Cardinality
Recap from Last Time
Domains and Codomains

- Every function $f$ has two sets associated with it: its domain and its codomain.
- A function $f$ can only be applied to elements of its domain. For any $x$ in the domain, $f(x)$ belongs to the codomain.
- We write $f : A \rightarrow B$ to indicate that $f$ is a function whose domain is $A$ and whose codomain is $B$.  

The function must be defined for each element of its domain.  

The output of the function must always be in the codomain, but not all elements of the codomain need to be producable.
Function Composition

- If $f : A \rightarrow B$ and $g : B \rightarrow C$ are functions, the \textit{composition of $f$ and $g$}, denoted $g \circ f$, is a function
  - whose domain is $A$,
  - whose codomain is $C$, and
  - which is evaluated as $(g \circ f)(x) = g(f(x))$. 

Injective Functions

- A function $f : A \rightarrow B$ is called **injective** (or **one-to-one**) if each element of the codomain has at most one element of the domain that maps to it.
  - A function with this property is called an **injection**.
  - Formally, $f : A \rightarrow B$ is an injection if this FOL statement is true:
    $\forall a_1 \in A. \forall a_2 \in A. (a_1 \neq a_2 \rightarrow f(a_1) \neq f(a_2))$
    (“If the inputs are different, the outputs are different”)
  - Equivalently:
    $\forall a_1 \in A. \forall a_2 \in A. (f(a_1) = f(a_2) \rightarrow a_1 = a_2)$
    (“If the outputs are the same, the inputs are the same”)
- **Theorem:** The composition of two injections is an injection.
Surjective Functions

• A function $f : A \rightarrow B$ is called **surjective** (or **onto**) if each element of the codomain is "covered" by at least one element of the domain.
  
  • A function with this property is called a **surjection**.
  
  • Formally, $f : A \rightarrow B$ is a surjection if this FOL statement is true:

    $$\forall b \in B. \exists a \in A. f(a) = b$$

    ("For every possible output, there's at least one possible input that produces it")

• **Theorem:** The composition of two surjections is a surjection.
Bijections

• A function that associates each element of the codomain with a unique element of the domain is called **bijective**.
  • Such a function is a **bijection**.

• Formally, a bijection is a function that is both **injective** and **surjective**.

• **Theorem**: The composition of two bijections is a bijection.
Cardinality Revisited
Cardinality

- Recall \textit{(from our first lecture!)} that the \textbf{cardinality} of a set is the number of elements it contains.

- If \( S \) is a set, we denote its cardinality by \(|S|\).

- For finite sets, cardinalities are natural numbers:
  - \(|\{1, 2, 3\}| = 3\)
  - \(|\{100, 200\}| = 2\)

- For infinite sets, we introduced \textit{infinite cardinals} to denote the size of sets:
  \[ |\mathbb{N}| = \aleph_0 \]
Defining Cardinality

- It is difficult to give a rigorous definition of what cardinalities actually are.
  - What is 4? What is $\aleph_0$?
  - (Take Math 161 for an answer!)
- **Idea:** Define cardinality as a *relation* between two sets rather than an absolute quantity.
Comparing Cardinalities

- Here is the formal definition of what it means for two sets to have the same cardinality:

\[ |S| = |T| \text{ if there exists a bijection } f : S \rightarrow T \]
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Fun with Cardinality
Terminology Refresher

- Let $a$ and $b$ be real numbers where $a \leq b$.
- The notation $[a, b]$ denotes the set of all real numbers between $a$ and $b$, inclusive.
  
  $[a, b] = \{ x \in \mathbb{R} \mid a \leq x \leq b \}$

- The notation $(a, b)$ denotes the set of all real numbers between $a$ and $b$, exclusive.
  
  $(a, b) = \{ x \in \mathbb{R} \mid a < x < b \}$
Consider the sets $[0, 1]$ and $[0, 2]$.

How do their cardinalities compare?
Home on the Range
$f : [0, 1] \to [0, 2]$

$f(x) = 2x$
**Theorem:** $|[0, 1]| = |[0, 2]|$

**Proof:**
Consider the function $f: [0, 1] \to [0, 2]$ defined as $f(x) = 2x$.

We will prove that $f$ is a bijection.

First, we will show that $f$ is a well-defined function. Choose any $x \in [0, 1]$. This means that $0 \leq x \leq 1$, so we know that $0 \leq 2x \leq 2$.
Consequently, we see that $0 \leq f(x) \leq 2$, so $f(x) \in [0, 2]$.

