Computational Decision Making
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• Many decisions we make involve uncertainty
  ▪ Can we use that fact to make better decisions?
• Probability
  ▪ Mathematical tool to formally quantify uncertainty
  ▪ Allows better reasoning about unknowns
• Decision analysis
  ▪ Term coined in 1964 by Ron Howard (Stanford Prof.)
  ▪ Normative framework for decision making
  ▪ General principle: maximize expected utility
  ▪ Note: not all decisions appropriate for formal analysis
• Have fun!
Sample Spaces and Events

- **Sample space**, $S$, is set of all possible outcomes of an experiment
  - Coin flip: $S = \{\text{Head, Tails}\}$
  - Flipping two coins: $S = \{(\text{H, H}), (\text{H, T}), (\text{T, H}), (\text{T, T})\}$
  - Rolling a 6-sided die: $S = \{1, 2, 3, 4, 5, 6\}$
  - Sum of two dice: $S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$
  - # emails in a day: $S = \{x \mid x \in \mathbb{Z}, x \geq 0\}$ (non-neg. ints)

- **Event**, $E$, is some subset of $S$ ($E \subseteq S$)
  - Coin flip is heads: $E = \{\text{Head}\}$
  - At least 1 head on 2 flips: $E = \{(\text{H, H}), (\text{H, T}), (\text{T, H})\}$
  - Rolling odd number (1 die): $E = \{1, 3, 5\}$
  - Rolling $\geq 8$ on 2 dice: $E = \{8, 9, 10, 11, 12\}$
  - # emails in a day $\leq 20$: $E = \{x \mid x \in \mathbb{Z}, 0 \leq x \leq 20\}$
Set operations on Events

- Say $E$ and $F$ are events in $S$
Set operations on Events

- Say E and F are events in S

Event that is in E or F

\[ E \cup F \]

- \( S = \{1, 2, 3, 4, 5, 6\} \) die roll outcome
- \( E = \{1, 2\} \) \( F = \{2, 3\} \) \( E \cup F = \{1, 2, 3\} \)
Set operations on Events

- Say E and F are events in S

Event that is in E and F

\[ E \cap F \text{ or } EF \]

- S = \{1, 2, 3, 4, 5, 6\} die roll outcome
- E = \{1, 2\} F = \{2, 3\} E F = \{2\}
- Note: mutually exclusive events means E F = \emptyset
Set operations on Events

- Say E and F are events in S

  Event that is not in E (called complement of E)
  \[ E^c \quad \text{or} \quad \sim E \]

- \( S = \{1, 2, 3, 4, 5, 6\} \) die roll outcome
- \( E = \{1, 2\} \quad E^c = \{3, 4, 5, 6\} \)
Set operations on Events

- Say $E$ and $F$ are events in $S$

DeMorgan’s Laws

\[(E \cup F)^c = E^c \cap F^c\]

\[(E \cap F)^c = E^c \cup F^c\]
Axioms of Probability

- Probability as relative frequency of event:
  \[ P(E) = \lim_{n \to \infty} \frac{n(E)}{n} \]

- Axiom 1: \( 0 \leq P(E) \leq 1 \)

- Axiom 2: \( P(S) = 1 \)

- Axiom 3: If \( E \) and \( F \) mutually exclusive \( (E \cap F = \emptyset) \), then \( P(E) + P(F) = P(E \cup F) \)

For any sequence of mutually exclusive events \( E_1, E_2, \ldots \)

\[ P\left( \bigcup_{i=1}^{\infty} E_i \right) = \sum_{i=1}^{\infty} P(E_i) \]
Implications of Axioms

- \( P(E^c) = 1 - P(E) \quad (= P(S) - P(E) \) 

- If \( E \subseteq F \), then \( P(E) \leq P(F) \)

- \( P(E \cup F) = P(E) + P(F) - P(EF) \)
  - This is the Inclusion-Exclusion Principle for Probability
  - Sometimes called the “Addition Rule”
Equally Likely Outcomes

- Some sample spaces have equally likely outcomes
  - Coin flip: \( S = \{ \text{Head, Tails} \} \)
  - Flipping two coins: \( S = \{(H, H), (H, T), (T, H), (T, T)\} \)
  - Roll of 6-sided die: \( S = \{1, 2, 3, 4, 5, 6\} \)

- \( P(\text{Each outcome}) = \frac{1}{|S|} \)

- In that case, \( P(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in } S} = \frac{|E|}{|S|} \)
Rolling Two Dice

• Roll two 6-sided dice.
  ▪ What is $P(\text{sum} = 7)$?

