Mathematical Logic
Part Two
Announcements

- Problem Set 2 and Checkpoint 3 graded.
  - Will be returned at end of lecture.
- Problem Set 3 due this Friday at 2:15PM.
  - Stop by office hours questions!
  - Email cs103-aut1213-staff@lists.stanford.edu with questions!
First-Order Logic
What is First-Order Logic?

- **First-order logic** is a logical system for reasoning about properties of objects.
- Augments the logical connectives from propositional logic with
  - predicates that describe properties of objects, and
  - functions that map objects to one another,
  - quantifiers that allow us to reason about multiple objects simultaneously.
The Universe of Propositional Logic
The Universe of Propositional Logic

\[ p \land q \rightarrow \neg r \lor \neg s \]
The Universe of Propositional Logic

\( p \land q \rightarrow \neg r \lor \neg s \)
The Universe of Propositional Logic

\[ p \land q \rightarrow \neg r \lor \neg s \]
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\[ p \land q \rightarrow \neg r \lor \neg s \]
Propositional Logic

• In propositional logic, each variable represents a **proposition**, which is either true or false.

• Consequently, we can directly apply connectives to propositions:
  - $p \rightarrow q$
  - $\neg p \land q$

• The truth or falsity of a statement can be determined by plugging in the truth values for the input propositions and computing the result.

• We can see all possible truth values for a statement by checking all possible truth assignments to its variables.
The Universe of First-Order Logic
The Universe of First-Order Logic
The Universe of First-Order Logic

The Sun
The Universe of First-Order Logic

- The Sun
- Venus
- The Moon
The Universe of First-Order Logic

The Sun

The Morning Star

Venus

The Moon
The Universe of First-Order Logic

The Sun

Venus

The Morning Star

The Moon

The Evening Star
First-Order Logic

- In first-order logic, each variable refers to some object in a set called the **domain of discourse**.
- Some objects may have multiple names.
- Some objects may have no name at all.
First-Order Logic

● In first-order logic, each variable refers to some object in a set called the **domain of discourse**.

● Some objects may have multiple names.

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Propositional vs. First-Order Logic

- Because propositional variables are either true or false, we can directly apply connectives to them.
  - \( p \rightarrow q \)
  - \( \neg p \leftrightarrow q \land r \)
- Because first-order variables refer to arbitrary objects, it does not make sense to apply connectives to them.
  - \( \textit{Venus} \rightarrow \textit{Sun} \)
  - \( 137 \leftrightarrow \neg 42 \)

\textit{This is not C!}
Reasoning about Objects

• To reason about objects, first-order logic uses **predicates**.

• Examples:
  • `GottaGetDownOn(Friday)`
  • `LookingForwardTo(Weekend)`
  • `ComesAfterwards(Sunday, Saturday)`

• Predicates can take any number of arguments, but each predicate has a fixed number of arguments (called its **arity**)

• Applying a predicate to arguments produces a proposition, which is either true or false.
First-Order Sentences

- Sentences in first-order logic can be constructed from predicates applied to objects:
  \[ \text{LikesToEat}(V, M) \land \text{Near}(V, M) \rightarrow \text{WillEat}(V, M) \]
  \[ \text{Cute}(t) \rightarrow \text{Dikdik}(t) \lor \text{Kitty}(t) \lor \text{Puppy}(t) \]
  \[ x < 8 \rightarrow x < 137 \]

The notation \( x < 8 \) is just a shorthand for something like \text{LessThan}(x, 8)\. Binary predicates in math are often written like this, but symbols like \(<\) are not a part of first-order logic.
Equality

• First-order logic is equipped with a special predicate $=$ that says whether two objects are equal to one another.
• Equality is a part of first-order logic, just as $\rightarrow$ and $\neg$ are.
• Examples:
  
  $\text{MorningStar} = \text{EveningStar}$
  
  $\text{Glenda} = \text{GoodWitchOfTheNorth}$
• Equality can only be applied to objects; to see if propositions are equal, use $\leftrightarrow$. 
For notational simplicity, define $\neq$ as

$$x \neq y \equiv \neg(x = y)$$
Expanding First-Order Logic

\[ x < 8 \land y < 8 \rightarrow x + y < 16 \]
Expanding First-Order Logic

\[ x < 8 \land y < 8 \rightarrow x + y < 16 \]

Why is this allowed?
Functions

• First-order logic allows functions that return objects associated with other objects.

• Examples:
  
  $x + y$
  
  $\text{LengthOf}(\text{path})$
  
  $\text{MedianOf}(x, y, z)$

• As with predicates, functions can take in any number of arguments, but each function has a fixed arity.

• Functions evaluate to objects, not propositions.

