Lecture 4
Natural response of first and second order systems

• first order systems

• second order systems
  – real distinct roots
  – real equal roots
  – complex roots
  – harmonic oscillator
  – stability
  – decay rate
  – critical damping
  – parallel & series RLC circuits
First order systems

\[ ay' + by = 0 \quad (\text{with } a \neq 0) \]

righthand side is zero:

- called *autonomous system*
- solution is called *natural* or *unforced response*

can be expressed as

\[ Ty' + y = 0 \quad \text{or} \quad y' + ry = 0 \]

where

- \( T = \frac{a}{b} \) is a *time* (units: seconds)
- \( r = \frac{b}{a} = \frac{1}{T} \) is a *rate* (units: \( 1/\text{sec} \))
Solution by Laplace transform

take Laplace transform of \( Ty' + y = 0 \) to get

\[
T \left( sY(s) - y(0) \right) + Y(s) = 0
\]

solve for \( Y(s) \) (algebra!)

\[
Y(s) = \frac{Ty(0)}{sT + 1} = \frac{y(0)}{s + 1/T}
\]

and so \( y(t) = y(0)e^{-t/T} \)
solution of $Ty' + y = 0$: $y(t) = y(0)e^{-t/T}$

if $T > 0$, $y$ decays exponentially

- $T$ gives time to decay by $e^{-1} \approx 0.37$
- $0.693T$ gives time to decay by half ($0.693 = \log 2$)
- $4.6T$ gives time to decay by $0.01$ ($4.6 = \log 100$)

if $T < 0$, $y$ grows exponentially

- $|T|$ gives time to grow by $e \approx 2.72$
- $0.693|T|$ gives time to double
- $4.6|T|$ gives time to grow by $100$
Examples

simple RC circuit:

\[
\begin{array}{c}
  R \\
  +
  \hline
  C \\
  v
  \hline
  -
\end{array}
\]

\[RCv' + v = 0\]

solution: \[v(t) = v(0)e^{-t/(RC)}\]

population dynamics:

- \(y(t)\) is population of some bacteria at time \(t\)
- growth (or decay if negative) rate is \(y' = by - dy\) where \(b\) is birth rate, \(d\) is death rate
- \(y(t) = y(0)e^{(b-d)t}\) (grows if \(b > d\); decays if \(b < d\))
thermal system:

- $y(t)$ is temperature of a body (above ambient) at $t$
- heat loss proportional to temp (above ambient): $ay$
- heat in body is $cy$, where $c$ is thermal capacity, so $cy' = -ay$
- $y(t) = y(0)e^{-at/c}$; $c/a$ is thermal time constant
Second order systems

\[ ay'' + by' + cy = 0 \]

assume \( a > 0 \) (otherwise multiply equation by \(-1\))

solution by Laplace transform:

\[
\begin{aligned}
& a(s^2Y(s) - sy(0) - y'(0)) + b(sY(s) - y(0)) + cY(s) = 0 \\
& L(y'') + bL(y') + cY(s) = 0
\end{aligned}
\]

solve for \( Y \) (just algebra!)

\[
Y(s) = \frac{asy(0) + ay'(0) + by(0)}{as^2 + bs + c} = \frac{\alpha s + \beta}{as^2 + bs + c}
\]

where \( \alpha = ay(0) \) and \( \beta = ay'(0) + by(0) \)
so solution of \( ay'' + by' + cy = 0 \) is

\[
y(t) = \mathcal{L}^{-1}\left(\frac{\alpha s + \beta}{as^2 + bs + c}\right)
\]

- \( \chi(s) = as^2 + bs + c \) is called characteristic polynomial of the system
- form of \( y = \mathcal{L}^{-1}(Y) \) depends on roots of characteristic polynomial \( \chi \)
- coefficients of numerator \( \alpha s + \beta \) come from initial conditions
Roots of $\chi$

(two) roots of characteristic polynomial $\chi$ are

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

i.e., we have $as^2 + bs + c = a(s - \lambda_1)(s - \lambda_2)$

three cases:

- roots are real and distinct: $b^2 > 4ac$

  $$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad \lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

