Cardinality and The Nature of Infinity
Recap from Last Time
Functions

- A **function** $f$ is a mapping such that every value in $A$ is associated with a single value in $B$.
  - For every $a \in A$, there exists some $b \in B$ with $f(a) = b$.
  - If $f(a) = b_0$ and $f(a) = b_1$, then $b_0 = b_1$.
- If $f$ is a function from $A$ to $B$, we call $A$ the **domain** of $f$ and $B$ the **codomain** of $f$.
- We denote that $f$ is a function from $A$ to $B$ by writing $f : A \rightarrow B$
Injective Functions

- A function $f : A \rightarrow B$ is called **injective** (or **one-to-one**; formal: *if* $f(x_0) = f(x_1)$, then $x_0 = x_1$) iff each element of the codomain has at most one element of the domain associated with it.
  - A function with this property is called an **injection**.
  - Formally:
    
    \[
    \text{If } f(x_0) = f(x_1), \text{ then } x_0 = x_1
    \]
  - An intuition: injective functions label the objects from $A$ using names from $B$. 

Surjective Functions

• A function $f : A \to B$ is called **surjective** (or **onto**) iff each element of the codomain has at least one element of the domain associated with it.
  
  • A function with this property is called a **surjection**.

• Formally:

  For any $b \in B$, there exists at least one $a \in A$ such that $f(a) = b$.

• An intuition: surjective functions cover every element of $B$ with at least one element of $A$. 
Bijections

- A function that associates each element of the codomain with a unique element of the domain is called **bijective**.
  - Such a function is a **bijection**.
- Formally, a bijection is a function that is both **injective** and **surjective**.
- A bijection is a one-to-one correspondence between two sets.
Comparing Cardinalities

- The relationships between set cardinalities are defined in terms of functions between those sets.

- \(|S| = |T|\) is defined using bijections.

\(|S| = |T| \text{ iff there is a bijection } f : S \to T\)
The Nature of Infinity
Infinite Cardinalities

\[ \mathbb{N} \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad \ldots \]

\[ \mathbb{Z} \quad \ldots \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad \ldots \]
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### Infinite Cardinalities

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Theorem: \(|\mathbb{Z}| = |\mathbb{N}|\).

Proof: We exhibit a bijection from \(\mathbb{Z}\) to \(\mathbb{N}\). Let \(f: \mathbb{Z} \to \mathbb{N}\) be defined as follows:

First, we prove this is a legal function from \(\mathbb{Z}\) to \(\mathbb{N}\). Consider any \(x \in \mathbb{Z}\). Note that if \(x \geq 0\), then \(f(x) = 2x\) is a natural number. Otherwise, if \(x < 0\), then \(f(x) = -2x + 1 = 2(-x) + 1\). Since \(x\) is a negative integer, \(-x\) is a positive integer. Thus \(2(-x) + 1\) is a positive integer, which is a natural number. Thus in all cases \(f(x)\) is a natural number.

Next, we prove \(f\) is injective. Suppose that \(f(x) = f(y)\). We will prove that \(x = y\). Note that, by construction, \(f(z)\) is even iff \(z\) is even. Since \(f(x) = f(y)\), we know that \(x\) and \(y\) must have the same parity. We consider two cases:

Case 1: \(x\) and \(y\) are even. Then \(f(x) = 2x\) and \(f(y) = 2y\). Since \(f(x) = f(y)\), we have \(2x = 2y\). Thus \(x = y\).

Case 2: \(x\) and \(y\) are odd. Then \(f(x) = -2x - 1\) and \(f(y) = -2y - 1\). Since \(f(x) = f(y)\), we have \(-2x - 1 = -2y - 1\), so \(x = y\).

Finally, we prove \(f\) is surjective. Consider any \(n \in \mathbb{N}\). We will prove that there is some \(x \in \mathbb{Z}\) such that \(f(x) = n\). We consider two cases:

Case 1: \(n\) is even. Then \(n/2\) is a nonnegative integer. Moreover, \(f(n/2) = 2(n/2) = n\).

Case 2: \(n\) is odd. Then \(-((n+1)/2)\) is a negative integer. Moreover, \(f(-((n+1)/2)) = -2(-((n+1)/2)) - 1 = n + 1 - 1 = n\).

Since \(f\) is injective and surjective, it is a bijection. Thus \(|\mathbb{Z}| = |\mathbb{N}|\). ■
Theorem: $|\mathbb{Z}| = |\mathbb{N}|$.

Proof: We exhibit a bijection from $\mathbb{Z}$ to $\mathbb{N}$. 

Let $f: \mathbb{Z} \rightarrow \mathbb{N}$ be defined as follows:

First, we prove this is a legal function from $\mathbb{Z}$ to $\mathbb{N}$.

Consider any $x \in \mathbb{Z}$. Note that if $x \geq 0$, then $f(x) = 2x$. Since in this case $x$ is a positive integer, $2x$ is a natural number. Otherwise, if $x < 0$, then $f(x) = -2x + 1 = 2(-x) + 1$. Since $x$ is a negative integer, $-x$ is a positive integer. Thus $2(-x) + 1$ is a positive integer, which is a natural number. Thus in all cases $f(x)$ is a natural number.

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Case 2: $n$ is odd. Then $-(n+1)/2$ is a negative integer. Moreover, $f(-(n+1)/2) = -2(-(n+1)/2) - 1 = n + 1 - 1 = n$. Since $f$ is injective and surjective, it is a bijection. Thus $|\mathbb{Z}| = |\mathbb{N}|$. ■
Theorem: $|\mathbb{Z}| = |\mathbb{N}|$.

Proof: We exhibit a bijection from $\mathbb{Z}$ to $\mathbb{N}$. Let $f: \mathbb{Z} \to \mathbb{N}$ be defined as follows:

$$f(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ -2x - 1 & \text{otherwise} \end{cases}$$

First, we prove this is a legal function from $\mathbb{Z}$ to $\mathbb{N}$. Consider any $x \in \mathbb{Z}$. Note that if $x \geq 0$, then $f(x) = 2x$. Since in this case $x$ is nonnegative, $2x$ is a natural number. Thus $f(x) \in \mathbb{N}$. Otherwise, $x < 0$, so $f(x) = -2x - 1 = 2(-x) - 1$. Since $x < 0$, we have $-x > 0$, so $-x \geq 1$. Then $f(x) = 2(-x) - 1 \geq 2 - 1 = 1$. Thus $f(x)$ is a positive integer, so $f(x) \in \mathbb{N}$. In either case $f(x) \in \mathbb{N}$, so $f: \mathbb{Z} \to \mathbb{N}$. 

Finally, we prove $f$ is injective. Suppose that $f(x) = f(y)$. We will prove that $x = y$. Note that, by construction, $f(z)$ is even iff $z$ is even. Since $f(x) = f(y)$, we know that $x$ and $y$ must have the same parity. We consider two cases:

Case 1: $x$ and $y$ are even. Then $f(x) = 2x$ and $f(y) = 2y$. Since $f(x) = f(y)$, we have $2x = 2y$. Thus $x = y$.

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Finally, we prove $f$ is surjective. Consider any $n \in \mathbb{N}$. We will prove that there is some $x \in \mathbb{Z}$ such that $f(x) = n$. We consider two cases:

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- **Case 2:** $x$ and $y$ are negative. Then $f(x) = -2x - 1$ and $f(y) = -2y - 1$. Since $f(x) = f(y)$, we have $-2x - 1 = -2y - 1$, so $x = y$.

Finally, we prove $f$ is surjective. Consider any $n \in \mathbb{N}$. We will prove that there is some $x \in \mathbb{Z}$ such that $f(x) = n$. We consider two cases:

- **Case 1:** $n$ is even.

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Theorem: $|\mathbb{Z}| = |\mathbb{N}|$.

Proof: We exhibit a bijection from $\mathbb{Z}$ to $\mathbb{N}$. Let $f : \mathbb{Z} \to \mathbb{N}$ be defined as follows:

$$f(x)=\begin{cases} 2x & \text{if} \ x \geq 0 \\ -2x - 1 & \text{otherwise} \end{cases}$$

First, we prove this is a legal function from $\mathbb{Z}$ to $\mathbb{N}$. Consider any $x \in \mathbb{Z}$. Note that if $x \geq 0$, then $f(x) = 2x$. Since in this case $x$ is nonnegative, $2x$ is a natural number. Thus $f(x) \in \mathbb{N}$. Otherwise, $x < 0$, so $f(x) = -2x - 1 = 2(-x) - 1$. Since $x < 0$, we have $-x > 0$, so $-x \geq 1$. Then $f(x) = 2(-x) - 1 \geq 2 - 1 = 1$. Thus $f(x)$ is a positive integer, so $f(x) \in \mathbb{N}$. In either case $f(x) \in \mathbb{N}$, so $f : \mathbb{Z} \to \mathbb{N}$.

