Midterm Review Problems III

Here's one last set of midterm review problems. Collectively, this means that you have three sets of review problems, plus tonight's practice midterm, to help you prepare for the exam. We hope that you find these extra problems useful!

We'll release solutions to these problems at the start of Wednesday's lecture.

**Problem One: Set Theory**

A set $S$ is called an *inductive set* if the follow two properties are true about $S$:

- $0 \in S$.
- For any number $x \in S$, the number $x + 1$ is also an element of $S$.

This question asks you to explore various properties of inductive sets.

i. Find two different examples of inductive sets.

ii. Prove that the intersection of any two inductive sets is also an inductive set.

iii. Prove that if $S$ is an inductive set, then $\mathbb{N} \subseteq S$. This proves that $\mathbb{N}$ is, in a sense, the most “fundamental” inductive set. In fact, in foundational mathematics, the set $\mathbb{N}$ is sometimes defined as the inductive set that's a subset of all inductive sets.

**Problem Two: Graph Theory**

A *simple path* in a graph $G$ is a path that does not repeat any nodes or edges.

i. Show that for every $n \geq 3$, there is an undirected graph $G$ with $n$ nodes where the maximum length of a simple path in $G$ is two.

ii. Show that for every $n \geq 3$, there is a directed graph $G$ with $n$ nodes where the maximum length of a simple path in $G$ is one.

iii. Prove or disprove the following statement: if $P_1$ is a simple path from $u$ to $v$ and $P_2$ is a simple path from $v$ to $x$, then the path formed by following $P_1$ and then $P_2$ is a simple path from $u$ to $x$. 
Problem Three: First-Order Logic

This question explores what happens when you interchange the order of quantifiers in a first-order logic statement.

Consider the following two statements in first-order logic:

\[
\exists x \forall y. \text{Loves}(x, y) \\
\forall y . \exists x. \text{Loves}(x, y)
\]

For simplicity, let's assume that the domains of discourse for these two statements are sets of people.

i. Translate these two statements into English.

ii. Prove that these statements are \textit{not} equivalent to one another. To do so, find an example of a group of people where one of these statements is true and the other is false.

iii. One of these statements implies the other. Figure out which statement implies the other, then prove it.