- roots are real and equal: $b^2 = 4ac$

  $$\lambda_1 = \lambda_2 = -b/(2a)$$
• roots are complex (and conjugates): $b^2 < 4ac$

$$\lambda_1 = \sigma + j\omega, \quad \lambda_2 = \sigma - j\omega,$$

where $\sigma = -b/(2a)$ and

$$\omega = \frac{\sqrt{4ac - b^2}}{2a} = \sqrt{\frac{c}{a} - (b/2a)^2}$$
Real distinct roots \((b^2 > 4ac)\)

\[
\chi(s) = a(s - \lambda_1)(s - \lambda_2) \quad (\lambda_1, \lambda_2 \text{ real})
\]

from page 4-6,

\[
Y(s) = \frac{\alpha s + \beta}{a(s - \lambda_1)(s - \lambda_2)}
\]

where \(\alpha, \beta\) depend on initial conditions

express \(Y\) as

\[
Y(s) = \frac{r_1}{s - \lambda_1} + \frac{r_2}{s - \lambda_2}
\]

where \(r_1\) and \(r_2\) are found from

\[
r_1 + r_2 = \frac{\alpha}{a}, \quad -\lambda_2 r_1 - \lambda_1 r_2 = \frac{\beta}{a}
\]

which yields

\[
r_1 = \frac{\lambda_1 \alpha + \beta}{\sqrt{b^2 - 4ac}}, \quad r_2 = \frac{-\lambda_2 \alpha - \beta}{\sqrt{b^2 - 4ac}}
\]
now we can find the inverse Laplace tranform . . .

\[ y(t) = r_1 e^{\lambda_1 t} + r_2 e^{\lambda_2 t} \]

a sum of two (real) exponentials

• coefficients of exponentials, *i.e.*, \( \lambda_1, \lambda_2 \), depend only on \( a, b, c \)
• associated time constants \( T_1 = 1/|\lambda_1|, T_2 = 1/|\lambda_2| \)
• \( r_1, r_2 \) depend (linearly) on the initial conditions \( y(0), y'(0) \)

• signs of \( \lambda_1, \lambda_2 \) determine whether solution grows or decays as \( t \to \infty \)
• magnitudes of \( \lambda_1, \lambda_2 \) determine growth rate (if positive) or decay rate (if negative)
Example: second-order RC circuit

\[ t = 0 \quad \begin{array}{c}
\text{1Ω} \\
\text{1Ω}
\end{array} \quad \begin{array}{c}
1F \\
1F
\end{array} \quad + \\
- \\
\text{y}
\]

at \( t = 0 \), the voltage across each capacitor is 1V

- for \( t \geq 0 \), \( y \) satisfies LCCODE (from page 2-18)

\[ y'' + 3y' + y = 0 \]

- initial conditions:

\[ y(0) = 1, \quad y'(0) = 0 \]

(at \( t = 0 \), voltage across righthand capacitor is one; current through righthand resistor is zero)
solution using Laplace transform

- characteristic polynomial: $\chi(s) = s^2 + 3s + 1$

- $b^2 = 9 > 4ac = 4$, so roots are real & distinct: $\lambda_1 = -2.62$, $\lambda_2 = -0.38$

- hence, solution has form

$$y(t) = r_1 e^{-2.62t} + r_2 e^{-0.38t}$$

- initial conditions determine $r_1$, $r_2$:

$$y(0) = r_1 + r_2 = 1, \quad y'(0) = -2.62r_1 - 0.38r_2 = 0$$

yields $r_1 = -0.17$, $r_2 = 1.17$,

$$y(t) = -0.17e^{-2.62t} + 1.17e^{-0.38t}$$
- first exponential decays fast, within 2sec ($T_1 = 1/|\lambda_1| = 0.38$)
- second exponential decays slower ($T_2 = 1/|\lambda_2| = 2.62$)

expanded scale, for $0 \leq t \leq 2$
Real equal roots \((b^2 = 4ac)\)

\[\chi(s) = a(s - \lambda)^2 \quad \text{with} \quad \lambda = -b/(2a)\]

from page 4-6,

\[Y(s) = \frac{\alpha s + \beta}{a(s - \lambda)^2}\]

express \(Y\) as

\[Y(s) = \frac{r_1}{s - \lambda} + \frac{r_2}{(s - \lambda)^2}\]

where \(r_1\) and \(r_2\) are found from \(r_1 = \alpha/a, \quad -\lambda r_1 + r_2 = \beta/a\), which yields

\[r_1 = \alpha/a, \quad r_2 = (\beta + \lambda \alpha)/a\]

inverse Laplace transform is

\[y(t) = r_1 e^{\lambda t} + r_2 te^{\lambda t}\]
Example: mass-spring-damper

mass $m = 1$, stiffness $k = 1$, damping $b = 2$

- LCCODE (from page 2-19):

