1 Introduction

Borsuk’s problem basically asks: how many pieces are needed to partition a set into subsets of strictly smaller diameter?

Despite the simplicity of the problem statement, the problem itself has been difficult to tackle. Borsuk conjectured that \( n + 1 \) pieces were enough in \( n \) dimensions, and this was proven true for \( n = 2, 3 \) by the 1940’s. For nearly five decades afterwards, no more results, positive or negative, were obtained. It was not until 1994 before the conjecture was proven false for large enough \( n \); since then the lowest dimension for which the conjecture is false has dropped, though it’s not even known whether the conjecture is true in dimension 4 or not.

What’s particularly fascinating is that the known counterexamples to Borsuk’s conjecture can always be taken to be finite sets. This suggests that the problem, despite being originally posed as problem about Euclidean geometry, is actually a problem in combinatorics. In particular, finding counterexamples to Borsuk’s conjecture can be reduced to finding (finite) graphs with certain properties.

In this talk I present a counterexample in 65 dimensions found by Bondarenko in 2013 which is rather easy to construct. On a personal note, I stumbled upon the topic of Borsuk’s conjecture and Bondarenko’s paper shortly after it was published in 2013 while I was still a first year in undergrad. I was pleasantly surprised to find that I could understand a math research paper presenting new results, and that the linear algebra used in the paper was actually useful. The paper was probably the first math paper I was able to fully understand.

2 Geometry of Borsuk’s Problem

**Definition:** Let \( E \subset \mathbb{R}^n \) be a bounded subset. The diameter of \( E \) is defined as

\[
\text{diam}(E) = \sup_{x, y \in E} |x - y|.
\]
Notice that the diameter of a singleton set is zero. By convention, we can let \( \text{diam}(\emptyset) = -\infty \).

Define the Borsuk number \( b(E) \) by

\[
b(E) = \min \left\{ n \in \mathbb{N} \mid \text{there exist } E_1, \ldots, E_n \text{ s.t. } E = \bigcup_{i=1}^{n} E_i \text{ and diam}(E_i) < \text{diam}(E) \text{ for all } i \right\},
\]

and let

\[
b(n) = \sup_{E \subset \mathbb{R}^n} b(E),
\]

with the supremum taken over all bounded subsets of \( \mathbb{R}^n \) with positive diameter.

Recall the following theorem from topology:

**Theorem** (Borsuk-Ulam): Let \( f : S^n \to \mathbb{R}^n \) be continuous. Then there exists \( x \in S^n \) such that \( f(x) = f(-x) \).

**Corollary**: \( b(S^{n-1}) \geq n + 1 \).

**Proof**: Suppose \( S^{n-1} = \bigcup_{i=1}^{n} E_i \), and take the \( E_i \) to be closed. Since \( \text{diam}(S^{n-1}) = 2 \), we wish to show \( \text{diam}(E_i) = 2 \) for some \( i \). Let \( f : S^{n-1} \to \mathbb{R}^{n-1} \) be defined by

\[
f(x) = (f_1(x), \ldots, f_{n-1}(x)), \quad f_i(x) = \text{diam}(x, E_i).
\]

Then \( f \) is continuous. Notice that if \( f_i(x) = 0 \) for some \( i \), then \( x \in E_i \), while if \( f_i(x) \neq 0 \) for all \( 1 \leq i \leq n-1 \), then \( x \not\in E_i \) for any \( 1 \leq i \leq n-1 \), i.e. \( x \in E_n \). By the Borsuk-Ulam theorem, there exists some \( x \in S^{n-1} \) such that \( f(x) = f(-x) \). Thus for both \( x \) and \( -x \), either \( f_i(\pm x) = 0 \) for some \( 1 \leq i \leq n-1 \), in which case \( x \) and \( -x \) are both in \( E_i \), or \( f_i(\pm x) \neq 0 \) for all \( 1 \leq i \leq n-1 \), in which case \( x \) and \( -x \) are both in \( E_n \). In any case, there is some \( 1 \leq i \leq n \) for which \( x \) and \( -x \) are both in \( E_i \), and since \( |x - (-x)| = 2 \), it follows that \( \text{diam}(E_i) = 2 \) for this particular \( i \). \hfill \( \blacksquare \)

