^n \). Assuming that this is well-defined, injectivity of \( I^n \) implies that \( f^n \) extends to a map \( C^n \to I^n \) that has the required properties. To check well-definedness, consider the following commutative diagram (in which the phantom map \( f^n \) is yet to be defined):

\[
\begin{array}{ccc}
C^{m-1} & \xrightarrow{d} & C^m \\
\downarrow{j_C} & & \downarrow{j_C} \\
C^{n-1} & \xrightarrow{d} & C^n \\
\downarrow{f^{n-1}} & & \downarrow{f^n} \\
I^{n-1} & \xrightarrow{d} & I^n \\
\downarrow{j_I} & & \downarrow{j_I} \\
I^{m-1} & \xrightarrow{d} & I^m
\end{array}
\]

(E.2.2)

We need to show for \( c \in C^{n-1} \), \( c' \in C^n \) satisfying \( dc = j_C(c') \) that \( df^{n-1}(c) = j_I f^n(c') \).

First, we claim \( c' = db' \) for some \( b' \in C^{n-1} \). By exactness of \( C' \) at \( C^m \) (as \( n > 0 \)), it suffices to show that \( dc' = 0 \). But \( j_C dc' = d j_C(c') = d^2 c = 0 \), since \( j_C^{n+1} \) is an inclusion, we see that \( dc' = 0 \) as desired. Our problem can now be recast in terms of \( b' \in C^{m-1} \) such that \( dc = d j_C(b') \): we claim that \( df^{n-1}(c) = j_I f^n db' \). Since \( c \) differs from \( j_C(b') \) by
a cocycle, and both the hypothesis and the desired conclusion are additive in \( c \) and \( b' \), it suffices to treat the cases when either \( c = j_C(b') \) or \( dc = 0 \) and \( b' = 0 \). That is, we want to show \( df^{n-1} j_C(b') = j_I f^m db' \) for \( b' \in C'^{n-1} \) and \( df^{n-1}(c) = 0 \) when \( dc = 0 \).

The first of these follows from the commutativity of (E.2.2). For the second, we have two subcases, depending on if \( n = 1 \) or \( n > 1 \) (since this impacts exactness properties at \( C'^{n-1} \)). If \( n = 1 \), then \( f^0 \) maps \( H^0(C) \) into \( H^0(I) \) (via \( g \)) by design, so we have what we need. If \( n > 1 \) then \( C^* \) is exact at \( C'^{n-1} \), so \( c = db \) for some \( b \in C'^{n-2} \). Therefore, \( df^{n-1}(c) = df^{n-1}(d(b)) = d^2 f^{n-2}(b) = 0 \). This completes the construction of \( f \).

Finally, we construct \( f'' \), once again proceeding inductively. The construction of \( f'' \) will differ that of \( f^0 \) because \( \ker \pi_C \neq 0 \) (in contrast with \( \ker j_C \)), though it will still use the commutativity of (E.2.1). Carrying out the usual procedure to try to make a well-defined construction on the span of \( \pi_C(C^0) \) and \( H^0(C'^0) \) inside \( C'^{n-1} \) (and then using that \( \pi'' \) is injective), we just have to show that if \( c \in C^0 \) satisfies \( c'' := \pi_C(c) \in H^0(C) \) then \( \pi_I(f''(c)) = g''(c'') \).

Since \( \pi_C(dc) = d\pi_C(c) = dc'' = 0 \) (as \( c'' \in H^0(C'^0) \) by assumption), we have \( dc = j_C(c') \) for some \( c' \in C'^n \) by left-exactness. Clearly \( j_C(c') = d j_C(c') = d^2 c = 0 \), so the vanishing of \( \ker j_C \) in all degrees yields that \( dc' = 0 \). By exactness of \( C'^* \) in all positive degrees, it follows that \( c' = db' \) for some \( b' \in C'^n \), so

\[
 dc = j_C(c') = j_C(db') = d j_C(b').
\]

In other words, \( b := c - j_C(b') \in H^0(C) \), so \( f^0(b) = g(b) \) by design of \( f^0 \). Thus,

\[
 \pi_I(f''(c)) = \pi_I(f''(j_C(b') + b)) = \pi_I(f''(j_C(b))) + \pi_I(g(b)) = \pi_I(j_I(f''(b))) + \pi_I(g(b))
\]

(as \( f \circ j_C = j_I \circ f' \) due to how \( f \) was built). But \( \pi_I \circ j_I = 0 \) and the commutativity of (E.2.1) gives that \( \pi_I(g(b)) = g''(\pi_C(b)) \). Hence, \( \pi_I(f''(c)) = g''(\pi_C(b)) \), so we want to show that \( g''(\pi_C(b)) = g''(c'') \). By definition, \( c'' = \pi_C(c) \). Since \( \pi_C(c) = \pi_C(j_C(b') + b) = \pi_C(b) \) (as \( \pi_C \circ j_C = 0 \)), the construction of \( f'' \) is complete.