$$y'' + 2y' + y = 0$$

- initial conditions

$$y(0) = 0, \quad y'(0) = 1$$
solution using Laplace transform

- characteristic polynomial: \( s^2 + 2s + 1 = (s + 1)^2 \)
- solution has form \( y(t) = r_1 e^{-t} + r_2 t e^{-t} \)
- initial conditions determine \( r_1, r_2 \): \( y(0) = r_1 = 0, \ y'(0) = -r_1 + r_2 = 1 \)
  yields \( r_1 = 0, \ r_2 = 1, \ i.e., \)

\[
y(t) = te^{-t}
\]

called \textit{critically damped} system (more later)
Complex roots \((b^2 < 4ac)\)

\[\chi(s) = a(s - \lambda)(s - \bar{\lambda}) \text{ with } \lambda = \sigma + j\omega, \quad \bar{\lambda} = \sigma - j\omega\]

from page 4-6,

\[Y(s) = \frac{\alpha s + \beta}{a(s - \lambda)(s - \bar{\lambda})}\]

express \(Y\) as

\[Y(s) = \frac{r_1}{s - \lambda} + \frac{r_2}{s - \bar{\lambda}}\]

where \(r_1\) and \(r_2\) follow from \(r_1 + r_2 = \alpha/a, \quad -r_1\bar{\lambda} - r_2\lambda = \beta/a:\)

\[r_1 = \frac{\alpha}{2a} + j\frac{\alpha b - 2a\beta}{4a^2\omega}, \quad r_2 = \bar{r}_1\]

inverse Laplace transform is

\[y(t) = r_1 e^{\lambda t} + \bar{r}_1 e^{\bar{\lambda} t}\]
other useful forms:

\[ y(t) = r_1 e^{\lambda t} + \overline{r}_1 e^{\overline{\lambda} t} \]
\[ = r_1 e^{\sigma t}(\cos \omega t + j \sin \omega t) + \overline{r}_1 e^{\sigma^* t}(\cos \omega t - j \sin \omega t) \]
\[ = (\Re(r_1) + j \Im(r_1)) e^{\sigma t}(\cos \omega t + j \sin \omega t) \]
\[ + (\Re(r_1) - j \Im(r_1)) e^{\sigma^* t} (\cos \omega t - j \sin \omega t) \]
\[ = 2 e^{\sigma t} (\Re(r_1) \cos \omega t - \Im(r_1) \sin \omega t) \]
\[ = A e^{\sigma t} \cos(\omega t + \phi) \]

where \( A = 2|r_1| \), \( \phi = \arctan(\Im(r_1)/\Re(r_1)) \)

- if \( \sigma > 0 \), \( y \) is an exponentially growing sinusoid; if \( \sigma < 0 \), \( y \) is an exponentially decaying sinusoid; if \( \sigma = 0 \), \( y \) is a sinusoid
- \( \Re \lambda = \sigma \) gives exponential rate of decay or growth; \( \Im \lambda = \omega \) gives oscillation frequency
- amplitude \( A \) and phase \( \phi \) determined by initial conditions
- \( A e^{\sigma t} \) is called the envelope of \( y \)
what are $\sigma$ and $\omega$ here?

- oscillation period is $2\pi/\omega$
- envelope decays exponentially with time constant $-1/\sigma$
• envelope gives $|y|$ when sinusoid term is ±1

• if $\sigma < 0$, envelope decays by $1/e$ in $-1/\sigma$ seconds

• if $\sigma > 0$, envelope doubles every $0.693/\sigma$ seconds

• growth/decay per period is $e^{2\pi(\sigma/\omega)}$

• if $\sigma < 0$, number of cycles to decay to 1% is

$$\left(\frac{4.6}{2\pi}\right)(\omega/|\sigma|) = 0.73(\omega/|\sigma|)$$
The harmonic oscillator

system described by LCCODE

\[ y'' + \omega^2 y = 0 \]

- characteristic polynomial is \( s^2 + \omega^2 \); roots are \( \pm j\omega \)
- solutions are sinusoidal: \( y(t) = A \cos(\omega t + \phi) \), where \( A \) and \( \phi \) come from initial conditions

LC circuit

- from \( i = Cv' \), \( v = -Li' \) we get

\[ v'' + (1/LC)v = 0 \]

- oscillation frequency is \( \omega = 1/\sqrt{LC} \)
mass-spring system

- \( my'' + ky = 0; \)
- oscillation frequency is \( \omega = \sqrt{k/m} \)
Stability of second order system

second order system

\[ ay'' + by' + cy = 0 \]

(recall assumption \( a > 0 \))

we say the system is stable if \( y(t) \to 0 \) as \( t \to \infty \) no matter what the initial conditions are

when is a 2nd order system stable?

