Mathematical Induction
Part Two
Announcements

- Problem Set 1 due Friday, October 4 at the start of class.
- Problem Set 1 checkpoints graded, will be returned at end of lecture.
  - Afterwards, will be available in the filing cabinets in the Gates Open Area near the submissions box.
The principle of mathematical induction states that if for some $P(n)$ the following hold:

- $P(0)$ is true
- For any $n \in \mathbb{N}$, we have $P(n) \rightarrow P(n + 1)$

then...

...and it stays true...

...then it's always true.

For any $n \in \mathbb{N}$, $P(n)$ is true.
Theorem: For any natural number $n$, $\sum_{i=0}^{n-1} 2^i = 2^n - 1$

Proof: By induction. Let $P(n)$ be $P(n) \equiv \sum_{i=0}^{n-1} 2^i = 2^n - 1$

For our base case, we need to show $P(0)$ is true, meaning that $\sum_{i=0}^{0} 2^i = 2^0 - 1$

Since $2^0 - 1 = 0$ and the left-hand side is the empty sum, $P(0)$ holds.

For the inductive step, assume that for some $n \in \mathbb{N}$, that $P(n)$ holds, so $\sum_{i=0}^{n-1} 2^i = 2^n - 1$

We need to show that $P(n + 1)$ holds, meaning that $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$

To see this, note that $\sum_{i=0}^{n} 2^i = (\sum_{i=0}^{n-1} 2^i) + 2^n = 2^n - 1 + 2^n = 2(2^n) - 1 = 2^{n+1} - 1$

Thus $P(n + 1)$ holds, completing the induction. ■
Induction in Practice

- Typically, a proof by induction will not explicitly state $P(n)$.
- Rather, the proof will describe $P(n)$ implicitly and leave it to the reader to fill in the details.
- Provided that there is sufficient detail to determine
  - what $P(n)$ is,
  - that $P(0)$ is true, and that
  - whenever $P(n)$ is true, $P(n + 1)$ is true,
the proof is usually valid.
Theorem: For any natural number \( n \), \( \sum_{i=0}^{n-1} 2^i = 2^n - 1 \)

Proof: By induction on \( n \). For our base case, if \( n = 0 \), note that

\[
\sum_{i=0}^{n-1} 2^i = 0 = 2^0 - 1
\]

and the theorem is true for 0.

For the inductive step, assume that for some \( n \) the theorem is true. Then we have that

\[
\sum_{i=0}^{n} 2^i = \sum_{i=0}^{n-1} i + 2^n = 2^n - 1 + 2^n = 2(2^n) - 1 = 2^{n+1} - 1
\]

so the theorem is true for \( n + 1 \), completing the induction. ■
Variations on Induction: Starting Later
Induction Starting at 0

- To prove that $P(n)$ is true for all natural numbers greater than or equal to 0:
  - Show that $P(0)$ is true.
  - Show that for any $n \geq 0$, that $P(n) \rightarrow P(n + 1)$.
  - Conclude $P(n)$ holds for all natural numbers greater than or equal to 0.
Induction Starting at $k$

- To prove that $P(n)$ is true for all natural numbers greater than or equal to $k$:
  - Show that $P(k)$ is true.
  - Show that for any $n \geq k$, that $P(n) \rightarrow P(n + 1)$.
  - Conclude $P(n)$ holds for all natural numbers greater than or equal to $k$.
- Pretty much identical to before, except that the induction begins at a later point.
Convex Polygons

- A **convex polygon** is a polygon where, for any two points in or on the polygon, the line between those points is contained within the polygon.
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Useful Fact

• **Theorem:** Any line drawn through a convex polygon splits that polygon into two convex polygons.
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Summing Angles

• Interesting fact: the sum of the angles in a convex polygon depends only on the number of vertices in the polygon, not the shape of that polygon.

• **Theorem:** For any convex polygon with $n$ vertices, the sum of the angles in that polygon is $(n - 2) \cdot 180^\circ$.
  
  • Angles in a triangle add up to 180°.
  • Angles in a quadrilateral add up to 360°.
  • Angles in a pentagon add up to 540°.
Theorem: The sum of the angles in any convex polygon with $n$ vertices is $(n - 2) \cdot 180^\circ$.