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**Proof:** We exhibit a bijection from \(\mathbb{Z}\) to \(\mathbb{N}\). Let \(f: \mathbb{Z} \rightarrow \mathbb{N}\) be defined as follows:

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Since $f$ is injective and surjective, it is a bijection.
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Since $f$ is injective and surjective, it is a bijection. Thus $|\mathbb{Z}| = |\mathbb{N}|$. ■
Why This Matters

• Note the thought process from this proof:
  • Start by drawing a picture to get an intuition.
  • Convert the picture into a mathematical object (here, a function).
  • Prove the object has the desired properties.

• This technique is at the heart of mathematics.

• We will use it extensively throughout the rest of this lecture.
Cantor's Theorem Revisited
Comparing Cardinalities

- We define $|S| \leq |T|$ as follows:
  
  $|S| \leq |T|$ iff there is an injection $f : S \rightarrow T$
Comparing Cardinalities

• Formally, we define $<$ on cardinalities as
\[ |S| < |T| \text{ iff } |S| \leq |T| \text{ and } |S| \neq |T| \]

• In other words:
  • There is an injection from $S$ to $T$.
  • There is no bijection between $S$ and $T$. 

\[ 
\begin{array}{c}
\text{\textbullet} \\
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\rightarrow 
\begin{array}{c}
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\]
Cantor's Theorem

- **Cantor's Theorem** states that
  \[ |S| < |\mathcal{P}(S)| \]

- This is how we concluded that there are more problems to solve than programs to solve them.

- We informally sketched a proof of this in the first lecture.

- Let's now formally prove Cantor's Theorem.
Lemma: For any set $S$, $|S| \leq |\wp(S)|$.  

Proof: Consider any set $S$. We show that there is an injection $f : S \to (\wp(S))$. Define $f(x) = \{x\}$. To see that $f(x)$ is a legal function from $S$ to $\wp(S)$, consider any $x \in S$. Then $\{x\} \subseteq S$, so $\{x\} \in (\wp(S))$. This means that $f(x) \in (\wp(S))$, so $f$ is a valid function from $S$ to $\wp(S)$. To see that $f$ is injective, consider any $x_0$ and $x_1$ such that $f(x_0) = f(x_1)$. We prove that $x_0 = x_1$. To see this, note that if $f(x_0) = f(x_1)$, then $\{x_0\} = \{x_1\}$. Since two sets are equal iff their elements are equal, this means that $x_0 = x_1$ as required. Thus $f$ is an injection from $S$ to $\wp(S)$, so $|S| \leq |\wp(S)|$. ■
Lemma: For any set $S$, $|S| \leq |\mathcal{P}(S)|$.

Proof: Consider any set $S$. 

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Lemma: For any set $S$, $|S| \leq |\mathcal{P}(S)|$.

Proof: Consider any set $S$. We show that there is an injection $f : S \to \mathcal{P}(S)$. 

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Lemma: For any set $S$, $|S| \leq |\mathcal{P}(S)|$.

Proof: Consider any set $S$. We show that there is an injection $f : S \rightarrow \mathcal{P}(S)$. 

![Diagram of geometric shapes]
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The Key Step

- We now need to show that
  \[ \text{For any set } S, \ |S| \neq |\wp(S)| \]
- By definition, \( |S| = |\wp(S)| \) iff there exists a bijection \( f : S \to \wp(S) \).
- This means that
  \[ |S| \neq |\wp(S)| \text{ iff there is no bijection } f : S \to \wp(S) \]
- Prove this by contradiction:
  - Assume that there is a bijection \( f : S \to \wp(S) \).
  - Derive a contradiction by showing that \( f \) is not a bijection.
$X_0$

$X_1$

$X_2$

$X_3$

$X_4$

$X_5$

...
\[ x_0 \rightarrow \{ x_0, x_2, x_4, \ldots \} \]
\[ x_1 \rightarrow \{ x_0, x_3, x_4, \ldots \} \]
\[ x_2 \rightarrow \{ x_4, \ldots \} \]
\[ x_3 \rightarrow \{ x_1, x_4, \ldots \} \]
\[ x_4 \rightarrow \{ x_0, x_5, \ldots \} \]
\[ x_5 \rightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \ldots \} \]
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$x_0 \xrightarrow{} \{ x_0, x_2, x_4, \ldots \}$

$x_1 \xrightarrow{} \{ x_0, x_3, x_4, \ldots \}$

$x_2 \xrightarrow{} \{ x_4, \ldots \}$

$x_3 \xrightarrow{} \{ x_1, x_4, \ldots \}$

$x_4 \xrightarrow{} \{ x_0, x_5, \ldots \}$

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Flip all Y’s to N’s and vice-versa to get a new set.
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Formalizing the Diagonal Argument

- Proof by contradiction; assume there is a bijection \( f : S \rightarrow \mathcal{P}(S) \).
- The diagonal argument shows that \( f \) cannot be a bijection:
  - Construct the table given the bijection \( f \).
  - Construct the complemented diagonal.
  - Show that the complemented diagonal cannot appear anywhere in the table.
  - Conclude, therefore, that \( f \) is not a bijection.

For finite sets this is fine, but what if the set is infinitely large?
Proof by contradiction; assume there is a bijection \( f : S \rightarrow \mathcal{P}(S) \).

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The **diagonal set** $D$ is the set

$$D = \{ \ x \in S \mid x \notin f(x) \ \}$$

There is no longer a dependence on the existence of the two-dimensional table.
Lemma: For any set $S$, $|S| \neq |\wp(S)|$. 

Proof: By contradiction; assume that there exists a set $S$ such that $|S| = |\wp(S)|$. This means that there exists a bijection $f: S \rightarrow \wp(S)$. Consider the set $D = \{ x \in S | x \notin f(x) \}$. Note that $D \subseteq S$, since by construction every $x \in D$ satisfies $x \in S$.

Since $f$ is a bijection, it is surjective, so there must be some $y \in S$ such that $f(y) = D$. Now, either $y \in f(y)$, or $y \notin f(y)$. We consider these cases separately:

Case 1: $y \in f(y)$.
By our definition of $D$, this means that $y \notin D$.
However, since $y \in f(y)$ and $f(y) = D$, we have $y \in D$.
We have reached a contradiction.

Case 2: $y \notin f(y)$.
By our definition of $D$, this means that $y \in D$.
However, since $y \notin f(y)$ and $f(y) = D$, we have $y \notin D$.
We have reached a contradiction.

In either case we reach a contradiction, so our assumption must have been wrong.
Thus for every set $S$, we have that $|S| \neq |\wp(S)|$. ■
Lemma: For any set $S$, $|S| \neq |\varnothing(S)|$.

Proof: By contradiction; assume that there exists a set $S$ such that $|S| = |\varnothing(S)|$.
**Lemma:** For any set $S$, $|S| \neq |\wp(S)|$.

**Proof:** By contradiction; assume that there exists a set $S$ such that $|S| = |\wp(S)|$. This means that there exists a bijection $f : S \to \wp(S)$. 
Lemma: For any set $S$, $|S| \neq |\wp(S)|$.

Proof: By contradiction; assume that there exists a set $S$ such that $|S| = |\wp(S)|$. This means that there exists a bijection $f : S \rightarrow \wp(S)$. Consider the set $D = \{ x \in S \mid x \notin f(x) \}$.
**Lemma:** For any set $S$, $|S| \neq |\wp(S)|$.

**Proof:** By contradiction; assume that there exists a set $S$ such that $|S| = |\wp(S)|$. This means that there exists a bijection $f : S \to \wp(S)$. Consider the set $D = \{ x \in S \mid x \notin f(x) \}$. Note that $D \subseteq S$, since by construction every $x \in D$ satisfies $x \in S$. 

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**Theorem (Cantor's Theorem):** For any set $S$, we have $|S| < |\mathcal{P}(S)|$.

**Proof:** Consider any set $S$. By our first lemma, we have that $|S| \leq |\mathcal{P}(S)|$. By our second lemma, we have that $|S| \neq |\mathcal{P}(S)|$. Thus $|S| < |\mathcal{P}(S)|$. ■
Why All This Matters

- The intuition behind a result is often more important than the result itself.
- Given the intuition, you can usually reconstruct the proof.
- Given just the proof, it is almost impossible to reconstruct the intuition.
- Think about compilation – you can more easily go from a high-level language to machine code than the other way around.
Cantor's *Other* Diagonal Argument
What is $|\mathbb{R}|$?
Theorem: $|\mathbb{N}| < |\mathbb{R}|$. 
Sketch of the Proof

- To prove that $|\mathbb{N}| < |\mathbb{R}|$, we will use a modification of the proof of Cantor's theorem.
- First, we will directly prove that $|\mathbb{N}| \leq |\mathbb{R}|$.
- Second, we will use a proof by diagonalization to show that $|\mathbb{N}| \neq |\mathbb{R}|$. 
Theorem: $|\mathbb{N}| \leq |\mathbb{R}|$. 
*Theorem:* \(|\mathbb{N}| \leq |\mathbb{R}|.\)

*Proof:* We will exhibit an injection \(f : \mathbb{N} \to \mathbb{R}\). Thus by definition, \(|\mathbb{N}| \leq |\mathbb{R}|.\)
Theorem: $|\mathbb{N}| \leq |\mathbb{R}|$.
Proof: We will exhibit an injection $f : \mathbb{N} \to \mathbb{R}$. Thus by definition, $|\mathbb{N}| \leq |\mathbb{R}|$.