Next, we'll show that $f$ is injective.
Pick any $x_1, x_2 \in [0, 1]$ where $f(x_1) = f(x_2)$. We will show that $x_1 = x_2$. To see this, notice that since $f(x_1) = f(x_2)$, we see that $2x_1 = 2x_2$, which in turn tells us that $x_1 = x_2$, as required.

Finally, we will show that $f$ is surjective. To do so, consider any $y \in [0, 2]$. We'll show that there is some $x \in [0, 1]$ where $f(x) = y$.

Let $x = y/2$. Since $y \in [0, 2]$, we know $0 \leq y \leq 2$, and therefore that $0 \leq y/2 \leq 1$. We picked $x = y/2$, so we know that $0 \leq x \leq 1$, which in turn means $x \in [0, 1]$. Moreover, notice that $f(x) = 2x = 2(y/2) = y$, so $f(x) = y$, as required. ■
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\textbf{Theorem:} \( |[0, 1]| = |[0, 2]| \)

\textbf{Proof:} Consider the function \( f : [0, 1] \to [0, 2] \) defined as \( f(x) = 2x \). We will prove that \( f \) is a bijection.

First, we will show that \( f \) is a well-defined function.

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Let $x = \frac{y}{2}$. Since $y \in [0, 2]$, we know $0 \leq y \leq 2$, and therefore that $0 \leq \frac{y}{2} \leq 1$. 

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Home on the Range

\[ f : [0, 1] \rightarrow [0, 2] \]

\[ f(x) = 2x \]
Home on the Range

\[ f : [0, 1] \rightarrow [0, 3] \]
\[ f(x) = 3x \]
Home on the Range

\[ f : [0, 1] \rightarrow [0, 137] \]
\[ f(x) = 137x \]
This means that *cardinality* (how many points there are) is a different idea than *mass* (how much those points weight). Look into *measure theory* if you’re curious to learn more!
And one more example, just for funzies.
Put a Ring On It

$f : (-\pi/2, \pi/2) \to \mathbb{R}$

$f(x) = \tan x$

$|(-\pi/2, \pi/2)| = |\mathbb{R}|$
Some Properties of Cardinality
Theorem: For any set $A$, we have $|A| = |A|$. 
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**Proof:**
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**Proof:** Consider any set $A$, and let $f : A \rightarrow A$ be the function defined as $f(x) = x$. 
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**Theorem:** For any set $A$, we have $|A| = |A|$.

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First, we’ll show that $f$ is a well-defined function. To see this, note that for any $x \in A$, we have $f(x) = x \in A$, as needed.
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**Theorem:** If $A$, $B$, and $C$ are sets where $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.
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Since $|A| = |B|$, we know that there is a some bijection $f : A \rightarrow B$. Similarly, since $|B| = |C|$ we know that there is at least one bijection $g : B \rightarrow C$. 

$\blacksquare$
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Consider the function $g \circ f : A \rightarrow C$. 
**Theorem:** If $A$, $B$, and $C$ are sets where $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

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Consider the function $g \circ f : A \to C$. Since $g$ and $f$ are bijections and the composition of two bijections is a bijection, we see that $g \circ f$ is a bijection from $A$ to $C$. 
**Theorem:** If $A$, $B$, and $C$ are sets where $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

**Proof:** Consider any sets $A$, $B$, and $C$ where $|A| = |B|$ and $|B| = |C|$. We need to prove that $|A| = |C|$. To do so, we need to show that there is a bijection from $A$ to $C$.

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**Theorem:** If $A$, $B$, and $C$ are sets where $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

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Consider the function $g \circ f : A \rightarrow C$. Since $g$ and $f$ are bijections and the composition of two bijections is a bijection, we see that $g \circ f$ is a bijection from $A$ to $C$. Thus $|A| = |C|$, as required. ■
Great exercise: Prove that if $A$ and $B$ are sets where $|A| = |B|$, then $|B| = |A|$. 
Time-Out for Announcements!
Problem Set Three

• The PS3 Checkpoint was due today at 2:30PM.
  • *Please read the solutions.* It never hurts to double-check that you solving things the right way.

• Solutions to Problem Set Two are now up on the course website.
  • *Please read the solutions.* Those questions were tricky, and it’s useful to see different ways to solve them.

• The remaining problems from PS3 are due this Friday at 2:30PM.
  • Have questions? Ask on Piazza or stop by office hours!
Midterm Exam Logistics

- Our first midterm exam is next **Monday, October 22nd**, from **7:00PM - 10:00PM**.
  - We’ll announce locations on Wednesday.
- You’re responsible for Lectures 00 – 05 and topics covered in PS0 – PS2. Later lectures (relations forward) and problem sets (PS3 onward) won’t be tested here.
- The exam is closed-book, closed-computer, and limited-note. You can bring a double-sided, 8.5” × 11” sheet of notes with you to the exam, decorated however you’d like.
- Students with OAE accommodations: please contact us **immediately** if you haven’t yet done so. We’ll ping you about setting up alternate exams.
Midterm Exam

- **We want you to do well on this exam.** We're not trying to weed out weak students. We're not trying to enforce a curve where there isn't one. We want you to show what you've learned up to this point so that you get a sense for where you stand and where you can improve.