Proof: By induction. Let $P(n)$ be "all convex polygons with $n$ vertices have angles that sum to $(n - 2) \cdot 180^\circ$." We will prove $P(n)$ holds for all $n \in \mathbb{N}$ where $n \geq 3$. As a base case, we prove $P(3)$: the sum of the angles in any convex polygon with three vertices is $180^\circ$. Any such polygon is a triangle, so its angles sum to $180^\circ$.

For the inductive step, assume for some $n \geq 3$ that $P(n)$ holds and all convex polygons with $n$ vertices have angles that sum to $(n - 2) \cdot 180^\circ$. We prove $P(n+1)$, that the sum of the angles in any convex polygon with $n+1$ vertices is $(n - 1) \cdot 180^\circ$. Let $A$ be an arbitrary convex polygon with $n+1$ vertices. Take any three consecutive vertices in $A$ and draw a line from the first to the third, as shown here:

The sum of the angles in $A$ is equal to the sum of the angles in triangle $B$ ($180^\circ$) and the sum of the angles in convex polygon $C$ (which, by the IH, is $(n - 2) \cdot 180^\circ$).

Therefore, the sum of the angles in $A$ is $(n - 1) \cdot 180^\circ$. Thus $P(n+1)$ holds, completing the induction. ■
Theorem: The sum of the angles in any convex polygon with \( n \) vertices is \((n - 2) \cdot 180°\).

Proof: By induction.

Let \( P(n) \) be "all convex polygons with \( n \) vertices have angles that sum to \((n - 2) \cdot 180°\)." We will prove \( P(n) \) holds for all \( n \in \mathbb{N} \) where \( n \geq 3 \). As a base case, we prove \( P(3) \): the sum of the angles in any convex polygon with three vertices is 180°. Any such polygon is a triangle, so its angles sum to 180°.

For the inductive step, assume for some \( n \geq 3 \) that \( P(n) \) holds and all convex polygons with \( n \) vertices have angles that sum to \((n - 2) \cdot 180°\).

We prove \( P(n+1) \), that the sum of the angles in any convex polygon with \( n+1 \) vertices is \((n - 1) \cdot 180°\). Let \( A \) be an arbitrary convex polygon with \( n+1 \) vertices. Take any three consecutive vertices in \( A \) and draw a line from the first to the third, as shown here:

\[ \text{The sum of the angles in } A \text{ is equal to the sum of the angles in triangle } B \text{ (180°) and the sum of the angles in convex polygon } C \text{ (which, by the IH, is } (n - 2) \cdot 180°\).} \]

Therefore, the sum of the angles in \( A \) is \((n - 1) \cdot 180°\). Thus \( P(n+1) \) holds, completing the induction. ■
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For the inductive step, assume for some \( n \geq 3 \) that \( P(n) \) holds and all convex polygons with \( n \) vertices have angles that sum to \( (n-2) \cdot 180^\circ \). We prove \( P(n+1) \), that the sum of the angles in any convex polygon with \( n+1 \) vertices is \( (n-1) \cdot 180^\circ \).
**Theorem:** The sum of the angles in any convex polygon with \( n \) vertices is \((n - 2) \cdot 180^\circ\).

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**Theorem:** The sum of the angles in any convex polygon with $n$ vertices is $(n - 2) \cdot 180^\circ$.

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![Diagram showing the proof](image)

The sum of the angles in $A$ is equal to the sum of the angles in triangle $B$ (180°) and the sum of the angles in convex polygon $C$ (which, by the IH, is $(n - 2) \cdot 180^\circ$).
Theorem: The sum of the angles in any convex polygon with \( n \) vertices is \((n - 2) \cdot 180^\circ\).

Proof: By induction. Let \( P(n) \) be “all convex polygons with \( n \) vertices have angles that sum to \((n - 2) \cdot 180^\circ\).” We will prove \( P(n) \) holds for all \( n \in \mathbb{N} \) where \( n \geq 3 \). As a base case, we prove \( P(3) \): the sum of the angles in any convex polygon with three vertices is \( 180^\circ \). Any such polygon is a triangle, so its angles sum to \( 180^\circ \).