Consider the function $f(n) = n$. Since all natural numbers are real numbers, this is a valid function from $\mathbb{N}$ to $\mathbb{R}$. Moreover, it is injective. To see this, consider any $n_0, n_1 \in \mathbb{N}$ such that $f(n_0) = f(n_1)$. We will prove that $n_0 = n_1$. To see this, note that $n_0 = f(n_0) = f(n_1) = n_1$. Thus $n_0 = n_1$, as required, so $f$ is injective. ■
\[\mathbb{N} \neq \mathbb{R}\]

- Now, we need to show that \(|\mathbb{N}| \neq |\mathbb{R}|\).
- To do this, we will use a proof by diagonalization similar to the one for Cantor's Theorem.
  - Assume there is a bijection \(f : \mathbb{N} \to \mathbb{R}\).
  - Construct a two-dimensional table from \(f\).
  - Construct a "diagonal number" from the table.
  - Show the diagonal number is not in the table.
  - Conclude \(f\) is not a bijection.
0 \leftrightarrow 8. 6 7 5 3 0 ... \\
1 \leftrightarrow 3. 1 4 1 5 9 ... \\
2 \leftrightarrow 0. 1 2 3 5 8 ... \\
3 \leftrightarrow -1. 0 0 0 0 0 0 0 0 0 ... \\
4 \leftrightarrow 2. 7 1 8 2 8 ... \\
5 \leftrightarrow 1. 6 1 8 0 3 ... \\
... \leftrightarrow ... ... ... ... ... ... ... ...
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| 0 | 8.    | 6.    | 7.    | 5.    | 3.    | 0.    |...
| 1 | 3.    | 1.    | 4.    | 1.    | 5.    | 9.    |...
| 2 | 0.    | 1.    | 2.    | 3.    | 5.    | 8.    |...
| 3 | -1.   | 0.    | 0.    | 0.    | 0.    | 0.    |...
| 4 | 2.    | 7.    | 1.    | 8.    | 2.    | 8.    |...
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Theorem: $|\mathbb{N}| \neq |\mathbb{R}|$. 
**Theorem:** $|\mathbb{N}| \neq |\mathbb{R}|$.

**Proof:** By contradiction; suppose that $|\mathbb{N}| = |\mathbb{R}|$. 

By contradiction; suppose that $|\mathbb{N}| = |\mathbb{R}|$. Then there exists a bijection $f: \mathbb{N} \rightarrow \mathbb{R}$. We introduce some new notation. For a real number $r$, let $r_0$ be the integer part of $r$, and let $r_n$ for $n \in \mathbb{N}$, $n > 0$, be the $n$th digit in the decimal representation of $r$. Now, define the real number $d$ as follows:

Since $d \in \mathbb{R}$, there must be some $n \in \mathbb{N}$ such that $f(n) = d$. So consider $f(n)$ and $d$. We consider two cases:

**Case 1:** $f(n) = 0$. Then by construction $d_n = 1$, meaning that $f(n) \neq d$.

**Case 2:** $f(n) \neq 0$. Then by construction $d_n = 0$, meaning that $f(n) \neq d$.

In either case, $f(n) \neq d$. This contradicts the fact that $f(n) = d$. We have reached a contradiction, so our assumption must have been wrong. Thus $|\mathbb{N}| \neq |\mathbb{R}|$. ■
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The Power of Diagonalization

• A large number of fundamental results in computability and complexity theory are based on diagonal arguments.

• We will see at least three of them in the remainder of the quarter.
An Interesting Historical Aside

- The diagonalization proof that $|\mathbb{N}| \neq |\mathbb{R}|$ was Cantor's original diagonal argument; he proved Cantor's theorem later on.

- However, this was *not* the first proof that $|\mathbb{N}| \neq |\mathbb{R}|$. Cantor had a different proof of this result based on infinite sequences.

- Come talk to me after class if you want to see the original proof; it's absolutely brilliant!
Cantor's *Other Other* Diagonal Argument

(This one is different!)
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<td>(1, 0)</td>
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<td>(1, 2)</td>
<td>(1, 3)</td>
<td>(1, 4)</td>
<td>...</td>
</tr>
<tr>
<td>2</td>
<td>(2, 0)</td>
<td>(2, 1)</td>
<td>(2, 2)</td>
<td>(2, 3)</td>
<td>(2, 4)</td>
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</tr>
<tr>
<td>3</td>
<td>(3, 0)</td>
<td>(3, 1)</td>
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<td>(3, 3)</td>
<td>(3, 4)</td>
<td>...</td>
</tr>
<tr>
<td>4</td>
<td>(4, 0)</td>
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<td>(4, 2)</td>
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</tr>
</tbody>
</table>
Diagonal 0
\( f(0, 0) = 0 \)

Diagonal 1
\( f(0, 1) = 1 \)
\( f(1, 0) = 2 \)

Diagonal 2
\( f(0, 2) = 3 \)
\( f(1, 1) = 4 \)
\( f(2, 0) = 5 \)

Diagonal 3
\( f(0, 3) = 6 \)
\( f(1, 2) = 7 \)
\( f(2, 1) = 8 \)
\( f(3, 0) = 9 \)

Diagonal 4
\( f(0, 4) = 10 \)
\( f(1, 3) = 11 \)
\( f(2, 2) = 12 \)
\( f(3, 1) = 13 \)
\( f(4, 0) = 14 \)

\[ f(a, b) = \text{The number of elements on all previous diagonals} + \text{The index of the current pair on its diagonal} \]
\[
\begin{align*}
\text{Diagonal 0} \\
\quad f(0, 0) &= 0 \\
\text{Diagonal 1} \\
\quad f(0, 1) &= 1 \\
\quad f(1, 0) &= 2 \\
\text{Diagonal 2} \\
\quad f(0, 2) &= 3 \\
\quad f(1, 1) &= 4 \\
\quad f(2, 0) &= 5 \\
\text{Diagonal 3} \\
\quad f(0, 3) &= 6 \\
\quad f(1, 2) &= 7 \\
\quad f(2, 1) &= 8 \\
\quad f(3, 0) &= 9 \\
\text{Diagonal 4} \\
\quad f(0, 4) &= 10 \\
\quad f(1, 3) &= 11 \\
\quad f(2, 2) &= 12 \\
\quad f(3, 1) &= 13 \\
\quad f(4, 0) &= 14
\end{align*}
\]

\[
f(a, b) = \sum_{i=1}^{a+b} i + \text{The index of the current pair on its diagonal}
\]
\[ f(a, b) = \frac{(a + b)(a + b + 1)}{2} \]

The index of the current pair on its diagonal

Diagonal 0
\[ f(0, 0) = 0 \]

Diagonal 1
\[ f(0, 1) = 1 \]
\[ f(1, 0) = 2 \]

Diagonal 2
\[ f(0, 2) = 3 \]
\[ f(1, 1) = 4 \]
\[ f(2, 0) = 5 \]

Diagonal 3
\[ f(0, 3) = 6 \]
\[ f(1, 2) = 7 \]
\[ f(2, 1) = 8 \]
\[ f(3, 0) = 9 \]

Diagonal 4
\[ f(0, 4) = 10 \]
\[ f(1, 3) = 11 \]
\[ f(2, 2) = 12 \]
\[ f(3, 1) = 13 \]
\[ f(4, 0) = 14 \]
\[
\begin{align*}
\text{Diagonal 0} \\
f(0, 0) &= 0 \\
\text{Diagonal 1} \\
f(0, 1) &= 1 \\
f(1, 0) &= 2 \\
\text{Diagonal 2} \\
f(0, 2) &= 3 \\
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\text{Diagonal 3} \\
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f(0, 4) &= 10 \\
f(1, 3) &= 11 \\
f(2, 2) &= 12 \\
f(2, 1) &= 13 \\
f(3, 0) &= 14
\end{align*}
\]

\[
f(a, b) = \frac{(a + b)(a + b + 1)}{2} + a
\]
Diagonal 0
\( f(0, 0) = 0 \)

Diagonal 1
\( f(0, 1) = 1 \)
\( f(1, 0) = 2 \)

Diagonal 2
\( f(0, 2) = 3 \)
\( f(1, 1) = 4 \)
\( f(2, 0) = 5 \)

Diagonal 3
\( f(0, 3) = 6 \)
\( f(1, 2) = 7 \)
\( f(2, 1) = 8 \)
\( f(3, 0) = 9 \)

Diagonal 4
\( f(0, 4) = 10 \)
\( f(1, 3) = 11 \)
\( f(2, 2) = 12 \)
\( f(3, 1) = 13 \)
\( f(4, 0) = 14 \)

\( f(a, b) = (a + b)(a + b + 1) / 2 + a \)

This function is called Cantor's Pairing Function.
\[
f(a, b) = (a + b)(a + b + 1) / 2 + a
\]
Theorem: $|\mathbb{N}^2| = |\mathbb{N}|$. 
Formalizing the Proof

- We need to show that this function $f$ is injective and surjective.
- These proofs are nontrivial, but have beautiful intuitions.
- I've included the proofs at the end of these slides if you're curious.
Next Time

- **The Pigeonhole Principle**
  - Pleasing and poignant pigeon-powered proofs!
Appendix: Proof that $|\mathbb{N}^2| = |\mathbb{N}|$
Proving Surjectivity

- Given just the definition of our function:
  \[ f(a, b) = \frac{(a + b)(a + b + 1)}{2} + a \]
  It is not at all clear that every natural number can be generated.

