Here we present yet another proof of Quadratic Reciprocity. The ideas behind it, however, are anything but new. It is based on the following simple formula.

**Lemma:** If $p$ is an odd prime and $\zeta = \zeta_p$ is a primitive $p$th root of unity (in some field), then $p^* = (-1)^{\frac{p-1}{2}} p.$

**Proof:**

\[
(\zeta^j - \zeta^{-j})^2 = -(2 - (\zeta^{2j} + \zeta^{-2j})) = -(1 - \zeta^{2j})(1 - \zeta^{-2j}).
\]

Therefore, since $2x$ varies over all of $\mathbb{F}_p^*$ as $x$ does, we have
\[
\prod_{j=1}^{p-1} (\zeta^j - \zeta^{-j})^2 = (-1)^{\frac{p-1}{2}} \prod_{j=1}^{p-1} (1 - \zeta^j) = p^*.
\]

We can now finish our proof of Quadratic Reciprocity. Let $p, q$ be distinct odd primes, and let $\zeta = \zeta_p$ be a primitive $p$th root of unity in a suitable extension of $\mathbb{F}_q$. Let $\tau = \prod_{j=1}^{p-1} (\zeta^j - \zeta^{-j})$. By the above lemma, $\tau^q = (\tau^2)^{\frac{q-1}{2}} \tau = p^* \frac{q-1}{2} \tau = (p^*/q) \tau$. On the other hand, $\tau^q = \prod_{j=1}^{q-1} (\zeta^{qj} - \zeta^{-qj}) = (q/p)\tau$ by Gauss’ Lemma. Therefore $(p^*/q) = (q/p)$, which completes the proof. ■