Notice in addition that \( b(S^{n-1}) \leq n + 1 \). Indeed, if we take an \( n \)-simplex (consisting of \( n + 1 \) vertices in \( \mathbb{R}^n \) which are equidistant from each other) centered at the origin, project the \( n - 1 \)-dimensional faces from the origin onto \( S^{n-1} \), and partition \( S^{n-1} \) into \( n + 1 \) subsets corresponding to the \( n + 1 \) vertices, then it can be verified that no subset’s closure contains a pair of antipodal points, and hence all subsets have strictly smaller diameter. Thus, \( b(S^{n-1}) = n + 1 \). This motivated Borsuk to conjecture that this was true for subsets of \( \mathbb{R}^n \) in general:

**Conjecture** (Borsuk): \( b(n) = n + 1 \).

The above example shows that \( b(n) \geq n + 1 \) by considering the sphere \( S^{n-1} \) (this can also be obtained more simply by considering the vertices of the \( n \)-simplex). The conjecture was proven true in the following cases\footnote{That is, if we redefine \( b(n) \) to only consider subsets of \( \mathbb{R}^n \) which are smooth convex bodies, then the theorem would be true. Similarly for the next two cases.}:

- \( n = 2 \) (Borsuk ’32)
- \( n = 3 \) (Perkal, Eggleston ’47,’55)
- For all smooth convex bodies\footnote{Recall the following theorem from topology:}

\( b(n) = n + 1 \).
• for all centrally symmetric sets (Riesling ’71)
• for all bodies of revolution (Dekster ’95)

Aside from \( n = 2, 3 \) and special classes of sets, no other results (either positive or negative) were obtained for several decades. In 1994, Kahn and Kalai proved the conjecture to be false for \( n \geq 2014 \), and moreover that for large \( n \) we have \( b(n) \geq (1.2)^{\sqrt{n}} \), i.e. that \( b(n) \) is not even of polynomial growth! In 2013, Bondarenko\(^1\) published a short proof which showed the conjecture was false for \( n \geq 65 \), and in 2014 Jenrich\(^2\) modified Bondarenko’s proof slightly to disprove the conjecture for \( n = 64 \) as well. As of January 2017, this is the lowest dimension where Borsuk’s conjecture is known to be false.

We will discuss Bondarenko’s 2013 paper disproving the conjecture for \( n \geq 65 \).

3 Strongly regular graphs

Definition: Let \( v, k, \lambda, \mu \) be nonnegative integers. A strongly regular graph with parameters \((v, k, \lambda, \mu)\) is a graph with the following properties:

- There are \( v \) vertices.
- Each vertex has \( k \) edges.
- Each pair of adjacent vertices have \( \lambda \) common neighbors.
- Each pair of nonadjacent vertices have \( \mu \) common neighbors.

Definition: Given a graph, the adjacency matrix \( A = (a_{ij}) \) is a \( v \times v \) matrix (where \( v \) is the number of vertices) where \( a_{ij} = 1 \) if \( v_i \sim v_j \), and is 0 otherwise. Here \( v_i \) denotes the \( i \)th vertex, and \( v_i \sim v_j \) if and only if vertices \( v_i \) and \( v_j \) are adjacent.

For ease of notation, we will write \( v_i \not\sim v_j \) to mean that vertices \( i \) and \( j \) are not adjacent and are also distinct.

For simple, undirected graphs, we have \( A^T = A \), and \( A \) has zeroes on its diagonal, so in particular \( \text{Tr}(A) = 0 \).

Let \( J_{m \times n} \) denote the \( m \times n \) matrix whose entries are all 1 (with the subscript dropped if the dimension is understood). The adjacency matrix \( A \) of a strongly regular graph with parameters \((v, k, \lambda, \mu)\) has the following properties:

- \( AJ_{v \times 1} = kJ_{v \times 1} \)
- \( AJ_{v \times v} = kJ_{v \times v} = J_{v \times v} A \)
- \( A^2 = A^T A = \begin{cases} k & i = j \\ \lambda & v_i \sim v_j = kI + \lambda A + \mu(J_{v \times v} - I - A) = (k - \mu)I + (\lambda - \mu)A + \mu J_{v \times v} \end{cases} \)