- However, given our intuition of how the function works (crawling along diagonals), we can start to formulate a proof of surjectivity.
Proving Surjectivity

\[ f(a, b) = \frac{(a + b)(a + b + 1)}{2} + a \]

- What pair of numbers maps to 137?
- We can figure this out by first trying to figure out what diagonal this would be in.
Proving Surjectivity

\[ f(a, b) = \frac{(a + b)(a + b + 1)}{2} + a \]

- What pair of numbers maps to 137?
- We can figure this out by first trying to figure out what diagonal this would be in.

\begin{array}{ccc}
  0 & 1 & 2 \\
  0 & (0, 0) & (0, 1) & (0, 2) \\
  1 & (1, 0) & (1, 1) & (1, 2) \\
  2 & (2, 0) & (2, 1) & (2, 2) \\
\end{array}
Proving Surjectivity

\[ f(a, b) = \frac{(a + b)(a + b + 1)}{2} + a \]

- What pair of numbers maps to 137?
- We can figure this out by first trying to figure out what diagonal this would be in.

\[
\begin{array}{ccc}
0 & 1 & 2 \\
0 & (0,0) & (0,1) & (0,2) \\
1 & (1,0) & (1,1) & (1,2) \\
2 & (2,0) & (2,1) & (2,2) \\
\end{array}
\]
Proving Surjectivity

\[ f(a, b) = (a + b)(a + b + 1) / 2 + a \]

- What pair of numbers maps to 137?
- We can figure this out by first trying to figure out what diagonal this would be in.

Total number of elements before

<table>
<thead>
<tr>
<th>Row</th>
<th>Elements before</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
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<tr>
<td>3</td>
<td>6</td>
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<tr>
<td>4</td>
<td>10</td>
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<tr>
<td>...</td>
<td></td>
</tr>
<tr>
<td>m</td>
<td>( m(m + 1) / 2 )</td>
</tr>
</tbody>
</table>
Proving Surjectivity

\( f(a, b) = (a + b)(a + b + 1) / 2 + a \)

- What pair of numbers maps to 137?
- We can figure this out by first trying to figure out what diagonal this would be in.
  - Answer: Diagonal 16, since there are 136 pairs that come before it.
- Now that we know the diagonal, we can figure out the index into that diagonal.
  - 137 - 136 = 1.
- So we'd expect the first entry of diagonal 16 to map to 137.

\[ f(1, 15) = 16 \times 17 / 2 + 1 = 136 + 1 = 137 \]
Generalizing Into a Proof

- We can generalize this logic as follows.
- To find a pair that maps to $n$:
  - Find which diagonal the number is in by finding the largest $d$ such that
    $$d(d + 1) / 2 \leq n$$
  - Find which index it is in by subtracting the starting position of that diagonal:
    $$k = n - d(d + 1) / 2$$
  - The $k$th entry of diagonal $d$ is the answer:
    $$f(k, d - k) = n$$
Lemma: Let $f(a, b) = (a + b)(a + b + 1) / 2 + a$ be a function from $\mathbb{N}^2$ to $\mathbb{N}$. Then $f$ is surjective.
**Lemma:** Let $f(a, b) = (a + b)(a + b + 1) / 2 + a$ be a function from $\mathbb{N}^2$ to $\mathbb{N}$. Then $f$ is surjective.

**Proof:** Consider any $n \in \mathbb{N}$.
Lemma: Let \( f(a, b) = (a + b)(a + b + 1) / 2 + a \) be a function from \( \mathbb{N}^2 \) to \( \mathbb{N} \). Then \( f \) is surjective.

Proof: Consider any \( n \in \mathbb{N} \). We will show that there exists a pair \( (a, b) \in \mathbb{N}^2 \) such that \( f(a, b) = n \).
Lemma: Let \( f(a, b) = (a + b)(a + b + 1) / 2 + a \) be a function from \( \mathbb{N}^2 \) to \( \mathbb{N} \). Then \( f \) is surjective.

Proof: Consider any \( n \in \mathbb{N} \). We will show that there exists a pair \( (a, b) \in \mathbb{N}^2 \) such that \( f(a, b) = n \).

Consider the largest \( d \in \mathbb{N} \) such that \( d(d + 1) / 2 \leq n \).

Intuitively, \( d \) is the diagonal containing \( n \).

Now, consider the value of \( f(k, d - k) \).

\[
\begin{align*}
\text{Intuitively, } d & \text{ is the diagonal containing } n. \\
\text{Consider the largest } d & \in \mathbb{N} \text{ such that } d(d + 1) / 2 \leq n. \\
\text{Consider the value of } f(k, d - k). \\
\end{align*}
\]
**Lemma:** Let \( f(a, b) = (a + b)(a + b + 1) / 2 + a \) be a function from \( \mathbb{N}^2 \) to \( \mathbb{N} \). Then \( f \) is surjective.

**Proof:** Consider any \( n \in \mathbb{N} \). We will show that there exists a pair \( (a, b) \in \mathbb{N}^2 \) such that \( f(a, b) = n \).

Consider the largest \( d \in \mathbb{N} \) such that \( d(d + 1) / 2 \leq n \). Then, let \( k = n - d(d + 1) / 2 \).

Intuition: \( k \) is the position within this diagonal.

Now, we need to rigorously establish that we came up with a legal pair, and that the pair actually maps to \( n \).
Lemma: Let \( f(a, b) = (a + b)(a + b + 1)/2 + a \) be a function from \( \mathbb{N}^2 \) to \( \mathbb{N} \). Then \( f \) is surjective.

Proof: Consider any \( n \in \mathbb{N} \). We will show that there exists a pair \( (a, b) \in \mathbb{N}^2 \) such that \( f(a, b) = n \).

Consider the largest \( d \in \mathbb{N} \) such that \( d(d + 1)/2 \leq n \). Then, let \( k = n - d(d + 1)/2 \). Since \( d(d + 1)/2 \leq n \), we have that \( k \in \mathbb{N} \).
Lemma: Let \( f(a, b) = (a + b)(a + b + 1) / 2 + a \) be a function from \( \mathbb{N}^2 \) to \( \mathbb{N} \). Then \( f \) is surjective.

Proof: Consider any \( n \in \mathbb{N} \). We will show that there exists a pair \((a, b) \in \mathbb{N}^2\) such that \( f(a, b) = n \).

Consider the largest \( d \in \mathbb{N} \) such that \( d(d + 1) / 2 \leq n \). Then, let \( k = n - d(d + 1) / 2 \). Since \( d(d + 1) / 2 \leq n \), we have that \( k \in \mathbb{N} \). We further claim that \( k \leq d \).

We need to formalize our intuition by showing that \( d \) gives an index on this diagonal.
Lemma: Let $f(a, b) = (a + b)(a + b + 1) / 2 + a$ be a function from $\mathbb{N}^2$ to $\mathbb{N}$. Then $f$ is surjective.

Proof: Consider any $n \in \mathbb{N}$. We will show that there exists a pair $(a, b) \in \mathbb{N}^2$ such that $f(a, b) = n$.

Consider the largest $d \in \mathbb{N}$ such that $d(d + 1) / 2 \leq n$. Then, let $k = n - d(d + 1) / 2$. Since $d(d + 1) / 2 \leq n$, we have that $k \in \mathbb{N}$. We further claim that $k \leq d$. To see this, suppose for the sake of contradiction that $k > d$. 


Lemma: Let \( f(a, b) = (a + b)(a + b + 1) / 2 + a \) be a function from \( \mathbb{N}^2 \) to \( \mathbb{N} \). Then \( f \) is surjective.

