Problem Set #8
Due: Thursday 10 March 2011 at 5 PM.
Last Problem Set

1. Convergence of Random Variables

Let $X_1, X_2, \ldots,$ be iid $\text{Exp} (\lambda)$ random variables and define four new sequences for $n \geq 1$

\[
W_n = \frac{1}{n} \sum_{i=1}^{n} X_i \\
V_n = \min(X_1, X_2, \ldots, X_n) \\
Y_n = \prod_{i=1}^{n} X_i \\
Z_n = \sqrt{n} \times (W_n - 1/\lambda)
\]

(a) Does the sequence $W_n; n = 1, 2, 3, \ldots$ converge in probability? If so, to what does it converge?

(b) Does the sequence $V_n; n = 1, 2, 3, \ldots$ converge in probability? If so, to what does it converge?

(c) Find the range of $\lambda > 0$ such that the sequence $Y_1, Y_2, Y_3 \ldots$ converges to zero in mean square.

(d) Does the sequence $Z_n; n = 1, 2, 3, \ldots$ converge in distribution? If so, to what does it converge?

You must demonstrate convergence to a particular value, not just state it.

2. Pulse amplitude modulation

This problem considers a simple special case of pulse amplitude modulation where a discrete time random process is converted into a continuous time random process by multiplying the former by pulses. For simplicity only a binary iid input process is considered.
Say that we are given an iid binary random process \( \{X_n\} \) with alphabet \( \pm 1 \), each having probability \( \frac{1}{2} \). We form a continuous time random process \( \{X(t)\} \) by assigning

\[
X(t) = X_n; \quad t \in [(n - 1)T, nT),
\]

for a fixed time \( T \). This process can also be described as follows: let \( p(t) \) be a pulse that is 1 for \( t \in [0, T) \) and 0 elsewhere. Define

\[
X(t) = \sum_k X_k p(t - kT).
\]

This is an example of pulse amplitude modulation (PAM). Observe that \( X(t) \) is not stationary, but we can force it to be at least weakly stationary by the trick of inserting a uniform random variable. Let \( U \) be a random variable, uniformly distributed on \( [0, T] \) and independent of the original iid process. Define the random process

\[
Y(t) = X(t - U).
\]

(a) The following integral will be needed. Find

\[
R_p(t, s) = \int_{-\infty}^{\infty} p(t - u)p(s - u)du
\]

(b) Find the mean and autocorrelation function of \( \{Y(t)\} \).

3. Linear estimation and sampling

A continuous time wide sense stationary (WSS) process \( X(t) \) has zero mean and power spectral density

\[
S_X(f) = \Pi(f) = \begin{cases} 1 & -0.5 \leq f \leq 0.5 \\ 0 & \text{otherwise} \end{cases}.
\]

Recall the Fourier Transform pair \( \mathcal{F} \left[ \frac{\sin(\pi t)}{\pi t} \right] = \Pi(f) \) which implies that

\[
R_X(\tau) = \frac{\sin(\pi \tau)}{\pi \tau} = \text{sinc}(\tau).
\]

(a) Given that \( X(0) = 2 \), find the best linear MSE estimate of \( X(1) \). What is the MSE?
(b) For the rest of the problem assume that $X(t)$ is a Gaussian process with the above autocorrelation function and power spectral density. Suppose that we fix two times $t$ and $s$. Find the pdf of the random variable $X(t) - X(s)$.

(c) Suppose that we fix a sampling period $T$ and then sample $X(t)$ to form a discrete-time random process $Y_n = X(nT)$ for all integer $n$. Find the autocorrelation function of the discrete-time process $Y_n$.
Is the process $Y_n$ wide sense stationary?
Is $Y_n$ a Gaussian process?
Can $T$ be chosen in such a way that $Y_n$ is iid? If so, how? If not, why not?