• There is no syntactic way to distinguish functions and predicates; you'll have to look at how they're used.
How would we translate the statement

“For any natural number $n$, $n$ is even iff $n^2$ is even”

into first-order logic?
Quantifiers

• The biggest change from propositional logic to first-order logic is the use of **quantifiers**.

• A **quantifier** is a statement that expresses that some property is true for some or all choices that could be made.

• Useful for statements like “for every action, there is an equal and opposite reaction.”
“For any natural number \( n \),
\( n \) is even iff \( n^2 \) is even”
“For any natural number $n$, $n$ is even iff $n^2$ is even”

\[ \forall n. (n \in \mathbb{N} \rightarrow (\text{Even}(n) \leftrightarrow \text{Even}(n^2))) \]
"For any natural number \( n \), \( n \) is even iff \( n^2 \) is even”

\[ \forall n. \ (n \in \mathbb{N} \rightarrow (\text{Even}(n) \leftrightarrow \text{Even}(n^2))) \]

\( \forall \) is the **universal quantifier**
and says “for any choice of \( n \),
the following is true.”
The Universal Quantifier

- A statement of the form $\forall x. \psi$ asserts that for every choice of $x$ in our domain, $\psi$ is true.
- Examples:

  \[
  \forall v. (\text{Velociraptor}(v) \rightarrow \text{WillEat}(v, \text{me}))
  \]
  \[
  \forall n. (n \in \mathbb{N} \rightarrow (\text{Even}(n) \leftrightarrow \neg \text{Odd}(n)))
  \]
  \[
  \text{Tallest}(x) \rightarrow \forall y. (x \neq y \rightarrow \text{IsShorterThan}(y, x))
  \]
Some velociraptor can open windows.
Some velociraptor can open windows.

$\exists v. (\text{Velociraptor}(v) \land \text{OpensWindows}(v))$
Some velociraptor can open windows.

\[ \exists v. \ (\text{Velociraptor}(v) \land \text{OpensWindows}(v)) \]
The Existential Quantifier

• A statement of the form $\exists x. \psi$ asserts that for some choice of $x$ in our domain, $\psi$ is true.

• Examples:

  $\exists x. (Even(x) \land Prime(x))$
  $\exists x. (TallerThan(x, me) \land LighterThan(x, me))$
  $(\exists x. Appreciates(x, me)) \rightarrow Happy(me)$
Operator Precedence (Again)

• When writing out a formula in first-order logic, the quantifiers ∀ and ∃ have precedence just below ¬.

• Thus

\[ \forall x. \, P(x) \lor R(x) \rightarrow Q(x) \]

is interpreted as

\[ (((\forall x. \, P(x))) \lor R(x)) \rightarrow Q(x) \]

rather than

\[ \forall x. \, ((P(x) \lor R(x)) \rightarrow Q(x)) \]
Combining Quantifiers

- Most interesting statements in first-order logic require a combination of quantifiers.
- Example: “Everyone loves someone else.”
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\[
\forall x. \exists y. (x \neq y \land Loves(x, y))
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Combining Quantifiers

- Most interesting statements in first-order logic require a combination of quantifiers.
- Example: “There is someone everyone else loves.”

\[ \exists y. \forall x. (x \neq y \rightarrow Loves(x, y)) \]
∀x. ∃y. (x \neq y \land Loves(x, y))
\exists y. \forall x. (x \neq y \rightarrow Loves(x, y))
∃y. ∀x. (x ≠ y → Loves(x, y))

This person does not love anyone else.
∀x. ∃y. (x ≠ y ∧ Loves(x, y))
\[ \forall x. \exists y. (x \neq y \land Loves(x, y)) \]

No one here is universally loved.
\((\forall x. \exists y. (x \neq y \land Loves(x, y))) \land \exists y. \forall x. (x \neq y \rightarrow Loves(x, y))\)
The statement

∀x. ∃y. P(x, y)

means “For any choice of x, there is some choice of y where P(x, y).”
The statement

\[ \exists y. \forall x. P(x, y) \]

means “There is some choice of \( y \) where for any choice of \( x \), \( P(x, y) \).”
Order matters when mixing existential and universal quantifiers!
A Note on the Checkpoints...
This Doesn't Work!

Theorem: If $R$ is transitive, then $R^{-1}$ is transitive.

Proof: Consider any $a$, $b$, and $c$ such that $aRb$ and $b Rc$. Since $R$ is transitive, we have $aRc$. Since $aRb$ and $bRc$, we have $bR^{-1}a$ and $cR^{-1}b$. Since we have $aRc$, we have $cR^{-1}a$. Thus $cR^{-1}b$, $bR^{-1}a$, and $cR^{-1}a$. ■

This proves

$\forall a. \forall b. \forall c. \ (aRb \land bRc \rightarrow cR^{-1}b \land bR^{-1}a \land cR^{-1}a)$

You need to show

$\forall a. \forall b. \forall c. \ (aR^{-1}b \land bR^{-1}c \rightarrow aR^{-1}c)$
Don't get tripped up by definitions!