Proof: Consider any \( n \in \mathbb{N} \). We will show that there exists a pair \( (a, b) \in \mathbb{N}^2 \) such that \( f(a, b) = n \).

Consider the largest \( d \in \mathbb{N} \) such that \( d(d + 1) / 2 \leq n \). Then, let \( k = n - d(d + 1) / 2 \). Since \( d(d + 1) / 2 \leq n \), we have that \( k \in \mathbb{N} \). We further claim that \( k \leq d \). To see this, suppose for the sake of contradiction that \( k > d \). Consequently, \( k \geq d + 1 \).

If \( m \) and \( n \) are natural numbers or integers, then \( m < n \) iff \( m + 1 \leq n \). This fact is remarkably useful in proofs on \( \mathbb{N} \) or \( \mathbb{Z} \).
Lemma: Let $f(a, b) = (a + b)(a + b + 1) / 2 + a$ be a function from $\mathbb{N}^2$ to $\mathbb{N}$. Then $f$ is surjective.

Proof: Consider any $n \in \mathbb{N}$. We will show that there exists a pair $(a, b) \in \mathbb{N}^2$ such that $f(a, b) = n$.

Consider the largest $d \in \mathbb{N}$ such that $d(d + 1) / 2 \leq n$. Then, let $k = n - d(d + 1) / 2$. Since $d(d + 1) / 2 \leq n$, we have that $k \in \mathbb{N}$. We further claim that $k \leq d$. To see this, suppose for the sake of contradiction that $k > d$. Consequently, $k \geq d + 1$. This means that $d + 1 \leq k$.
Lemma: Let $f(a, b) = (a + b)(a + b + 1) / 2 + a$ be a function from $\mathbb{N}^2$ to $\mathbb{N}$. Then $f$ is surjective.

Proof: Consider any $n \in \mathbb{N}$. We will show that there exists a pair $(a, b) \in \mathbb{N}^2$ such that $f(a, b) = n$.

Consider the largest $d \in \mathbb{N}$ such that $d(d + 1) / 2 \leq n$. Then, let $k = n - d(d + 1) / 2$. Since $d(d + 1) / 2 \leq n$, we have that $k \in \mathbb{N}$. We further claim that $k \leq d$. To see this, suppose for the sake of contradiction that $k > d$. Consequently, $k \geq d + 1$. This means that

\[
\begin{align*}
  d + 1 &\leq k \\
  d + 1 &\leq n - d(d + 1) / 2 
\end{align*}
\]
Lemma: Let \( f(a, b) = (a + b)(a + b + 1) / 2 + a \) be a function from \( \mathbb{N}^2 \) to \( \mathbb{N} \). Then \( f \) is surjective.

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d + 1 & \leq k \\
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d + 1 + d(d + 1)/2 & \leq n
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Lemma: Let \( f(a, b) = (a + b)(a + b + 1) / 2 + a \) be a function from \( \mathbb{N}^2 \) to \( \mathbb{N} \). Then \( f \) is surjective.

Proof: Consider any \( n \in \mathbb{N} \). We will show that there exists a pair \( (a, b) \in \mathbb{N}^2 \) such that \( f(a, b) = n \).

Consider the largest \( d \in \mathbb{N} \) such that \( d(d + 1) / 2 \leq n \). Then, let \( k = n - d(d + 1) / 2 \). Since \( d(d + 1) / 2 \leq n \), we have that \( k \in \mathbb{N} \). We further claim that \( k \leq d \). To see this, suppose for the sake of contradiction that \( k > d \). Consequently, \( k \geq d + 1 \). This means that

\[

d + 1 \leq k \\
-d + 1 \leq n - d(d + 1) / 2 \\
d + 1 + d(d + 1) / 2 \leq n \\
(2(d + 1) + d(d + 1)) / 2 \leq n
\]
Lemma: Let \( f(a, b) = (a + b)(a + b + 1) / 2 + a \) be a function from \( \mathbb{N}^2 \) to \( \mathbb{N} \). Then \( f \) is surjective.

Proof: Consider any \( n \in \mathbb{N} \). We will show that there exists a pair \((a, b) \in \mathbb{N}^2\) such that \( f(a, b) = n \).

Consider the largest \( d \in \mathbb{N} \) such that \( d(d + 1) / 2 \leq n \). Then, let \( k = n - d(d + 1) / 2 \). Since \( d(d + 1) / 2 \leq n \), we have that \( k \in \mathbb{N} \). We further claim that \( k \leq d \). To see this, suppose for the sake of contradiction that \( k > d \). Consequently, \( k \geq d + 1 \). This means that

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Lemma: Let $f(a, b) = (a + b)(a + b + 1) / 2 + a$ be a function from $\mathbb{N}^2$ to $\mathbb{N}$. Then $f$ is surjective.

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Consider the largest $d \in \mathbb{N}$ such that $d(d + 1) / 2 \leq n$. Then, let $k = n - d(d + 1) / 2$. Since $d(d + 1) / 2 \leq n$, we have that $k \in \mathbb{N}$. We further claim that $k \leq d$. To see this, suppose for the sake of contradiction that $k > d$. Consequently, $k \geq d + 1$. This means that

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  d + 1 &\leq n - d(d + 1) / 2 \\
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  (d + 1)(d + 2) / 2 &\leq n
\end{align*}
\]

But this means that $d$ is not the largest natural number satisfying the inequality $d(d + 1) / 2 \leq n$, a contradiction.
Lemma: Let \( f(a, b) = (a + b)(a + b + 1) / 2 + a \) be a function from \( \mathbb{N}^2 \) to \( \mathbb{N} \). Then \( f \) is surjective.

Proof: Consider any \( n \in \mathbb{N} \). We will show that there exists a pair \((a, b) \in \mathbb{N}^2\) such that \( f(a, b) = n \).

Consider the largest \( d \in \mathbb{N} \) such that \( d(d + 1) / 2 \leq n \). Then, let \( k = n - d(d + 1) / 2 \). Since \( d(d + 1) / 2 \leq n \), we have that \( k \in \mathbb{N} \). We further claim that \( k \leq d \). To see this, suppose for the sake of contradiction that \( k > d \). Consequently, \( k \geq d + 1 \). This means that

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\begin{align*}
  d + 1 & \leq k \\
  d + 1 & \leq n - d(d + 1) / 2 \\
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  (2(d + 1) + d(d + 1)) / 2 & \leq n \\
  (d + 1)(d + 2) / 2 & \leq n
\end{align*}
\]

But this means that \( d \) is not the largest natural number satisfying the inequality \( d(d + 1) / 2 \leq n \), a contradiction. Thus our assumption must have been wrong, so \( k \leq d \).
Lemma: Let $f(a, b) = (a + b)(a + b + 1) / 2 + a$ be a function from $\mathbb{N}^2$ to $\mathbb{N}$. Then $f$ is surjective.

Proof: Consider any $n \in \mathbb{N}$. We will show that there exists a pair $(a, b) \in \mathbb{N}^2$ such that $f(a, b) = n$.

Consider the largest $d \in \mathbb{N}$ such that $d(d + 1) / 2 \leq n$. Then, let $k = n - d(d + 1) / 2$. Since $d(d + 1) / 2 \leq n$, we have that $k \in \mathbb{N}$. We further claim that $k \leq d$. To see this, suppose for the sake of contradiction that $k > d$. Consequently, $k \geq d + 1$. This means that

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d + 1 & \leq n - d(d + 1) / 2 \\
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(d + 1)(d + 2) / 2 & \leq n
\end{align*}
\]

But this means that $d$ is not the largest natural number satisfying the inequality $d(d + 1) / 2 \leq n$, a contradiction. Thus our assumption must have been wrong, so $k \leq d$.

Since $k \leq d$, we have that $0 \leq k - d$, so $k - d \in \mathbb{N}$.

We have a valid pair! All that's left to do now is to show that index $k$ on diagonal $d$ maps to $n$. 
Lemma: Let \( f(a, b) = (a + b)(a + b + 1) / 2 + a \) be a function from \( \mathbb{N}^2 \) to \( \mathbb{N} \). Then \( f \) is surjective.

Proof: Consider any \( n \in \mathbb{N} \). We will show that there exists a pair \( (a, b) \in \mathbb{N}^2 \) such that \( f(a, b) = n \).

Consider the largest \( d \in \mathbb{N} \) such that \( d(d + 1) / 2 \leq n \). Then, let \( k = n - d(d + 1) / 2 \). Since \( d(d + 1) / 2 \leq n \), we have that \( k \in \mathbb{N} \). We further claim that \( k \leq d \). To see this, suppose for the sake of contradiction that \( k > d \). Consequently, \( k \geq d + 1 \). This means that

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  d + 1 + d(d + 1) / 2 & \leq n \\
  (2(d + 1) + d(d + 1)) / 2 & \leq n \\
  (d + 1)(d + 2) / 2 & \leq n
\end{align*}
\]

But this means that \( d \) is not the largest natural number satisfying the inequality \( d(d + 1) / 2 \leq n \), a contradiction. Thus our assumption must have been wrong, so \( k \leq d \).

