Exercise 1 (Penalization method) Let \((X, \| \cdot \|_X)\) and \((M, \| \cdot \|_M)\) be two Hilbert spaces. Denote by \((\cdot, \cdot)_X\) and \((\cdot, \cdot)_M\) the scalar products associated to the norms \(\| \cdot \|_X\) and \(\| \cdot \|_M\). For \(f \in X'\) and \(g \in M'\), consider the following problem: search for \((u, p) \in X \times M\) such that for all \((v, q) \in X \times M\):

\[
(P) \begin{cases}
  a(u, v) + b(v, p) = < f, v >, \\
  b(u, q) = < g, q >.
\end{cases}
\]

Assume that \(a(\cdot, \cdot)\) and \(b(\cdot, \cdot)\) are bilinear continuous forms on \(X \times X\) and \(X \times M\) respectively. Assume there exists \(\beta > 0\) such that:

\[
\forall q \in M, \exists v \in X, v \neq 0, b(v, q) \geq \beta \|v\|_X \|q\|_M.
\]

and assume that \(a(\cdot, \cdot)\) is \(\alpha\)-coercive.

Let \(c(\cdot, \cdot)\) be a bilinear continuous and \(\gamma\)-coercive form on \(M \times M\) and let \(C \in \mathcal{L}(M, M')\) be defined by

\[
<f, p, q > = c(p, q), \quad \forall p, q \in M.
\]

Operator \(A\) and \(B\) are defined accordingly, associated to \(a(\cdot, \cdot)\) and \(b(\cdot, \cdot)\) respectively.

1) Prove that problem (P) is well-posed.

For \(0 < \varepsilon < 1\), consider the following problem: search for \((u_\varepsilon, p_\varepsilon) \in X \times M\) such that for all \((v, q) \in X \times M\),

\[
(P_\varepsilon) \begin{cases}
  a(u_\varepsilon, v) + b(v, p_\varepsilon) = < f, v >, \\
  -\varepsilon c(p_\varepsilon, q) + b(u_\varepsilon, q) = < g, q >.
\end{cases}
\]

2) Prove that \((P_\varepsilon)\) is equivalent to finding \((u_\varepsilon, p_\varepsilon) \in X \times M\) such that

\[
a(u_\varepsilon, v) + \frac{1}{\varepsilon} < C^{-1} Bu_\varepsilon, Bv > = < f, v > + \frac{1}{\varepsilon} < C^{-1} g, Bv >, \forall v \in X
\]

\[
p_\varepsilon = \frac{1}{\varepsilon} C^{-1} (Bu_\varepsilon - g).
\]

3) Prove that the problem of the previous question has a unique solution. Is condition (1) necessary to prove that result?

4) Prove the following inequalities:

\[
\|p_\varepsilon - p\|_M \leq \frac{\|a\|}{\beta} \|u_\varepsilon - u\|_X,
\]

\[
a(u_\varepsilon - u, u_\varepsilon - u) \leq \varepsilon \frac{\|a\| \|c\|}{\beta} \|u_\varepsilon - u\|_X \|p\|_M,
\]

5) Conclude that there exists \(C_2 > 0\) independent of \(\varepsilon\) such that

\[
\|u - u_\varepsilon\|_X + \|p - p_\varepsilon\|_M \leq C_2 \varepsilon (\|f\|_{X'} + \|g\|_{M'}).
\]

6) Let \(\varepsilon > 0\). Consider the problem: find \((u_\varepsilon, p_\varepsilon) \in (H^1_0(\Omega))^3 \times L^2_0(\Omega)\) such that:

\[
\begin{aligned}
-\nu \Delta u_\varepsilon + \nabla p_\varepsilon &= f, \\
p_\varepsilon &= -\frac{1}{\varepsilon} \text{div} u_\varepsilon.
\end{aligned}
\]

Prove that this problem is well-posed and that, when \(\varepsilon\) goes to 0, its solution goes to the solution of a problem to be determined.

7) Prove that (7) can be written as a minimization problem. Discuss the advantages and the drawbacks of the resolution of this problem.
Exercise 2 (Augmented Lagrangian method) Let $M \in \mathbb{N}$ and $N \in \mathbb{N}$ with $M < N$. Denote by $|\cdot|$ the Euclidian norm in $\mathbb{R}^M$ or $\mathbb{R}^N$ and $(\cdot,\cdot)$ the associated scalar product. Let $A$ be a $N \times N$ symmetric positive definite matrix and $b \in \mathbb{R}^N$. Let $B$ be a full-rank matrix $M \times N$. Define $\mathcal{L}$ from $\mathbb{R}^N \times \mathbb{R}^M$ on $\mathbb{R}$ by:

$$
\mathcal{L}(v,q) = J(v) + (q,Bv)
$$

with $J(v) = \frac{1}{2}(Av,v) - (b,v)$. Let us denote by $(u,p)$ a saddle-point of $\mathcal{L}$: for all $(v,q) \in \mathbb{R}^N \times \mathbb{R}^M$:

$$
\mathcal{L}(u,q) \leq \mathcal{L}(u,p) \leq \mathcal{L}(v,p).
$$

Let $r > 0$, let us define

$$
\mathcal{L}_r(v,q) = \mathcal{L}(v,q) + \frac{r}{2}|Bv|^2.
$$

Define the sequences $(u_n)_{n \in \mathbb{N}}$ and $(p_n)_{n \in \mathbb{N}}$ as follows: let $p_0 \in \mathbb{R}^M$, for $n \geq 0$, assuming $p_n$ is known, compute $u_n \in \mathbb{R}^N$ solution to

$$
\mathcal{L}_r(u_n,p_n) \leq \mathcal{L}_r(v,p_n), \forall v \in \mathbb{R}^N,
$$

then, set

$$
p_{n+1} = p_n + \rho_nBu_n,
$$

where $\rho_n$ is a given positive number. Assume that $\forall n, 0 < \alpha \leq \rho_n \leq 2r$, where $\alpha$ is given. Set $\delta u_n = u - u_n$ and $\delta p_n = p - p_n$.

1) Show that (8) is equivalent to

$$(A + rB^T B)u_n + B^Tp_n = b$$

2) Show that

$$
|\delta p_n|^2 - |\delta p_{n+1}|^2 = 2\rho_n(A\delta u_n, \delta u_n) + \rho_n(2r - \rho_n)|B\delta u_n|^2
$$

3) Show that $\delta p_n$ converges. Deduce that $u_n \to u$ and $p_n \to p$ as $n \to \infty$.

4) Show that any saddle point of $\mathcal{L}$ is a saddle point of $\mathcal{L}_r$ and conversely.