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Theorem: The sum of the angles in any convex polygon with \( n \) vertices is \((n - 2) \cdot 180^\circ\).

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A Different Proof Approach
A Different Proof Approach
Using Induction

- Many proofs that work by induction can be written non-inductively by using similar arguments.
- Don't feel that you *have* to use induction; it's one of many tools in your proof toolbox!
Variations on Induction: Bigger Steps
Subdividing a Square
Subdividing a Square
Subdividing a Square
Subdividing a Square
Subdividing a Square
For what values of $n$ can a square be subdivided into $n$ squares?
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The Key Insight
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The Key Insight
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• If we can subdivide a square into \( n \) squares, we can also subdivide it into \( n + 3 \) squares.

• Since we can subdivide a bigger square into 6, 7, and 8 squares, we can subdivide a square into \( n \) squares for any \( n \geq 6 \):
  - For multiples of three, start with 6 and keep adding three squares until \( n \) is reached.
  - For numbers congruent to one modulo three, start with 7 and keep adding three squares until \( n \) is reached.
  - For numbers congruent to two modulo three, start with 8 and keep adding three squares until \( n \) is reached.
Theorem: For any $n \geq 6$, it is possible to subdivide a square into $n$ squares.
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Proof: By induction.
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Proof: By induction. Let \( P(n) \) be “a square can be subdivided into \( n \) squares.”
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Proof: By induction. Let \( P(n) \) be “a square can be subdivided into \( n \) squares.” We will prove \( P(n) \) holds for all \( n \geq 6 \).
Theorem: For any $n \geq 6$, it is possible to subdivide a square into $n$ squares.

Proof: By induction. Let $P(n)$ be “a square can be subdivided into $n$ squares.” We will prove $P(n)$ holds for all $n \geq 6$.

As our base cases, we prove $P(6)$, $P(7)$, and $P(8)$, that a square can be subdivided into 6, 7, and 8 squares.
**Theorem:** For any $n \geq 6$, it is possible to subdivide a square into $n$ squares.

**Proof:** By induction. Let $P(n)$ be “a square can be subdivided into $n$ squares.” We will prove $P(n)$ holds for all $n \geq 6$.

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As our base cases, we prove $P(6)$, $P(7)$, and $P(8)$, that a square can be subdivided into 6, 7, and 8 squares. This is shown here:

For the inductive step, assume that for some $n \geq 6$ that $P(n)$ is true and a square can be subdivided into $n$ squares.
**Theorem:** For any $n \geq 6$, it is possible to subdivide a square into $n$ squares.

**Proof:** By induction. Let $P(n)$ be “a square can be subdivided into $n$ squares.” We will prove $P(n)$ holds for all $n \geq 6$.

As our base cases, we prove $P(6)$, $P(7)$, and $P(8)$, that a square can be subdivided into 6, 7, and 8 squares. This is shown here:

![Subdivisions](image)

For the inductive step, assume that for some $n \geq 6$ that $P(n)$ is true and a square can be subdivided into $n$ squares. We prove $P(n + 3)$, that a square can be subdivided into $n + 3$ squares.
Theorem: For any $n \geq 6$, it is possible to subdivide a square into $n$ squares.

Proof: By induction. Let $P(n)$ be “a square can be subdivided into $n$ squares.” We will prove $P(n)$ holds for all $n \geq 6$.

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For the inductive step, assume that for some $n \geq 6$ that $P(n)$ is true and a square can be subdivided into $n$ squares. We prove $P(n + 3)$, that a square can be subdivided into $n + 3$ squares. To see this, obtain a subdivision of a square into $n$ squares.
*Theorem:* For any $n \geq 6$, it is possible to subdivide a square into $n$ squares.

*Proof:* By induction. Let $P(n)$ be “a square can be subdivided into $n$ squares.” We will prove $P(n)$ holds for all $n \geq 6$.