Since \( k \leq d \), we have that \( 0 \leq k - d \), so \( k - d \in \mathbb{N} \). Now, consider the value of \( f(k, d - k) \).
Lemma: Let \( f(a, b) = (a + b)(a + b + 1) / 2 + a \) be a function from \( \mathbb{N}^2 \) to \( \mathbb{N} \). Then \( f \) is surjective.

Proof: Consider any \( n \in \mathbb{N} \). We will show that there exists a pair \( (a, b) \in \mathbb{N}^2 \) such that \( f(a, b) = n \).

Consider the largest \( d \in \mathbb{N} \) such that \( d(d + 1) / 2 \leq n \). Then, let \( k = n - d(d + 1) / 2 \). Since \( d(d + 1) / 2 \leq n \), we have that \( k \in \mathbb{N} \). We further claim that \( k \leq d \). To see this, suppose for the sake of contradiction that \( k > d \). Consequently, \( k \geq d + 1 \). This means that

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\begin{align*}
    d + 1 & \leq k \\
    d + 1 & \leq n - d(d + 1) / 2 \\
    d + 1 + d(d + 1) / 2 & \leq n \\
    (2(d + 1) + d(d + 1)) / 2 & \leq n \\
    (d + 1)(d + 2) / 2 & \leq n
\end{align*}
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Proving Injectivity

- Given the function

\[ f(a, b) = \frac{(a + b)(a + b + 1)}{2} + a \]

- It is not at all obvious that \( f \) is injective.

- We'll have to use our intuition to figure out why this would be.
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Proving Injectivity

\[ f(a, b) = \frac{(a + b)(a + b + 1)}{2} + a \]

- Suppose that \( f(a, b) = f(c, d) \). We need to prove \( (a, b) = (c, d) \).
- Our proof will proceed in two steps:
  - First, we'll prove that \( (a, b) \) and \( (c, d) \) have to be in the same diagonal.
  - Next, using the fact that they're in the same diagonal, we'll show that they're at the same position within that diagonal.
  - From this, we can conclude \( (a, b) = (c, d) \).
Lemma: Suppose \( f(a, b) = (a + b)(a + b + 1) / 2 + a \). Then the largest \( m \in \mathbb{N} \) for which \( m(m + 1) / 2 \leq f(a, b) \) is given by \( m = a + b \).

The point of this lemma is to let us “read off” what diagonal we are in just by looking at \( a \) and \( b \). We will need this in a second.
Lemma: Suppose \( f(a, b) = (a + b)(a + b + 1)/2 + a \). Then the largest \( m \in \mathbb{N} \) for which \( m(m + 1)/2 \leq f(a, b) \) is given by \( m = a + b \).

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So \( m \) satisfies the inequality.

Next, we will show that any \( m' \in \mathbb{N} \) with \( m' > a + b \) will not satisfy the inequality. Take any \( m' \in \mathbb{N} \) where \( m' > a + b \). This means that \( m' \geq a + b + 1 \). Consequently, we have

\[
\begin{align*}
m'(m' + 1) / 2 &\geq (a + b + 1)(a + b + 2) / 2 \\
&= ((a + b)(a + b + 2) + 2(a + b + 1)) / 2 \\
&= (a + b)(a + b + 1) / 2 + a + b + 1 \\
&> (a + b)(a + b + 1) / 2 + a \\
&= f(a, b)
\end{align*}
\]

Thus \( m' \) does not satisfy the inequality.
Lemma: Suppose \( f(a, b) = (a + b)(a + b + 1)/2 + a \). Then the largest \( m \in \mathbb{N} \) for which \( m(m + 1)/2 \leq f(a, b) \) is given by \( m = a + b \).

Proof: First, we show that \( m = a + b \) satisfies the above inequality. Note that if \( m = a + b \), we have

\[
f(a, b) = (a + b)(a + b + 1)/2 + a \\
\geq (a + b)(a + b + 1)/2 \\
= m(m + 1)/2
\]

So \( m \) satisfies the inequality.

Next, we will show that any \( m' \in \mathbb{N} \) with \( m' > a + b \) will not satisfy the inequality. Take any \( m' \in \mathbb{N} \) where \( m' > a + b \). This means that \( m' \geq a + b + 1 \). Consequently, we have

\[
m'(m' + 1)/2 \geq (a + b + 1)(a + b + 2)/2 \\
= ((a + b)(a + b + 2) + 2(a + b + 1))/2 \\
= (a + b)(a + b + 1)/2 + a + b + 1 \\
> (a + b)(a + b + 1)/2 + a \\
= f(a, b)
\]

Thus \( m' \) does not satisfy the inequality. Consequently, \( m = a + b \) is the largest natural number satisfying the inequality.
Lemma: Suppose \( f(a, b) = (a + b)(a + b + 1) / 2 + a \). Then the largest \( m \in \mathbb{N} \) for which \( m(m + 1) / 2 \leq f(a, b) \) is given by \( m = a + b \).

Proof: First, we show that \( m = a + b \) satisfies the above inequality. Note that if \( m = a + b \), we have

\[
\begin{align*}
f(a, b) &= (a + b)(a + b + 1) / 2 + a \\
&\geq (a + b)(a + b + 1) / 2 \\
&= m(m + 1) / 2
\end{align*}
\]

So \( m \) satisfies the inequality.

Next, we will show that any \( m' \in \mathbb{N} \) with \( m' > a + b \) will not satisfy the inequality. Take any \( m' \in \mathbb{N} \) where \( m' > a + b \). This means that \( m' \geq a + b + 1 \). Consequently, we have

\[
\begin{align*}
m'(m' + 1) / 2 &\geq (a + b + 1)(a + b + 2) / 2 \\
&= ((a + b)(a + b + 2) + 2(a + b + 1)) / 2 \\
&= (a + b)(a + b + 1) / 2 + a + b + 1 \\
&> (a + b)(a + b + 1) / 2 + a \\
&= f(a, b)
\end{align*}
\]

Thus \( m' \) does not satisfy the inequality. Consequently, \( m = a + b \) is the largest natural number satisfying the inequality. \( \square \)
Theorem: Let \( f(a, b) = (a + b)(a + b + 1) / 2 + a \). Then \( f \) is injective.
Theorem: Let $f(a, b) = (a + b)(a + b + 1)/2 + a$. Then $f$ is injective.

Proof: Consider any $(a, b), (c, d) \in \mathbb{N}^2$ such that $f(a, b) = f(c, d)$.
**Theorem:** Let \( f(a, b) = (a + b)(a + b + 1) / 2 + a \). Then \( f \) is injective.

**Proof:** Consider any \((a, b), (c, d) \in \mathbb{N}^2\) such that \( f(a, b) = f(c, d) \). We will show that \((a, b) = (c, d)\).

First, we will show that \( a + b = c + d \). To do this, assume for the sake of contradiction that \( a + b \neq c + d \). Then either \( a + b < c + d \) or \( a + b > c + d \). Assume without loss of generality that \( a + b < c + d \).

By our lemma, we know that \( m = a + b \) is the largest natural number such that \( f(a, b) \leq m(m + 1) / 2 \). Since \( a + b < c + d \), this means that \( f(a, b) < (c + d)(c + d + 1) / 2 = f(c, d) \), contradicting that \( f(a, b) = f(c, d) \).

We have reached a contradiction, so our assumption must have been wrong. Thus \( a + b = c + d \).

Given this, we have that \( f(a, b) = f(c, d) \) since \( (a, b) = (c, d) \), as required. ■
Theorem: Let \( f(a, b) = (a + b)(a + b + 1)/2 + a \). Then \( f \) is injective.

Proof: Consider any \((a, b), (c, d) \in \mathbb{N}^2\) such that \( f(a, b) = f(c, d) \). We will show that \((a, b) = (c, d)\).

First, we will show that \( a + b = c + d \).

Intuitively, this proves that \((a, b)\) and \((c, d)\) belong to the same diagonal.
Theorem: Let \( f(a, b) = (a + b)(a + b + 1) / 2 + a \). Then \( f \) is injective.

Proof: Consider any \((a, b), (c, d) \in \mathbb{N}^2\) such that \( f(a, b) = f(c, d) \). We will show that \((a, b) = (c, d)\).

First, we will show that \( a + b = c + d \). To do this, assume for the sake of contradiction that \( a + b \neq c + d \).