Now suppose that \( n > 0 \) and that \( \{f^m\}_{m<n} \) has been constructed satisfying all desired properties. The requirements on \( f^m \) are:

- \( f^m(dc') = df^{m-1}(c') \) for \( c' \in C'^{m-1} \),
- \( f^m \pi_C(c) = \pi_I f^m(c) \) for \( c \in C^m \).

Once again we want to define \( f^m \) to satisfy these on the span of \( d(C'^{m-1}) \) and \( \pi_C(C^m) \) inside \( C'^{m} \). Assuming this is well-defined, we can use the injectivity of \( f^m \) to extend \( f^m \) to a map \( C'^{m} \to I^m \) (again denoted \( f^m \)) that meets our needs. It therefore only remains to check this well-definedness.

We need to show that if members \( c' \in C'^{m-1} \) and \( c \in C^m \) satisfy \( dc'' = \pi_C(c) \), then \( df^{m-1}(c') = \pi_I f^m(c) \). We first claim that \( c = db + j_C(c') \) for some \( b \in C'^{m-1} \), \( c' \in C'^m \). This
follows from an easy diagram chase in the following commutative diagram with exact rows and columns, using the fact that $\pi_C(c) \in d(C^{m-1})$:

\[
\begin{array}{c}
C^m \to C^{m+1} \to C^{m+2} \\
\downarrow \pi_C \quad \downarrow j_C \quad \downarrow \pi_C \\
C^{m-1} \to C^m \to C^{m+1} \\
\downarrow j_C \\
C^{m-1} \to C^m \to C^{m+1}
\end{array}
\]

Namely, $\pi_C(dc) = d\pi_C(c) \in d^2(C^{m-1}) = 0$, so $dc = j_C(b')$ for some $b' \in C^{m+1}$. Necessarily $db' = 0$ since $j_C(db') = dj_C(b') = d^2c = 0$ and $\ker j_C = 0$, so $b' = dc'$ for some $c' \in C^m$ since $C^m$ is exact in positive degrees. It follows that $dj_C(c') = j_C(dc') = j_C(b') = dc$, so $c - j_C(c') \in \ker d^n = \im d^{m-1}$, which is to say $c = db + j_C(c')$ for some $b \in C^{n-1}$ and $c' \in C^m$ as desired.

With $c$ now expressed in the asserted form $db + j_C(c')$, the hypothesis linking $c'$ and $c$ can be rewritten as $dc'' = \pi_C(db) = d\pi_C(b)$ (since $\pi_C \circ j_C = 0$) and the desired conclusion can be rewritten as

\[
df^{m-1}(c'') = \pi_I df^n(db + j_C(c')) = d\pi_I f^{n-1}(b) + \pi_I j_I df^n(c') = d\pi_I f^{n-1}(b)
\]

since $\pi_I \circ j_I = 0$. In other words, $c'' - \pi_C(b) \in \ker d_C^{n-1}$ and we want to show that $f^{m-1}(c'') - \pi_I f^{n-1}(b) \in \ker d_I^{n-1}$. But $\pi_I f^{n-1} = f^{m-1} \pi_C$, so the desired conclusion says $f^{m-1}(c'' - \pi_C(b)) \in \ker d_I^{n-1}$. Hence, it suffices to show that $f^{m-1}$ carries $\ker d_C^{n-1}$ into $\ker d_I^{n-1}$. If $n = 1$ then we use that $f^0$ carries $H^0(C^m)$ into $H^0(I^m)$ (via $g''$) by design. If $n > 1$ then by exactness of $C^m$ and $I^m$ in positive degrees it is the same to show that $f^{m-1}$ carries $\im d_C^{m-2}$ into $\im d_I^{m-2}$. But this is obvious because $f^{m-1}d = df^{m-2}$. This completes the construction of $f''$ and the proof of the lemma.

We can now complete the proof of Proposition E.2.1. By standard homological algebra, we may construct a short exact sequence of complexes

\[
0 \to \mathcal{J}'^\bullet \to \mathcal{J}^\bullet \to \mathcal{J}''^\bullet \to 0
\]

such that $\mathcal{J}'^\bullet, \mathcal{J}''^\bullet$ are injective resolutions of $\mathcal{F}'$, $\mathcal{F}$, $\mathcal{F}''$ respectively, and such that the maps on 0th homologies are the given maps between these sheaves. For the left-exact sequence of Čech complexes

\[
0 \to \mathcal{C}^\bullet(U, \mathcal{F}) \to \mathcal{C}^\bullet(U, \mathcal{F}') \to \mathcal{C}^\bullet(U, \mathcal{F}'')
\]