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For the inductive step, assume that for some $n \geq 6$ that $P(n)$ is true and a square can be subdivided into $n$ squares. We prove $P(n + 3)$, that a square can be subdivided into $n + 3$ squares. To see this, obtain a subdivision of a square into $n$ squares. Then, choose a square and split it into four equal squares.
**Theorem:** For any $n \geq 6$, it is possible to subdivide a square into $n$ squares.

**Proof:** By induction. Let $P(n)$ be “a square can be subdivided into $n$ squares.” We will prove $P(n)$ holds for all $n \geq 6$.

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**Theorem:** For any \( n \geq 6 \), it is possible to subdivide a square into \( n \) squares.

**Proof:** By induction. Let \( P(n) \) be “a square can be subdivided into \( n \) squares.” We will prove \( P(n) \) holds for all \( n \geq 6 \).

As our base cases, we prove \( P(6) \), \( P(7) \), and \( P(8) \), that a square can be subdivided into 6, 7, and 8 squares. This is shown here:

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Why This Works

- This induction has three consecutive base cases and takes steps of size three.
- Thinking back to our “induction machine” analogy:

\[ P(n) \rightarrow P(n+3) \]
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\[ P(n) \rightarrow P(n+3) \]

\[ P(8) \quad P(9) \quad P(10) \]
Why This Works

- This induction has three consecutive base cases and takes steps of size three.
- Thinking back to our “induction machine” analogy:
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\[ P(n) \rightarrow P(n+3) \]

\[ P(9) \]

\[ P(10) \]

\[ P(11) \]
Why This Works

- This induction has three consecutive base cases and takes steps of size three.
- Thinking back to our “induction machine” analogy:
Generalizing Induction

- When doing a proof by induction:
  - Feel free to use multiple base cases.
  - Feel free to take steps of sizes other than one.
- Just be careful to make sure you cover all the numbers you think that you're covering!
Some Announcements
Variations on Induction: Complete Induction
Review: Induction as a Machine

\[ P(n) \rightarrow P(n + 1) \]

\[ P(0) \]
Review: Induction as a Machine

\[ P(0) \]

\[ P(n) \rightarrow P(n + 1) \]
Review: Induction as a Machine

\[ P(n) \rightarrow P(n + 1) \]
Review: Induction as a Machine

\[ P(n) \rightarrow P(n + 1) \]

\[ P(1) \]
Review: Induction as a Machine

\[ P(1) \]

\[ P(n) \rightarrow P(n + 1) \]
Review: Induction as a Machine

\[ P(n) \rightarrow P(n + 1) \]

\[ P(2) \]
Review: Induction as a Machine

\[ P(n) \rightarrow P(n + 1) \]
Review: Induction as a Machine

\[ P(2) \]

\[ P(n) \rightarrow P(n + 1) \]
Review: Induction as a Machine

\[ P(n) \rightarrow P(n + 1) \]
An Observation
An Observation

\[ P(n) \rightarrow P(n + 1) \]
An Observation

$P(0)$

$P(n) \rightarrow P(n + 1)$

$P(0)$
An Observation

\[ P(n) \rightarrow P(n + 1) \]
An Observation

\[ \text{Observation: } P(n) \to P(n + 1) \]
An Observation

\[ P(n) \rightarrow P(n + 1) \]
An Observation

$P(n) \rightarrow P(n + 1)$

$P(0)$

$P(1)$

$P(2)$
An Observation

$P(n) \rightarrow P(n + 1)$

$P(0)$  $P(1)$  $P(2)$
An Observation

\[ P(n) \rightarrow P(n + 1) \]
An Observation

\[ P(n) \rightarrow P(n + 1) \]
An Observation

$P(n) \rightarrow P(n + 1)$
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\[ P(n) \rightarrow P(n + 1) \]
An Observation

$P(n) \rightarrow P(n + 1)$

$P(0) \quad P(1) \quad P(2) \quad P(3) \quad P(4)$
An Observation

$$P(n) \rightarrow P(n + 1)$$

$P(0)$  $P(1)$  $P(2)$  $P(3)$  $P(4)$
An Observation

\[ P(n) \rightarrow P(n + 1) \]
An Observation

\[ P(n) \rightarrow P(n + 1) \]
An Observation

• In a proof by induction, the inductive step works as follows:
  • Assume that for some particular \( n \) that \( P(n) \) is true.
  • Prove that \( P(n + 1) \) is true.

