Extra Practice Problems 2

Here are some extra practice problems on topics that were popular on the Google Moderator site. We'll release solutions to these problems on Monday.

**Problem One: Binary Relations**

i. Let $\Sigma$ be some alphabet and $L$ be a language over $\Sigma$. Two strings $x, y \in \Sigma^*$ are called *indistinguishable* relative to $L$, denoted $x \equiv_L y$, iff for every $w \in \Sigma^*$ we have $xw \in L$ iff $yw \in L$. This relation arises in a more complete version of the Myhill-Nerode theorem. Prove that $\equiv_L$ is an equivalence relation over $\Sigma^*$.

ii. Prove that a binary relation is a total order iff it is total, antisymmetric, and transitive.

iii. How many equivalence relations are there over the set $\{a, b, c\}$?

**Problem Two: Injections, Surjections, and Bijections**

i. Find functions $f : A \to B$ and $g : B \to C$ where $g \circ f$ is a bijection but neither $f$ nor $g$ are bijections.

ii. An *involution* is a function $f : A \to A$ where $f(f(x)) = x$. Prove that all involutions are bijections.

**Problem Three: Regular and Nonregular Languages**

i. Let $\Sigma = \{a, b\}$ and let $L = \{a^n b^m \mid n, m \in \mathbb{N} \text{ and } n \neq m\}$. Prove that $L$ is not regular.

ii. Let $\Sigma = \{a, b\}$ and let $L = \{w \in \Sigma^* \mid w \text{ is a palindrome}\}$. Prove that $L$ is not regular.

iii. Let $\Sigma = \{a\}$ and let $L = \{w \in \Sigma^* \mid w \text{ is a palindrome}\}$. Prove that $L$ is regular.
Problem Four: Closure Properties and Nonregular Languages

The regular languages are closed under intersection; if $L_1$ and $L_2$ are regular, then $L_1 \cap L_2$ is regular as well.

i. Is the converse of this statement true? That is, if $L_1 \cap L_2$ is regular, then are $L_1$ and $L_2$ regular? Prove or disprove this statement.

The fact that regular languages are closed under intersection can be used to prove that various languages are not regular without using the Myhill-Nerode theorem. For example, let $\Sigma = \{a, b\}$ and let $L = \{ w \in \Sigma^* \mid w \text{ has the same number of } a's \text{ and } b's \}$. This language is similar to the language $L' = \{ a^n b^n \mid n \in \mathbb{N} \}$, which we know isn't regular. Using this fact, it's possible to prove that $L$ can't be regular.

ii. Find a regular language $R$ such that $L \cap R = L'$.

iii. Using your result from (ii), prove that $L$ is not regular.

Problem Five: R, RE, co-RE Languages

i. Prove that the $R$ languages are closed under intersection.

ii. Prove that the $RE$ languages are closed under intersection.

iii. Prove that the co-$RE$ languages are closed under intersection.

Problem Six: R, RE, and co-RE Languages II

In lecture, we sketched a proof that if $M$ is a recognizer for $L$, then the machine $M'$ formed by swapping the accept and reject states of $M$ is a co-recognizer for $\overline{L}$. However, the machine $M'$ won't in general be a recognizer for $L$.

Prove that the machine $M'$ formed by swapping the accept and reject states of $M$ is a recognizer for the language $\overline{L}$ iff $M$ is a decider.

Problem Seven: Mapping Reducibility

i. A nontrivial language is a language other than $\emptyset$ and $\Sigma^*$. Prove that all nontrivial decidable languages are mapping reducible to one another.

ii. Find an example of an $RE$ language and a co-$RE$ language that are mapping reducible to one another.

iii. Prove that $L \leq_M \Sigma^*$ iff $L = \Sigma^*$.