To prove the first property, notice that the entries of \( AJ_{v \times 1} \) are just the sums of the entries in the rows of \( A \), and each row of \( A \) contains \( k \)’s (and the rest 0’s) since each vertex is adjacent to \( k \) other vertices. The second property is proven similarly, noting that \( A \) and \( J_{v \times v} \) commute since each column of \( A \) also has \( k \)’s as its nonzero entries. To prove the third property, notice that
the \((i,j)\) entry of \(A^TA\) is the dot product of columns \(i\) and \(j\). Since the nonzero entries of each column are 1’s, the dot product just counts for how many rows the columns both have ones. This is equivalent to counting the number of vertices adjacent to both vertex \(i\) and \(j\). This is \(k\) if \(v_i\) and \(v_j\) are the same; \(\lambda\) if \(v_i\) and \(v_j\) are adjacent, and \(\mu\) if not. The next equality follows since \(I\) is the “indicator matrix” of the condition \(i = j\) (i.e. the entry is 1 if \(i = j\) and 0 otherwise); \(A\) is the indicator matrix of \(v_i \sim v_j\), and \(J - I - A\) covers the remaining case of \(v_i \not\sim v_j\).

We can rewrite the last property as follows:

\[ A^2 + (\mu - \lambda)A + (\mu - k)I = \mu J_{v \times v}. \]

If we let \(r\) and \(s\) be the roots of the quadratic \(x^2 + (\mu - \lambda)x + (\mu - k)\), then the left-hand side can be rewritten as \((A - rI)(A - sI)\). Since

\[ (A - kI)J_{v \times v} = AJ_{v \times v} - kJ_{v \times v} = 0 \]

by the second property above, we have

\[ (A - kI)(A - rI)(A - sI) = (A - kI)\mu J_{v \times v} = 0. \]

This means that the minimal polynomial of \(A\) divides \((A - kI)(A - rI)(A - sI)\), and hence the possible eigenvalues of \(A\) are \(k\), \(r\), and \(s\). In particular, there can be at most three distinct eigenvalues!

It is not hard to see that the multiplicity of \(k\) is 1 (with \(J_{v \times 1}\) being the corresponding eigenvector), since \(A\) only has \(k\) 1’s in each entry. If we let \(f\) and \(g\) be the multiplicities of \(r\) and \(s\), then \(f\) and \(g\) must satisfy the equations

\[
\begin{align*}
1 + f + g &= v \\
k + rf + sg &= 0.
\end{align*}
\]

The first equation follows from the fact that \(A\) is symmetric and hence has a full set of eigenvectors by the Spectral Theorem, i.e. it has \(v\) eigenvalues in total. The second equation follows because the trace of \(A\) is zero. Thus, \(f\) and \(g\) are uniquely determined by the values of \(v\), \(k\), \(r\) and \(s\) (which are themselves determined by \(\mu\), \(\lambda\), and \(k\)) if \(r \neq s\); if \(r = s\) then the multiplicity of \(r = s\) is automatically \(v - 1\). Of course, \(f\) and \(g\) must be integers, which puts severe restrictions on which tuples can be parameters for a strongly regular graph.

In addition, since \(A^2\) can be written as a linear combination of \(A\), \(I\), and \(J\), and \(\alpha I\) and \(\beta I\) all commute with each other, it follows that the linear span \(\text{span}(A, I, J)\) is actually closed under multiplication; in fact, it is a commutative subalgebra of the algebra of \(v \times v\) matrices which also lies in the subspace of the symmetric matrices. In particular, if \(Z \in \text{span}(A, I, J)\), then \(Z^T Z = Z^2 = \alpha A + \beta I + \gamma J\) for some \(\alpha, \beta, \gamma\), and hence if \(Z_i\) denotes the \(i\)th column of \(Z\), then \(Z_i \cdot Z_j\) is the \((i,j)\) entry of \(\alpha A + \beta I + \gamma J\), i.e.

\[
Z_i \cdot Z_j = \begin{cases} 
\beta + \gamma & i = j \\
\alpha + \gamma & v_i \sim v_j \\
\gamma & v_i \not\sim v_j 
\end{cases}
\]