• Notice: When trying to prove \( P(n + 1) \), we already know \( P(0), P(1), P(2), \ldots, P(n) \) but only assume \( P(n) \) is true.

• Why are we discarding all the intermediary results?
Complete Induction

• If the following are true:
  • $P(0)$ is true, and
  • If $P(0)$, $P(1)$, $P(2)$, ..., $P(n)$ are true, then $P(n+1)$ is true as well.

• Then $P(n)$ is true for all $n \in \mathbb{N}$.

• This is called the principle of complete induction or the principle of strong induction.

  • (A note: this also works starting from a number other than 0; just modify what you're assuming appropriately.)
Proof by Complete Induction

• State that your proof works by complete induction.
• State your choice of $P(n)$.
• Prove the base case: state what $P(0)$ is, then prove it using any technique you'd like.
• Prove the inductive step:
  • State that for some arbitrary $n \in \mathbb{N}$ that you're assuming $P(0)$, $P(1)$, $\ldots$, $P(n)$ (that is, $P(n')$ for all natural numbers $0 \leq n' \leq n$.)
  • State that you are trying to prove $P(n + 1)$ and what $P(n + 1)$ means.
  • Prove $P(n + 1)$ using any technique you'd like.
Example: Polygon Triangulation
Polygon Triangulation

• Given a convex polygon, an elementary triangulation of that polygon is a way of connecting the vertices with lines such that
  • No two lines intersect, and
  • The polygon is converted into a set of triangles.

• Question: How many lines do you have to draw to elementarily triangulate a convex polygon?
Elementary Triangulations
Elementary Triangulations
Elementary Triangulations
Elementary Triangulations
Elementary Triangulations
Elementary Triangulations
Elementary Triangulations
Elementary Triangulations
Elementary Triangulations
Elementary Triangulations
Some Observations

- Every elementary triangulation of the same convex polygon seems to require the same number of lines.
- The number of lines depends on the number of vertices:
  - 5 vertices: 2 lines
  - 6 vertices: 3 lines
  - 8 vertices: 5 lines
- **Conjecture:** Every elementary triangulation of an $n$-vertex convex polygon requires $n - 3$ lines.
Elementary Triangulations
Elementary Triangulations
Elementary Triangulations

\[ n - k + 2 \] vertices

\[ k \] vertices
Theorem: Every elementary triangulation of a convex polygon with \( n \) vertices requires \( n - 3 \) lines.
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Proof: By complete induction.
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Proof: By complete induction. Let \( P(n) \) be “every elementary triangulation of a convex polygon requires \( n-3 \) lines.”
**Theorem:** Every elementary triangulation of a convex polygon with \( n \) vertices requires \( n - 3 \) lines.

**Proof:** By complete induction. Let \( P(n) \) be “every elementary triangulation of a convex polygon requires \( n - 3 \) lines.” We prove \( P(n) \) holds for all \( n \geq 3 \).
Theorem: Every elementary triangulation of a convex polygon with \( n \) vertices requires \( n - 3 \) lines.

Proof: By complete induction. Let \( P(n) \) be “every elementary triangulation of a convex polygon requires \( n - 3 \) lines.” We prove \( P(n) \) holds for all \( n \geq 3 \). As a base case, we prove \( P(3) \): elementarily triangulating a convex polygon with three vertices requires no lines.
Theorem: Every elementary triangulation of a convex polygon with \( n \) vertices requires \( n - 3 \) lines.

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For the inductive step, assume for some \( n \geq 3 \) that \( P(3), P(4), \ldots, P(n) \) are true.
Theorem: Every elementary triangulation of a convex polygon with \( n \) vertices requires \( n - 3 \) lines.

Proof: By complete induction. Let \( P(n) \) be “every elementary triangulation of a convex polygon requires \( n-3 \) lines.” We prove \( P(n) \) holds for all \( n \geq 3 \). As a base case, we prove \( P(3) \): elementarily triangulating a convex polygon with three vertices requires no lines. Any polygons with three vertices is a triangle, so any elementary triangulation of it will have no lines.