Lemma E.2.2 inserts it into a commutative diagram of complexes

\[
\begin{array}{c}
0 \to \mathcal{J}'^\bullet \to \mathcal{J}^\bullet \to \mathcal{J}''^\bullet \to 0 \\
\downarrow \\
0 \to \mathcal{C}^\bullet(U, \mathcal{F}) \to \mathcal{C}^\bullet(U, \mathcal{F}') \to \mathcal{C}^\bullet(U, \mathcal{F}'')
\end{array}
\]
This yields a commutative diagram

\[
\begin{array}{ccccccc}
0 & \longrightarrow & \Gamma(X, \mathcal{E}^\bullet(U, \mathcal{F})) & \longrightarrow & \Gamma(X, \mathcal{E}^\bullet(U, \mathcal{F}')) & \longrightarrow & \Gamma(X, \mathcal{E}^\bullet(U, \mathcal{F}'')) \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma(X, \mathcal{I}^\bullet) & \longrightarrow & \Gamma(X, \mathcal{I}^\bullet) & \longrightarrow & \Gamma(X, \mathcal{I}''^\bullet) & \longrightarrow & 0
\end{array}
\]  

(E.2.3)

(exactness at the right on the bottom due to the injectivity of the sheaves \(\mathcal{I}^m\)). The connecting map in derived functor cohomology is obtained by applying the snake lemma construction to the bottom sequence in this diagram, and the snake lemma construction between the top and bottom sequences is clearly compatible. Thus, Proposition E.2.1 follows immediately upon applying Proposition E.1.1.
Appendix F

Characteristic 0

Here we include a short discussion of how the proofs of our main results must be modified in characteristic 0 (that is, for p-adic and number fields). Our goal is not to give full proofs, but rather to indicate how these arguments generally go, and why they are typically much simpler. The proofs in the characteristic 0 case are in fact generally much easier. Let us first remark that we may replace fppf with étale cohomology throughout in characteristic 0. This actually doesn’t really change the content of any of the results, thanks to Remark 2.2.20. Let us also remark that the topological issues that appear when stating Theorem 1.2.4 don’t show up in characteristic 0. In fact, we have the following simpler statement.

**Theorem F.0.1.** Let $k$ be a local field of characteristic 0, $G$ an affine commutative $k$-group scheme of finite type. Then $H^1(k, G)$ and $H^1(k, \hat{G})$ are finite, and the cup product pairing

\[ H^1(k, G) \times H^1(k, \hat{G}) \to H^2(k, G_m) \xrightarrow{\text{inv}} \mathbb{Q}/\mathbb{Z} \]

is perfect.

The proof proceeds via dévissage from the finite case, which is part of the classical results of Poitou-Tate, [Ser1] §5, Prop. 14].

Let us also remark that the local Theorems 1.2.1–1.2.4 also hold for $\mathbb{R}$ if one replaces ordinary cohomology with Tate cohomology.

Now we discuss how the arguments typically proceed in order to prove our main results for an affine commutative $G$ of finite type. One typically applies a dévissage from the finite (necessarily étale!) case to reduce to the case when $G$ is connected (and necessarily smooth). Such $G$ is of the form $T \times G^n_a$ for some $k$-torus $T$ and some $n \geq 0$. We are therefore reduced to the cases when $G$ is a torus or $G_a$. The proofs for tori are typically the same as in the function field case, reducing to the case $T = G_m$ by applying Lemma 2.1.3(iv) or by using the injection $T \hookrightarrow R_{k'/k}(\hat{T}_{k'})$ for some finite separable extension $k'/k$ splitting $T$. 

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For $G_a$, on the other hand, the proofs are much simpler. Indeed, we have $H^i(k, G_a) = H^i(A, G_a) = 0$ for all $i > 0$, since $G_a$ is a quasi-coherent sheaf and the higher étale cohomology of quasi-coherent sheaves on any affine scheme vanishes. Further, we have $H^i_{	ext{ét}}(k, \hat{G}_a) = H^i_{\text{ét}}(A, \hat{G}_a) = 0$ for all $i$, since $\hat{G}_a = 0$ as an étale sheaf on these schemes, because $G_a$ has no nontrivial characters over a reduced ring. So the cohomology of $G_a$ and $\hat{G}_a$ (particularly $H^2(k, \hat{G}_a)$) is both simpler and easier to compute.

To prove Theorem 1.2.2 for $G = G_a$ over a characteristic-0 local field $k$, for example, one must show that $k_{\text{pro}} = 0$. But this is clear, since $k$ is divisible, hence has no nontrivial finite quotients (since any such would be finite and divisible). Similarly, if $k$ is a number field, then $k_{\text{pro}} = 0$, so in order to prove that the sequence in Theorem 1.2.8 is exact at $G_a(A)$, for example, one must show that $A_{\text{pro}} = 0$, and this once again follows from the fact that $A$ is divisible.
Bibliography


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