• $S = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6),
  (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6),
  (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6),
  (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6),
  (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6),
  (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$

• $E = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$

• $P(\text{sum} = 7) = |E|/|S| = 6/36 = 1/6$
Permutations

- Def.: A permutation is an ordered arrangement of distinct objects

- Number of ways $n$ objects can be permuted is:
  \[
  n \times (n - 1) \times (n - 2) \times \ldots \times 2 \times 1 = n!
  \]

- Select 1st object out of $n$, then select 2nd object out of $n - 1$ remaining objects, etc.)
Combinations

• Definition: A **combination** is an *unordered* selection of *k* distinct objects from a set of *n* distinct objects

• Written as: \( \binom{n}{k} \) (pronounced “n choose k”)

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}
\]

• Example: Number of ways of drawing 5 cards for a standard deck of 52 cards: \( \binom{52}{5} = 2,598,960 \)
Any Straight in Poker

- Consider 5 card poker hands.
  - “straight” is 5 consecutive rank cards of any suit
  - What is \( P(\text{straight}) \)?

- \(|S| = \binom{52}{5}\)

- \(|E| = 10 \left( \binom{4}{1} \right)^5\)

- \( P(\text{straight}) = \frac{10 \left( \binom{4}{1} \right)^5}{\binom{52}{5}} \approx 0.00394\)
“Official” Straight in Poker

• Consider 5 card poker hands.
  • “straight” is 5 consecutive rank cards of any suit
  • “straight flush” is 5 consecutive rank cards of same suit
  • What is \( P(\text{straight, but not straight flush}) \)?

• \(|S| = \binom{52}{5}\)

• \(|E| = 10\binom{4}{1}^5 - 10\binom{4}{1}\)

• \(P(\text{straight}) = \frac{10\binom{4}{1}^5 - 10\binom{4}{1}}{\binom{52}{5}} \approx 0.00392\)
Card Flipping

• 52 card deck. Cards flipped one at a time.
  ▪ After first ace (of any suit) appears, consider next card
  ▪ Is $P(\text{next card} = \text{Ace Spades}) < P(\text{next card} = \text{2 Clubs})$?
  ▪ Consider the two cases:

• First note: $|S| = 52!$ (all cards shuffled)

• Case 1: Take Ace Spades out of deck
  ▪ Shuffle left over 51 cards, add Ace Spades after first ace
  ▪ $|E| = 51! \times 1$ (only 1 place Ace Spades can be added)

• Case 2: Do same as case 1, but...
  ▪ Replace “Ace Spades” with “2 Clubs” in description
  ▪ $|E|$ and $|S|$ are the same as case 1
  ▪ So $P(\text{next card} = \text{Ace Spade}) = P(\text{next card} = \text{2 Clubs})$
Communication Networks

- Say 28% of all students use Snapchat
  - 7% of students use Instagram
  - 5% of students use Snapchat and Instagram
- What percentage of students use neither Snapchat nor Instagram
  - Let $A = \text{event that a random student uses Snapchat}$
  - Let $B = \text{event that a random student uses Instagram}$
  - $1 - P(A \cup B) = 1 - [P(A) + P(B) - P(AB)]$
    - $= 1 - (0.28 + 0.07 - 0.05) = 0.7 \rightarrow 70\%$
- What percent use Instagram, but not Snapchat?
  - $P(A^c B) = P(B) - P(AB) = 0.07 - 0.05 = 0.02 \rightarrow 2\%$
Birthdays

- What is the probability that of \( n \) people, none share the same birthday (regardless of year)?
  - \(|S| = (365)^n\)
  - \(|E| = (365)(364)...(365 - n + 1)\)
  - \(P(\text{no matching birthdays})\)
    \[= \frac{(365)(364)...(365 - n + 1)}{(365)^n}\]

- Interesting values of \( n \)
  - \(n = 17\): \(P(\text{no matching birthdays}) \approx 68\%\) (our class)
  - \(n = 23\): \(P(\text{no matching birthdays}) < 50\%\) (least such \(n\))
  - \(n = 50\): \(P(\text{no matching birthdays}) < 3\%\)
  - \(n = 61\): \(P(\text{no matching birthdays}) < 0.5\%\)
  - \(n = 80\): \(P(\text{no matching birthdays}) < 0.01\%\)
Birthdays

• What is the probability that of $n$ other people, none of them share the same birthday as you?
  - $|S| = (365)^n$
  - $|E| = (364)^n$
  - $P(\text{no birthdays matching yours}) = (364)^n/(365)^n$

• Interesting values of $n$
  - $n = 16$: $P(\text{no matching birthdays}) \approx 95.7\%$
  - $n = 23$: $P(\text{no matching birthdays}) \approx 93.9\%$
  - $n = 150$: $P(\text{no matching birthdays}) \approx 66.3\%$
  - $n = 253$: $P(\text{no matching birthdays}) \approx 50.0\%$
    - Least such $n$ for which $P(\text{no matching birthdays}) < \frac{1}{2}$

• Why are these probabilities much higher than before?
  - Anyone born on May 10th?