For the inductive step, assume for some \( n \geq 3 \) that \( P(3), P(4), \ldots, P(n) \) are true. This means any elementary triangulation of an \( n' \)-vertex convex polygon, where \( 3 \leq n' \leq n \), uses \( n'-3 \) lines.
Theorem: Every elementary triangulation of a convex polygon with \( n \) vertices requires \( n - 3 \) lines.

Proof: By complete induction. Let \( P(n) \) be “every elementary triangulation of a convex polygon requires \( n - 3 \) lines.” We prove \( P(n) \) holds for all \( n \geq 3 \). As a base case, we prove \( P(3) \): elementarily triangulating a convex polygon with three vertices requires no lines. Any polygons with three vertices is a triangle, so any elementary triangulation of it will have no lines.

For the inductive step, assume for some \( n \geq 3 \) that \( P(3) \), \( P(4) \), ..., \( P(n) \) are true. This means any elementary triangulation of an \( n' \)-vertex convex polygon, where \( 3 \leq n' \leq n \), uses \( n' - 3 \) lines. We prove \( P(n+1) \): any elementary triangulation of any \((n+1)\)-vertex convex polygon uses \( n - 2 \) lines.
Theorem: Every elementary triangulation of a convex polygon with \( n \) vertices requires \( n - 3 \) lines.

Proof: By complete induction. Let \( P(n) \) be “every elementary triangulation of a convex polygon requires \( n - 3 \) lines.” We prove \( P(n) \) holds for all \( n \geq 3 \). As a base case, we prove \( P(3) \): elementarily triangulating a convex polygon with three vertices requires no lines. Any polygons with three vertices is a triangle, so any elementary triangulation of it will have no lines.

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Let \( A \) be an arbitrary convex polygon with \( n+1 \) vertices.
Theorem: Every elementary triangulation of a convex polygon with \( n \) vertices requires \( n - 3 \) lines.

Proof: By complete induction. Let \( P(n) \) be "every elementary triangulation of a convex polygon requires \( n - 3 \) lines." We prove \( P(n) \) holds for all \( n \geq 3 \). As a base case, we prove \( P(3) \): elementarily triangulating a convex polygon with three vertices requires no lines. Any polygons with three vertices is a triangle, so any elementary triangulation of it will have no lines.

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Let \( A \) be an arbitrary convex polygon with \( n + 1 \) vertices. Pick any elementary triangulation of \( A \) and select an arbitrary line in that triangulation.
Theorem: Every elementary triangulation of a convex polygon with \( n \) vertices requires \( n - 3 \) lines.

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Let \( A \) be an arbitrary convex polygon with \( n+1 \) vertices. Pick any elementary triangulation of \( A \) and select an arbitrary line in that triangulation. This line splits \( A \) into two smaller convex polygons \( B \) and \( C \), which are also triangulated.
Theorem: Every elementary triangulation of a convex polygon with \( n \) vertices requires \( n - 3 \) lines.

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Theorem: Every elementary triangulation of a convex polygon with $n$ vertices requires $n - 3$ lines.

Proof: By complete induction. Let $P(n)$ be “every elementary triangulation of a convex polygon requires $n-3$ lines.” We prove $P(n)$ holds for all $n \geq 3$. As a base case, we prove $P(3)$: elementarily triangulating a convex polygon with three vertices requires no lines. Any polygons with three vertices is a triangle, so any elementary triangulation of it will have no lines.

For the inductive step, assume for some $n \geq 3$ that $P(3), P(4), \ldots, P(n)$ are true. This means any elementary triangulation of an $n'$-vertex convex polygon, where $3 \leq n' \leq n$, uses $n'-3$ lines. We prove $P(n+1)$: any elementary triangulation of any $(n+1)$-vertex convex polygon uses $n-2$ lines.

Let $A$ be an arbitrary convex polygon with $n+1$ vertices. Pick any elementary triangulation of $A$ and select an arbitrary line in that triangulation. This line splits $A$ into two smaller convex polygons $B$ and $C$, which are also triangulated. Let $k$ be the number of vertices in $B$, meaning $C$ has $(n+1)-k+2 = n-k+3$ vertices. By our inductive hypothesis, any triangulations of $B$ and $C$ must use $k-3$ and $n-k$ lines, respectively.
Theorem: Every elementary triangulation of a convex polygon with \( n \) vertices requires \( n - 3 \) lines.