- The purpose of this midterm is to give you a chance to show what you've learned in the past few weeks. It is not designed to assess your “mathematical potential” or “innate mathematical ability.”
Extra Practice Problems

• By the end of the evening, up on the course website, you’ll find
  • Extra Practice Problems 1 (a set of cumulative review problems), and
  • three practice midterm exams, each of which is a (slightly modified) version of a real exam we’ve given out in a previous quarter.

• **Use these resources strategically.** Give these problems your best effort, and, importantly, **have the course staff review your work.** Ask for polite but honest feedback. ☺
Practice Midterm Exam

- To help you prepare for the midterm, we'll be holding a practice midterm exam on **Wednesday, October 17th from 7PM - 10PM** in room **320-105**.
  - The exam we’ll use isn’t one of the ones posted up on the course website, so feel free to use those as practice in the meantime.
- Course staff will be on hand to answer your questions.
- Can't make it? We'll release that practice exam and solutions online. Set up your own practice exam time with a small group and work through it under realistic conditions!
Preparing for the Exam

- We've released a handout (Handout 20) containing advice about how to prepare for the exam, along with advice from previous CS103 students.
- Read over it... there's good advice there!
Your Questions
“What are some good places for a birthday dinner around here?”

Hold a portable potluck! Borrow or rent a car (you get a discount rate as a student) and drive up Page Mill Road toward Monte Bello Open Space Preserve. Find a pretty place to pull over and stop where you have an epic view of the hills and the sunset. Park, pop open your trunk, set up paper plates and the like, and have a happy birthday!
Back to CS103!
Unequal Cardinalities

• Recall: \(|A| = |B|\) if the following statement is true:

   \textbf{There exists a bijection }f : A \to B

• What does it mean for \(|A| \neq |B|\) to be true?

   \textbf{Every function }f : A \to B \textbf{ is not a bijection.}

• This is a strong statement! To prove \(|A| \neq |B|\), we need to show that \textit{no possible function} from \(A\) to \(B\) can be injective and surjective.

\begin{align*}
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\end{align*}
Unequal Cardinalities

• Recall: \(|A| = |B|\) if the following statement is true:
  
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Cantor’s Theorem Revisited
Cantor’s Theorem

• In our very first lecture, we sketched out a proof of \textit{Cantor’s theorem}, which says that

\[ \text{If } S \text{ is a set, then } |S| < |\wp(S)|. \]

• That proof was visual and pretty hand-wavy. Let’s see if we can go back and formalize it!
Where We’re Going

• Today, we’re going to formally prove the following result:

\[
\text{If } S \text{ is a set, then } |S| \neq |\wp(S)|.
\]

• We’ve released an online Guide to Cantor’s Theorem, which will go into way more depth than what we’re going to see here.

• The goal for today will be to see how to start with our picture and turn it into something rigorous.

• On the next problem set, you’ll explore the proof in more depth and see some other applications.
The Roadmap

• We’re going to prove this statement:
  
  \[ \text{If } S \text{ is a set, then } |S| \neq |\wp(S)|. \]

• Here’s how this will work:
  
  • Pick an arbitrary set \( S \).
  • Pick an arbitrary function \( f : S \to \wp(S) \).
  • Show that \( f \) is not surjective using a diagonal argument.
  • Conclude that there are no bijections from \( S \) to \( \wp(S) \).
  • Conclude that \( |S| \neq |\wp(S)| \).
The Roadmap

We’re going to prove this statement:

If $S$ is a set, then $|S| \neq |\wp(S)|$.

Here’s how this will work:

Pick an arbitrary set $S$.
Pick an arbitrary function $f : S \to \wp(S)$.