The upshot is this: we can look at a Euclidean representation of \(Z\), i.e. view the columns of \(Z\) as vectors in \(\mathbb{R}^n\) where \(n = \text{rank}(Z)\). Then there is a bijective correspondence between these vectors and the vertices of the strongly regular graph. The above calculation shows that the length of each
is constant (it is $\beta + \gamma$, independent of $i$), so the distance between two vectors depends only on their dot product. For different vectors the dot product can achieve one of two values, and furthermore the choice of value depends on whether or not the corresponding vertices are adjacent or not. So, for example, if the longer distance is attained by columns whose corresponding vertices are not adjacent, then if we can show that sufficiently large subgraphs contain vertices that are not adjacent, then it follows that sufficiently large subsets of the vectors attain the diameter of the collection of vectors, thus requiring many subsets to partition the collection into subsets of smaller diameter.

4 Bondarenko’s proof

We use the following result from graph theory:

**Theorem**: There exists a strongly regular graph with parameters $(416, 100, 36, 20)$. Moreover, the clique size of this graph is at most 5 (i.e. any collection of 6 vertices in this graph must contain a pair of nonadjacent vertices).

This graph is known as the $G_{2}(4)$ graph. It can be explicitly constructed\(^2\) using $PG(2, 16)$, the projective plane on the field of order 16.

For this graph, the adjacency matrix $A$ satisfies $A^{2} - 16A - 80I = 20J$, and hence the eigenvalues of $A$ are 100 (of multiplicity 1), as well as the roots $r, s$ of $x^{2} - 16x - 80$, i.e. $r = 20$ and $s = -4$. They have multiplicities $f = 65$ and $g = 350$, respectively.

We now prove Bondarenko’s result:

**Theorem** (Bondarenko ’13): There exists $E \subset \mathbb{R}^{65}$ such that $b(E) \geq 84$, and hence in particular Borsuk’s conjecture is false for $n = 65$. Furthermore, we can take $E$ to be a subset of $S^{64}$ with $\text{diam}(E) > \sqrt{2}$.

**Proof**: Let $A$ be the adjacency matrix of $G_{2}(4)$, and let

$$Z = (A - sI)(I - \frac{1}{v}J)$$

with $s = -4$ and $v = 416$ as above. Notice that the factors $A - sI$ and $I - \frac{1}{v}J$ commute, and $A - sI$ kills all eigenvectors of $A$ of eigenvalue $s$, while $I - \frac{1}{v}J$ kills the vector $J_{v \times 1}$, which is an eigenvector of $A$ of eigenvalue $r$. It follows that the only eigenvalues of $A$ not killed by $Z$ are the eigenvectors of eigenvalue $r$, and since the eigenvectors of $A$ form a basis of $\mathbb{R}^{v}$, it follows that $\text{rank}(Z) \leq g = 65$. Thus, if we let $Z_{i}$ be the columns of $Z$, then $\{Z_{i}\}$ span a 65-dimensional subspace of $\mathbb{R}^{65}$, and hence we can take the $Z_{i}$ to lie in $\mathbb{R}^{65}$.

Thus, let $E = \{Z_{i}\} \subset \mathbb{R}^{65}$. I claim that $b(E) \geq 84$. Indeed, we have

$$Z = (A - sI)(I - \frac{1}{v}J) = (A + 4I)(I - \frac{1}{416}J)$$

$$= A + 4I - \frac{1}{416}AJ - \frac{4}{416}J$$

$$= A + 4I - \frac{100}{416}J - \frac{4}{416}J = A + 4I - \frac{1}{4}J$$
and hence
\[ Z^T Z = Z^2 = (A + 4I - \frac{1}{4}J)^2 = A^2 + 8A + 16I - \frac{1}{2}AJ - 2J + \frac{1}{16}J^2 \]
\[ = (16A + 80I + 20J) + 8A + 16I - \frac{1}{2}(100J) - 2J + \frac{1}{16}(416J) \]
\[ = 24A + 96I - 6J. \]

This implies that
\[ Z_i \cdot Z_j = \begin{cases} 96 - 6 & i = j \\ 24 - 6 & v_i \sim v_j \\ -6 & v_i \not\sim v_j \end{cases} \]