• for real distinct roots, solutions have the form \( y(t) = r_1 e^{\lambda_1 t} + r_2 e^{\lambda_2 t} \)

  for stability, we need

  \[ \lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} < 0, \quad \lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} < 0, \]

  we must have \( b > 0 \) and \( 4ac > 0 \), i.e., \( c > 0 \)
• for real equal roots, solutions have the form \( y(t) = r_1e^{\lambda t} + r_2te^{\lambda t} \)

for stability, we need

\[ \lambda = -\frac{b}{2a} < 0 \]

i.e., \( b > 0 \); since \( b^2 = 4ac \), we also have \( c > 0 \)

• for complex roots, solutions have the form \( y(t) = Ae^{\sigma t} \cos(\omega t + \phi) \)

for stability, we need

\[ \sigma = \Re \lambda = -\frac{b}{2a} < 0 \]

i.e., \( b > 0 \); since \( b^2 < 4ac \) we also have \( c > 0 \)

**summary:** second order system with \( a > 0 \) is stable when

\[ b > 0 \text{ and } c > 0 \]
Decay rate

assume system \( ay'' + by' + cy = 0 \) is stable \((a, b, c > 0)\); how fast do the solutions decay?

- **real distinct roots** \((b^2 > 4ac)\)

  since \( \lambda_1 > \lambda_2 \), for \( t \) large,

  \[
  |r_1 e^{\lambda_1 t}| \gg |r_2 e^{\lambda_2 t}|
  \]

  (assuming \( r_1 \) is nonzero); hence asymptotic decay rate is given by

  \[
  -\lambda_1 = \frac{b - \sqrt{b^2 - 4ac}}{2a}
  \]
• real equal roots \((b^2 = 4ac)\)

solution is \(r_1 e^{\lambda t} + r_2 t e^{\lambda t}\) which decays like \(e^{\lambda t}\), so decay rate is

\[-\lambda = b / (2a) = \sqrt{c/a}\]

• complex roots \((b^2 < 4ac)\)

solution is \(A e^{\sigma t} \cos(\omega t + \phi)\), so decay rate is

\[-\sigma = -\Re(\lambda) = b / (2a)\]
Critical damping

**question:** given $a > 0$ and $c > 0$, what value of $b > 0$ gives maximum decay rate?

**answer:**

$$b = 2\sqrt{ac}$$

which corresponds to equal roots, and decay rate $\sqrt{c/a}$

- $b = 2\sqrt{ac}$ is called *critically damped* (real, equal roots)
- $b > 2\sqrt{ac}$ is called *overdamped* (real, distinct roots)
- $b < 2\sqrt{ac}$ is called *underdamped* (complex roots)

**justification:**

- if the system is underdamped, the decay rate is worse than $\sqrt{c/a}$ because
  $$\frac{b}{2a} < \sqrt{\frac{c}{a}},$$
  if $b < 2\sqrt{ac}$
• if the system is overdamped, the decay rate is worse than $\sqrt{c/a}$ because

$$\frac{b - \sqrt{b^2 - 4ac}}{2a} < \sqrt{c/a}$$

to prove this, multiply by $2a$ and re-arrange to get

$$b - 2\sqrt{ac} < \sqrt{b^2 - 4ac}$$

rewrite as

$$b - 2\sqrt{ac} < \sqrt{(b - 2\sqrt{ac})(b + 2\sqrt{ac})}$$

divide by $b - 2\sqrt{ac}$ to get

$$1 < \frac{\sqrt{b + 2\sqrt{ac}}}{\sqrt{b - \sqrt{ac}}}$$

which is true . . .
Parallel RLC circuit

\[ i \]

\[ L \quad R \quad C \]

\[ + \quad v \quad - \]

we have \( v = -Li' \) and \( Cv' = i - v/R \), so

\[ v'' + \frac{1}{RC}v' + \frac{1}{LC}v = 0 \]

- stable (assuming \( L, R, C > 0 \))
- overdamped if \( R < \sqrt{L/(4C)} \)
- critically damped if \( R = \sqrt{L/(4C)} \)
- underdamped if \( R > \sqrt{L/4C} \); oscillation frequency is

\[ \omega = \sqrt{1/LC - (1/2RC)^2} \]
by KVL, $Ri + Li' + v = 0$; also, $i = Cv'$, so

$$v'' + \frac{R}{L}v' + \frac{1}{LC}v = 0$$

- stable (assuming $L, R, C > 0$)
- overdamped if $R > 2\sqrt{L/C}$
- critically damped if $R = 2\sqrt{L/C}$
- underdamped if $R < 2\sqrt{L/C}$; oscillation frequency is

$$\omega = \sqrt{1/LC - (R/2L)^2}$$