- **Show that $f$ is not surjective using a diagonal argument.**

Conclude that there are no bijections from $S$ to $\wp(S)$.
Conclude that $|S| \neq |\wp(S)|$. 
\( x_0 \)
\( x_1 \)
\( x_2 \)
\( x_3 \)
\( x_4 \)
\( x_5 \)
\( \ldots \)
This is a drawing of our function $f : S \rightarrow \mathcal{P}(S)$.

\[
\begin{align*}
\mathbf{x}_0 & \rightarrow \{ x_0, x_2, x_4, \ldots \} \\
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\mathbf{x}_3 & \rightarrow \{ x_1, x_4, \ldots \} \\
\mathbf{x}_4 & \rightarrow \{ x_0, x_5, \ldots \} \\
\mathbf{x}_5 & \rightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \ldots \} \\
& \ldots
\end{align*}
\]
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\begin{align*}
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  \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
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\end{array}
\end{align*}

\ldots
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|   | \( x_0 \) | \( x_1 \) | \( x_2 \) | \( x_3 \) | \( x_4 \) | \( x_5 \) | ...
|---|---|---|---|---|---|---|---
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| 2 |   |   |   |   |   |   |   
| 3 |   |   |   |   |   |   |   
| 4 |   |   |   |   |   |   |   
| 5 |   |   |   |   |   |   |   

\( x_0 \) \rightarrow \{ x_0, x_3, x_4, \ldots \}

\( x_1 \) \rightarrow \{ x_4, \ldots \}

\( x_2 \) \rightarrow \{ x_4, \ldots \}

\( x_3 \) \rightarrow \{ x_1, x_4, \ldots \}

\( x_4 \) \rightarrow \{ x_0, x_5, \ldots \}

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$x_3$ → \{ $x_1$, $x_4$, ... \}

$x_4$ → \{ $x_0$, $x_5$, ... \}

$x_5$ → \{ $x_0$, $x_1$, $x_2$, $x_3$, $x_4$, $x_5$, ... \}

...
This is a drawing of our function \( f : S \rightarrow \wp(S) \).

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This is a drawing of our function $f : S \rightarrow \mathcal{P}(S)$.

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$\{x_0, x_5, \ldots\}$
This is a drawing of our function $f : S \to \wp(S)$.

Flip all Y's to N's and vice-versa to get a new set.

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This is a drawing of our function $f : S \to \wp(S)$. Flipping all Y's to N's and vice-versa gives a new set.
This is a drawing of our function $f : S \rightarrow \mathcal{P}(S)$.

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Flip all Y’s to N’s and vice-versa to get a new set.
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This is a drawing of our function $f : S \to \wp(S)$. Which row in the table is paired with this set?
This is a drawing of our function $f : S \rightarrow \wp(S)$.

The table shows the function $f$ applied to different elements $x_0, x_1, x_2, x_3, x_4, x_5, \ldots$ of a set $S$. Each row corresponds to a different element of $S$, and the values in the columns indicate whether the element is in the range of $f$. The set $\wp(S)$ represents the power set of $S$.

Which row in the table is paired with this set?
This is a drawing of our function $f : S \rightarrow \wp(S)$.

Which row in the table is paired with this set?
This is a drawing of our function $f : S \to \mathcal{P}(S)$.

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Which row in the table is paired with this set?
Which row in the table is paired with this set?

This is a drawing of our function $f : S \to \wp(S)$. 

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This is a drawing of our function $f : S \to \mathcal{P}(S)$. Which row in the table is paired with this set?
This is a drawing of our function \( f : S \rightarrow \mathcal{P}(S) \).

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Which row in the table is paired with this set?
Which row in the table is paired with this set?

This is a drawing of our function $f : S \rightarrow \mathcal{P}(S)$.

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N Y Y Y Y Y N ...
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What set is this?
This is a drawing of our function $f : S \to \wp(S)$. What set is this?
This is a drawing of our function \( f : S \to \wp(S) \).
This is a drawing of our function $f : S \rightarrow \wp(S)$.

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This is a drawing of our function $f : S \rightarrow \wp(S)$. 

$f(x_0)$
This is a drawing of our function $f : S \to \wp(S)$.

$x_0 \in f(x_0)$?
This is a drawing of our function $f : S \rightarrow \mathcal{P}(S)$.

$x_0 \notin f(x_0)$?
This is a drawing of our function $f : S \to \wp(S)$.

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$f(x_1)$
This is a drawing of our function \( f : S \to \wp(S) \).

\[
\begin{array}{ccccccc}
  & x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & \ldots \\
\hline
x_0 & Y & N & Y & N & Y & N & \ldots \\
x_1 & Y & N & N & Y & Y & N & \ldots \\
x_2 & N & N & N & N & Y & N & \ldots \\
x_3 & N & Y & N & N & Y & N & \ldots \\
x_4 & Y & N & N & N & N & N & Y & \ldots \\
x_5 & Y & Y & Y & Y & Y & Y & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{array}
\]

\( x_1 \in f(x_1) ? \)
This is a drawing of our function $f : S \rightarrow \wp(S)$. 