Thus, \( E \) is a two-distance set, where the distance between two distinct elements of \( E \) is the shorter distance if the corresponding vertices are adjacent, and longer otherwise. In particular, the diameter of \( E \) is attained between any two elements whose corresponding vertices are not adjacent. Hence, any subset of \( E \) which has diameter strictly less than that of \( E \) must have the corresponding vertices all be adjacent to each other; i.e. the subset corresponds to a clique in the graph. The fact that the clique size of the graph is at most 5 implies that any diameter-reducing subset of \( E \) must have size at most 5, and hence if \( E \) is partitioned into subsets all of which are diameter-reducing, then each such subset has size at most 5. By the Pigeonhole Principle, at least \( \lceil \frac{416}{5} \rceil = 84 \) subsets are thus needed.

Finally, notice that scaling \( E \) preserves its Borsuk number, so we may scale \( E \) to lie on \( S^{64} \). In that case, the diameter of \( E \) is at least \( \sqrt{2} \), since the dot product of elements corresponding to nonadjacent vertices is negative.

While this example disproves Borsuk’s conjecture only for \( n = 65 \), the same is true for any \( n \geq 65 \). The crucial lemma is the following:

**Lemma:** Let \( E \subset S^{n-1} \) with \( \text{diam}(E) > \sqrt{2} \). Then there exists \( E' \subset S^n \) such that \( b(E') \geq b(E) + 1 \) and \( \text{diam}(E') > \sqrt{2} \).

This disproves the Borsuk’s conjecture for \( n \geq 65 \) by induction.

**Proof:** We may assume \( E \) is closed without loss of generality. Let \( x > 0 \) satisfy
\[ \sqrt{1 - x^2} \text{diam}(E) = \sqrt{(1 + x)^2 + (1 - x^2)} = \sqrt{2x + 2}. \]

Notice that such \( x \) exist by the Intermediate Value Theorem since at \( x = 0 \) we have \( LHS = \text{diam}(E) > \sqrt{2} = RHS \), while at \( x = 1 \) we have \( LHS = 0 < RHS \). If we let \( \{e_1, \ldots, e_{n+1}\} \) be an orthonormal basis for \( \mathbb{R}^{n+1} \), and view \( S^{n-1} \subset \mathbb{R}^n = \text{span}(e_1, \ldots, e_n) \subset \mathbb{R}^{n+1} \), let
\[ E' = (\sqrt{1 - x^2}E - xe_{n+1}) \cup \{e_{n+1}\}. \]

That is, \( E' \) consists of the vertices of \( E \) scaled by \( \sqrt{1 - x^2} \) and translated \( x \) units to the left in the \( n+1 \)-th direction, along with \( e_{n+1} \). Then \( E' \subset S^n \), and if we let \( E'' = \sqrt{1 - x^2}E - xe_{n+1} \), we see that \( \text{diam}(E'') = \sqrt{1 - x^2} \text{diam}(E) \), while the distance between any vertex of \( E' \) and \( e_{n+1} \) is
\[ \sqrt{(1 + x)^2 + (\sqrt{1 - x^2})^2} = \sqrt{2x + 2} = \sqrt{1 - x^2} \text{diam}(E). \]
It follows that
\[ \text{diam}(E') = \text{diam}(E'') = \sqrt{1 - x^2} \text{diam}(E) = \sqrt{(1 + x)^2 + (1 - x^2)} = \sqrt{2x + 2} > \sqrt{2}. \]

\(^2\)Of course, one can obtain \( x \) by solving the corresponding quadratic equation.
In particular, the diameter of $E'$ is attained between $e_{n+1}$ and any vertex of $E''$. It follows that if $E'$ is partitioned into subsets of strictly smaller diameter, then the subset containing $e_{n+1}$ cannot contain any other vertex. Furthermore, the remaining subsets not containing $e_{n+1}$ form a partition of $E''$ where each subset has diameter less than $\text{diam}(E') = \text{diam}(E'')$. This corresponds to a diameter-reducing partition of $E$, which requires at least $b(E)$ subsets. It follows that the number of subsets needed for a diameter-reducing partition of $E'$ is at least $b(E) + 1$, i.e. $b(E') \geq b(E) + 1$, as desired.

References