4. Suppose that $\{Z_n\}$ and $\{W_n\}$ are two mutually independent two-sided zero-mean iid Gaussian processes with variances $\sigma_Z^2$ and $\sigma_W^2$, respectively. $Z_n$ is put into a linear time-invariant filter to form an output process $\{X_n\}$ defined by

$$X_n = Z_n - rZ_{n-1},$$

where $0 < r < 1$. (Such a filter is sometimes called a pre-emphasis filter in speech processing.) This process is then used to form a new process

$$Y_n = X_n + W_n,$$

which can be viewed as a noisy version of the pre-emphasized $Z_n$ process. Lastly, the $Y_n$ process is put through a “de-emphasis filter” to form an output process $U_n$ defined by

$$U_n = rU_{n-1} + Y_n.$$

(a) Find the autocorrelation $R_Z$ and the power spectral density $S_Z$. Recall that for a weakly stationary discrete time process with zero mean $R_Z(k) = E(Z_nZ_{n+k})$ and

$$S_Z(f) = \sum_{k=-\infty}^{\infty} R_Z(k)e^{-j2\pi fk},$$

the discrete time Fourier transform of $R_Z$.

(b) Find the autocorrelation $R_X$ and the power spectral density $S_X$.

(c) Find the autocorrelation $R_Y$ and the power spectral density $S_Y$. 

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(d) Find the conditional pdf $f_{Y|X}(y|x)$.
(e) Find the overall mean squared error $E[(U_n - Z_n)^2]$.

5. A discrete time random process $\{X_n; n \in \mathbb{Z}\}$ is iid and Gaussian, with mean 0 and variance 1. It is the input process for a linear time-invariant (LTI) causal filter with $\delta$-responses $h$ defined by

$$h_k = \begin{cases} 1/K & k = 0, 1, \ldots, K - 1, \\ 0 & \text{otherwise} \end{cases},$$

so that the output process $\{Y_n\}$ is defined by

$$Y_n = \sum_{k=0}^{K-1} \frac{1}{K} X_{n-k}.$$

This filter (an FIR filter) is often referred to as a *comb filter.*

A third process $\{W_n\}$ is defined by

$$W_n = Y_n - Y_{n-1}.$$

(a) What are the mean $E(Y_n)$ and the power spectral density $S_Y(f)$ of the process $\{Y_n\}$?

*Hint:* It is much easier to find the power spectral density directly using the formula $S_Y(f) = S_X(f)|H(f)|^2$ than it is to first find the autocorrelation $R_Y(\tau)$ and then take the Fourier transform.

(b) Find the characteristic function $M_{Y_n}(ju)$ and the marginal pdf $f_{Y_n}(y)$.

(c) Find the Kronecker delta response (discrete time impulse response) $g$ of an LTI filter for which

$$W_n = \sum_k g_k X_{n-k}.$$

(d) Find the mean function and autocovariance function of $\{W_n\}$.

(e) Does $\frac{1}{n} \sum_{k=0}^{n-1} W_k$ converge in probability as $n \to \infty$? If so, to what?

*Hint:* First show that if a sequence of random variables $S_n$ converges to a random variable $S$ in mean square and another sequence of random variables $R_n$ converges to a random variable $R$ in mean square, and the random variables $S_n, n = 1, 2, 3, \ldots$ and $S$ are independent of the random variables $R_n, n = 1, 2, 3, \ldots$ and $R$, then for any constants $a, b$ the random variables $aS_n + bR_n$ will converge to $aS + bR$ in mean square and hence also in probability.
6. *Filtered Poisson Processes*

Let \( \{N_t\} \) be a Poisson counting process. Let \( i(t) \) be the deterministic waveform defined by

\[
i(t) = \begin{cases} 
1 & \text{if } t \in [0, \delta] \\
0 & \text{otherwise}
\end{cases}
\]

— that is, a flat pulse of duration \( \delta \). For \( k = 1, 2, \ldots \), let \( t_k \) denote the time of the \( k \)th jump in the counting process (that is, \( t_k \) is the smallest value of \( t \) for which \( N_t = k \)). Define the random process \( \{Y(t)\} \) by

\[
Y(t) = \sum_{k=1}^{N_t} i(t - t_k).
\]

This is a special case of a class of processes known as *filtered Poisson processes*. This particular example is a model for shot noise in electronic devices.

Find \( M_{Y(t)}(ju) \) and \( p_{Y(t)}(n) \).

*Hint:* you need not consider any properties of the random variables \( \{t_k\} \) to solve this problem. It may be useful to sketch sample waveforms of \( Y(t) \) to see how the process behaves.