But this means that \( f(a, b) < f(c, d) \), contradicting that \( f(a, b) = f(c, d) \). We have reached a contradiction, so our assumption must have been wrong. Thus \( a + b = c + d \). Given this, we have that \( f(a, b) = f(c, d) \) and \( a = c \) and \( a + b = c + d \), so we have that \( b = d \). Thus \((a, b) = (c, d)\), as required. ■
Theorem: Let \( f(a, b) = (a + b)(a + b + 1) / 2 + a \). Then \( f \) is injective.

Proof: Consider any \((a, b), (c, d) \in \mathbb{N}^2\) such that \( f(a, b) = f(c, d) \). We will show that \((a, b) = (c, d)\).

First, we will show that \( a + b = c + d \). To do this, assume for the sake of contradiction that \( a + b \neq c + d \). Then either \( a + b < c + d \) or \( a + b > c + d \).
Theorem: Let \( f(a, b) = (a + b)(a + b + 1) / 2 + a \). Then \( f \) is injective.

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First, we will show that \( a + b = c + d \). To do this, assume for the sake of contradiction that \( a + b \neq c + d \). Then either \( a + b < c + d \) or \( a + b > c + d \). Assume without loss of generality that \( a + b < c + d \).
Theorem: Let \( f(a, b) = (a + b)(a + b + 1) / 2 + a. \) Then \( f \) is injective.

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By our lemma, we know that \( m = a + b \) is the largest natural number such that \( f(a, b) \leq m(m + 1) / 2 \).
Theorem: Let $f(a, b) = (a + b)(a + b + 1) / 2 + a$. Then $f$ is injective.

Proof: Consider any $(a, b), (c, d) \in \mathbb{N}^2$ such that $f(a, b) = f(c, d)$. We will show that $(a, b) = (c, d)$.

First, we will show that $a + b = c + d$. To do this, assume for the sake of contradiction that $a + b \neq c + d$. Then either $a + b < c + d$ or $a + b > c + d$. Assume without loss of generality that $a + b < c + d$.

By our lemma, we know that $m = a + b$ is the largest natural number such that $f(a, b) \leq m(m + 1) / 2$. Since $a + b < c + d$, this means that

$$f(a, b) = (a + b)(a + b + 1) / 2 + a$$
Theorem: Let \( f(a, b) = (a + b)(a + b + 1) / 2 + a \). Then \( f \) is injective.

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First, we will show that \( a + b = c + d \). To do this, assume for the sake of contradiction that \( a + b \neq c + d \). Then either \( a + b < c + d \) or \( a + b > c + d \). Assume without loss of generality that \( a + b < c + d \).

By our lemma, we know that \( m = a + b \) is the largest natural number such that \( f(a, b) \leq m(m + 1) / 2 \). Since \( a + b < c + d \), this means that

\[
\begin{align*}
f(a, b) &= (a + b)(a + b + 1) / 2 + a \\
&< (c + d)(c + d + 1) / 2
\end{align*}
\]

This step works because we know that any number \( n \) bigger than \( a + b \) doesn't satisfy

\[ n(n + 1) / 2 \leq f(a, b) \]

This means that

\[ f(a, b) < n(n + 1) / 2. \]
Theorem: Let \( f(a, b) = (a + b)(a + b + 1) / 2 + a \). Then \( f \) is injective.

Proof: Consider any \((a, b), (c, d) \in \mathbb{N}^2\) such that \( f(a, b) = f(c, d) \). We will show that \((a, b) = (c, d)\).

First, we will show that \( a + b = c + d \). To do this, assume for the sake of contradiction that \( a + b \neq c + d \). Then either \( a + b < c + d \) or \( a + b > c + d \). Assume without loss of generality that \( a + b < c + d \).

By our lemma, we know that \( m = a + b \) is the largest natural number such that \( f(a, b) \leq m(m + 1) / 2 \). Since \( a + b < c + d \), this means that

\[
\begin{align*}
f(a, b) &= (a + b)(a + b + 1) / 2 + a \\
&< (c + d)(c + d + 1) / 2 \\
&\leq (c + d)(c + d + 1) / 2 + c
\end{align*}
\]
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First, we will show that \( a + b = c + d \). To do this, assume for the sake of contradiction that \( a + b \neq c + d \). Then either \( a + b < c + d \) or \( a + b > c + d \). Assume without loss of generality that \( a + b < c + d \).

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  f(a, b) &= (a + b)(a + b + 1) / 2 + a \\
  &< (c + d)(c + d + 1) / 2 \\
  \leq (c + d)(c + d + 1) / 2 + c \\
  &= f(c, d)
\end{align*}
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Theorem: Let \( f(a, b) = (a + b)(a + b + 1) / 2 + a \). Then \( f \) is injective.

Proof: Consider any \( (a, b), (c, d) \in \mathbb{N}^2 \) such that \( f(a, b) = f(c, d) \). We will show that \( (a, b) = (c, d) \).

First, we will show that \( a + b = c + d \). To do this, assume for the sake of contradiction that \( a + b \neq c + d \). Then either \( a + b < c + d \) or \( a + b > c + d \). Assume without loss of generality that \( a + b < c + d \).

By our lemma, we know that \( m = a + b \) is the largest natural number such that \( f(a, b) \leq m(m + 1) / 2 \). Since \( a + b < c + d \), this means that

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\begin{align*}
f(a, b) &= (a + b)(a + b + 1) / 2 + a \\
&< (c + d)(c + d + 1) / 2 \\
&\leq (c + d)(c + d + 1) / 2 + c \\
&= f(c, d)
\end{align*}
\]

But this means that \( f(a, b) < f(c, d) \), contradicting that \( f(a, b) = f(c, d) \).
Theorem: Let \( f(a, b) = (a + b)(a + b + 1) / 2 + a \). Then \( f \) is injective.

Proof: Consider any \((a, b), (c, d) \in \mathbb{N}^2\) such that \( f(a, b) = f(c, d) \). We will show that \((a, b) = (c, d)\).

First, we will show that \( a + b = c + d \). To do this, assume for the sake of contradiction that \( a + b \neq c + d \). Then either \( a + b < c + d \) or \( a + b > c + d \). Assume without loss of generality that \( a + b < c + d \).

By our lemma, we know that \( m = a + b \) is the largest natural number such that \( f(a, b) \leq m(m + 1) / 2 \). Since \( a + b < c + d \), this means that

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  f(a, b) &= (a + b)(a + b + 1) / 2 + a \\
  &< (c + d)(c + d + 1) / 2 \\
  &\leq (c + d)(c + d + 1) / 2 + c \\
  &= f(c, d)
\end{align*}
\]

But this means that \( f(a, b) < f(c, d) \), contradicting that \( f(a, b) = f(c, d) \). We have reached a contradiction, so our assumption must have been wrong.
**Theorem:** Let \( f(a, b) = (a + b)(a + b + 1) / 2 + a \). Then \( f \) is injective.

**Proof:** Consider any \((a, b), (c, d) \in \mathbb{N}^2\) such that \( f(a, b) = f(c, d) \). We will show that \((a, b) = (c, d)\).

First, we will show that \( a + b = c + d \). To do this, assume for the sake of contradiction that \( a + b \neq c + d \). Then either \( a + b < c + d \) or \( a + b > c + d \). Assume without loss of generality that \( a + b < c + d \).

By our lemma, we know that \( m = a + b \) is the largest natural number such that \( f(a, b) \leq m(m + 1) / 2 \). Since \( a + b < c + d \), this means that

\[
\begin{align*}
  f(a, b) &= (a + b)(a + b + 1) / 2 + a \\
           &< (c + d)(c + d + 1) / 2 \\
           &\leq (c + d)(c + d + 1) / 2 + c \\
           &= f(c, d)
\end{align*}
\]

But this means that \( f(a, b) < f(c, d) \), contradicting that \( f(a, b) = f(c, d) \). We have reached a contradiction, so our assumption must have been wrong. Thus \( a + b = c + d \).

Now that we've got these points in the same diagonal, we just need to show that they have the same index.
Theorem: Let $f(a, b) = (a + b)(a + b + 1) / 2 + a$. Then $f$ is injective.

Proof: Consider any $(a, b), (c, d) \in \mathbb{N}^2$ such that $f(a, b) = f(c, d)$. We will show that $(a, b) = (c, d)$.

First, we will show that $a + b = c + d$. To do this, assume for the sake of contradiction that $a + b \neq c + d$. Then either $a + b < c + d$ or $a + b > c + d$. Assume without loss of generality that $a + b < c + d$.

By our lemma, we know that $m = a + b$ is the largest natural number such that $f(a, b) \leq m(m + 1) / 2$. Since $a + b < c + d$, this means that

$$f(a, b) = (a + b)(a + b + 1) / 2 + a$$
$$< (c + d)(c + d + 1) / 2$$
$$\leq (c + d)(c + d + 1) / 2 + c$$
$$= f(c, d)$$

But this means that $f(a, b) < f(c, d)$, contradicting that $f(a, b) = f(c, d)$. We have reached a contradiction, so our assumption must have been wrong. Thus $a + b = c + d$. Given this, we have that

$$f(a, b) = f(c, d)$$
Theorem: Let \( f(a, b) = (a + b)(a + b + 1) / 2 + a \). Then \( f \) is injective.