$x_1 \notin f(x_1)$?
This is a drawing of our function $f : S \rightarrow \wp(S)$.

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$f(x_2)$
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$x_2 \in f(x_2)$?
This is a drawing of our function $f : S \rightarrow \mathcal{P}(S)$.

$X_0$  |  $X_1$  |  $X_2$  |  $X_3$  |  $X_4$  |  $X_5$  |  ...
---|---|---|---|---|---|---
$X_0$ | $Y$ | $N$ | $Y$ | $N$ | $Y$ | $N$ | ...
$X_1$ | $Y$ | $N$ | $N$ | $Y$ | $Y$ | $N$ | ...
$X_2$ | $N$ | $N$ | $N$ | $N$ | $N$ | $Y$ | $N$ | ...
$X_3$ | $N$ | $Y$ | $N$ | $N$ | $N$ | $Y$ | $N$ | ...
$X_4$ | $Y$ | $N$ | $N$ | $N$ | $N$ | $N$ | $Y$ | ...
$X_5$ | $Y$ | $Y$ | $Y$ | $Y$ | $Y$ | $Y$ | ...
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$x_2 \notin f(x_2)$?
This is a drawing of our function $f: S \to \mathcal{P}(S)$.

This is a drawing of our function $f: S \to \mathcal{P}(S)$.

$x_3 \notin f(x_3)$?
This is a drawing of our function $f : S \to \wp(S)$.

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$x_4 \notin f(x_4)$?

N  Y  Y  Y  Y  Y  N  ...
This is a drawing of our function \( f : S \rightarrow \mathcal{P}(S) \).

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\( x_5 \notin f(x_5) \)?
This is a drawing of our function $f : S \to \wp(S)$.

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<td>$x_5$</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
<td>$\ldots$</td>
</tr>
</tbody>
</table>

$x \not\in f(x)$?
This is a drawing of our function $f : S \rightarrow \wp(S)$.

<table>
<thead>
<tr>
<th>$x_0$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y</td>
<td>N</td>
<td>Y</td>
<td>N</td>
<td>Y</td>
<td>N</td>
<td>...</td>
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</tbody>
</table>

{ $x \in S$ | $x \notin f(x)$ }
The Diagonal Set

• For any set $S$ and function $f : S \to \mathcal{P}(S)$, we can define a set $D$ as follows:

$$D = \{ x \in S \mid x \notin f(x) \}$$

(“The set of all elements $x$ where $x$ is not an element of the set $f(x)$.”)

• This is a formalization of the set we found in the previous picture.

• Using this choice of $D$, we can formally prove that no function $f : S \to \mathcal{P}(S)$ is a bijection.
Theorem: If $S$ is a set, then $|S| \neq |\wp(S)|$. 
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**Proof:** Let $S$ be an arbitrary set. We will prove that $|S| \neq |\wp(S)|$ by showing that there are no bijections from $S$ to $\wp(S)$. 

Starting with $f: S \to \wp(S)$, we define the set $D = \{ x \in S \mid x \notin f(x) \}$. (1)

We will show that there is no $y \in S$ such that $f(y) = D$. To do so, we proceed by contradiction. Suppose that there is some $y \in S$ such that $f(y) = D$. By the definition of $D$, we know that $y \in D$ if $y \notin f(y)$. (2)

By assumption, $f(y) = D$. Combined with (2), this tells us $y \in D$ if $y \notin D$. (3)

This is impossible. We have reached a contradiction, so our assumption must have been wrong. Therefore, there is no $y \in S$ such that $f(y) = D$, so $f$ is not surjective. This means that $f$ is not a bijection, and since our choice of $f$ was arbitrary, we conclude that there are no bijections between $S$ and $\wp(S)$. Thus $|S| \neq |\wp(S)|$, as required. ■
**Theorem:** If $S$ is a set, then $|S| \neq |\wp(S)|$.

**Proof:** Let $S$ be an arbitrary set. We will prove that $|S| \neq |\wp(S)|$ by showing that there are no bijections from $S$ to $\wp(S)$. To do so, choose an arbitrary function $f : S \to \wp(S)$. Starting with $f$, we define the set $D = \{ x \in S \mid x \notin f(x) \}$.

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The Big Recap

- We define equal cardinality in terms of bijections between sets.
- Lots of different sets of infinite size have the same cardinality.
- Cardinality acts like an equivalence relation – but only because we can prove specific properties of how it behaves by relying on properties of function.
- Cantor’s theorem can be formalized in terms of surjectivity.
Next Time

• **Graphs**
  • A ubiquitous, expressive, and flexible abstraction!

• **Properties of Graphs**
  • Building high-level structures out of lower-level ones!