Proof: By complete induction. Let \( P(n) \) be “every elementary triangulation of a convex polygon requires \( n-3 \) lines.” We prove \( P(n) \) holds for all \( n \geq 3 \). As a base case, we prove \( P(3) \): elementarily triangulating a convex polygon with three vertices requires no lines. Any polygons with three vertices is a triangle, so any elementary triangulation of it will have no lines.

For the inductive step, assume for some \( n \geq 3 \) that \( P(3), P(4), \ldots, P(n) \) are true. This means any elementary triangulation of an \( n' \)-vertex convex polygon, where \( 3 \leq n' \leq n \), uses \( n'-3 \) lines. We prove \( P(n+1) \): any elementary triangulation of any \( (n+1) \)-vertex convex polygon uses \( n-2 \) lines.

Let \( A \) be an arbitrary convex polygon with \( n+1 \) vertices. Pick any elementary triangulation of \( A \) and select an arbitrary line in that triangulation. This line splits \( A \) into two smaller convex polygons \( B \) and \( C \), which are also triangulated. Let \( k \) be the number of vertices in \( B \), meaning \( C \) has \( (n+1)-k+2 = n-k+3 \) vertices. By our inductive hypothesis, any triangulations of \( B \) and \( C \) must use \( k-3 \) and \( n-k \) lines, respectively. Therefore, the total number of lines in the triangulation of \( A \) is \( n-k+k-3+1 = n-2 \).
Theorem: Every elementary triangulation of a convex polygon with \( n \) vertices requires \( n - 3 \) lines.

Proof: By complete induction. Let \( P(n) \) be “every elementary triangulation of a convex polygon requires \( n - 3 \) lines.” We prove \( P(n) \) holds for all \( n \geq 3 \). As a base case, we prove \( P(3) \): elementarily triangulating a convex polygon with three vertices requires no lines. Any polygons with three vertices is a triangle, so any elementary triangulation of it will have no lines.

For the inductive step, assume for some \( n \geq 3 \) that \( P(3), P(4), \ldots, P(n) \) are true. This means any elementary triangulation of an \( n' \)-vertex convex polygon, where \( 3 \leq n' \leq n \), uses \( n' - 3 \) lines. We prove \( P(n+1) \): any elementary triangulation of any \((n+1)\)-vertex convex polygon uses \( n - 2 \) lines.

Let \( A \) be an arbitrary convex polygon with \( n+1 \) vertices. Pick any elementary triangulation of \( A \) and select an arbitrary line in that triangulation. This line splits \( A \) into two smaller convex polygons \( B \) and \( C \), which are also triangulated. Let \( k \) be the number of vertices in \( B \), meaning \( C \) has \((n+1)-k+2 = n-k+3\) vertices. By our inductive hypothesis, any triangulations of \( B \) and \( C \) must use \( k - 3 \) and \( n - k \) lines, respectively. Therefore, the total number of lines in the triangulation of \( A \) is \( n-k+k-3+1 = n-2 \). Thus \( P(n+1) \) holds, completing the induction. ■
Using Complete Induction

- When is it appropriate to use complete induction in contrast to standard induction?
- Depends on the proof approach:
  - Typically, standard induction is used when a problem of size \( n + 1 \) is reduced to a simpler problem of size \( n \).
  - Typically, complete induction is used when the problem of size \( n + 1 \) is split into multiple subproblems of unknown but smaller sizes.
- It is never “wrong” to use complete induction. It just might be unnecessary. We suggest writing drafts of your proofs just in case.
Summary

- Induction doesn't have to start at 0. It's perfectly fine to start induction later on.
- Induction doesn't have to take steps of size 1. It's not uncommon to see other step sizes.
- Induction doesn't have to have a single base case.
- Complete induction lets you assume all prior results, not just the last result.
Next Time

- **Graphs**
  - Representing relationships between objects.
  - Connectivity in graphs.
  - Planar graphs.