Proof: Consider any \((a, b), (c, d) \in \mathbb{N}^2\) such that \( f(a, b) = f(c, d) \). We will show that \((a, b) = (c, d)\).

First, we will show that \( a + b = c + d \). To do this, assume for the sake of contradiction that \( a + b \neq c + d \). Then either \( a + b < c + d \) or \( a + b > c + d \). Assume without loss of generality that \( a + b < c + d \).

By our lemma, we know that \( m = a + b \) is the largest natural number such that \( f(a, b) \leq m(m + 1) / 2 \). Since \( a + b < c + d \), this means that

\[
\begin{align*}
  f(a, b) &= (a + b)(a + b + 1) / 2 + a \\
  &< (c + d)(c + d + 1) / 2 \\
  &\leq (c + d)(c + d + 1) / 2 + c \\
  &= f(c, d)
\end{align*}
\]

But this means that \( f(a, b) < f(c, d) \), contradicting that \( f(a, b) = f(c, d) \). We have reached a contradiction, so our assumption must have been wrong. Thus \( a + b = c + d \). Given this, we have that

\[
\begin{align*}
  f(a, b) &= f(c, d) \\
  (a + b)(a + b + 1) / 2 + a &= (c + d)(c + d + 1) / 2 + c
\end{align*}
\]
**Theorem:** Let $f(a, b) = (a + b)(a + b + 1) / 2 + a$. Then $f$ is injective.

**Proof:** Consider any $(a, b), (c, d) \in \mathbb{N}^2$ such that $f(a, b) = f(c, d)$. We will show that $(a, b) = (c, d)$.

First, we will show that $a + b = c + d$. To do this, assume for the sake of contradiction that $a + b \neq c + d$. Then either $a + b < c + d$ or $a + b > c + d$. Assume without loss of generality that $a + b < c + d$.

By our lemma, we know that $m = a + b$ is the largest natural number such that $f(a, b) \leq m(m + 1) / 2$. Since $a + b < c + d$, this means that

$$f(a, b) = (a + b)(a + b + 1) / 2 + a < (c + d)(c + d + 1) / 2 \leq (c + d)(c + d + 1) / 2 + c = f(c, d)$$

But this means that $f(a, b) < f(c, d)$, contradicting that $f(a, b) = f(c, d)$. We have reached a contradiction, so our assumption must have been wrong. Thus $a + b = c + d$. Given this, we have that

$$f(a, b) = f(c, d)$$

$$(a + b)(a + b + 1) / 2 + a = (c + d)(c + d + 1) / 2 + c$$

$$(a + b)(a + b + 1) / 2 + a = (a + b)(a + b + 1) / 2 + c$$
Theorem: Let \( f(a, b) = (a + b)(a + b + 1) / 2 + a \). Then \( f \) is injective.

Proof: Consider any \((a, b), (c, d) \in \mathbb{N}^2\) such that \( f(a, b) = f(c, d) \). We will show that \((a, b) = (c, d)\).

First, we will show that \( a + b = c + d \). To do this, assume for the sake of contradiction that \( a + b \neq c + d \). Then either \( a + b < c + d \) or \( a + b > c + d \). Assume without loss of generality that \( a + b < c + d \).

By our lemma, we know that \( m = a + b \) is the largest natural number such that \( f(a, b) \leq m(m + 1) / 2 \). Since \( a + b < c + d \), this means that

\[
\begin{align*}
    f(a, b) &= (a + b)(a + b + 1) / 2 + a \\
    &< (c + d)(c + d + 1) / 2 \\
    &\leq (c + d)(c + d + 1) / 2 + c \\
    &= f(c, d)
\end{align*}
\]

But this means that \( f(a, b) < f(c, d) \), contradicting that \( f(a, b) = f(c, d) \). We have reached a contradiction, so our assumption must have been wrong. Thus \( a + b = c + d \). Given this, we have that

\[
\begin{align*}
    f(a, b) &= f(c, d) \\
    (a + b)(a + b + 1) / 2 + a &= (c + d)(c + d + 1) / 2 + c \\
    (a + b)(a + b + 1) / 2 + a &= (a + b)(a + b + 1) / 2 + c \\
    a &= c
\end{align*}
\]
Theorem: Let \( f(a, b) = (a + b)(a + b + 1) / 2 + a \). Then \( f \) is injective.

Proof: Consider any \((a, b), (c, d) \in \mathbb{N}^2\) such that \( f(a, b) = f(c, d) \). We will show that \((a, b) = (c, d)\).

First, we will show that \( a + b = c + d \). To do this, assume for the sake of contradiction that \( a + b \neq c + d \). Then either \( a + b < c + d \) or \( a + b > c + d \). Assume without loss of generality that \( a + b < c + d \).

By our lemma, we know that \( m = a + b \) is the largest natural number such that \( f(a, b) \leq m(m + 1) / 2 \). Since \( a + b < c + d \), this means that

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f(a, b) &= (a + b)(a + b + 1) / 2 + a \\
&< (c + d)(c + d + 1) / 2 \\
&\leq (c + d)(c + d + 1) / 2 + c \\
&= f(c, d)
\end{align*}
\]

But this means that \( f(a, b) < f(c, d) \), contradicting that \( f(a, b) = f(c, d) \). We have reached a contradiction, so our assumption must have been wrong. Thus \( a + b = c + d \). Given this, we have that

\[
\begin{align*}
f(a, b) &= f(c, d) \\
(a + b)(a + b + 1) / 2 + a &= (c + d)(c + d + 1) / 2 + c \\
(a + b)(a + b + 1) / 2 + a &= (a + b)(a + b + 1) / 2 + c \\
&\quad \quad a = c
\end{align*}
\]

Since \( a = c \) and \( a + b = c + d \), we have that \( b = d \).
Theorem: Let \( f(a, b) = (a + b)(a + b + 1) / 2 + a \). Then \( f \) is injective.

Proof: Consider any \((a, b), (c, d) \in \mathbb{N}^2\) such that \( f(a, b) = f(c, d) \). We will show that \((a, b) = (c, d)\).

First, we will show that \( a + b = c + d \). To do this, assume for the sake of contradiction that \( a + b \neq c + d \). Then either \( a + b < c + d \) or \( a + b > c + d \). Assume without loss of generality that \( a + b < c + d \).

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\[
\begin{align*}
f(a, b) &= (a + b)(a + b + 1) / 2 + a \\
&< (c + d)(c + d + 1) / 2 \\
&\leq (c + d)(c + d + 1) / 2 + c \\
&= f(c, d)
\end{align*}
\]

But this means that \( f(a, b) < f(c, d) \), contradicting that \( f(a, b) = f(c, d) \). We have reached a contradiction, so our assumption must have been wrong. Thus \( a + b = c + d \). Given this, we have that

\[
\begin{align*}
f(a, b) &= f(c, d) \\
(a + b)(a + b + 1) / 2 + a &= (c + d)(c + d + 1) / 2 + c \\
(a + b)(a + b + 1) / 2 + a &= (a + b)(a + b + 1) / 2 + c \\
a &= c
\end{align*}
\]

Since \( a = c \) and \( a + b = c + d \), we have that \( b = d \). Thus \((a, b) = (c, d)\), as required.
Theorem: Let $f(a, b) = \frac{(a + b)(a + b + 1)}{2} + a$. Then $f$ is injective.

Proof: Consider any $(a, b), (c, d) \in \mathbb{N}^2$ such that $f(a, b) = f(c, d)$. We will show that $(a, b) = (c, d)$.

First, we will show that $a + b = c + d$. To do this, assume for the sake of contradiction that $a + b \neq c + d$. Then either $a + b < c + d$ or $a + b > c + d$. Assume without loss of generality that $a + b < c + d$.

By our lemma, we know that $m = a + b$ is the largest natural number such that $f(a, b) \leq m(m + 1) / 2$. Since $a + b < c + d$, this means that

$$f(a, b) = \frac{(a + b)(a + b + 1)}{2} + a$$
$$< \frac{(c + d)(c + d + 1)}{2}$$
$$\leq \frac{(c + d)(c + d + 1)}{2} + c$$
$$= f(c, d)$$

But this means that $f(a, b) < f(c, d)$, contradicting that $f(a, b) = f(c, d)$. We have reached a contradiction, so our assumption must have been wrong. Thus $a + b = c + d$. Given this, we have that

$$f(a, b) = f(c, d)$$
$$\frac{(a + b)(a + b + 1)}{2} + a = \frac{(c + d)(c + d + 1)}{2} + c$$
$$a = c$$

Since $a = c$ and $a + b = c + d$, we have that $b = d$. Thus $(a, b) = (c, d)$, as required. ■