To directly prove that $p \rightarrow q$, assume $p$ and prove $q$. 
A Correct Proof

\[ \forall a. \forall b. \forall c. \ (aR^{-1}b \land bR^{-1}c \rightarrow aR^{-1}c) \]

**Theorem:** If \( R \) is transitive, then \( R^{-1} \) is transitive.

**Proof:** Consider any \( a, b, \) and \( c \) such that \( aR^{-1}b \) and \( bR^{-1}c \). We will prove \( aR^{-1}c \). Since \( aR^{-1}b \) and \( bR^{-1}c \), we have that \( bRa \) and \( cRb \). Since \( cRb \) and \( bRa \), by transitivity we know \( cRa \). Since \( cRa \), we have \( aR^{-1}c \), as required. \( \blacksquare \)
Back to First-Order Logic...
A Bad Translation

Everyone who can outrun velociraptors won't get eaten.

∀x. (FasterThanVelociraptors(x) ∧ ¬WillBeEaten(x))
A Bad Translation

Everyone who can outrun velociraptors won't get eaten.

\[ \forall x. (\text{FasterThanVelociraptors}(x) \land \neg \text{WillBeEaten}(x)) \]

What happens if \( x \) refers to someone slower than velociraptors who does get eaten?
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What happens if x refers to someone slower than velociraptors who does get eaten?
“Whenever $P(x)$, then $Q(x)$” translates as

$\forall x. (P(x) \rightarrow Q(x))$
Another Bad Translation

There is some velociraptor that can open windows and eat me.

∀x. (Velociraptor(x) ∧ OpensWindows(x) → EatsMe(x))
Another Bad Translation

There is some velociraptor that can open windows and eat me.

\[ \exists x. (Velociraptor(x) \land \text{OpensWindows}(x) \rightarrow \text{EatsMe}(x)) \]

What happens if

1. The above statement is false, but
2. \( x \) refers to me (I'm not a velociraptor!)
Another Bad Translation

There is some velociraptor that can open windows and eat me.

$$\exists x. \ (\text{Velociraptor}(x) \land \text{OpensWindows}(x) \rightarrow \text{EatsMe}(x))$$

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What happens if

1. The above statement is false, but
2. $x$ refers to me (I’m not a velociraptor!)
“There is some $P(x)$ where $Q(x)$” translates as

$\exists x. (P(x) \land Q(x))$
The Takeaway Point

- Be careful when translating statements into first-order logic!
- $\forall$ is usually paired with $\to$.
- $\exists$ is usually paired with $\land$. 
Quantifying Over Sets

- The notation
  \[ \forall x \in S. \ P(x) \]
  means “for any element \( x \) of set \( S \), \( P(x) \) holds.”
- This is not technically a part of first-order logic; it is a shorthand for
  \[ \forall x. \ (x \in S \rightarrow P(x)) \]
- How might we encode this concept?

  Answer: \[ \exists x \in S. \ P(x) \]

  Note the use of \( \land \) instead of \( \rightarrow \) here.
Quantifying Over Sets

- The syntax
  \[ \forall x \in S. \varphi \]
  \[ \exists x \in S. \varphi \]
  is allowed for quantifying over sets.

- In CS103, please do not use variants of this syntax.

- Please don't do things like this:
  \[ \forall x \text{ with } P(x). \ Q(x) \]
  \[ \forall y \text{ such that } P(y) \land Q(y). \ R(y). \]
Translating into First-Order Logic

• First-order logic has great expressive power and is often used to formally encode mathematical definitions.

• Let's go provide rigorous definitions for the terms we've been using so far.
Set Theory

“Two sets are equal iff they contain the same elements.”

\[ S = T \iff \forall x. (x \in S \iff x \in T) \]

Every possible element is either in both \( S \) and \( T \), or it's in neither \( S \) nor \( T \).
Set Theory

“Two sets are equal iff they contain the same elements.”

\[ S = T \iff \forall x. (x \in S \iff x \in T) \]

Is something missing?
Set Theory

“Two sets are equal iff they contain the same elements.”

\[
\forall S. \forall T. (S = T \leftrightarrow \forall x. (x \in S \leftrightarrow x \in T))
\]

These quantifiers are critical here, but they don’t appear anywhere in the English. Many statements asserting a general claim is true are implicitly universally quantified.